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On a characterization of m-subharmonic functions with weak singularities

PER ÅHAG (Umeå) and RAFAŁ CZYŻ (Kraków)

In memory of Professor Józef Siciak

Abstract. We provide a characterization of functions in Cegrell's energy class $\mathcal{E}_{p,m}$.

1. Introduction. In 1998, Cegrell published an article [15] about plurisubharmonic functions with weak singularities. He originally set out to solve the Dirichlet problem for the complex Monge–Ampère operator for plurisubharmonic functions with finite 1-pluricomplex energy, but was forced at the time to make a detour to functions with finite pluricomplex p-energy. For $p > 0$, let $\mathcal{E}_p$ be Cegrell's energy class containing negative plurisubharmonic functions with finite pluricomplex p-energy defined on a bounded hyperconvex domain $\Omega \subset \mathbb{C}^n$, and let

$$\mathcal{M}_p = \{\mu : \mu \text{ is a non-negative Radon measure on } \Omega \text{ such that } (dd^c u)^n = \mu \text{ for some } u \in \mathcal{E}_p\}$$

(see Section 2 for definitions and further details). Cegrell proved in [15], among other things, that the following conditions are equivalent:

(1) $\mu \in \mathcal{M}_p$;
(2) there exists a constant $A \geq 0$ such that
$$\int_\Omega (-u)^p \, d\mu \leq A e_p(u)^{\frac{p}{p+n}} \quad \text{for all } u \in \mathcal{E}_p;$$
(3) $\mathcal{E}_p \subset L^p(\mu)$.

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Here we have used the notation $e_p(u)=\int_\Omega(-u)^p(dd^cu)^n$. Cegrell's energy classes $\mathcal{E}_p$, and their generalizations, are not only a well of inspiration and tools for researchers interested in the model itself [9, 12, 17, 21, 25, 27, 35], but also for applications [3, 10, 20, 23, 24, 36].

Motivated by the study of the dual space of $\mathcal{E}_p$, as explained in more detail in Section 3, we aim to prove the following characterization of Cegrell's class $\mathcal{E}_p$:

(1) $u\in\mathcal{E}_p$;
(2) there exists a constant $B\geq 0$ such that
$$\int_\Omega(-u)^p\,d\mu\leq Be_p(u_\mu)^{\frac{n}{p+n}}\quad\text{ for all }\mu\in\mathcal{M}_p,$$
where $u_\mu\in\mathcal{E}_p$ is the unique solution of the equation $(dd^cu_\mu)^n=\mu$;
(3) $u\in L^p(\mu)$ for all $\mu\in\mathcal{M}_p$.

We shall not prove this theorem within classical pluripotential theory, but in the extended energy classes, $\mathcal{E}_{p,m}$, for the so called m-subharmonic functions. Further details are presented in Section 2.

2. Preliminaries. In this section we give some necessary background on the theory of m-subharmonic functions. For further information we refer to [1, 2, 13, 22, 28, 30]. Let $\Omega\subset\mathbb{C}^n$ be a bounded domain, $1\leq m\leq n$, and define $\mathbb{C}_{(1,1)}$ to be the set of $(1,1)$-forms with constant coefficients. With this notation we define
$$\Gamma_m=\{\alpha\in\mathbb{C}_{(1,1)}:\alpha\wedge\beta^{n-1}\geq 0,\dots,\alpha^m\wedge\beta^{n-m}\geq 0\},$$
where $\beta=dd^c|z|^2$ is the canonical Kähler form in $\mathbb{C}^n$.

Definition 2.1. Assume that $\Omega\subset\mathbb{C}^n$ is a bounded domain, and let u be a subharmonic function defined on Ω. Then we say that u is *m-subharmonic*, $1\leq m\leq n$, if
$$dd^cu\wedge\alpha_1\wedge\dots\wedge\alpha_{m-1}\wedge\beta^{n-m}\geq 0$$
in the sense of currents for all $\alpha_1,\dots,\alpha_{m-1}\in\Gamma_m$.

Remark. Note that m-subharmonic functions are just plurisubharmonic functions if $m=n$, and subharmonic functions if $m=1$.

Let Ω be a bounded domain in $\mathbb{C}^n$. We say that Ω is *m-hyperconvex* if it admits a negative and m-subharmonic *exhaustion function*, i.e. the closure of the set $\{z\in\Omega:\varphi(z)<c\}$ is compact in Ω for every $c\in(-\infty,0)$. An immediate consequence is that n-hyperconvex domains are standard hyperconvex domains in pluripotential theory, and 1-hyperconvex domains are regular domains in potential theory. Throughout this note we assume that Ω is m-hyperconvex. We refer to [7] for further information about the geometry of m-hyperconvex domains.

Let $p > 0$, and let $1 \le m \le n$. We say that a m-subharmonic function φ defined on a bounded m-hyperconvex domains Ω belongs to:

(i) $\mathcal{E}_{0,m}$ if φ is bounded,
$$\lim_{z\to\xi} \varphi(z) = 0 \quad \text{for every } \xi \in \partial\Omega,$$
and
$$\int_\Omega (dd^c\varphi)^m \wedge (dd^c|z|^2)^{n-m} < \infty;$$

(ii) $\mathcal{E}_{p,m}$ if there exists a decreasing sequence, $\{u_j\}$, $u_j \in \mathcal{E}_{0,m}$, that converges pointwise to u on Ω as j tends to ∞, and
$$\sup_j e_{p,m}(u_j) = \sup_j \int_\Omega (-u_j)^p (dd^c u_j)^m \wedge (dd^c|z|^2)^{n-m} < \infty.$$

In [29, 31], it was proved that for $u \in \mathcal{E}_{p,m}$ the complex Hessian operator, $\mathrm{H}_m(u)$, is well-defined, where
$$\mathrm{H}_m(u) = (dd^c u)^m \wedge (dd^c|z|^2)^{n-m}.$$

Theorem 2.2 below is essential when working with $\mathcal{E}_{p,m}$.

THEOREM 2.2. *Let $p > 0$ and $u_0, u_1, \ldots, u_n \in \mathcal{E}_{p,m}$. If $n \ge 2$, then*
$$\int_\Omega (-u_0)^p dd^c u_1 \wedge \cdots \wedge dd^c u_m \wedge (dd^c|z|^2)^{n-m} \le C e_p(u_0)^{\frac{p}{p+m}} e_p(u_1)^{\frac{1}{p+m}} \cdots e_p(u_m)^{\frac{1}{p+m}},$$
where $C \ge 1$ depends only on p, m, n and Ω.

Proof. See e.g. Lu [29, 31] and Nguyen [32]. For the case $m = n$ see [34, Theorem 3.4] (see also [6, 15, 18]). ■

REMARK. If $p \ne 1$, then $C > 1$ (see [4, 5, 19]).

As a consequence of Theorem 2.2 we have the following estimate: for any $u_1, \ldots, u_k \in \mathcal{E}_{p,m}$,
$$e_{p,m}(u_1 + \cdots + u_k)^{\frac{1}{p+m}} \le \sum_{j=1}^{k} C^{j/m} e_{p,m}(u_j)^{\frac{1}{p+m}}, \tag{2.1}$$
where C is the same constant as in Theorem 2.2. We shall use (2.1) in the proof of Theorem 3.1. For $p \ge 0$, the following set of measures will be used in Theorem 2.3 as well as in Theorem 3.1:
$$\mathcal{M}_{p,m} = \{\mu : \mu \text{ is a positive Radon measure on } \Omega \text{ such that } \mathrm{H}_m(u) = \mu \text{ for some } u \in \mathcal{E}_{p,m}\}.$$

Theorem 2.3 is due to Lu [29, 31] and Nguyen [32], and it is a generalization of Cegrell's work from [15].

Theorem 2.3. *Let $p > 0$, $1 \leq m \leq n$, and let Ω be a bounded m-hyperconvex domain in $\mathbb{C}^n$. Then the following conditions are equivalent:*

(1) $\mu \in \mathcal{M}_{p,m}$;
(2) *there exists a constant $A \geq 0$ such that*

$$\int_\Omega (-u)^p\, d\mu \leq A\Big(\int_\Omega (-u)^p (dd^c u)^m \wedge (dd^c|z|^2)^{n-m}\Big)^{\frac{p}{p+m}} \qquad \text{for all } u \in \mathcal{E}_{0,m};$$

(3) *there exists a constant $A \geq 0$ such that*

$$\int_\Omega (-u)^p\, d\mu \leq A\Big(\int_\Omega (-u)^p (dd^c u)^m \wedge (dd^c|z|^2)^{n-m}\Big)^{\frac{p}{p+m}} \qquad \text{for all } u \in \mathcal{E}_{p,m};$$

(4) $\mathcal{E}_{p,m} \subset L^p(\mu)$.

3. Motivation and the main result. Before stating Theorem 3.1, let us give the motivation behind our interest in this result. Let $\mathcal{K}$ be a convex cone of functions. If one want to study $\mathcal{K}$ in a vector space, then a natural candidate is $\mathcal{K} - \mathcal{K} = \delta\mathcal{K}$, the so-called delta-$\mathcal{K}$ functions. The delta-subharmonic functions originate from the 1953 work of Arsove [8], and the delta-plurisubharmonic functions were first considered by Cegrell [14] and Kiselman [26]. We are interested in the vector space

$$\delta\mathcal{E}_{p,m} = \mathcal{E}_{p,m} - \mathcal{E}_{p,m}.$$

For $u \in \delta\mathcal{E}_{p,m}$ we define

$$\|u\|_{p,m} = \inf_{\substack{u_1 - u_2 = u\\ u_1, u_2 \in \mathcal{E}_{p,m}}} \Big(\int_\Omega (-(u_1+u_2))^p (dd^c(u_1+u_2))^m \wedge (dd^c|z|^2)^{n-m}\Big)^{\frac{1}{p+m}}.$$

It was proved in [32] that $(\delta\mathcal{E}_{p,m}, \|\cdot\|_{p,m})$ is a quasi-Banach space, and for $p = 1$ a Banach space. Similarly, Nguyen considered

$$\delta\mathcal{M}_{p,m} = \mathcal{M}_{p,m} - \mathcal{M}_{p,m}.$$

Furthermore, for $\mu \in \delta\mathcal{M}_{p,m}$ let $u^+, u^- \in \mathcal{E}_{p,m}$ be the unique m-subharmonic functions such that

$$\mathrm{H}_m(u^+) = \mu^+ = \tfrac{1}{2}(|\mu| + \mu) \quad \text{and} \quad \mathrm{H}_m(u^-) = \mu^- = \tfrac{1}{2}(|\mu| - \mu).$$

Now we can define

$$|\mu|_{p,m} = \|u^+\|_{p,m}^m + \|u^-\|_{p,m}^m.$$

Then it was proved also in [32] that $(\delta\mathcal{M}_{p,m}, |\cdot|_{p,m})$ is a quasi-Banach space, and for $p = 1$ a Banach space. The main result of [32] was that the space $(\delta\mathcal{M}_{p,m}, |\cdot|_{p,m})$ is dense in the dual space $(\delta\mathcal{E}_{p,m}, \|\cdot\|_{p,m})'$, and that $(\delta\mathcal{E}_{p,m}, \|\cdot\|_{p,m})$ is dense in $(\delta\mathcal{M}_{p,m}, |\cdot|_{p,m})'$. The article [32], due to Nguyen, is a generalization of [5] where the authors proved the result in the classical case $m = n$.

With these duality results at hand together with Theorem 2.3, we arrive at our main result.

THEOREM 3.1. *Let $p > 0$ and $1 \le m \le n$. Let Ω be a bounded m-hyperconvex domain in $\mathbb{C}^n$, and let u be a non-positive m-subharmonic function defined on Ω. The following conditions are then equivalent:*

(1) $u \in \mathcal{E}_{p,m}$;

(2) *there exists a constant $B \ge 0$ such that*
$$\int_\Omega (-u)^p \, d\mu \le B\Big(\int_\Omega (-u_\mu)^p (dd^c u_\mu)^m \wedge (dd^c|z|^2)^{n-m}\Big)^{\frac{m}{p+m}}$$
for all $\mu \in \mathcal{M}_{0,m}$, where $u_\mu \in \mathcal{E}_{0,m}$ is the unique solution of the equation $\mathrm{H}_m(u_\mu) = \mu$;

(3) *there exists a constant $B \ge 0$ such that*
$$\int_\Omega (-u)^p \, d\mu \le B\Big(\int_\Omega (-u_\mu)^p (dd^c u_\mu)^m \wedge (dd^c|z|^2)^{n-m}\Big)^{\frac{m}{p+m}}$$
for all $\mu \in \mathcal{M}_{p,m}$, where $u_\mu \in \mathcal{E}_{p,m}$ is the unique solution of the equation $\mathrm{H}_m(u_\mu) = \mu$;

(4) $u \in L^p(\mu)$ *for all* $\mu \in \mathcal{M}_{p,m}$.

Proof. The implication (3) ⇒ (2) is immediate, and the following implications follow from Theorem 2.3: (1) ⇒ (2), (1) ⇒ (3), and (1) ⇒ (4).

To prove (4) ⇒ (2) assume that there exists no constant B such that (2) holds. Then for each $j \in \mathbb{N}$, we can find $u_j \in \mathcal{E}_{0,m}(\Omega)$ with

$$\int_\Omega (-u)^p (dd^c u_j)^m \wedge (dd^c|z|^2)^{n-m} \ge 2^{mj} C^j \quad \text{and} \quad e_{p,m}(u_j) = 1, \tag{3.1}$$

where C is the constant from Theorem 2.2. Define
$$\psi_k = \sum_{j=1}^k \alpha_j u_j,$$
where $\alpha_j = 2^{-j} C^{-j/m}$, and C is still the constant from Theorem 2.2. Then $\psi_k \in \mathcal{E}_{0,m}$, and by (2.1) we get

$$\begin{aligned} e_{p,m}(\psi_k)^{\frac{1}{p+m}} = e_{p,m}\Big(\sum_{j=1}^k \alpha_j u_j\Big)^{\frac{1}{p+m}} &\le \sum_{j=1}^k C^{j/m} e_{p,m}(\alpha_j u_j)^{\frac{1}{p+m}} \\ &= \sum_{j=1}^k 2^{-j} e_{p,m}(u_j)^{\frac{1}{p+m}} \le 1. \end{aligned} \tag{3.2}$$

From [29, 31] it follows that there exists $w \in \mathcal{E}_{0,m}$ such that $\mathrm{H}_m(w) = dV_{2n}$, where dV_{2n} is the Lebesgue measure on $\mathbb{C}^n$. Thanks to Theorem 2.2, and (3.2), we arrive at the estimate

$$\int_\Omega (-\psi_k)^p\, dV_{2n} = \int_\Omega (-\psi_k)^p (dd^c w)^m \wedge (dd^c|z|^2)^{m-n}$$
$$\leq C e_{p,m}(\psi_k)^{\frac{p}{p+m}} e_{p,m}(w)^{\frac{m}{p+m}} \leq C e_{p,m}(w)^{\frac{m}{p+m}} < \infty.$$

This means that there exists an m-subharmonic function φ such that $\psi_k \searrow \varphi$, and by (3.2) we deduce that $\varphi \in \mathcal{E}_{p,m}$. Let $\mu = \mathrm{H}_m(\varphi) \in \mathcal{M}_{p,m}$. Then by (3.1) we get

$$\int_\Omega (-u)^p\, d\mu \geq \sum_{j=1}^{\infty} \int_\Omega (-u)^p (dd^c \alpha_j u_j)^m \wedge (dd^c|z|^2)^{n-m} \geq \sum_{j=1}^{\infty} \alpha_j^m 2^{mj} C^j = \infty,$$

which is impossible by our assumption (4).

Finally, to prove (2) ⇒ (1) note that by [7, Theorem 5.2] there exists a decreasing sequence $u_j \in \mathcal{E}_{0,m}$ such that $\lim_{j\to\infty} u_j = u$. Let $\mu_j = \mathrm{H}_m(u_j) \in \mathcal{M}_{0,m}$. By assumption (2) we get

$$\int_\Omega (-u_j)^p\, d\mu_j \leq \int_\Omega (-u)^p\, d\mu_j$$
$$\leq B\Big(\int_\Omega (-u_j)^p (dd^c u_j)^m \wedge (dd^c|z|^2)^{n-m}\Big)^{\frac{m}{p+m}}.$$

Hence,

$$\int_\Omega (-u_j)^p (dd^c u_j)^m \wedge (dd^c|z|^2)^{n-m} \leq B^{\frac{p+m}{p}},$$

and therefore $u \in \mathcal{E}_{p,m}$. ■

Remark. We point out that Theorem 3.1 also provides a new characterization of $\mathcal{E}_{p,m}$ in the case $m = n$.

Before finishing this note, let us recall that a function v is in $\mathcal{F}_m$ if it is an m-subharmonic function defined on a bounded m-hyperconvex domain Ω, and there exists a decreasing sequence $\{\varphi_j\}$, $\varphi_j \in \mathcal{E}_{0,m}$, that converges pointwise to v on Ω as j tends to ∞, and

$$\sup_j \int_\Omega (dd^c \varphi_j)^m \wedge (dd^c|z|^2)^{n-m} < \infty.$$

We shall also need the following version of Błocki's inequality [11].

Lemma 3.2. *Let Ω be a bounded m-hyperconvex domain in $\mathbb{C}^n$, and let $u \in \mathcal{E}_{0,m}$ and $v \in \mathcal{F}_m$. Then*

$$\int_\Omega (-v)^m (dd^c u)^m \wedge (dd^c|z|^2)^{n-m} \leq m!\|u\|_\infty^k \int_\Omega (dd^c v)^m \wedge (dd^c|z|^2)^{n-m}.$$

Proof. See [29, 31, 32, 33] for the m-subharmonic case, and [11, 16] for the plurisubharmonic case. ■

Finally, let us present an example that shows that the set $\mathcal{M}_{p,m}$ in Theorem 3.1(4) cannot be replaced by $\mathcal{M}_{0,m}$.

Example 3.3. Let $\Omega = \mathbb{B}$ be the unit ball in $\mathbb{C}^n$, and let $1 \le m \le n$. For $z \in \mathbb{B}$ define

$$v(z) = \begin{cases} \log\|z\| & \text{if } m = n, \\ -\|z\|^{2-2n/m} & \text{if } m < n. \end{cases}$$

It is clear that $v \in \mathcal{F}_m$, and

$$(dd^c v)^m \wedge (dd^c|z|^2)^{n-m} = c(n,m)\delta_0,$$

where $c(n,m)$ is a constant depending only on n and m, and δ_0 is the Dirac delta measure concentrated at the origin. Now take any $u \in \mathcal{E}_{0,m}$. Lemma 3.2 yields

$$\int_\Omega (-v)^m (dd^c u)^m \wedge (dd^c|z|^2)^{n-m} \le m!\|u\|_\infty^m \int_\Omega (dd^c v)^m \wedge (dd^c|z|^2)^{n-m} = c(n,m)m!\|u\|_\infty^m.$$

Thus, we have proved that $v \in L^m(\mu)$ for all $\mu \in \mathcal{M}_{0,m}$ but $v \notin \mathcal{E}_{m,m}$.

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Per Åhag
Department of Mathematics
and Mathematical Statistics
Umeå University
SE-901 87 Umeå, Sweden
E-mail: Per.Ahag@math.umu.se

Rafał Czyż
Institute of Mathematics
Faculty of Mathematics and Computer Science
Jagiellonian University
Łojasiewicza 6
30-348 Kraków, Poland
E-mail: Rafal.Czyz@im.uj.edu.pl