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Topological entropy and IE-tuples of indecomposable continua

by

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Abstract. For a graph G we define a new notion of "free tracing property by free G-chains" on G-like continua and we prove that a positive topological entropy homeomorphism f of a G-like continuum X admits a Cantor set Z in X and an indecomposable subcontinuum H of X satisfying the following conditions:

(1) Z has the free tracing property by free G-chains,
(2) H is the unique minimal subcontinuum of X containing Z and no two points of Z belong to the same composant of H,
(3) any sequence $(z_1, \ldots, z_n)$ of points in Z is an IE-tuple of f, and
(4) f is Li–Yorke chaotic on Z.

1. Introduction. During the last thirty years or so, many interesting connections between dynamical systems and continuum theory have been studied by many authors (see [1, 2, 5–7, 9–15, 17, 19, 22–25, 27, 28]). We are interested in the idea that chaotic topological dynamics should imply the existence of complicated topological structures of the underlying spaces. In many cases, indecomposable continua appear as chaotic attractors of dynamical systems. Also, the composants of indecomposable continua are often strongly related to stable or unstable (connected) sets of the dynamics. For instance, both in continuum theory and in the theory of dynamical systems, the Knaster continuum (= Smale's horseshoe), the pseudo-arc, solenoids and Wada's lakes (= Plykin attractors) are well-known indecomposable continua.

By using ergodic theory, Blanchard, Glasner, Kolyada and Maass [3] proved that if a map $f : X \to X$ of a compact metric space has positive

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topological entropy, then there is an uncountable δ-scrambled subset of X for some $\delta > 0$ and hence the dynamics (X, f) is Li–Yorke chaotic. Huang and Ye [8] studied local entropy theory and gave a characterization of positive topological entropy by use of entropy tuples. Kerr and Li [18] developed local entropy theory and gave a new proof of the theorem of Blanchard, Glasner, Kolyada and Maass. Moreover, they proved that X contains a Cantor set Z which yields more chaotic behaviors (see [18, Theorem 3.18]). Barge and Diamond [1] showed that for piecewise monotone surjections of graphs, the conditions of having positive topological entropy, containing a horseshoe, and the inverse limit space containing an indecomposable subcontinuum are all equivalent. Mouron [24] proved that if X is an arc-like continuum which admits a homeomorphism with positive topological entropy, then X contains an indecomposable subcontinuum. In [6, Theorem 5.5], extending Mouron's theorem, we proved that if X is a G-like continuum for a graph G and X admits a homeomorphism f with positive topological entropy, then X contains an indecomposable subcontinuum. Moreover, if G is a tree, then there are distinct points x and y of X such that (x, y) is an IE-pair of f and the irreducible continuum between x and y in X is an indecomposable subcontinuum.

In this paper, for any graph G we define a new notion of "free tracing property by free G-chains" on G-like continua and we prove that a positive topological entropy homeomorphism f of a G-like continuum X admits a Cantor set Z in X such that any sequence $(z_1, \ldots, z_n)$ of points in Z is an IE-tuple of f and Z has the free tracing property by free G-chains. Our main result is Theorem 3.2 which strengthens [6, Theorem 5.5] and whose proof gives a new proof of the latter. Also, we show that a similar result can be obtained for positive topological entropy "monotone" maps.

2. Definitions and notations. In this paper, we assume that all spaces considered are separable metric spaces and all maps are continuous. Let $\mathbb{N} = \{1, 2, \ldots\}$. A *graph* means a 1-dimensional compact connected polyhedron.

Let X be a compact metric space and $\mathcal{U}, \mathcal{V}$ be two covers of X. Put

$$\mathcal{U} \vee \mathcal{V} = \{U \cap V \mid U \in \mathcal{U},\ V \in \mathcal{V}\}.$$

Let $\mathrm{N}(\mathcal{U})$ denote the minimal cardinality of a subcover of $\mathcal{U}$. Let $f : X \to X$ be a map and let $\mathcal{U}$ be an open cover of X. Put

$$h(f,\mathcal{U}) = \lim_{n\to\infty} \frac{\log \mathrm{N}(\mathcal{U} \vee f^{-1}(\mathcal{U}) \vee \cdots \vee f^{-n+1}(\mathcal{U}))}{n}.$$

The *topological entropy* of f, denoted by $h(f)$, is the supremum of $h(f,\mathcal{U})$ over all open covers $\mathcal{U}$ of X. The reader may refer to [3, 4, 6, 8, 18, 22–25, 27, 28] for important facts concerning topological entropy. Positive topological entropy of a map is one of generally accepted definitions of chaos.

We say that a set $I \subseteq \mathbb{N}$ has *positive density* if

$$\liminf_{n\to\infty} \frac{|I \cap \{1, \ldots, n\}|}{n} > 0.$$

Let X be a compact metric space and $f : X \to X$ a map. Let $\mathcal{A}$ be a collection of subsets of X. We say that a set $I \subset \mathbb{N}$ is an *independence set* for $\mathcal{A}$ [18] if for all finite sets $J \subseteq I$, and all $(Y_j) \in \prod_{j\in J} \mathcal{A}$, we have

$$\bigcap_{j\in J} f^{-j}(Y_j) \neq \emptyset.$$

We now recall the definition of IE-tuple which is related to independence sets in $\mathbb{N}$ and (topological) entropy (see [8] and [18]). We say that a sequence $(x_1, \ldots, x_n)$ of points in X is an *IE-tuple* of f if whenever $A_1, \ldots, A_n$ are open sets containing $x_1, \ldots, x_n$, respectively, the collection $\mathcal{A} = \{A_1, \ldots, A_n\}$ has an independence set with positive density. For $n = 2$, we use the term *IE-pair*. We use IE_k to denote the set of all IE-tuples of length k.

Let $f : X \to X$ be a map of a compact metric space X with metric d and let $\delta > 0$. A subset S of X is a *δ-scrambled set* of f if $|S| \geq 2$ and for any distinct $x, y \in S$,

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0 \quad \text{and} \quad \limsup_{n\to\infty} d(f^n(x), f^n(y)) \geq \delta.$$

We say that $f : X \to X$ is *Li–Yorke chaotic* if there is an uncountable subset S of X such that for any distinct $x, y \in S$,

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0 \quad \text{and} \quad \limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0.$$

A *continuum* is a compact connected metric space. We say that a continuum is *nondegenerate* if it has more than one point. A continuum is *indecomposable* (see [20]) if it is nondegenerate and it is not the union of two proper subcontinua. For any continuum H and $p \in H$, the set $c(p)$ of all points of H which can be joined to p by a proper subcontinuum of H is said to be the *composant* of p (see [20, p. 208]). Note that for an indecomposable continuum H and $p, q \in H$, the following conditions are equivalent:

(1) p, q belong to the same composant of H;
(2) $c(p) \cap c(q) \neq \emptyset$;
(3) $c(p) = c(q)$.

If H is an indecomposable continuum, then the family

$$\{c(p) \mid p \in H\}$$

of all composants of H is a family of uncountable mutually disjoint sets which are connected and dense F_σ-sets in H (see [20, p. 212, Theorem 6]). Note that a (nondegenerate) continuum X is indecomposable if and only if there are three distinct points of X such that any subcontinuum of X containing any two of these points coincides with X, i.e., X is irreducible between any two of the three points.

Let H be an indecomposable continuum. We say that a subset Z of H is *vertically embedded with respect to composants of* H if no two points of Z belong to the same composant of H, i.e., whenever x, y are distinct points of Z and E is any subcontinuum of H containing x and y, then $E = H$.

Let X_i $(i \in \mathbb{N})$ be a sequence of compact metric spaces and let $f_{i,i+1} : X_{i+1} \to X_i$ be a map for each $i \in \mathbb{N}$. The *inverse limit* of the inverse sequence $\{X_i, f_{i,i+1}\}_{i=1}^{\infty}$ is the space

$$\varprojlim\{X_i, f_{i,i+1}\} = \left\{ (x_i)_{i=1}^{\infty} \in \prod_{i=1}^{\infty} X_i \;\middle|\; x_i = f_{i,i+1}(x_{i+1}) \text{ for each } i \in \mathbb{N} \right\}$$

with the topology inherited from the product space $\prod_{i=1}^{\infty} X_i$.

If $f : X \to X$ is a map, then we use $\varprojlim(X, f)$ to denote the inverse limit of X with f as the bonding maps, i.e.,

$$\varprojlim(X, f) = \{(x_i)_{i=1}^{\infty} \in X^{\mathbb{N}} \mid x_i = f(x_{i+1}) \ (i \in \mathbb{N})\}.$$

Let $\sigma_f : \varprojlim(X, f) \to \varprojlim(X, f)$ be the *shift homeomorphism* defined by

$$\sigma_f(x_1, x_2, \ldots) = (x_2, x_3, \ldots).$$

A map g from X onto G is an *ϵ-map* $(\epsilon > 0)$ if $\operatorname{diam} g^{-1}(y) < \epsilon$ for every $y \in G$. Let $\mathcal{P}$ be a collection of graphs. A continuum X is *$\mathcal{P}$-like* if for any $\epsilon > 0$ there exist $G \in \mathcal{P}$ and an ϵ-map from X onto G. A continuum X is *G-like* if $\mathcal{P} = \{G\}$ and X is $\mathcal{P}$-like.

Note the following facts:

(1) A continuum X is 1-dimensional if and only if there is a (countable) collection $\mathcal{P}$ of graphs such that X is $\mathcal{P}$-like, i.e., X is homeomorphic to the inverse limit of an inverse sequence $\{G_i, f_{i,i+1}\}_{i=1}^{\infty}$ such that $G_i \in \mathcal{P}$ $(i \in \mathbb{N})$ and the bonding maps $f_{i,i+1}$ are onto maps.

(2) For any graph G, a continuum X is G-like if and only if X is homeomorphic to the inverse limit of an inverse sequence $\{G_i, f_{i,i+1}\}_{i=1}^{\infty}$ such that $G_i = G$ $(i \in \mathbb{N})$ and the bonding maps $f_{i,i+1}$ are onto maps.

Arc-like continua are those which are G-like for $G = [0, 1]$.

If $\mathcal{U}$ is a collection of subsets of X, then the *nerve* $N(\mathcal{U})$ of $\mathcal{U}$ is the polyhedron whose vertices are elements of $\mathcal{U}$ and there is a simplex $\langle U_1, \ldots, U_k \rangle$ with distinct vertices $U_1, \ldots, U_k$ if

$$\bigcap_i U_i \neq \emptyset.$$

In this paper, we consider only the case where nerves are graphs.

If $\{C_1, \ldots, C_n\}$ is a subcollection of $\mathcal{U}$ we call it a *chain* if $C_i \cap C_{i+1} \neq \emptyset$ for $1 \leq i < n$ and $\overline{C_i} \cap \overline{C_j} \neq \emptyset$ implies $|i-j| \leq 1$. We say that $\{C_1, \ldots, C_n\}$ is a *free chain in* $\mathcal{U}$ if it is a chain and moreover for all $1 < i < n$ the condition $C \in \mathcal{U}$ with $\overline{C} \cap \overline{C_i} \neq \emptyset$ implies that $C = C_i$, $C = C_{i-1}$ or $C = C_{i+1}$. By the *mesh* of a finite collection $\mathcal{U}$ of sets, denoted by $\mathrm{mesh}(\mathcal{U})$, we mean the largest diameter of an element of $\mathcal{U}$. Note that for a graph G, a continuum X is G-like if and only if for any $\epsilon > 0$, there is a finite open cover $\mathcal{U}$ of X such that $N(\mathcal{U})$ is homeomorphic to G and $\mathrm{mesh}(\mathcal{U}) < \epsilon$.

The Knaster continuum (= Smale's horseshoe) and the pseudo-arc are arc-like continua, solenoids are circle-like continua, and Plykin attractors are $(S_1 \vee \cdots \vee S_m)$-like continua, where $S_1 \vee \cdots \vee S_m$ $(m \geq 3)$ denotes the one-point union of m circles S_i. Such spaces are typical indecomposable continua. The reader may refer to [20] and [26] for standard facts concerning continuum theory.

Let X be a continuum and $m \in \mathbb{N}$. Suppose that A_i $(1 \leq i \leq m)$ are m (nonempty) open sets in X and x_i $(1 \leq i \leq m)$ are m distinct points of X. We identify the order $A_1 \to \cdots \to A_m$ and the converse order $A_m \to \cdots \to A_1$, so we consider the equivalence class

$$[A_1 \to \cdots \to A_m] = \{A_1 \to \cdots \to A_m,\ A_m \to \cdots \to A_1\}.$$

Suppose that $\mathcal{U}$ is a finite open cover of X. We say that a chain $\{C_1, \ldots, C_n\} \subseteq \mathcal{U}$ *follows the pattern* $[A_1 \to \cdots \to A_m]$ if there exist

$$1 \leq k_1 < \cdots < k_m \leq n \quad \text{or} \quad 1 \leq k_m < \cdots < k_1 \leq n$$

such that $C_{k_i} \subset A_i$ for each $i = 1, \ldots, m$. In this case, we say more precisely that the chain $[C_{k_1} \to \cdots \to C_{k_m}]$ follows the pattern $[A_1 \to \cdots \to A_m]$. Similarly, we say that a chain $\{C_1, \ldots, C_n\} \subseteq \mathcal{U}$ *follows the pattern* $[x_1 \to \cdots \to x_m]$ if there exist

$$1 \leq k_1 < \cdots < k_m \leq n \quad \text{or} \quad 1 \leq k_m < \cdots < k_1 \leq n$$

such that $x_i \in C_{k_i}$ for each $i = 1, \ldots, m$, where

$$[x_1 \to \cdots \to x_m] = \{x_1 \to \cdots \to x_m,\ x_m \to \cdots \to x_1\}.$$

More precisely, we say that the chain $[C_{k_1} \to \cdots \to C_{k_m}]$ follows the pattern $[x_1 \to \cdots \to x_m]$.

Let G be a graph and X a G-like continuum. We say that a subset Z of X has *the free tracing property by* (*free*) G*-chains* if for any $\epsilon > 0$, any $m \in \mathbb{N}$ and any order $x_1 \to \cdots \to x_m$ of any m distinct points x_i $(i = 1, \ldots, m)$ of Z, there is an open cover $\mathcal{U}$ of X such that $\mathrm{mesh}(\mathcal{U}) < \epsilon$, the nerve $N(\mathcal{U})$ is homeomorphic to G and there is a (free) chain in $\mathcal{U}$ which follows the pattern $[x_1 \to \cdots \to x_m]$.

EXAMPLE 1. (1) Let $X = [0, 1]$ be the unit interval and D a subset of X. If $|D| \geq 3$, then D does not have the free tracing property by $[0, 1]$-chains.

(2) Let $X = S^1$ be the unit circle and D a subset of X. If $|D| \leq 3$, then D has the free tracing property by S^1-chains. If $|D| \geq 4$, then D does not have the free tracing property by S^1-chains.

3. Topological entropy on G-like continua and Cantor sets which have the free tracing property by free G-chains. In [18], by using local entropy theory (IE-tuples), Kerr and Li proved the following general theorem.

THEOREM 3.1 ([18, Theorem 3.18]). *Suppose that $f : X \to X$ is a positive topological entropy map of a compact metric space X, and $x_1, \ldots, x_m$ ($m \geq 2$) are distinct points of X such that $(x_1, \ldots, x_m)$ is an IE-tuple of f. If A_i ($i = 1, \ldots, m$) is any neighborhood of x_i, then there are Cantor sets $Z_i \subset A_i$ such that*

(1) *any sequence $(z_1, \ldots, z_n)$ of points in the Cantor set $Z = \bigcup_i Z_i$ is an IE-tuple of f, and*
(2) *for any $k \in \mathbb{N}$, any distinct $y_1, \ldots, y_k \in Z$ and any $z_1, \ldots, z_k \in Z$,*

$$\liminf_{n\to\infty} \max\{d(f^n(y_i), z_i) \mid 1 \leq i \leq k\} = 0.$$

In particular, Z is a δ-scrambled set of f for some $\delta > 0$.

In this paper, we consider the case when a positive topological entropy map $f : X \to X$ is a homeomorphism and X is a G-like continuum for some graph G. Our main result is a structure theorem for such an f:

MAIN THEOREM 3.2. *In the setting of Theorem 3.1, assume additionally that X is a G-like continuum for a graph G and $f : X \to X$ is a homeomorphism. Then the Cantor sets $Z_i \subset A_i$ ($i = 1, \ldots, m$) can be chosen to satisfy, in addition to (1) and (2), also the following conditions:*

(3) *$Z = \bigcup_{i=1}^m Z_i$ has the free tracing property by free G-chains, and*
(4) *the unique minimal subcontinuum H of X containing Z is indecomposable and Z is vertically embedded with respect to composants of H.*

REMARK 1. (1) Let $g : Z \to Z$ be a homeomorphism of a Cantor set Z which has positive topological entropy. Let $X = \mathrm{Cone}(Z)$ be the cone of Z and let $f : X \to X$ be a homeomorphism which is the natural extension of g. Then $h(f) > 0$ and X is $\mathcal{T}$-like, where $\mathcal{T} = \{T_i \mid i \in \mathbb{N}\}$ and $T_i = \mathrm{Cone}(\{a_1, \ldots, a_i\})$ is the cone of an i-point set $\{a_1, \ldots, a_i\}$. Note that X is not G-like for any graph G, and X contains no indecomposable subcontinua.

(2) In the statement of Theorem 3.2, one cannot relax the hypothesis of positive topological entropy to Li–Yorke chaos. Boroński and Oprocha [5] construct an onto map $f : I \to I$ of the interval $I = [0, 1]$ such that the shift homeomorphism $\sigma_f : \varprojlim(I, f) \to \varprojlim(I, f)$ is Li–Yorke chaotic and the arc-like continuum $\varprojlim(I, f)$ contains no indecomposable subcontinua.

To prove Theorem 3.2, we need the following notations and results. Let $m \geq 2$ and let $\{1,\dots,m\}^n$ be the set of all functions from $\{1,\dots,n\}$ to $\{1,\dots,m\}$. For $\sigma \in \{1,\dots,m\}^n$ $(m \geq 2)$, we write $\sigma = (\sigma(1),\dots,\sigma(n))$, where $\sigma(i) \in \{1,\dots,m\}$. Note that $|\{1,\dots,m\}^n| = m^n$.

PROPOSITION 3.3 (cf. [6, Proposition 3.3]). *Let $m,n \in \mathbb{N}$, and let $\sigma_1,\dots,\sigma_{[(m-1)n+1][(m-1)^n+1]}$ be a sequence of distinct elements of $\{1,\dots,m\}^n$. Then there are $1 \leq i \leq n$ and*

$$1 \leq k_1 < \dots < k_m \leq [(m-1)n+1][(m-1)^n+1]$$

such that $\sigma_{k_j}(i) = j$ for $j = 1,\dots,m$.

Proof. First, we prove the following claim:

(∗) If $B \subset \{1,\dots,m\}^n$ with $|B| = (m-1)^n + 1$, then there is $1 \leq i \leq n$ such that

$$B(i) = \{1,\dots,m\}, \quad \text{where} \quad B(i) = \{\sigma(i) \mid \sigma \in B\}.$$

Suppose, on the contrary, that $|B(j)| \leq m-1$ for each $1 \leq j \leq n$. Then we may consider B to be a subset of $B(1)\times\dots\times B(n)$ of cardinality $\leq (m-1)^n$, a contradiction.

To prove the proposition, we divide the given sequence

$$\sigma_1,\dots,\sigma_{[(m-1)n+1][(m-1)^n+1]}$$

into $(m-1)n+1$ subsequences as follows. Let

$$\begin{aligned} B_1 &= \{\sigma_1,\dots,\sigma_{(m-1)^n+1}\},\\ B_2 &= \{\sigma_{(m-1)^n+2},\dots,\sigma_{2[(m-1)^n+1]}\}, \ \dots,\\ B_{(m-1)n+1} &= \{\sigma_{[(m-1)n][(m-1)^n+1]+1},\dots,\sigma_{[(m-1)n+1][(m-1)^n+1]}\}. \end{aligned}$$

Since $|B_s| = (m-1)^n + 1$ $(s = 1,\dots,(m-1)n+1)$, the above claim (∗) implies that for each B_s, there is $1 \leq i_s \leq n$ with $B_s(i_s) = \{\sigma(i_s) \mid \sigma \in B_s\} = \{1,\dots,m\}$. Define $F : \mathcal{B} = \{B_s \mid 1 \leq s \leq (m-1)n+1\} \to \{1,\dots,n\}$ by $F(B_s) = i_s$. Since $|\mathcal{B}| = (m-1)n+1$, we can find $1 \leq i \leq n$ such that $|F^{-1}(i)| \geq m$. Then we can choose $1 \leq s_1 < \dots < s_m \leq (m-1)n+1$ such that $B_{s_j}(i) = \{1,\dots,m\}$ $(j = 1,\dots,m)$. Further, we can choose $\sigma_{k_j} \in B_{s_j}$ such that $\sigma_{k_j}(i) = j$ for $j = 1,\dots,m$. Consequently,

$$1 \leq k_1 < \dots < k_m \leq [(m-1)n+1][(m-1)^n+1]$$

and $\sigma_{k_j}(i) = j$ for $j = 1,\dots,m$. ∎

PROPOSITION 3.4 ([6, Proposition 3.1]). *Let $I \subseteq \mathbb{N}$ be a set with positive density and $n \in \mathbb{N}$. Then there is a set $F \subseteq I$ with $|F| = n$ and a positive density set B such that $F + B \subseteq I$.*

PROPOSITION 3.5 ([6, Proposition 3.2]). *Let $f : X \to X$ be a map of a compact metric space X and let $\mathcal{A}$ be a collection of subsets of X which* *has an independence set $I \subseteq \mathbb{N}$ with positive density. Suppose that F is a finite set and B is a positive density set such that $F + B \subseteq I$. Then B is an independence set for*

$$\mathcal{A}_F = \Big\{ \bigcap_{i \in F} f^{-i}(Y_i) : Y_i \in \mathcal{A} \Big\}.$$

Let $f : X \to X$ be a map and let I be an independence set with positive density for a collection $\mathcal{A}$ of subsets of X. Note that for any $k \in \mathbb{N}$, the set $(-k + I) \cap \mathbb{N}$ is also an independence set for $\mathcal{A}$ with positive density. Hence we may assume that I satisfies the following additional condition: for any $J \subset I$, all $(Y_j) \in \prod_{j \in J} \mathcal{A}$ and any $Y_0 \in \mathcal{A}$,

$$Y_0 \cap \bigcap_{j \in J} f^{-j}(Y_j) \neq \emptyset. \tag{kl}$$

LEMMA 3.6. *Let $f : X \to X$ be a map of a compact metric space X. Suppose that $(A_1, \ldots, A_k)$ is a tuple of closed subsets of X which has an independent set of positive density. Then there is a tuple $(A'_1, \ldots, A'_k)$ of closed subsets of X which has an independent set with positive density such that $A'_j \subset A_j$ $(j = 1, \ldots, k)$, and if $h : \{1, \ldots, k\} \to \{1, \ldots, k\}$ is any function, then there is $n_h \in \mathbb{N}$ such that $f^{n_h}(A'_j) \subset A_{h(j)}$ for each $j = 1, \ldots, k$.*

Proof. Suppose that $(A_1, \ldots, A_k)$ has an independent set $I \subset \mathbb{N}$ with positive density satisfying (kl). Let $K = \{1, \ldots, k\}^k$. Since $|K| = k^k$ $(=: p)$, we can put $K = \{h_1, \ldots, h_p\}$. By Proposition 3.4, there is a set $F \subset I$ with $|F| = p$ and a positive density set B such that $F + B \subset I$. Let $F = \{i_1, \ldots, i_p\}$. For each $1 \leq j \leq k$, we put

$$A'_j = A_j \cap \bigcap_{i_s \in F} f^{-i_s}(A_{h_s(j)}).$$

Then B is an independence set of positive density for $\mathcal{A}' = \{A'_j \mid 1 \leq j \leq k\}$ (see Proposition 3.5). Also $\mathcal{A}'$ is the desired family, since $f^{i_s}(A'_j) \subset A_{h_s(j)}$ for each $j = 1, \ldots, k$. ∎

We will freely use the following facts from local entropy theory.

PROPOSITION 3.7 ([8] and [18, Propositions 3.8 and 3.9]). *Let X be a compact metric space and let $f : X \to X$ be a map.*

1. *Let $(A_1, \ldots, A_k)$ be a tuple of closed subsets of X which has an independent set of positive density. Then there is an IE-tuple $(x_1, \ldots, x_k)$ with $x_i \in A_i$ for $1 \leq i \leq k$.*
2. *$h(f) > 0$ if and only if f has an IE-pair (x_1, x_2) with $x_1 \neq x_2$.*
3. *IE_k is a closed and $f \times \cdots \times f$-invariant subset of X^k.*

(4) *If* $(A_1, \ldots, A_k)$ *has an independence set with positive density and, for* $1 \le i \le k$, $\mathcal{A}_i$ *is a finite collection of sets such that* $A_i \subseteq \bigcup \mathcal{A}_i$, *then there is* $A'_i \in \mathcal{A}_i$ *such that* $(A'_1, \ldots, A'_k)$ *has an independence set with positive density.*

The following lemma is the key lemma to prove Theorem 3.2.

Lemma 3.8. *Let* G *be a graph and let* $f : X \to X$ *be a homeomorphism of a* G*-like continuum* X *with positive topological entropy. Suppose that* $\mathcal{A}$ *is a finite open collection of subsets of* X *which has an independence set with positive density, any two distinct elements of* $\mathcal{A}$ *are disjoint, and* $|\mathcal{A}| = m \ge 2$. *Then for any* $\epsilon > 0$ *and any order* $A_1 \to \cdots \to A_m$ *of all elements of* $\mathcal{A}$, *there exists a finite open cover* $\mathcal{V}$ *of* X *satisfying the following conditions:*

(1) $\mathrm{mesh}(\mathcal{V}) < \epsilon$,
(2) *the nerve* $N(\mathcal{V})$ *is homeomorphic to* G,
(3) *for each* $A \in \mathcal{A}$ *there is a shrink* $s(A) \in \mathcal{V}$ *with* $s(A) \subset A$ *such that*

$$s(\mathcal{A}) = \{s(A) \mid A \in \mathcal{A}\}$$

has an independence set with positive density, and
(4) *there is a free chain* $[s(A_1) \to \cdots \to s(A_m)]$ *in* $\mathcal{V}$ *which follows the pattern* $[A_1 \to \cdots \to A_m]$.

Proof. We put

$$\mathcal{A} = \{A_1, \ldots, A_m\}.$$

For each $A \in \mathcal{A}$, we can choose an open set $A' \subset \overline{A'} \subset A$ so that

$$\mathcal{A}' = \{A' \mid A \in \mathcal{A}\}$$

has an independence set I with positive density (see Proposition 3.7(1)). We choose $0 < \epsilon' < \epsilon$ so small that $d(\overline{A'}, X - A) > \epsilon'$ for each $A \in \mathcal{A}$. Let $E(G)$ be the set of all edges of the graph G and let $|E(G)|$ be the cardinality of $E(G)$. We can choose $n \in \mathbb{N}$ so large that

$$m^n > |E(G)| \cdot [(m-1)n + 1][(m-1)^n + 1].$$

By Proposition 3.4, there are F with $|F| = n$ and B satisfying the condition of Proposition 3.4. Put $F = \{j_1, \ldots, j_n\}$. Recall

$$\mathcal{A}'_F = \Big\{ \bigcap_{j \in F} f^{-j}(Y_j) \;\Big|\; Y_j \in \mathcal{A}' \Big\} = \Big\{ \bigcap_{i=1}^{n} f^{-j_i}(A_{\sigma(i)}) \;\Big|\; \sigma \in \{1, \ldots, m\}^n \Big\}.$$

Note that $|\mathcal{A}'_F| = m^n$ and B is an independence set for $\mathcal{A}'_F$ with positive density (see Proposition 3.5). Since F is a finite set, we can choose a sufficiently small $\delta > 0$ such that any distinct elements of $\mathcal{A}'_F$ are at least δ apart and if $U \subseteq X$ with $\operatorname{diam} U < \delta$, then $\operatorname{diam} f^i(U) < \epsilon'$ $(i \in F)$. Since X is G-like, we can choose an open cover $\mathcal{U}$ of X such that $N(\mathcal{U})$ is homeomorphic to G and $\mathrm{mesh}(\mathcal{U}) < \delta$. Since δ is so small, we see that each element of $\mathcal{U}$ intersects at most one element of $\mathcal{A}'_F$. By Proposition 3.7(4), we obtain a subcollection $\mathcal{U}'$ of $\mathcal{U}$ such that each element of $\mathcal{A}'_F$ intersects only one element of $\mathcal{U}'$, $|\mathcal{U}'| = m^n$ and the family $\mathcal{U}'$ has an independent set of positive density. Then we can choose a free chain $\mathcal{C}$ in $\mathcal{U}$ that contains at least

$$m^n/|E(G)| \geq [(m-1)n+1][(m-1)^n+1]$$

elements of $\mathcal{U}'$. Put $\mathcal{C} = \{C_1, \ldots, C_p\}$. Note that each $U' \in \mathcal{U}'$ determines the element $\sigma \in \{1, \ldots, m\}^n$. By Proposition 3.3, we can choose $i \in F$ such that there is a sequence

$$1 \leq k_1 < \cdots < k_m \leq [(m-1)n+1][(m-1)^n+1]$$

such that $C_{k_j} \in \mathcal{U}'$ and

$$f^i(C_{k_j}) \cap A'_j \neq \emptyset$$

for each $j = 1, \ldots, m$. By the choice of ϵ', $f^i(C_{k_j}) \subset A_j$ for each $j = 1, \ldots, m$. Then the free chain

$$[f^i(C_{k_1}) \to \cdots \to f^i(C_{k_m})]$$

in $f^i(\mathcal{U})$ follows the pattern $[A_1 \to \cdots \to A_m]$. Put $s(A_j) = f^i(C_{k_j})$ and $\mathcal{V} = f^i(\mathcal{U})$. Then

$$s(\mathcal{A}) = \{s(A) \mid A \in \mathcal{A}\}$$

is the desired family. ■

Finally, we need the following.

PROPOSITION 3.9. *Let X be a G-like continuum for a graph G and let T be a Cantor set in X.*

(1) *Suppose that T has the free tracing property by G-chains. Then any minimal continuum H in X containing T is indecomposable and there is $s \in \mathbb{N}$ such that $|c \cap T| \leq s$ for any composant c of H. Also, there is a subset Z of T such that Z is a Cantor set and Z is vertically embedded with respect to composants of H.*
(2) *Moreover, if T has the free tracing property by free G-chains, then there is a unique minimal continuum H in X containing T, and T itself is vertically embedded with respect to composants of H.*

Proof. (1) For the graph G, we can find a sufficiently large $s \in \mathbb{N}$ such that G does not contain s simple closed curves.

Consider the family $\mathcal{K}$ of all subcontinua of X containing T. By Zorn's lemma, there is a minimal element H of $\mathcal{K}$. We will show that H is indecomposable. Suppose, on the contrary, that there are two proper subcontinua A and B of H such that $H = A \cup B$. Since H is a minimal continuum (= irreducible continuum) containing T, there are $x, y \in T$

with $x \in A - B$ and $y \in B - A$. Since T is perfect, $T \cap (A - B)$ and $T \cap (B - A)$ are infinite sets. Then there are distinct points $a_i \in T \cap (A - B)$ $(i = 0, 1, \ldots, s)$ and $b_i \in T \cap (B - A)$ $(i = 1, \ldots, s)$. Let $\epsilon > 0$ be so small that $d(A, \{b_i \mid i = 1, \ldots, s\}) > \epsilon$. Since T has the free tracing property by G-chains, there is an open cover $\mathcal{U}$ of X such that mesh$(\mathcal{U}) < \epsilon$, $N(\mathcal{U})$ is homeomorphic to G and there is a chain in $\mathcal{U}$ which follows the pattern

$$[a_0 \to b_1 \to a_1 \to b_1 \to a_2 \to b_2 \to \cdots \to a_{s-1} \to b_s \to a_s].$$

Since A is connected, for any $z, z' \in A$ there a chain $\{C_1, \ldots, C_n\}$ in $\mathcal{U}$ from z to z' such that $C_j \cap A \neq \emptyset$. By using these facts, we can easily find s distinct simple closed curves in $N(\mathcal{U})$ which is homeomorphic to G, a contradiction. By similar arguments, we see that $|c \cap T| \leq s$ for any composant c of H. To find the desired Cantor set $Z \subset T$, we consider the following subset of T^2:

$$R = \{(x, y) \in T^2 \mid \text{there is a proper subcontinuum } F \text{ of } H \text{ with } x, y \in F\}.$$

Let $\{U_i \mid i \in \mathbb{N}\}$ be an open base of H and let

$$R_i = \{(x, y) \in T^2 \mid \text{there is a subcontinuum } F \text{ in } H - U_i \text{ with } x, y \in F\}.$$

Note that R_i is a closed set of T^2 and

$$R = \bigcup \{R_i \mid i \in \mathbb{N}\}.$$

Since $|c \cap T| \leq s$ for any composant c of H, we see that R is a nowhere dense F_σ-set in T^2 (see [21, p. 71, Application 2]). By [21, p. 70, Corollary 3], there is a Cantor set Z in T such that Z is independent in R, i.e., if $x, y \in Z$ and $x \neq y$, then $(x, y) \notin R$. Thus we see that Z is vertically embedded with respect to composants of H.

(2) Assume that T has the free tracing property by free G-chains. We will show that there is a unique minimal continuum H in X containing T. Suppose that H_1 and H_2 are minimal continua in X containing T. We must show $H_1 = H_2$. Let $\epsilon > 0$. Since T is a Cantor set, we can choose distinct points $a_i, b_{i,j}$ $(i = 1, 2, 3, 4;\ j = 1, \ldots, n)$, a_5 of T such that for each $i = 1, 2, 3, 4$,

$$d_H(T, \{a_i, b_{i,j} \mid j = 1, \ldots, n\}) < \epsilon,$$

where d_H denotes the Hausdorff metric. Then there is an open cover $\mathcal{U}$ of X such that mesh$(\mathcal{U}) < \epsilon$, $N(\mathcal{U})$ is homeomorphic to G and there is a free chain $[C(a_1) \to C(b_{1,1}) \to \cdots \to C(b_{4,n}) \to C(a_5)]$ in $\mathcal{U}$ which follows the pattern

$$[a_1 \to b_{1,1} \to b_{1,2} \to \cdots \to b_{1,n} \to a_2 \to b_{2,1} \to b_{2,2} \to \cdots \to b_{2,n} \to$$
$$a_3 \to b_{3,1} \to b_{3,2} \to \cdots \to b_{3,n} \to a_4 \to b_{4,1} \to b_{4,2} \to \cdots \to b_{4,n} \to a_5].$$

Since H_1 is a continuum containing a_1, a_3, a_5 and the above chain is free, one of the following cases holds:

Case (i): $[C(a_1) \to C(b_{1,1}) \to \cdots \to C(a_3)]$ is a chain in

$$\mathcal{U}|H_1 = \{U \in \mathcal{U} \mid U \cap H_1 \neq \emptyset\}.$$

Case (ii): $[C(a_3) \to C(b_{3,1}) \to \cdots \to C(a_5)]$ is a chain in $\mathcal{U}|H_1$.

We may assume that Case (i) holds. Since H_2 is a continuum containing a_1, a_2, a_3 and the chain $[C(a_1) \to C(b_{1,1}) \to \cdots C(a_2) \cdots \to C(a_3)]$ is free, one of the following cases holds:

Case (i)$'$: $[C(a_1) \to C(b_{1,1}) \to \cdots \to C(a_2)]$ is a chain in

$$\mathcal{U}|H_2 = \{U \in \mathcal{U} \mid U \cap H_2 \neq \emptyset\}.$$

Case (ii)$'$: $[C(a_2) \to C(b_{2,1}) \to \cdots \to C(a_3)]$ is a chain in $\mathcal{U}|H_2$.

Consequently, there is a subchain $\mathcal{D}$ in

$$[C(a_1) \to C(b_{1,1}) \to \cdots \to C(b_{4,n}) \to C(a_5)]$$

such that each element of $\mathcal{D}$ intersects both H_1 and H_2, and $d_H(T, D \cap T) < \epsilon$, where $D = \bigcup \mathcal{D}$. Hence we have a sequence of chains $\mathcal{D}_i$ ($i \in \mathbb{N}$) of open covers $\mathcal{U}_i$ of X such that $\mathrm{mesh}(\mathcal{U}_i) < 1/i$, each element of $\mathcal{D}_i$ intersects both H_1 and H_2, and $d_H(T, D_i \cap T) < 1/i$, where $D_i = \bigcup \mathcal{D}_i$. Since $\mathcal{D}_i$ are chains and $\mathrm{mesh}(\mathcal{D}_i) \to 0$, we may assume that $\overline{D_i}$ converges to a continuum H. Then $H \subset H_1 \cap H_2$ and $T \subset H$. Since H_1 and H_2 are minimal continua containing T, we have $H = H_1 = H_2$.

Finally, we will show that T is vertically embedded with respect to composants of H. Suppose, on the contrary, that there is a composant c of H containing two distinct points $a_1, a_3 \in T$. We can choose a proper subcontinuum E in c containing a_1, a_3. Let $a_2, a_4 \in T$ be such that a_i ($i = 1, 2, 3, 4$) are distinct and $a_2, a_4 \notin E$. Since T has the free tracing property by free G-chains, there is an open cover $\mathcal{U}$ of X such that $\mathrm{mesh}(\mathcal{U})$ is sufficiently small, $N(\mathcal{U})$ is homeomorphic to G and there is a free chain $[C_1 \to C_2 \to C_3 \to C_4]$ in $\mathcal{U}$ which follows the pattern $[a_1 \to a_2 \to a_3 \to a_4]$. Since E is connected and $\mathrm{mesh}(\mathcal{U})$ is sufficiently small, we can find a chain from C_1 to C_3 in $\mathcal{U}$ which is not a subchain of $[C_1 \to C_2 \to C_3 \to C_4]$. This contradicts $[C_1 \to C_2 \to C_3 \to C_4]$ being a free chain in $\mathcal{U}$. Hence T itself is vertically embedded with respect to composants of H. ∎

Proof of Theorem 3.2. Let $\mathcal{A}_1 = \{A_i \mid i = 1, \ldots, m\}$. We may assume that $\mathcal{A}_1$ has an independence set with positive density and the closures of any two distinct elements of $\mathcal{A}_1$ are disjoint. Also, we may assume $|\mathcal{A}_1| = m$ $(= m_1) \geq 3$ and $\mathrm{mesh}(\mathcal{A}_1) < \epsilon_1 \in (0, 1/2)$. Lemma 3.6 yields a collection

$$\mathcal{A}'_1 = \{A'_1, \ldots, A'_{m_1}\}$$

of open sets which has an independent set with positive density and satisfies the following condition:

(KL) $A'_i \subset A_i$ $(i = 1, \ldots, m_1)$, and if $h : \{1, \ldots, m_1\} \to \{1, \ldots, m_1\}$ is any function, then there is $n_h \in \mathbb{N}$ such that $f^{n_h}(A'_i) \subset A_{h(i)}$ for each $i = 1, \ldots, m_1$.

Consider the set $\operatorname{Ord} \mathcal{A}'_1(m_1)$ of orders (= permutations) of all elements of $\mathcal{A}'_1$. Note that the cardinality of $\operatorname{Ord} \mathcal{A}'_1(m_1)$ is $m_1!$. We consider the set $[\operatorname{Ord} \mathcal{A}'_1(m_1)]$ of equivalence classes of elements of $\operatorname{Ord} \mathcal{A}'_1(m_1)$, i.e.,

$$[\operatorname{Ord} \mathcal{A}'_1(m_1)] = \{[A^i_1 \to A^i_2 \to \cdots \to A^i_{m_1}] \mid i = 1, \ldots, q_1\},$$

where $q_1 = m_1!/2$. By Lemma 3.8, there exists a finite open cover $\mathcal{U}_1$ of X such that $\operatorname{mesh}(\mathcal{U}_1) < \epsilon_1$ and

(1) $N(\mathcal{U}_1)$ is homeomorphic to G,
(2) for each $A \in \mathcal{A}_1$ there is $s_1(A) \in \mathcal{U}_1$ such that $s_1(A) \subset A' \subset A$, the family $\{s_1(A) \mid A \in \mathcal{A}_1\}$ has an independence set with positive density, and we have a free chain
$$[s_1(A^1_1) \to s_1(A^1_2) \to \cdots \to s_1(A^1_{m_1})]$$
in $\mathcal{U}_1$ which follows the pattern
$$[A^1_1 \to A^1_2 \to \cdots \to A^1_{m_1}].$$
This is the case $i = 1$. By induction on $i = 1, \ldots, q_1$, we obtain a sequence $\mathcal{U}_1, \ldots, \mathcal{U}_{q_1}$ of finite open covers of X and $s_i(A) \in \mathcal{U}_i$ $(A \in \mathcal{A}_1, i = 1, \ldots, q_1)$ such that
(3) $N(\mathcal{U}_i)$ is homeomorphic to G,
(4) $\mathcal{U}_{i+1}$ is a refinement of $\mathcal{U}_i$,
(5) for each $A \in \mathcal{A}_1$, $s_i(A) \in \mathcal{U}_i$ $(i = 1, \ldots, q_1)$ satisfies $A \supset A' \supset s_i(A) \supset s_{i+1}(A)$ and for each $i = 1, \ldots, q_1$, the family $\{s_i(A) \mid A \in \mathcal{A}_1\}$ has an independence set with positive density, and there is a free chain
$$[s_i(A^i_1) \to s_i(A^i_2) \to \cdots \to s_i(A^i_{m_1})]$$
in $\mathcal{U}_i$ which follows the pattern
$$[s_{i-1}(A^i_1) \to s_{i-1}(A^i_2) \to \cdots \to s_{i-1}(A^i_{m_1})] \quad (i = 1, \ldots, q_1),$$
where $s_0(A^1_j) = A^1_j$, etc.

Note that the following fact holds:

$(*)_1$ Let $x_A \in s_{q_1}(A)$ be any point for each $A \in \mathcal{A}_1$. For any order $x_1 \to \cdots \to x_{m_1}$ of distinct points of the set $\{x_A \mid A \in \mathcal{A}_1\}$, there is a finite open cover $\mathcal{U}$ of X such that $\mathcal{U} = \mathcal{U}_j$ for some $1 \leq j \leq q_1$ and there is a free chain in $\mathcal{U}$ which follows the pattern $[x_1 \to \cdots \to x_{m_1}]$.

By Proposition 3.5, for each $A \in \mathcal{A}_1$, we can choose nonempty open sets $s_{q_1}(A)^+$ and $s_{q_1}(A)^-$ in $s_{q_1}(A)$ with disjoint closures such that the collection

$$\mathcal{A}_2 = \{s_{q_1}(A)^+, s_{q_1}(A)^- \mid A \in \mathcal{A}_1\}$$

has an independence set with positive density.

Let $|\mathcal{A}_2| = m_2$ $(= 2m_1)$ and $0 < \epsilon_2 \leq \frac{1}{2} \cdot \epsilon_1$. By Lemma 3.6, for $\mathcal{A}_2$ we can choose a collection $\mathcal{A}'_2$ such that $\mathrm{mesh}(\mathcal{A}'_2) < \epsilon_2$ and $\mathcal{A}'_2$ satisfies condition (KL) for $\mathcal{A}_2$ as above. Also, we consider the set $\mathrm{Ord}\,\mathcal{A}'_2(m_2)$ of permutations and the set $[\mathrm{Ord}\,\mathcal{A}'_2(m_2)]$ as above.

By repeated use of Lemma 3.8, we obtain the desired families

$$\{s_i(A) \mid A \in \mathcal{A}_2\} \qquad (i = 1, \ldots, q_2)$$

as above, where $q_2 = m_2!/2$.

Then the following fact holds:

$(*)_2$ Let $x_A \in s_{q_2}(A)$ be any point for each $A \in \mathcal{A}_2$. For any order $x_1 \to \cdots \to x_{m_2}$ of distinct points of $\{x_A \mid A \in \mathcal{A}_2\}$, there is a finite open cover $\mathcal{U}$ of X such that $N(\mathcal{U})$ is homeomorphic to G, $\mathrm{mesh}(\mathcal{U}) < \epsilon_2$ and there is a free chain in $\mathcal{U}$ which follows the pattern $[x_1 \to \cdots \to x_{m_2}]$.

From $\mathcal{A}_2$, we obtain

$$\mathcal{A}_3 = \{s_{q_2}(A)^+, s_{q_2}(A)^- \mid A \in \mathcal{A}_2\}$$

as above. Note that $|\mathcal{A}_3| = m_3$ $(= 2^2 \cdot m_1)$.

If we continue this procedure, we get a sequence ϵ_i $(i \in \mathbb{N})$ of positive numbers and sequences of families $\mathcal{A}_i$ and $\mathcal{A}'_i$ of open subsets of X satisfying the following conditions;

(6) $\epsilon_i > \epsilon_{i+1}$ $(i \in \mathbb{N})$ and $\lim_{i\to\infty} \epsilon_i = 0$,
(7) the closures of any two distinct elements of $\mathcal{A}_i$ are disjoint, each $A \in \mathcal{A}_i$ contains the closures of two elements of $\mathcal{A}_{i+1}$, and $\mathrm{mesh}(\mathcal{A}_i) < \epsilon_i$,
(8) $\mathcal{A}_i$ and $\mathcal{A}'_i$ have independence sets with positive density,
(9) $\mathcal{A}'_i$ satisfies condition (KL) for $\mathcal{A}_i$, and
(10) the fact $(*)_i$ holds,

where

$(*)_i$ Let $x_A \in s_{q_i}(A)$ be any point for each $A \in \mathcal{A}_i$. For any order $x_1 \to \cdots \to x_{m_i}$ of distinct points of $\{x_A \mid A \in \mathcal{A}_i\}$, there is a finite open cover $\mathcal{U}$ of X such that $N(\mathcal{U})$ is homeomorphic to G, $\mathrm{mesh}(\mathcal{U}) < \epsilon_i$ and there is a free chain in $\mathcal{U}$ which follows the pattern $[x_1 \to \cdots \to x_{m_i}]$.

We put

$$T_i = \bigcup \mathcal{A}_i \qquad (i \in \mathbb{N}),$$

$$Z = \bigcap_{i\in\mathbb{N}} T_i,$$

$$Z_i = Z \cap A_i \qquad (A_i \in \mathcal{A}_1),\ i = 1, \ldots, m.$$

Then Z is a Cantor set. We will show that Z has the free tracing property by free G-chains. Let $x_1 \to \cdots \to x_k$ be any order of k distinct points of Z, and $\epsilon > 0$. We choose $i \in \mathbb{N}$ so large that no two points x_j $(j = 1, \dots, k)$ belong to the same element of $\mathcal{A}_i$, and $\epsilon_i < \epsilon$. Note that

$$Z \subset T_{i+1} = \bigcup \mathcal{A}_{i+1} \subset \bigcup_{A \in \mathcal{A}_i} s_{q_i}(A)$$

and no two points x_j belong to the same element of $\{s_{q_i}(A) \mid A \in \mathcal{A}_i\}$. By $(*)_i$, there is a finite open cover $\mathcal{U}$ of X such that $N(\mathcal{U})$ is homeomorphic to G, $\mathrm{mesh}(\mathcal{U}) < \epsilon_i$ and there is a free chain in $\mathcal{U}$ which follows the pattern $[x_1 \to \cdots \to x_k]$. Hence Z has the free tracing property by free G-chains. By Proposition 3.9(2), the unique minimal continuum H in X containing Z is indecomposable and Z is vertically embedded with respect to composants of H. Also, by the construction, Z satisfies conditions (1) and (2) of Theorem 3.2. This completes the proof. ∎

Example 2. Let $f : X \to X$ be a positive topological entropy homeomorphism of X, where X is the Knaster continuum [20], a solenoid [26] or a Plykin attractor [27]. Then the graph G in Theorem 3.2 is an arc, a circle or a bouquet of finitely many circles, respectively, and $H = X$. (The latter follows from the fact that no proper subcontinuum of X is indecomposable.)

Remark 2. Let G be any graph, Y a G-like continuum and $g : Y \to Y$ an onto map with positive topological entropy. With $X = \varprojlim(Y, g)$ and $f = \sigma_g$, one then has $h(\sigma_g) = h(g) > 0$ (see e.g. [28]) and hence the assertions of Theorems 3.1 and 3.2 hold true for the pair (X, f).

An onto map $f : X \to Y$ of continua is *monotone* if $f^{-1}(y)$ is connected for any $y \in Y$. In [16], we proved that if G is any graph and $f : X \to X$ is a monotone map of a G-like continuum X with positive topological entropy, then X contains an indecomposable subcontinuum. Here we give the following more precise result.

Theorem 3.10. *Let X be a G-like continuum, where G is a graph. If $f : X \to X$ is a monotone map with positive topological entropy, then there exists a Cantor set Z in X satisfying conditions* (1) *and* (2) *of Theorem* 3.1 *and vertically embedded with respect to composants of an indecomposable subcontinuum H of X. Moreover, H can be taken to be the unique minimal subcontinuum of X containing Z.*

To prove Theorem 3.10, we need the following notations and lemma. A continuum E is an *n-od* $(2 \le n < \infty)$ if E contains a subcontinuum A such that the complement of A in E is the union n nonempty mutually separated sets, i.e.,

$$E - A = \bigcup \{E_i \mid i = 1, \dots, n\}$$

for some subsets E_i satisfying $\overline{E_i} \cap E_j = \emptyset$ $(i \ne j)$. For any continuum X, let

$$T(X) = \sup\{n \mid \text{there is an } n\text{-od in } X\}.$$

Note that if X is a G-like continuum for a graph G, then $T(X) < \infty$.

Lemma 3.11 (cf. [16, Lemma 2.3]). *Let X and Y be continua with $T(X) < \infty$. Suppose that $f : X \to Y$ is a monotone map, H' is an indecomposable subcontinuum of X, and Z' is a Cantor set which is vertically embedded with respect to composants of H'. If $H = f(H')$ is nondegenerate, then H is an indecomposable subcontinuum of Y and there is a subset Z of $f(Z')$ that is a Cantor set and is vertically embedded with respect to composants of H.*

Proof. As $f : X \to Y$ is monotone, $f^{-1}(C)$ is connected for any subcontinuum C in Y. By using this fact, we can see that $T(Y) \leq T(X) < \infty$. For each $x \in Z'$, let $c(x)$ denote the composant of H' containing $x \in Z'$. Let

$$\mathrm{Comp}(Z'; H') = \{c(x) \mid x \in Z'\}.$$

Since Z' is vertically embedded with respect to composants of H', the family $\mathrm{Comp}(Z'; H')$ consists of mutually disjoint dense connected subsets of H'. For $x, y \in Z'$, we write $x \sim_f y$ if $f(c(x)) \cap f(c(y)) \neq \emptyset$. Also, we define $x \sim y$ if there is a finite sequence $x = x_1, x_2, \ldots, x_s = y$ of $x_i \in Z'$ such that $x_i \sim_f x_{i+1}$ for each $i = 1, \ldots, s-1$. Then $\sim$ is an equivalence relation on Z'. Note that $f(c(x)) \cap f(c(y)) \neq \emptyset$ if and only if there is $z \in Y$ with

$$f^{-1}(z) \cap c(x) \neq \emptyset \neq f^{-1}(z) \cap c(y).$$

Let $[x]$ denote the $\sim$-equivalence class containing $x \in Z'$. Since $f^{-1}(z)$ is a subcontinuum of X for each $z \in Y$, we can conclude that $|[x]| \leq T(X)$. In particular, $f|Z' : Z' \to f(Z')$ is a finite-to-one map and hence $f(Z')$ is a perfect set, i.e., $f(Z')$ has no isolated point.

Since Z' is uncountable, we can choose an uncountable subset Z'' of Z' such that $\{f(c(x)) \mid x \in Z''\}$ is a family of mutually disjoint subsets of $H = f(H')$.

We will prove that $H = f(H')$ is indecomposable. Suppose, on the contrary, that H is decomposable. There is a proper subcontinuum A of $H = f(H')$ with

$$\mathrm{Int}_H(A) \neq \emptyset.$$

Since each composant of H' is dense in H' and hence $f(c(x))$ is dense in H for any $x \in Z''$, we have $f(c(x)) \cap A \neq \emptyset$. This implies $|T(Y)| = \infty$, a contradiction. Hence $H = f(H')$ is indecomposable.

We show that $|c \cap f(Z')| \leq T(X)$ for each composant c of H. Suppose, on the contrary, that there is a proper subcontinuum C of H such that $|C \cap f(Z')| \geq T(X) + 1$. Then $f^{-1}(C)$ is a continuum which intersects $T(X) + 1$ composants of H', a contradiction.

Since $f(Z')$ is perfect, by the proof of Proposition 3.9 we can find a Cantor set Z in $f(Z')$ such that Z is vertically embedded with respect to composants of H. ∎

Proof of Theorem 3.10. We consider the inverse $\tilde{f} : \varprojlim(X, f) \to \varprojlim(X, f)$ of the shift map σ_f, i.e.,

$$\tilde{f}(x_1, x_2, x_3, \ldots) = (f(x_1), x_1, x_2, \ldots).$$

Note that $h(f) = h(\tilde{f}) > 0$ [28]. By Theorem 3.2, we can find an indecomposable subcontinuum H' and a Cantor set Z' for $\tilde{f}$ as in Theorem 3.2. Since f is monotone, so is the projection $p_n : \varprojlim(X, f) \to X_n = X$ to the nth coordinate. For sufficiently large n, $H = p_n(H')$ is nondegenerate. By the above lemma, H is indecomposable and there is a Cantor set $Z \subset p_n(Z')$ such that Z is vertically embedded with respect to composants of H. Note that the projection p_n preserves the properties of IE-tuples and (2) of Theorem 3.1.

Finally, we show that H is a unique minimal subcontinuum of X containing Z. Suppose that H_i $(i = 1, 2)$ are any minimal subcontinua of X containing Z. Take a Cantor set Z'' in Z' such that

$$Z'' \subset Z' \cap p_n^{-1}(Z) \subset p_n^{-1}(H_1) \cap p_n^{-1}(H_2).$$

Since $Z'' \subset Z'$, Z'' also has the free tracing property by free G-chains. By Proposition 3.9(2), there is a unique minimal subcontinuum H'' of X containing Z''. Note that the $p_n^{-1}(H_i)$ are continua because p_n is monotone. Then $H'' \subset p_n^{-1}(H_1) \cap p_n^{-1}(H_2)$. Also, since $Z'' \subset Z'$, we have $H'' \subset H'$. Thus $H'' = H'$ because no two points of Z'' belong to the same composant of H'. Then

$$H = p_n(H') = p_n(H'') \subset H_1 \cap H_2$$

and hence $H_1 = H_2 = H$. Consequently, Z and H are as desired. ∎

Note. In Theorem 3.10, we do not know whether Z may be taken to have the free tracing property by free G-chains.

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