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Closedness of convex sets in Orlicz spaces with applications to dual representation of risk measures

by

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Abstract. We study various types of closedness of convex sets in an Orlicz space L^Φ and its heart H^Φ and their relations to a natural version of the Krein–Šmulian property. Let L^Ψ be the conjugate Orlicz space and H^Ψ be the heart of L^Ψ. Precisely, we show that the following statements are equivalent:

(i) Every order closed convex set in L^Φ is $\sigma(L^\Phi, L^\Psi)$-closed.
(ii) Every boundedly a.s. closed convex set in H^Φ is $\sigma(H^\Phi, H^\Psi)$-closed.
(iii) Every $\sigma(L^\Phi, L^\Psi)$-sequentially closed convex set in L^Φ is $\sigma(L^\Phi, L^\Psi)$-closed.
(iv) Every $\sigma(H^\Phi, H^\Psi)$-sequentially closed convex set in H^Φ is $\sigma(H^\Phi, H^\Psi)$-closed.
(v) $\sigma(L^\Phi, L^\Psi)$ (respectively, $\sigma(H^\Phi, H^\Psi)$) has the Krein–Šmulian property.
(vi) Either Φ or its conjugate Ψ satisfies the Δ_2-condition.

The implication (i)⇒(vi) solves an open question raised by Owari (2014) and has applications in the dual representation theory of risk measures.

1. Introduction and notation. Let (Φ, Ψ) be a conjugate pair of Orlicz functions. Let L^Φ be the Orlicz space over a fixed nonatomic probability space, and L^Ψ be the conjugate space. Recall that L^Ψ is the *order continuous dual* of L^Φ, i.e., it consists of all random variables Y such that $\mathbb{E}[X_nY] \to \mathbb{E}[XY]$ whenever (X_n) is dominated in L^Φ and $X_n \xrightarrow{\text{a.s.}} X$. A set $\mathcal{C} \subset L^\Phi$ is said to be *order closed* in L^Φ if it contains the limits of all dominated a.s.-convergent sequences with elements in $\mathcal{C}$. A well-known problem arising from the theory of risk measures on Orlicz spaces (see, e.g., [15, p. 3585]) is whether

$(*)$ Every order closed convex set $\mathcal{C}$ in L^Φ is $\sigma(L^\Phi, L^\Psi)$-closed.

If Φ satisfies the Δ_2-condition, then $(L^\Phi)^* = L^\Psi$. Thus, since order closed

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sets are automatically norm closed and therefore weakly closed, $(*)$ holds. In [7], it is proved that $(*)$ also holds if Ψ satisfies the Δ_2-condition.

In Section 2, we give a complete answer to the above problem by showing that if $(*)$ holds then either Φ or Ψ satisfies the Δ_2-condition. We begin by showing that order closedness and $\sigma(L^\Phi, L^\Psi)$-closedness coincide for all norm bounded convex sets in *any* L^Φ (Theorem 2.2). As a corollary, a convex set $\mathcal{C}$ in L^Φ is order closed if and only if it is $\sigma(L^\Phi, L^\Psi)$-sequentially closed, if and only if $\mathcal{C}\cap k\mathcal{B}$ is $\sigma(L^\Phi, L^\Psi)$-closed for all $k \geq 1$, where $\mathcal{B}$ is the closed unit ball in L^Φ. The last condition naturally introduces the following Krein–Šmulian property. A locally convex topology τ on a Banach space $\mathcal{X}$ is said to have the *Krein–Šmulian property* if every convex set $\mathcal{C}$ is τ-closed whenever $\mathcal{C}\cap k\mathcal{B}$ is τ-closed for all $k \geq 1$, where $\mathcal{B}$ is the closed unit ball of $\mathcal{X}$. Recall that the well-known *Krein–Šmulian Theorem* states that the weak-* topology has the Krein–Šmulian property. In Theorem 2.5, we show that $\sigma(L^\Phi, L^\Psi)$ has the Krein–Šmulian property if and only if $(*)$ holds, if and only if either Φ or Ψ satisfies the Δ_2-condition.

In Section 3, we investigate closedness of convex sets in H^Φ and the Krein–Šmulian property of $\sigma(H^\Phi, H^\Psi)$. In this dual pair, H^Ψ consists of all random variables Y such that $\mathbb{E}[X_nY] \to \mathbb{E}[XY]$ whenever (X_n) is norm bounded in H^Φ and $X_n \xrightarrow{\text{a.s.}} X$. That is, H^Ψ is the *uo-dual* of H^Φ; the notion of uo-dual has been introduced and studied in Banach lattices in [11]. The last condition motivates the following counterpart of order closedness in H^Φ. A set $\mathcal{C} \subset H^\Phi$ is said to be *boundedly a.s. closed* in H^Φ if it contains the limits of all norm bounded a.s.-convergent sequences with elements in $\mathcal{C}$. In view of the results of Section 2, we begin the analysis by asking whether *every boundedly a.s. closed convex set in H^Φ is $\sigma(H^\Phi, H^\Psi)$-closed.* In Theorem 3.4, we show that this statement is equivalent to $(*)$ and is valid if and only if $\sigma(H^\Phi, H^\Psi)$ has the Krein–Šmulian property. It deserves mentioning that for the Banach lattice pairs $(\mathcal{X}, \mathcal{Y})$ considered in Sections 2 and 3, we know that $\sigma(\mathcal{X}, \mathcal{Y})$ has the Krein–Šmulian property only if $\mathcal{Y} = \mathcal{X}^*$ or $\mathcal{X} = \mathcal{Y}^*$. It remains an open question to us whether this property is true for any Banach lattice pair, or under what conditions it holds true.

In Section 4, we present an application to the theory of risk measures. In particular, the technical Lemmas 2.4 and 3.3 enable us to construct the first known examples of coherent risk measures with the (respectively, strong) Fatou property on L^Φ (respectively, H^Φ) that do not admit a Fenchel–Moreau dual representation via L^Ψ (respectively, H^Ψ). These results complement known representation results of risk measures in Orlicz spaces and Orlicz hearts, as so far only positive results have been obtained.

We adopt [1] and [8, 16] as standard references for unexplained terminology and facts on Banach lattices and Orlicz spaces, respectively. Recall that a function $\Phi : [0,\infty) \to [0,\infty)$ is called an *Orlicz function* if it is convex, increasing, and $\Phi(0) = 0$. Define the *conjugate function* of Φ by

$$\Psi(s) = \sup\{ts - \Phi(t) : t \geq 0\}$$

for all $s \geq 0$. If $\lim_{t\to\infty} \Phi(t)/t = \infty$ (equivalently, if Ψ is finite-valued), then Ψ is also an Orlicz function, and its conjugate is Φ. Throughout this paper, (Φ, Ψ) stands for an Orlicz pair such that $\lim_{t\to\infty} \Phi(t)/t = \infty$ and $\Phi(t) > 0$ for any $t > 0$. Since we will only work on nonatomic probability spaces, the restrictions on Φ are minor as they only eliminate the case where L^Φ coincides with L^1 or L^∞, in which cases our main results are either trivial or known.

Throughout this paper, $(\Omega, \Sigma, \mathbb{P})$ stands for a nonatomic probability space. The *Orlicz space* $L^\Phi := L^\Phi(\Omega, \Sigma, \mathbb{P})$ is the space of all real-valued random variables X (modulo a.s. equality) such that

$$\|X\|_\Phi := \inf\{\lambda > 0 : \mathbb{E}[\Phi(|X|/\lambda)] \leq 1\} < \infty.$$

The norm $\|\cdot\|_\Phi$ on L^Φ is called the *Luxemburg norm*. The subspace of L^Φ consisting of all $X \in L^\Phi$ such that

$$\mathbb{E}[\Phi(|X|/\lambda)] < \infty \quad \text{for all } \lambda > 0$$

is conventionally called the *Orlicz heart*, or the subspace of finite elements, of L^Φ and is denoted by H^Φ. It is well known that $L^\infty \subset H^\Phi \subset L^\Phi \subset L^1$ and that H^Φ is a norm closed subspace of L^Φ. We always endow the conjugate Orlicz space L^Ψ and the conjugate Orlicz heart H^Ψ with the *Orlicz norm*

$$\|Y\|_\Psi := \sup_{X\in L^\Phi,\, \|X\|_\Phi\leq 1} |\mathbb{E}[XY]|$$

for all $Y \in L^\Psi$, which is equivalent to the Luxemburg norm on L^Ψ.

An Orlicz function Φ satisfies the Δ_2-*condition* if there exist $t_0 \in (0,\infty)$ and $k \in \mathbb{R}$ such that $\Phi(2t) < k\Phi(t)$ for all $t \geq t_0$. It is well known that $L^\Psi = (H^\Phi)^*$ and that L^Ψ, being the order continuous dual of L^Φ, is a lattice ideal in $(L^\Phi)^*$. Moreover, the following conditions are equivalent:

(1) $L^\Psi = (L^\Phi)^*$.
(2) $L^\Phi = H^\Phi$.
(3) The Orlicz function Φ satisfies the Δ_2-condition.
(4) L^Φ *has order continuous norm*, i.e.,

$$X_\alpha \downarrow,\ \inf_\alpha X_\alpha = 0 \text{ in } L^\Phi \implies \inf_\alpha \|X_\alpha\|_\Phi = 0.$$

A sequence (X_n) in L^Φ is *order bounded* (or dominated) if there exists $X \in L^\Phi$ such that $|X_n| \leq X$ a.s. for all n. A sequence (X_n) in L^Φ *order converges* to $X \in L^\Phi$, written $X_n \xrightarrow{o} X$ in L^Φ, if $X_n \xrightarrow{\text{a.s.}} X$ and (X_n) is order bounded in L^Φ. If L^Φ has order continuous norm, then

$$X_n \xrightarrow{o} X \text{ in } L^\Phi \implies \|X_n - X\|_\Phi \to 0.$$

2. Closed convex sets in L^Φ. For a subset $\mathcal{C}$ of L^Φ, define its *order closure* in L^Φ to be the set

$$\overline{\mathcal{C}}^o := \{X \in L^\Phi : X_n \xrightarrow{o} X \text{ for some sequence } (X_n) \text{ in } \mathcal{C}\}.$$

Note that in this definition we can equivalently use nets instead of sequences. Indeed, since L^Φ has the countable sup property, if $X_\alpha \xrightarrow{o} X$ in L^Φ then there exist countably many (α_n) such that $X_{\alpha_n} \xrightarrow{o} X$ in L^Φ. We say that $\mathcal{C}$ is *order closed* in L^Φ if $\mathcal{C} = \overline{\mathcal{C}}^o$. In spite of the terminology, $\overline{\mathcal{C}}^o$ *is not necessarily order closed.* By the Dominated Convergence Theorem,

$$X_n \xrightarrow{o} X \text{ in } L^\Phi \implies \mathbb{E}[X_nY] \to \mathbb{E}[XY] \text{ for any } Y \in L^\Psi.$$

Thus

$$\overline{\mathcal{C}}^o \subset \overline{\mathcal{C}}^{\sigma_s(L^\Phi,L^\Psi)} \subset \overline{\mathcal{C}}^{\sigma(L^\Phi,L^\Psi)}$$

for any set $\mathcal{C} \subset L^\Phi$, where $\overline{\mathcal{C}}^{\sigma_s(L^\Phi,L^\Psi)}$ denotes the $\sigma(L^\Phi, L^\Psi)$-*sequential* closure of $\mathcal{C}$. In particular, every $\sigma(L^\Phi, L^\Psi)$-closed set is order closed. The following result shows that order closures and $\sigma(L^\Phi, L^\Psi)$-closures are in general different.

Proposition 2.1. $\overline{\mathcal{C}}^o = \overline{\mathcal{C}}^{\sigma(L^\Phi,L^\Psi)}$ *for every convex set $\mathcal{C}$ in L^Φ if and only if Φ satisfies the Δ_2-condition.*

Proof. If Φ satisfies the Δ_2-condition, then $L^\Phi = H^\Phi$ and thus $\sigma(L^\Phi, L^\Psi)$ is the weak topology on L^Φ. By Mazur's Theorem, $\overline{\mathcal{C}}^{\sigma(L^\Phi,L^\Psi)} = \overline{\mathcal{C}}^{\|\cdot\|}$. Since every norm convergent sequence admits a subsequence that order converges to the same limit (cf. [8, Theorem 2.1.10(6)]), $\overline{\mathcal{C}}^{\sigma(L^\Phi,L^\Psi)} \subset \overline{\mathcal{C}}^o$. Therefore, $\overline{\mathcal{C}}^o = \overline{\mathcal{C}}^{\sigma(L^\Phi,L^\Psi)}$.

Conversely, suppose that Φ fails the Δ_2-condition. By [16, p. 139, Theorem 5] (or [1, Theorem 4.51]), there exist a sequence (X_n) of pairwise disjoint positive random variables in L^Φ and a closed sublattice $\mathcal{X}$ of L^Φ such that the map $\mathcal{T} : \ell^\infty \to \mathcal{X}$ defined by $\mathcal{T}\big((a_n)_n\big) = \sum_n a_nX_n$ (pointwise sum) is a Banach lattice isomorphism. Denote by e the identically 1 sequence in ℓ^∞. If $Y \in L^\Psi$, then

$$\sum_n \mathbb{E}[X_nY] \le \sum_n \mathbb{E}[X_n|Y|] = \mathbb{E}[\mathcal{T}e|Y|] < \infty.$$

Hence $(\mathbb{E}[X_nY]) \in \ell^1$. Thus, if $w = (a_n) \in \ell^\infty$, then

$$\mathbb{E}[(\mathcal{T}w)Y] = \mathbb{E}\Big[\Big(\sum a_nX_n\Big)Y\Big] = \sum a_n\mathbb{E}[X_nY] = \langle(\mathbb{E}[X_nY]), w\rangle. \tag{1}$$

By Ostrovskiĭ's Theorem (cf. [13, Theorem 2.34]), there exist a subspace $\mathcal{W}$ of ℓ^∞ and $w \in \overline{\mathcal{W}}^{\sigma(\ell^\infty,\ell^1)}$ such that w is not the $\sigma(\ell^\infty, \ell^1)$-limit of any sequence in $\mathcal{W}$. Let $\mathcal{C} = \mathcal{T}(\mathcal{W})$. Obviously, $\mathcal{C}$ is a convex set in L^Φ. Take a net $(w_\alpha) \subset \mathcal{W}$ that $\sigma(\ell^\infty, \ell^1)$-converges to $w \in \ell^\infty$. Let $Y \in L^\Psi$. By (1),

$$\mathbb{E}[(\mathcal{T}w_\alpha)Y] = \langle (\mathbb{E}[X_nY]), w_\alpha\rangle \to \langle (\mathbb{E}[X_nY]), w\rangle = \mathbb{E}[(\mathcal{T}w)Y].$$

Thus $(\mathcal{T}w_\alpha)$ $\sigma(L^\Phi, L^\Psi)$-converges to $\mathcal{T}w$ and $\mathcal{T}w \in \overline{\mathcal{C}}^{\sigma(L^\Phi,L^\Psi)}$. Suppose, if possible, that $\mathcal{T}w \in \overline{\mathcal{C}}^o$. Take a sequence (w_k) in $\mathcal{W}$ such that $\mathcal{T}w_k \xrightarrow{o} \mathcal{T}w$ in L^Φ. Clearly, $(\mathcal{T}w_k)$, being order bounded, is norm bounded in L^Φ, so that (w_k) is norm bounded in ℓ^∞. Write $w_k = (a_n^k)_n$ and $w = (a_n)_n$. Since the X_n's are disjoint and $\mathcal{T}w_k = \sum_n a_n^k X_n \xrightarrow{\text{a.s.}} Tw = \sum_n a_n X_n$, $\lim_k a_n^k = a_n$ for each n. It follows that (w_k) $\sigma(\ell^\infty, \ell^1)$-converges to w, contrary to the choice of w. Hence, $\overline{\mathcal{C}}^o \neq \overline{\mathcal{C}}^{\sigma(L^\Phi,L^\Psi)}$. ∎

However, the equality $\overline{\mathcal{C}}^o = \overline{\mathcal{C}}^{\sigma(L^\Phi,L^\Psi)}$ does hold in general for *norm bounded* convex sets $\mathcal{C} \subset L^\Phi$.

THEOREM 2.2. *Let $\mathcal{C}$ be any norm bounded convex set in L^Φ. Then $\overline{\mathcal{C}}^o = \overline{\mathcal{C}}^{\sigma(L^\Phi,L^\Psi)}$. In particular, $\overline{\mathcal{C}}^o$ is order closed, and $\mathcal{C}$ is order closed if and only if it is $\sigma(L^\Phi, L^\Psi)$-closed.*

Proof. As has been observed, the inclusion $\overline{\mathcal{C}}^o \subset \overline{\mathcal{C}}^{\sigma(L^\Phi,L^\Psi)}$ always holds. To prove the reverse inclusion, it suffices to show that if $\mathcal{C}$ is a convex subset of the unit ball of L^Φ such that $0 \in \overline{\mathcal{C}}^{\sigma(L^\Phi,L^\Psi)}$, then $0 \in \overline{\mathcal{C}}^o$. We divide the proof into three steps.

STEP I. *For each $n \geq 1$ and each $0 \leq Y \in L^\Psi$, there exists a pair of disjoint random variables $Z_{Y,n}$ and $W_{Y,n}$ satisfying*

(1) $Z_{Y,n} \in L^\Phi$ *and* $W_{Y,n} \in H^\Phi$,
(2) $Z_{Y,n} + W_{Y,n} \in \mathcal{C}$,
(3) $\mathbb{E}[\Phi(|Z_{Y,n}|)] \leq 1/2^n$,
(4) $\mathbb{E}[|W_{Y,n}|Y] \leq 1$.

Since L^Ψ is a lattice ideal in $(L^\Phi)^*$, by [1, Theorem 3.50] the topological dual of L^Φ under $|\sigma|(L^\Phi, L^\Psi)$ is precisely L^Ψ. Thus, by Mazur's Theorem,

$$0 \in \overline{\mathcal{C}}^{\sigma(L^\Phi,L^\Psi)} = \overline{\mathcal{C}}^{|\sigma|(L^\Phi,L^\Psi)},$$

so that there exists $X \in \mathcal{C}$ such that $\mathbb{E}[|X|Y] \leq 1$. Since $\mathcal{C}$ is in the unit ball, we have $\mathbb{E}[\Phi(|X|)] \leq 1$. Now take $k \geq 1$ such that

$$\mathbb{E}[\mathbb{1}_{\{|X|>k\}}\Phi(|X|)] \leq 1/2^n.$$

Set

$$Z_{Y,n} = X\mathbb{1}_{\{|X|>k\}} \quad \text{and} \quad W_{Y,n} = X\mathbb{1}_{\{|X|\leq k\}}.$$

Clearly, $Z_{Y,n}$ and $W_{Y,n}$ are disjoint. Conditions (1)–(3) are easily verified. Condition (4) holds because

$$\mathbb{E}[|W_{Y,n}|Y] \leq \mathbb{E}[|X|Y] \leq 1.$$

Step II. *There exist sequences* (Z_n) *and* (W_n) *in* L^Φ *such that for each* $n \geq 1$,

(1) $X_n := Z_n + W_n \in \mathcal{C}$,
(2) $\mathbb{E}[\Phi(|Z_n|)] \leq 1/2^n$,
(3) $\|W_n\|_\Phi \leq 1/2^n$.

Keep the notation of Step I. For any $n \geq 1$, define $\mathcal{A}_n = \{W_{Y,n} : 0 \leq Y \in L^\Psi\} \subset H^\Phi$. Let $Y_1, \dots, Y_k \in L^\Psi$ and let $\varepsilon > 0$ be given. Set $Y = \varepsilon^{-1} \sum_{i=1}^k |Y_i| \in (L^\Psi)_+$. By Step I,

$$|\mathbb{E}[W_{Y,n} Y_i]| \leq \mathbb{E}[|W_{Y,n} Y_i|] \leq \varepsilon \mathbb{E}[|W_{Y,n}| Y] \leq \varepsilon.$$

This shows that 0 lies in the $\sigma(H^\Phi, L^\Psi)$-closed convex hull of $\mathcal{A}_n$. Since $\sigma(H^\Phi, L^\Psi)$ is the weak topology on H^Φ, 0 lies in the norm closed convex hull of $\mathcal{A}_n$.

Now take $W_{Y_i,n} \in \mathcal{A}_n$, $1 \leq i \leq k$, and a convex combination $W_n = \sum_{i=1}^k c_i W_{Y_i,n}$ such that $\|W_n\|_\Phi \leq 1/2^n$. Put $Z_n = \sum_{i=1}^k c_i Z_{Y_i,n}$. Then

$$X_n := Z_n + W_n = \sum_{i=1}^k c_i (Z_{Y_i,n} + W_{Y_i,n}) \in \mathcal{C}$$

by convexity of $\mathcal{C}$. Moreover, since Φ is a convex function,

$$\mathbb{E}[\Phi(|Z_n|)] \leq \sum_{i=1}^k c_i \mathbb{E}[\Phi(|Z_{Y_i,n}|)] \leq 1/2^n.$$

Step III. *In the notation of Step II, a subsequence of* (X_n) *order converges to* 0. *Thus* $0 \in \overline{\mathcal{C}}^o$.

From Step II, we know that $\|W_n\|_\Phi \leq 1/2^n$ for all n and hence $\sum_{n=1}^\infty |W_n| \in L^\Phi$. Also, since Φ is continuous and increasing,

$$\mathbb{E}\Big[\Phi\Big(\sup_n |Z_n|\Big)\Big] = \mathbb{E}\Big[\sup_n \Phi(|Z_n|)\Big] \leq \sum_{n=1}^\infty \mathbb{E}[\Phi(|Z_n|)] \leq \sum \frac{1}{2^n} = 1,$$

from which it follows that $\sup_n |Z_n| \in L^\Phi$. Therefore,

$$\widetilde{X} := \sup_n |Z_n| + \sum_{n=1}^\infty |W_n| \in L^\Phi.$$

Obviously, $|X_n| \leq \widetilde{X}$ for all $n \geq 1$. Thus (X_n) is an order bounded sequence in L^Φ. By Markov's Inequality,

$$\Phi(\varepsilon) \mathbb{P}\{|Z_n| > \varepsilon\} \leq \mathbb{E}[\Phi(|Z_n|)] \leq 1/2^n$$

for any $\varepsilon > 0$. It follows that (Z_n) converges to 0 in probability. Since $\sum_{n=1}^\infty |W_n| \in L^\Phi$, (W_n) converges to 0 a.s. Therefore, a subsequence of (X_n)

converges to 0 a.s., and thus in order, since the whole sequence (X_n) is order bounded. ∎

Theorem 2.2 allows us to characterize general order closed convex sets in L^{Φ} in terms of the topology $\sigma(L^{\Phi}, L^{\Psi})$.

COROLLARY 2.3. *Denote by $\mathcal{B}$ the closed unit ball in L^{Φ}. For a convex set $\mathcal{C}$ in L^{Φ}, the following statements are equivalent:*

(1) *$\mathcal{C}$ is order closed.*
(2) *$\mathcal{C}$ is $\sigma(L^{\Phi}, L^{\Psi})$-sequentially closed.*
(3) *$\mathcal{C} \cap k\mathcal{B}$ is $\sigma(L^{\Phi}, L^{\Psi})$-closed for all $k \geq 1$.*

Proof. The implication (2)⇒(1) follows from the observation at the beginning of this section. Theorem 2.2 gives (1)⇒(3). The implication (3)⇒(2) follows from the fact that every $\sigma(L^{\Phi}, L^{\Psi})$-convergent sequence is norm bounded. ∎

Corollary 2.3 leads to the natural question of characterizing the pairs (Φ, Ψ) such that $\sigma(L^{\Phi}, L^{\Psi})$ has the Krein–Šmulian property. The next lemma is the key construction to solving this question. A set $\mathcal{C} \subset L^{\Phi}$ is said to be

(i) *monotone* if $X_1 \geq X_2 \in \mathcal{C}$ implies $X_1 \in \mathcal{C}$,
(ii) *positively homogeneous* if $\lambda\mathcal{C} \subset \mathcal{C}$ for any $\lambda \geq 0$,
(iii) *additive* if $\mathcal{C} + \mathcal{C} \subset \mathcal{C}$.

A set that is positively homogeneous and additive is clearly convex.

LEMMA 2.4. *If Φ and Ψ both fail the Δ_2-condition, then L^{Φ} admits a monotone, positively homogeneous and additive subset $\mathcal{C}$ which is order closed but not $\sigma(L^{\Phi}, L^{\Psi})$-closed. Furthermore, for any $X \in L^{\Phi}$, there exists $k \in \mathbb{R}$ such that $X - k\mathbb{1} \notin \mathcal{C}$.*

Proof. Assume that both Φ and Ψ fail the Δ_2-condition. We claim that there are a norm bounded set of disjoint positive random variables $\{X_n\}_{n\geq 1} \cup \{W_0\} \cup \{W_{ij}\}_{i,j\geq 1}$ in L^{Φ} and a norm bounded set of disjoint positive random variables $\{Y_n\}_{n\geq 1} \cup \{Z_0\} \cup \{Z_{ij}\}_{i,j\geq 1}$ in L^{Ψ} such that

(a) $\operatorname{supp} Y_n \subset \operatorname{supp} X_n$, $\operatorname{supp} W_0 \subset \operatorname{supp} Z_0$ and $\operatorname{supp} W_{ij} \subset \operatorname{supp} Z_{ij}$ for all $n, i, j \geq 1$,
(b) The pointwise sums $\widetilde{X} := \sum_n X_n$ and $\widetilde{Z} := \sum_{i,j} Z_{ij}$ belong to L^{Φ} and L^{Ψ} respectively,
(c) $\mathbb{E}[X_n Y_n] = \mathbb{E}[W_0 Z_0] = \mathbb{E}[W_{ij} Z_{ij}] = 1$ for all $n, i, j \geq 1$.

Since $\mathbb{P}$ is nonatomic, there are three disjoint measurable subsets $\Omega_1, \Omega_2, \Omega_3$ of Ω, each of which is atomless and has positive measure. Choose any $0 \leq W_0 \in L^{\Phi}(\Omega_2)$ and $0 \leq Z_0 \in L^{\Psi}(\Omega_2)$ such that $\operatorname{supp} W_0 \subset \operatorname{supp} Z_0$ and $\mathbb{E}[W_0 Z_0] = 1$. Since Φ fails the Δ_2-condition, we may apply [16, p. 139, Theorem 5] to $L^{\Phi}(\Omega_1)$ to obtain a sequence (X_n) of normalized disjoint positive random variables in L^Φ such that $\widetilde{X} := \sum_n X_n \in L^\Phi$. Choose a norm bounded sequence $(Y_n) \subset L^\Psi$ so that $\operatorname{supp} Y_n \subset \operatorname{supp} X_n$ and that $\mathbb{E}[X_n Y_n] = 1$ for all n. Similarly, since Ψ fails the Δ_2-condition, there is a normalized disjoint positive sequence $(Z_{ij})_{i,j\geq 1} \subset L^\Psi(\Omega_3)$ such that $\widetilde{Z} := \sum_{i,j} Z_{ij} \in L^\Psi$. Then choose $(W_{ij}) \subset L^\Phi$ with the desired properties.

For any $X \in L^\Phi$,

$$\sum_{i,j} |\mathbb{E}[XZ_{ij}]| \leq \mathbb{E}[|X\widetilde{Z}|] \leq \|X\|_\Phi \|\widetilde{Z}\|_\Psi.$$

Thus the map $\mathcal{T}$ defined by

$$\mathcal{T}X = (\mathbb{E}[XY_n])_n \oplus \mathbb{E}[XZ_0] \oplus (\mathbb{E}[XZ_{ij}])_{ij}$$

is a bounded linear operator from L^Φ into $\ell^\infty \oplus \mathbb{R} \oplus \ell^1(\mathbb{N}\times\mathbb{N})$. Clearly, $\mathcal{T}$ is a positive operator. Define the *summing operator* $\mathcal{S} : \ell^1 \to \ell^\infty$ by

$$\mathcal{S}((a_j)_j) = \Big(\sum_{j=1}^n a_j\Big)_n.$$

For any $y = (y(i,j))_{i,j\geq 1} \in \ell^1(\mathbb{N}\times\mathbb{N})$, put $y_i = (y(i,j))_j \in \ell^1$ for any $i \geq 1$. Let $\mathcal{C}$ be the subset of L^Φ consisting of all functions $X \in L^\Phi$ for which, if we write $\mathcal{T}X = u \oplus a \oplus v$, there are $\lambda \in \mathbb{R}$ and $y \in \ell^1(\mathbb{N}\times\mathbb{N})$ such that

$$\lambda \geq 0, \qquad y \geq 0 \qquad \text{and} \qquad \sum_i 2^i \|y_i\|_1 = 1,$$

$$a \geq -\lambda, \qquad v \geq \lambda y \qquad \text{and} \qquad u \geq \lambda \sum_{i=1}^l 4^i \mathcal{S} y_i \quad \text{for all } l \geq 1.$$

If the above occurs, we write $X \sim (\lambda, y)$.

Claim I. *$\mathcal{C}$ is monotone, positively homogeneous and additive.*

Indeed, if $X' \geq X \in \mathcal{C}$ and $X \sim (\lambda, y)$, then clearly $X' \sim (\lambda, y)$ and $\mu X \sim (\mu\lambda, y)$ for any $\mu \geq 0$, so that $X', \mu X \in \mathcal{C}$. Now suppose $X \sim (\lambda, y)$ and $X' \sim (\lambda', y')$. Since $y, y' \geq 0$, it follows that

$$\sum_i 2^i \|\lambda y_i + \lambda' y_i'\|_1 = \lambda \sum_i 2^i \|y_i\|_1 + \lambda' \sum_i 2^i \|y_i'\|_1 = \lambda + \lambda'.$$

Thus we can find $0 \leq y'' \in \ell^1(\mathbb{N}\times\mathbb{N})$ such that $\sum_i 2^i \|y_i''\|_1 = 1$ and that

$$(\lambda + \lambda')y'' = \lambda y + \lambda' y'.$$

Let us show that $X + X' \sim (\lambda + \lambda', y'')$. Indeed, write $\mathcal{T}X = u \oplus a \oplus v$ and $\mathcal{T}X' = u' \oplus a' \oplus v'$; then

$$\mathcal{T}(X + X') = (u + u') \oplus (a + a') \oplus (v + v').$$

Now

$$a + a' \geq (-\lambda) + (-\lambda') = -(\lambda + \lambda'), \qquad v + v' \geq \lambda y + \lambda' y' = (\lambda + \lambda')y'',$$

and

$$u+u' \geq \lambda \sum_{i=1}^{l} 4^i \mathcal{S} y_i + \lambda' \sum_{i=1}^{l} 4^i \mathcal{S} y_i' = \sum_{i=1}^{l} 4^i \mathcal{S}(\lambda y_i + \lambda' y_i')$$
$$= (\lambda+\lambda') \sum_{i=1}^{l} 4^i \mathcal{S} y_i'' \quad \text{for all } l \geq 1.$$

This proves that $X+X' \sim (\lambda+\lambda', y'')$, so that $X+X' \in \mathcal{C}$, as desired.

Since order intervals in ℓ^1 are norm compact, for any norm convergent positive sequence (v_p) in ℓ^1 the set $\bigcup_p [0,v_p]$ is relatively norm compact in ℓ^1.

Claim II. *$\mathcal{C}$ is order closed in L^Φ.*

Let (U_p) be a sequence in $\mathcal{C}$ that order converges to some $U \in L^\Phi$. We want to show that $U \in \mathcal{C}$. Write

$$\mathcal{T} U_p = u_p \oplus a_p \oplus v_p \quad \text{and} \quad \mathcal{T} U = u \oplus a \oplus v.$$

For each $n \geq 1$, denote by $x(n)$ the nth coordinate of a vector x in ℓ^∞. By the Dominated Convergence Theorem, for any $n \geq 1$,

$$\lim_p u_p(n) = \lim_p \mathbb{E}[U_p Y_n] = \mathbb{E}[U Y_n] = u(n).$$

Moreover, since (U_p) is order bounded, and therefore norm bounded, in L^Φ, it is easy to see that (u_p) is norm bounded in ℓ^∞. It follows that (u_p) $\sigma(\ell^\infty, \ell^1)$-converges to u. Similarly, (a_p) converges to a. For any $(b_{ij}) \in \ell^\infty(\mathbb{N}\times\mathbb{N})$,

$$\Big|\sum_{i,j} b_{i,j} \mathbb{E}[(U_p - U) Z_{ij}]\Big| \leq \mathbb{E}\Big[|U_p - U| \sum_{i,j} |b_{ij}| Z_{ij}\Big]$$
$$\leq \sup_{i,j} |b_{ij}| \cdot \mathbb{E}[|U_p - U| \widetilde{Z}] \to 0.$$

So (v_p) converges to v with respect to the topology $\sigma(\ell^1(\mathbb{N}\times\mathbb{N}), \ell^\infty(\mathbb{N}\times\mathbb{N}))$. Since $\ell^1(\mathbb{N}\times\mathbb{N})$ has the Schur property, (v_p) norm converges to v.

For each p, suppose that $U_p \sim (\lambda_p, y_p)$ and write $y_{pi} = (y_p(i,j))_{j=1}^\infty$ for each i. Choose M so that $\|u_p\|_\infty \leq M$ for all p. If $l \geq 1$, then

$$\text{(2)} \qquad M \geq u_p(n) \geq \lambda_p \sum_{i=1}^{l} 4^i \mathcal{S} y_{pi}(n) \to \lambda_p \sum_{i=1}^{l} 4^i \|y_{pi}\|_1 \quad \text{as } n \to \infty.$$

In particular, $M \geq \lambda_p \sum_i 2^i \|y_{pi}\|_1 = \lambda_p \geq 0$, so that (λ_p) is a bounded sequence. Take a subsequence if necessary to assume that (λ_p) converges to some $\lambda \geq 0$. If $\lambda = 0$, let y be any positive element in $\ell^1(\mathbb{N}\times\mathbb{N})$ such that $\sum_i 2^i \|y_i\|_1 = 1$, where $y_i = (y(i,j))_{j=1}^\infty$. Then it is easy to see that $a \geq -\lambda$, $v \geq \lambda y$ and $u \geq \lambda \sum_{i=1}^{l} 4^i \mathcal{S} y_i$ for all l. Hence, $U \sim (\lambda, y)$, and $U \in \mathcal{C}$.

For the rest of the proof, assume that $\lambda > 0$. Since $v_p \geq \lambda_p y_p \geq 0$ for all p, and (v_p) is norm convergent in $\ell^1(\mathbb{N}\times\mathbb{N})$, it follows that $(\lambda_p y_p)$

is relatively norm compact in $\ell^1(\mathbb{N}\times\mathbb{N})$. Passing to a subsequence again, we may assume that $(\lambda_p y_p)_p$ converges in norm to some z in $\ell^1(\mathbb{N}\times\mathbb{N})$. Set $y = z/\lambda$. Then $y \geq 0$, and (y_p) converges to y in norm. To complete the proof, we will verify that $U \sim (\lambda, y)$. Clearly, for any $i \geq 1$, we have $2^i\|y_{pi}\|_1 \to 2^i\|y_i\|_1$ as $p \to \infty$. Choose p_0 such that $\lambda_p \geq \frac{\lambda}{2}$ for all $p \geq p_0$. By (2), if $p \geq p_0$, then $0 \leq 2^i\|y_{pi}\|_1 \leq M/(\lambda 2^{i-1})$ for any $i \geq 1$. It follows from the Dominated Convergence Theorem that

$$\sum_i 2^i\|y_i\|_1 = \lim_p \sum_i 2^i\|y_{pi}\|_1 = 1. \tag{3}$$

Furthermore,

$$a = \lim a_p \geq -\lim \lambda_p = -\lambda, \tag{4}$$

$$v = \lim_p v_p \geq \lim_p \lambda_p y_p = \lambda y. \tag{5}$$

Finally, for each n and each i, $\mathcal{S}y_{pi}(n) \to \mathcal{S}y_i(n)$ as $p \to \infty$. So, for any l,

$$u(n) = \lim_p u_p(n) \geq \lim_p \lambda_p \sum_{i=1}^{l} 4^i \mathcal{S}y_{pi}(n) = \lambda \sum_{i=1}^{l} 4^i \mathcal{S}y_i(n).$$

This proves that

$$u \geq \lambda \sum_{i=1}^{l} 4^i \mathcal{S}y_i \quad \text{for any } l. \tag{6}$$

Clearly, (3)–(6) show that $U \sim (\lambda, y)$, as desired.

Claim III. $-W_0 \in \overline{\mathcal{C}}^{\sigma(L^{\Phi},L^{\Psi})} \setminus \mathcal{C}$ *and thus* $\mathcal{C}$ *is not* $\sigma(L^{\Phi},L^{\Psi})$*-closed.*

Clearly

$$\mathcal{T}(-W_0) = 0 \oplus -1 \oplus 0.$$

If $-W_0 \in \mathcal{C}$, then there would exist $\lambda \geq 0$ and $0 \leq y \in \ell^1(\mathbb{N}\times\mathbb{N})$ such that $-1 \geq -\lambda$, $0 \geq \lambda y$, and $\sum_i 2^i\|y_i\|_1 = 1$, where $y_i = (y(i,j))_j$. It follows that $\lambda \geq 1$, forcing $y = 0$, which is impossible. This proves that $-W_0 \notin \mathcal{C}$.

Next, we show that $-W_0 \in \overline{\mathcal{C}}^{\sigma(L^{\Phi},L^{\Psi})}$. Let $V_1, \ldots, V_l \in L^{\Psi}$ and $\varepsilon > 0$ be given. Set $V = \frac{1}{\varepsilon}\sum_{t=1}^{l}|V_t|$. Since $\sup_{ij}\mathbb{E}[W_{ij}V] \leq (\sup_{ij}\|W_{ij}\|_{\Phi})\|V\|_{\Psi} < \infty$, there exists $s \geq 1$ large enough so that

$$\mathbb{E}[W_{ij}V] < 2^{s-1}$$

for all i, j. Since $\sum_n \mathbb{E}[X_n V] \leq \mathbb{E}[\widetilde{X}V] < \infty$, there exists $r \geq 1$ such that

$$\sum_{n=r}^{\infty} \mathbb{E}[X_n V] < 1/2^{s+1}.$$

Let $y \in \ell^1(\mathbb{N}\times\mathbb{N})$ be defined by $y(i,j) = 1/2^s$ if $(i,j) = (s,r)$ and 0 otherwise. Simple computations show that if

$$X := 2^s \sum_{n=r}^{\infty} X_n - W_0 + W_{sr}/2^s,$$

then $X \sim (1, y)$, so that $X \in \mathcal{C}$. Moreover, if $1 \le t \le l$, then

$$\begin{aligned}|\mathbb{E}[XV_t] - \mathbb{E}[(-W_0)V_t]| &\le \varepsilon \mathbb{E}[|X + W_0|V] \\ &\le \varepsilon 2^s \sum_{n=r}^{\infty} \mathbb{E}[X_n V] + \frac{\varepsilon}{2^s}\mathbb{E}[W_{sr}V] < \varepsilon.\end{aligned}$$

This proves that $-W_0 \in \overline{\mathcal{C}}^{\sigma(L^\Phi, L^\Psi)}$.

Finally, suppose that there exists $X \in L^\Phi$ such that $X - k\mathbb{1} \in \mathcal{C}$ for all $k \in \mathbb{R}$. Let u be the first component of $\mathcal{T}X$. Then the first component of $\mathcal{T}(X - k\mathbb{1})$ is $u - k(\mathbb{E}[Y_n])_n$. Since $X - k\mathbb{1} \in \mathcal{C}$, $u - k(\mathbb{E}[Y_n])_n \ge 0$. Thus

$$(\mathbb{E}[Y_n])_n \le \frac{u}{k} \quad \text{for all } k \ge 1,$$

from which it follows that $\mathbb{E}[Y_n] = 0$ for all n, contrary to the choice of Y_n. ∎

We are now ready to present the main result of this section.

THEOREM 2.5. *The following statements are equivalent for an Orlicz space L^Φ defined on a nonatomic probability space:*

(1) *Every order closed convex set in L^Φ is $\sigma(L^\Phi, L^\Psi)$-closed.*
(2) *Every $\sigma(L^\Phi, L^\Psi)$-sequentially closed convex set in L^Φ is $\sigma(L^\Phi, L^\Psi)$-closed.*
(3) *$\sigma(L^\Phi, L^\Psi)$ has the Krein–Šmulian property.*
(4) *Either Φ or Ψ satisfies the Δ_2-condition.*

Proof. By Corollary 2.3, we have (1)⇔(2)⇔(3). If Φ satisfies the Δ_2-condition, then $\sigma(L^\Phi, L^\Psi)$ is the weak topology, which has the Krein–Šmulian property. If Ψ satisfies the Δ_2-condition, then L^Φ is the norm dual of L^Ψ, and $\sigma(L^\Phi, L^\Psi)$ is the weak* topology, which has the Krein–Šmulian property by the Krein–Šmulian Theorem. This shows that (4)⇒(3). Finally, Lemma 2.4 shows that (1)⇒(4). ∎

3. Closed convex sets in H^Φ. The *bounded a.s. closure* of a subset $\mathcal{C} \subset H^\Phi$ is defined as follows:

$$\overline{\mathcal{C}}^{\text{buo}} := \{X \in H^\Phi : X_n \xrightarrow{\text{a.s.}} X \text{ for some norm bounded sequence } (X_n) \text{ in } \mathcal{C}\}.$$

We say that $\mathcal{C}$ is *boundedly a.s. closed* if $\mathcal{C} = \overline{\mathcal{C}}^{\text{buo}}$. The notion of a.s.-convergence is the same as uo-convergence in Banach lattices, and in view of this, bounded a.s. closure and closedness can be studied in the general framework of Banach lattices; see [11]. The next two results are counterparts of Theorem 2.2 and Corollary 2.3, respectively. They are special cases of the corresponding results on general Banach lattices that have been obtained in [11, Theorem 4.1 and Corollary 4.4] (see also [11, Corollary 4.8]).

Theorem 3.1. *Let $\mathcal{C}$ be any norm bounded convex set $\mathcal{C}$ in H^Φ. Then* $\overline{\mathcal{C}}^{\mathrm{buo}} = \overline{\mathcal{C}}^{\sigma(H^\Phi, H^\Psi)}$.

Corollary 3.2. *Denote by $\mathcal{B}$ the closed unit ball of H^Φ. For a convex set $\mathcal{C}$ in H^Φ, the following are equivalent:*

(1) *$\mathcal{C}$ is boundedly a.s. closed.*
(2) *$\mathcal{C}$ is $\sigma(H^\Phi, H^\Psi)$-sequentially closed.*
(3) *$\mathcal{C} \cap k\mathcal{B}$ is $\sigma(H^\Phi, H^\Psi)$-closed for all $k \geq 1$.*

Lemma 3.3. *Assume that Φ and Ψ both fail the Δ_2-condition. There is a monotone, positively homogeneous and additive subset $\mathcal{C}$ of H^Φ which is boundedly a.s. closed but fails to be $\sigma(H^\Phi, H^\Psi)$-closed. Furthermore, for any $X \in L^\Phi$, there exists $k \in \mathbb{R}$ such that $X - k\mathbb{1} \notin \mathcal{C}$.*

Sketch of proof. Observe that, for any positive random variable X in L^Φ (respectively, L^Ψ), the truncation $X\mathbb{1}_{\{X\leq k\}}$ lies in H^Φ (respectively, H^Ψ). Moreover, $\|X\mathbb{1}_{\{X\leq k\}}\| \uparrow_k \|X\|$. As in the proof of Lemma 2.4, and applying suitable truncations, we find a norm bounded set of disjoint, positive functions $\{X_n\}_{n\geq1} \cup \{W_0\} \cup \{W_i\}_{i\geq1}$ in H^Φ and a norm bounded set of disjoint, positive functions $\{Y_n\}_{n\geq1} \cup \{Z_0\} \cup \{Z_i\}_{i\geq1}$ in H^Ψ such that

(a) $\operatorname{supp} Y_n \subset \operatorname{supp} X_n$, $\operatorname{supp} W_0 \subset \operatorname{supp} Z_0$ and $\operatorname{supp} W_i \subset \operatorname{supp} Z_i$ for all $n, i \geq 1$,
(b) the pointwise sums satisfy $\widetilde{X} = \sum_n X_n \in L^\Phi$ and $\widetilde{Z} = \sum_i Z_i \in L^\Psi$,
(c) $\mathbb{E}[X_nY_n] = \mathbb{E}[W_0Z_0] = \mathbb{E}[W_iZ_i] = 1$ for all $n, i \geq 1$.

Since Y_n's are disjoint, we have $Y_n \xrightarrow{\text{a.s.}} 0$, so that $\lim_n \mathbb{E}[XY_n] = 0$ for any $X \in H^\Phi$. Thus, $(\mathbb{E}[XY_n])_n \in c_0$. Also, since

$$\sum_i |\mathbb{E}[XZ_i]| \leq \mathbb{E}[|X|\widetilde{Z}] < \infty,$$

we have $(\mathbb{E}[XZ_i])_i \in \ell^1$. Therefore, the map

$$\mathcal{T} : X \mapsto (\mathbb{E}[XY_n])_n \oplus \mathbb{E}[XZ_0] \oplus (\mathbb{E}[XZ_i])_i$$

is a bounded positive linear operator from H^Φ into $c_0 \oplus \mathbb{R} \oplus \ell^1$.

For any $y = (y(i,j))_{i,j\geq1} \in \ell^1(\mathbb{N} \times \mathbb{N})$, put $y_i = (y(i,j))_j \in \ell^1$ for any $i \geq 1$. Let $(s_j)_j$ be the summing basis for c_0, i.e.,

$$s_j = (1, \dots, 1, 0, \dots),$$

with 1 occurring in the first j coordinates, and let (w_j) be the standard basis for ℓ^1, i.e.,

$$w_j = (0, \dots, 0, 1, 0, \dots),$$

with 1 occurring in the jth coordinate. Let $\mathcal{C}$ be the subset of H^Φ consisting of all random variables $X \in H^\Phi$ for which, if we write $\mathcal{T}X = u \oplus a \oplus v$, there are $\lambda \in \mathbb{R}$ and $y \in \ell^1(\mathbb{N} \times \mathbb{N})$ such that

$$\lambda \geq 0, \quad y \geq 0, \quad \sum_i 2^i \|y_i\|_1 = 1, \quad \text{and} \quad \sum_i 4^i \|y_i\|_1 < \infty,$$
$$a \geq -\lambda, \quad v \geq \lambda \sum_j \Big(\sum_i 4^i y(i,j)\Big) w_j, \quad u \geq \lambda \sum_j \Big(\sum_i y(i,j)\Big) s_j.$$

If the above occurs, we write $X \sim (\lambda, y)$.

We omit the verifications that $\mathcal{C}$ is monotone, positively homogeneous and additive and the last statement. For $j \geq 1$, define the projection $\mathcal{P}_j$ on ℓ^1 by

$$\mathcal{P}_j(b_1, b_2, \ldots) = (0, \ldots, 0, b_j, b_{j+1}, \ldots).$$

Then the condition on u is equivalent to

$$u \geq \lambda \Big(\sum_i \|\mathcal{P}_1 y_i\|_1, \sum_i \|\mathcal{P}_2 y_i\|_1, \ldots\Big). \tag{7}$$

Claim I. *$U \in \mathcal{C}$ whenever there exists a norm bounded sequence $(U_p)_p$ in $\mathcal{C}$ such that $(U_p)_p$ converges a.s. to $U \in H^\Phi$.*

Suppose that

$$\mathcal{T}U_p = u_p \oplus a_p \oplus v_p \quad \text{and} \quad \mathcal{T}U = u \oplus a \oplus v.$$

Write $u_p = (u_p(j))_j$ and $u = (u(j))_j$. Then

$$u_p(j) = \mathbb{E}[U_p Y_j] \to \mathbb{E}[U Y_j] = u(j)$$

for each j. Moreover, since (U_p) is norm bounded in H^Φ, (u_p) is norm bounded in c_0. Thus, $u_p \xrightarrow{\sigma(c_0,\ell^1)} u$. There is a sequence of convex combinations, $(\sum_{j=p_{n-1}+1}^{p_n} c_j u_j)$, $0 = p_0 < p_1 < p_2 < \cdots$, which converges to u in the norm of c_0. By replacing U_p's with the corresponding convex combinations (which also converge a.s. to U), we may assume that (u_p) converges to u in c_0-norm. Similarly, $v_p \to v$ coordinatewise and

$$\|v_p\|_1 \leq \sum_i \mathbb{E}[|U_p| Z_i] = \mathbb{E}[|U_p| \widetilde{Z}] \leq \|U_p\|_\Phi \|\widetilde{Z}\|_\Psi.$$

Therefore, $v_p \xrightarrow{\sigma(\ell^1,c_0)} v$. Clearly, $a_p \to a$.

For each p, let $U_p \sim (\lambda_p, y_p)$. Write $y_{pi} = (y_p(i,j))_{j=1}^\infty$ for each i. Then there exists some $M > 0$ such that

$$M \geq \|v_p\|_1 \geq \lambda_p \sum_{i,j} 4^i y_p(i,j) = \lambda_p \sum_i 4^i \|y_{pi}\|_1. \tag{8}$$

In particular, $M \geq \lambda_p \sum_i 2^i \|y_{pi}\|_1 = \lambda_p \geq 0$. Thus (λ_p) is a bounded sequence. Passing to a subsequence, we may assume that (λ_p) converges to some $\lambda \geq 0$. If $\lambda = 0$, set y to be any positive element in $\ell^1(\mathbb{N}\times\mathbb{N})$ such that $\sum_i 2^i\|y_i\|_1 = 1$ and $\sum_i 4^i\|y_i\|_1 < \infty$. It is easily checked that $U \sim (0, y)$ and hence $U \in \mathcal{C}$. Assume that $\lambda > 0$. By (8), for all sufficiently large p,

$$\frac{\lambda}{2}\|\mathcal{P}_j(2^i y_{pi})\|_1 \leq \lambda_p\|2^i y_{pi}\|_1 \leq \frac{M}{2^i}$$

for all $i, j \geq 1$. It follows that, for each j, the sequence $((\|\mathcal{P}_j(2^i y_{pi})\|_1)_i)_{p\geq 1}$, being contained in an interval of ℓ^1, is relatively norm compact in ℓ^1. By passing to a subsequence, we may assume that there exists $(b_{ij})_i \in \ell^1$ such that

$$\lim_p (\|\mathcal{P}_j(2^i y_{pi})\|_1)_i = (b_{ij})_i \quad \text{in } \ell^1\text{-norm}$$

for each $j \geq 1$. In particular,

$$b_{i,j+1} \leq b_{ij} \quad \text{and} \quad \sum_i b_{i1} = \lim_p \sum_i \|2^i y_{pi}\|_1 = 1.$$

Set

$$y(i,j) = \frac{b_{ij} - b_{i,j+1}}{2^i} \qquad \text{for all } i, j \geq 1,$$

and

$$y := (y(i,j)) \geq 0.$$

We claim that $U \sim (\lambda, y)$ and hence $U \in \mathcal{C}$. Note that

$$b_{ij} - b_{i,j+1} = \lim_p (\|\mathcal{P}_j(2^i y_{pi})\|_1 - \|\mathcal{P}_{j+1}(2^i y_{pi})\|_1) = \lim_p 2^i y_p(i,j).$$

Thus $y_p(i,j) \to y(i,j)$ for any $i, j \geq 1$. It follows from Fatou's Lemma and (8) that $\sum_i 4^i\|y_i\|_1 < \infty$. Since (u_p) converges to u in c_0, $\lim_j u_p(j) = 0$ uniformly in p. By condition (7) for u_p, $\lim_j \sum_i \|\mathcal{P}_j y_{pi}\|_1 = 0$ uniformly in p. In particular, for each i, $\lim_j \|\mathcal{P}_j y_{pi}\|_1 = 0$ uniformly in p, so that $\lim_j b_{ij} = 0$ for each i. Therefore, $\|2^i y_i\|_1 = \sum_j (b_{ij} - b_{i,j+1}) = b_{i1}$ for all i, and

$$\sum_i \|2^i y_i\|_1 = \sum_i b_{i1} = 1.$$

Fatou's Lemma also implies that

$$v(j) = \lim_p v_p(j) \geq \liminf_p \lambda_p \sum_i 4^i y_p(i,j) \geq \lambda \sum_i 4^i y(i,j),$$

and that (using (7) for u_p)

$$u(j) = \lim_p u_p(j) \geq \liminf_p \lambda_p \sum_i \|P_j y_{pi}\|_1 \geq \lambda \sum_i \|P_j y_i\|_1.$$

This completes the proof that $U \sim (\lambda, y)$.

Claim II. *$\mathcal{C}$ is not $\sigma(H^\Phi, H^\Psi)$-closed. Precisely,* $-W_0 \in \overline{\mathcal{C}}^{\sigma(H^\Phi, H^\Psi)} \setminus \mathcal{C}$.

Since $\mathcal{T}(-W_0) = 0 \oplus -1 \oplus 0$, it is easy to see that $-W_0 \notin \mathcal{C}$. On the other hand, observe that, for any $s, r \geq 1$,

$$X_{sr} := \frac{1}{2^s} \sum_{n=1}^{r} X_n - W_0 + 2^s W_r \sim (1, y),$$

where $y(i,j) = 1/2^s$ if $(i,j) = (s,r)$ and 0 otherwise, so that $X_{sr} \in \mathcal{C}$. As in the proof of Lemma 2.4, one can show that $-W_0$ lies in the $\sigma(H^\Phi, H^\Psi)$-closure of the double sequence $(X_{sr})_{s,r\geq 1}$. ■

THEOREM 3.4. *The following statements are equivalent for an Orlicz heart H^Φ defined on a nonatomic probability space:*

(1) *Every boundedly a.s. closed convex set in H^Φ is $\sigma(H^\Phi, H^\Psi)$-closed.*
(2) *Every $\sigma(H^\Phi, H^\Psi)$-sequentially closed convex set in H^Φ is $\sigma(H^\Phi, H^\Psi)$-closed.*
(3) *$\sigma(H^\Phi, H^\Psi)$ has the Krein–Šmulian property.*
(4) *Either Φ or Ψ satisfies the Δ_2-condition.*

Proof. By Corollary 3.2, we have that (1)⇔(2)⇔(3). If Φ satisfies the Δ_2-condition, then $H^\Phi = L^\Phi$ is the dual space of H^Ψ, so that $\sigma(H^\Phi, H^\Psi)$ is the weak* topology, which has the Krein–Šmulian property by the Krein–Šmulian Theorem. If Ψ satisfies the Δ_2-condition, then $H^\Psi = L^\Psi$ is the norm dual of H^Φ, so that $\sigma(H^\Phi, H^\Psi)$ is the weak topology and also has the Krein–Šmulian property. This proves (4)⇒(3). By applying Lemma 3.3 we get (1)⇒(4). ■

4. Application to risk measures. In the paper [2], a theoretical foundation was laid for the problem of quantifying the risk of a financial position in terms of coherent risk measures. In this axiomatic treatment of risk measures, financial positions are modeled by a vector space $\mathcal{X}$, which includes constants, of random variables on a probability space $(\Omega, \Sigma, \mathbb{P})$. A *coherent risk measure* on $\mathcal{X}$ is a functional $\rho : \mathcal{X} \to (-\infty, \infty]$ that is proper (i.e., not identically ∞) and satisfies the following properties:

(1) (subadditive) $\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$ for all $X_1, X_2 \in \mathcal{X}$,
(2) (monotone) $\rho(X_1) \leq \rho(X_2)$ if $X_1, X_2 \in \mathcal{X}$ and $X_1 \geq X_2$ a.s.,
(3) (cash additive) $\rho(X + m\mathbb{1}) = \rho(X) - m$ for any $X \in \mathcal{X}$ and any $m \in \mathbb{R}$,
(4) (positively homogeneous) $\rho(\lambda X) = \lambda\rho(X)$ for any $X \in \mathcal{X}$ and any $0 < \lambda \in \mathbb{R}$.

An important topic is to determine when a risk measure on $\mathcal{X}$ admits a representation with respect to some duality involving $\mathcal{X}$. The first major result in this direction was obtained by Delbaen [6], who used as model space $\mathcal{X}$ the space of all bounded random variables $L^\infty(\mathbb{P})$ and considered the duality (L^∞, L^1).

THEOREM 4.1 (Delbaen). *The following are equivalent for every coherent risk measure ρ on $L^\infty(\mathbb{P})$:*

(1) *There is a set $\mathcal{Q}$ of nonnegative random variables with expectation* 1 *such that*
$$\rho(X) = \sup_{Y\in\mathcal{Q}} \mathbb{E}[-XY] \quad \text{for any } X \in L^\infty.$$

(2) *ρ satisfies the* (L^∞-)Fatou property*:*
$$\rho(X) \le \liminf_n \rho(X_n) \text{ whenever } (X_n) \text{ is bounded in } L^\infty \text{ and } X_n \xrightarrow{\text{a.s.}} X.$$

The representation in (1) of Theorem 4.1 is connected with $\sigma(L^\infty, L^1)$-lower semicontinuity of ρ via the Fenchel–Moreau Duality Theorem in convex analysis. Here, $\sigma(L^\infty, L^1)$-*lower semicontinuity* of ρ refers to the property that the sublevel sets
$$\{\rho \le \lambda\} = \{X \in L^\infty(\mathbb{P}) : \rho(X) \le \lambda\}$$
are $\sigma(L^\infty, L^1)$-closed for any $\lambda \in \mathbb{R}$. On the other hand, condition (2) in Theorem 4.1 is equivalent to the sets $\{\rho \le \lambda\}$ being order closed in L^∞.

In Theorem 4.1, as in other early framework for risk measures, the model space consists of bounded financial positions. The reader may refer to Föllmer and Schied [9] for a comprehensive treatment of the main results in this setting. More realistic models of financial positions may involve unbounded random variables; this motivates the study of risk measures on model spaces beyond L^∞. The Orlicz spaces L^Φ and Orlicz hearts H^Φ, being natural classes of Banach function spaces that generalize the L^p spaces, have emerged as important model spaces of (unbounded) financial positions. An important outstanding problem in the theory of dual representation of risk measures is the generalization of Theorem 4.1 to the Orlicz-space framework. See [3, 5, 7, 10, 12].

For the following, we fix a conjugate pair (Φ, Ψ) of Orlicz functions, and let ρ be a coherent risk measure $\rho : \mathcal{X} \to (-\infty, \infty]$, where $\mathcal{X} = L^\Phi$ or H^Φ. We say that ρ satisfies the (*strong*) *Fatou property* if
$$\rho(X) \le \liminf_n \rho(X_n) \quad \text{whenever} \quad (X_n) \text{ is } \textit{order } (\textit{norm}) \textit{ bounded} \text{ in } \mathcal{X} \text{ and } X_n \xrightarrow{\text{a.s.}} X.$$
ρ is said to admits a *dual representation* via L^Ψ (resp., H^Ψ) if there is a set $\mathcal{Q}$ of nonnegative random variables in L^Ψ (resp., H^Ψ), with expectation 1 each, such that
$$\rho(X) = \sup_{Y\in\mathcal{Q}} \mathbb{E}[-XY] \quad \text{for any } X \in \mathcal{X}.$$

Suppose that $\mathcal{X} = L^\Phi$. If ρ admits a dual representation via L^Ψ, it is easy to see that ρ has the Fatou property. The converse is shown true, in [3, 5] if

Φ satisfies the Δ_2-condition and in [7] if Ψ satisfies the Δ_2-condition. This equivalence breaks down for a general Orlicz pair (Φ, Ψ), in particular we have the following surprising result.

THEOREM 4.2. *Let (Φ, Ψ) be a conjugate pair of Orlicz functions such that both fail the Δ_2-condition. Then there exists a coherent risk measure $\rho : L^\Phi \to (-\infty, \infty]$ with the Fatou property that does not admit a dual representation via L^Ψ.*

Proof. Let $\mathcal{C}$ be the set in L^Φ obtained by applying Lemma 2.4. Define $\rho : L^\Phi \to (-\infty, \infty]$ by

$$\rho(X) = \inf\{m \in \mathbb{R} : X + m\mathbb{1} \in \mathcal{C}\}.$$

Using the properties of $\mathcal{C}$, it is standard to check that ρ is a coherent risk measure. Clearly, $\mathcal{C} \subset \{\rho \leq 0\}$ and $\{\rho < 0\} \subset \mathcal{C}$ by monotonicity of $\mathcal{C}$. It follows from the order closedness of $\mathcal{C}$ that $X \in \mathcal{C}$ if $\rho(X) = 0$. Therefore, $\{\rho \leq 0\} = \mathcal{C}$, so that

$$\{\rho \leq m\} = \mathcal{C} - m\mathbb{1} \quad \text{for any } m \in \mathbb{R}.$$

Thus $\{\rho \leq m\}$ is order closed for all m, from which it follows that ρ has the Fatou property. Indeed, let (X_n) be a sequence in L^Φ that order converges to $X \in L^\Phi$. If $\liminf_n \rho(X_n) = \infty$, then $\rho(X) \leq \liminf_n \rho(X_n)$ trivially. Otherwise, let $m \in \mathbb{R}$ be such that $m > \liminf_n \rho(X_n)$. Choose a subsequence (X_{n_k}) so that $\rho(X_{n_k}) < m$ for all k. Then $X_{n_k} \in \{\rho \leq m\}$ for all k and $X_{n_k} \xrightarrow{o} X$. Thus $X \in \{\rho \leq m\}$ and $\rho(X) \leq m$. As this applies to any $m > \liminf_n \rho(X_n)$, $\rho(X) \leq \liminf_n \rho(X_n)$. If ρ admits a representation via L^Ψ, then there exists a set $\mathcal{Q}$ of nonnegative random variables in L^Ψ such that

$$\rho(X) = \sup_{Y \in \mathcal{Q}} \mathbb{E}[-XY] \quad \text{for any } X \in L^\Phi;$$

it is well known from the Fenchel–Moreau duality theory, and is also readily checked, that $\mathcal{C} = \{\rho \leq 0\}$ is $\sigma(L^\Phi, L^\Psi)$-closed, contrary to the construction of $\mathcal{C}$. ∎

Despite the above result, if $\rho : L^\Phi \to (-\infty, \infty]$ satisfies the strong Fatou property then it always admits a dual representation via H^Ψ ([12]). However, if ρ is defined only on H^Φ, the representation may fail. In particular we have the following result that can be obtained by applying Lemma 3.3 and following the construction in the proof of Theorem 4.2

THEOREM 4.3. *Let (Φ, Ψ) be a conjugate pair of Orlicz functions such that both fail the Δ_2-condition. Then there exists a coherent risk measure $\rho : H^\Phi \to (-\infty, \infty]$ with the strong Fatou property that does not admit a dual representation via H^Ψ.*

We end with a remark on the Orlicz pairs (Φ, Ψ) both of which fail the Δ_2-condition.

REMARK 4.4 ([16]). One can easily see from [16, Thm. 3, p. 22 and Cor. 4, p. 26] that Φ and Ψ both failing the Δ_2-condition basically means

$$\limsup_{x\to\infty} \frac{x\Phi'(x)}{\Phi(x)} = \infty \quad \text{and} \quad \liminf_{x\to\infty} \frac{x\Phi'(x)}{\Phi(x)} = 1,$$

where Φ' is the left derivative of Φ. A well-known example of such a pair is due to Krasnosel'skiĭ and Rutitskiĭ and can be found in [14, pp. 28–29].

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