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Application of L^2 methods to the Levi problem on complex manifolds

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To the memory of Józef Siciak

Abstract. L^2 methods for extending holomorphic sections of semipositive bundles are applied to show the holomorphic convexity of complex spaces under several conditions.

1. Introduction. Let X be a (connected) complex manifold of dimension n. A classical question in several complex variables, known as the *Levi problem*, asks about geometric conditions for X to be holomorphically convex. A typical situation is when X is *weakly* 1*-complete*, i.e. when X admits a C^∞ plurisubharmonic exhaustion function. (See [O] and [G-1, G-2] for the classical results.) In [N], [F-N] and [F], some Levi problems on weakly 1-complete manifolds were solved to characterize certain modifications of analytic spaces such as the inverse of the monoidal transform. By extending this method, Takayama [TY] has shown that a weakly 1-complete manifold X is holomorphically convex if its canonical bundle K_X is negative (see also [Oh-2] for the case $n = 2$). Takayama's result becomes false if K_X is only assumed to be seminegative (cf. [G-3]). On the other hand, Takegoshi [TG] showed that a two-dimensional weakly 1-complete manifold is holomorphically convex if it admits a nonconstant holomorphic function.

The purpose of the present article is to continue the study of the Levi problem in the direction suggested by the results of Takayama and Takegoshi. We shall do it in the following way at first.

Theorem 1.1. *Let X be a weakly* 1*-complete manifold which contains a Stein open set whose complement is a complex analytic subset of X, say A. Assume that X has a nonconstant holomorphic function f. Then, for any*

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holomorphic line bundle F over X such that $K_X \otimes F^*$ is seminegative, every holomorphic section of F over $f^{-1}(w)$ is holomorphically extendable to X if $f^{-1}(w) \cap A$ is nowhere dense in $f^{-1}(w)$ and there exist no critical points of f contained in $f^{-1}(w)$. In particular, a projective (i.e. projectively embeddable) weakly 1-complete manifold X with seminegative canonical bundle is holomorphically convex if X admits a nonconstant holomorphic function without critical points.

The proof is a straightforward application of an L^2 extension theorem which the author has formulated in [Oh-4] after [Oh-T].

This proof can be generalized to give a constructive proof of some special case of a theorem of Knorr and Schneider [K-S] asserting that surjective holomorphic maps with strongly pseudoconvex (= 1-convex) fibers are locally approximated by holomorphically convex maps (see Theorem 4.1 for the precise statement). This result is of basic importance in the theory of complex spaces because it characterizes proper modifications of complex spaces as a relative variant of Grauert's theorem of [G-1, G-2] for point modifications. We note that the proof in [K-S] relies on a deep result of Knorr [K] and Siu [S] on the coherence of higher direct images of coherent analytic sheaves for 1-convex maps. Although restricting ourselves to the three-dimensional case, we will deduce local holomorphic convexity of 1-convex maps directly from a refined variant of Kodaira–Nakano's method of extending holomorphic functions on weakly 1-complete manifolds (see Theorem 2.2).

We shall also strengthen Theorem 1.1 for the case $n = 3$ and prove the following.

THEOREM 1.2. *A 3-dimensional weakly 1-complete manifold with seminegative canonical bundle is holomorphically convex if it admits a nonconstant holomorphic function with holomorphically convex fibers.*

2. L^2 extension theorems. Let us briefly recall the basic terminology. Let X be a complex manifold of dimension n. Let $B \to X$ be a holomorphic line bundle equipped with a C^∞ fiber metric h, which is naturally identified with a positive section of $B^* \otimes \overline{B}^*$, where $\overline{B}$ denotes the complex conjugate of B and $\overline{B}^*$ its dual. With respect to an open covering $\{U_j\}_{j\in J}$ of X such that $B|_{U_j} \cong \mathbb{C} \times U_j$, and a system $\{e_{jk}\}_{j,k}$ ($e_{jk} : U_j \cap U_k \to \mathbb{C}$) of transition functions of B, h is identified with a system $\{h_j\}$ of positive C^∞ functions on U_j satisfying $|e_{jk}|^2 h_j = h_k$ on $U_j \cap U_k$. Then the curvature form Θ_h of h is defined as $\{-\partial\bar\partial \log h_j\}$, which is a well-defined $(1,1)$-form on X. The form Θ_h and the pair (B, h) are said to be (*semi*)*positive* (resp. (*semi*)*negative*) if the Hermitian matrices

$$\left(-\frac{\partial^2 \log h_j}{\partial z_\alpha \partial \bar z_\beta} \right)_{1\le\alpha,\beta\le n}$$

are (semi)positive (resp. (semi)negative) with respect to the local coordinates $(z_1, \dots, z_n)$ of X. The bundle B is called (semi)positive (resp. (semi)negative) if it admits a fiber metric whose curvature form is so.

The following is a slight generalization of an extension theorem in [Oh-T]. For the proof, see [Oh-4]. See also [B], [G-Z] and [Oh-5] for sharper variants.

THEOREM 2.1 (cf. [Oh-4]). *Let M be a Stein manifold of dimension n, $S \subset M$ a closed complex submanifold of codimension m, and (B, h) a semi-positive line bundle over M. Let Φ be any plurisubharmonic function on M and let $s_1, \dots, s_m$ be holomorphic functions on M vanishing on S. Then, given a holomorphic B-valued $(n-m)$-form g on S with*

$$\Big|\int_S e^{-\Phi} h(g) \wedge \overline{g}\Big| < \infty,$$

there exists, for any $\epsilon > 0$, a holomorphic B-valued n-form G_ϵ on M which coincides with $g \wedge ds_1 \wedge \dots \wedge ds_m$ on S and satisfies

$$\Big|\int_M e^{-\Phi}(1+|s|^2)^{-m-\epsilon} h(G_\epsilon) \wedge \overline{G}_\epsilon\Big| \le \epsilon^{-1} C_m \Big|\int_S e^{-\Phi} h(g) \wedge \overline{g}\Big|, \tag{2.1}$$

where $|s|^2 = \sum_{i=1}^m |s_i|^2$ and C_m is a positive number which depends only on m.

Proof of Theorem 1.1. Let X, f, w and F be as in Theorem 1.1, let φ be a C^∞ plurisubharmonic exhaustion function on X, and let g be any holomorphic section of F over $f^{-1}(w)$.

Since f has no critical points along $f^{-1}(w)$, *df* naturally induces an isomorphism between $K_{f^{-1}(w)}$ and $K_X|_{f^{-1}(w)}$ by exterior multiplication.

In order to extend g to X, we first identify g with a $K_X^* \otimes F$-valued $(n-1)$-form on $f^{-1}(w)$ by exploiting the isomorphism $K_X|_{f^{-1}(w)} \cong K_{f^{-1}(w)}$. Since there exists a fiber metric of $K_X^* \otimes F$ whose curvature form is semipositive, one can apply Theorem 2.1 to extend g to X, by choosing as M a dense Stein domain of X with analytic complement and $\lambda(\varphi)$ as Φ for some convex increasing function $\lambda : \mathbb{R} \to \mathbb{R}$, because nowhere dense complex analytic subsets are negligible with respect to L^2 holomorphic functions. If f has no critical points and the fibers of f are holomorphically convex, that X is holomorphically convex is a straighforward consequence of the estimate (2.1). ∎

The following variant of Theorem 2.1 is a consequence of the L^2 vanishing theorem of Demailly [Dm-1] (see also [Dm-2]). It is somewhat weaker than Theorem 2.1 because it works only for bundles with singular fiber metrics with strictly positive curvature current. Nevertheless, it is also useful because it can be applied to extend holomorphic functions from analytic subsets which may not be Stein.

THEOREM 2.2. *Let X be a weakly 1-complete Kähler manifold of dimension n equipped with a C^∞ plurisubharmonic exhaustion function φ, let $f : X \to \mathbb{C}$ be a holomorphic function and let (B, h) be a line bundle over X with a singular fiber metric h whose curvature current is strictly positive. Then, for any $c \in \mathbb{R}$, there exist a convex increasing function $\lambda : (-\infty, c) \to \mathbb{R}$, a positive integer m_0 and a constant $C > 0$ such that, for any integer $m > m_0$, for any convex increasing function $\mu : (-\infty, c) \to \mathbb{R}$ and for any holomorphic section g of B^m over $f^{-1}(0)$ satisfying*

$$\int_{f^{-1}(0)} e^{-\lambda(\varphi)-\mu(\varphi)} |g|^2_{h^m} (i\Theta_h)^{n-1} < \infty,$$

one can find a holomorphic extension G of g to $X_c := \{c \in X; \varphi(x) < c\}$ satisfying

$$\int_{X_c} e^{-\lambda(\varphi)-\mu(\varphi)} |G|^2_{h^m} (i\Theta_h)^n \le C \int_{f^{-1}(0)} e^{-\lambda(\varphi)-\mu(\varphi)} |g|^2_{h^m} (i\Theta_h)^{n-1},$$

where $|\cdot|_{h^m}$ denotes length with respect to h^m.

3. Holomorphically convex spaces. For the further applications of L^2 extension theorems, we shall recall well-known basic facts from the classical theory of several complex variables. The proofs are not given here because they are either elementary or contained in the textbooks such as [B-S].

A (possibly nonreduced) complex space Z is said to be *weakly* 1-*complete* (resp. 1-*complete*) if there exists a C^∞ plurisubharmonic (resp. strictly plurisubharmonic) exhaustion function on Z. Further, Z is called 1-*convex* or *strongly pseudoconvex* if Z admits a C^∞ plurisubharmonic exhaustion function which is strictly plurisubharmonic outside a compact subset of Z.

PROPOSITION 3.1. *Every* 1-*complete space is Stein and vice versa.*

PROPOSITION 3.2. *Every* 1-*convex space is holomorphically convex.*

Let Z be a weakly 1-complete space equipped with a C^∞ plurisubharmonic exhaustion function φ. We put $Z_c = \{z \in Z; \varphi(z) < c\}$.

PROPOSITION 3.3. *A weakly* 1-*complete space Z is holomorphically convex if and only if the sets Z_c are all holomorphically convex.*

A morphism (= a holomorphic map) π from a complex space Z to a complex space T is said to be *locally Stein* (resp. *locally pseudoconvex*) if every point $t \in T$ has a neighborhood U such that $\pi^{-1}(U)$ is a Stein space (resp. a weakly 1-complete space).

PROPOSITION 3.4. *Suppose that a weakly* 1-*complete space Z admits a locally Stein holomorphic map to a Stein space. Then Z is Stein.*

PROPOSITION 3.5. *For any holomorphically convex space Z, there exist a Stein space $\hat{Z}$ and a proper holomorphic map $\beta : Z \to \hat{Z}$ with nonempty connected fibers. Moreover, there exists a Stein space Z^* with a proper and surjective map $\alpha : Z \to Z^*$ with connected fibers such that for any $(\hat{Z}, \beta)$ as above there exists a holomorphic map $\gamma : Z^* \to \hat{Z}$ satisfying $\gamma \circ \alpha = \beta$.*

Z^* is called the *Remmert reduction* of Z.

Combining Theorem 2.2 with Propositions 3.3–3.5, it is easy to deduce the following.

THEOREM 3.1. *Let X be a weakly 1-complete manifold with a C^∞ plurisubharmonic exhaustion function φ and a nonconstant holomorphic map $f : X \to \mathbb{C}^m$ ($m \in \mathbb{N}$) whose level sets are holomorphically convex. Assume that there exists an effective divisor D on X such that the associated line bundle $[D] \to X$ carries a singular fiber metric whose curvature current is strictly negative, $f|_{|D|}$ is proper and $\varphi|_{X \setminus |D|}$ is strictly plurisubharmonic. Then X is holomorphically convex. Here $|D|$ denotes the support of D.*

4. Application to relative exceptional sets. Let Z and T be complex spaces such that there exists a surjective holomorphic map $\alpha : Z \to T$. We say that α is 1-*convex* if every point $t \in T$ has a neighborhood U such that there exists a C^∞ plurisubharmonic function ψ on $\alpha^{-1}(U)$ such that $\alpha|_{\{x;\,\psi(x) \le c\}}$ is proper for every $c \in \mathbb{R}$ and $\psi|_{\{x;\,\psi(x) > c\}}$ is strictly plurisubharmonic for some $c \in \mathbb{R}$. It is easy to see that 1-convex maps are locally pseudoconvex. We say that α is *locally holomorphically convex* if every point $t \in T$ has a neighborhood U such that $\alpha^{-1}(U)$ is holomorphically convex. The following deep result is due to Knorr and Scheider [K-S].

THEOREM 4.1. 1-*convex maps are locally holomorphically convex.*

Theorem 4.1 is the relative variant of Grauert's solution of the Levi problem [G-1]. We recall that the existence of a 1-convex map is crucial for the characterization of relative exceptional sets (see a counterexample in [F]).

By virtue of [H] and [H-R], one can apply Theorem 3.1 to prove the following. It is contained in Theorem 4.1, but might be of some interest because of the proof which is independent of the coherence of direct image sheaves.

THEOREM 4.2. *Let X be a weakly 1-complete manifold of dimension 3 and let $\alpha : X \to \mathbb{D} := \{z \in \mathbb{C};\ |z| < 1\}$ be a 1-convex map with two-dimensional fibers. Suppose that there exists an analytic set $A \subset X$ such that $\alpha|_A$ is proper and there exists a strictly plurisubharmonic function on $X \setminus A$. Then α is locally holomorphically convex.*

Proof. Let $A_0 = \alpha^{-1}(0)$. By a proper modification we may assume that A_0 is a divisor of simple normal crossings. Moreover by a base change we may assume that the codimension of $(d\alpha)^{-1}(0)$ is at least two. In this situation, let D_1 be the union of the noncompact irreducible components of $\alpha^{-1}(0)$ and let D_2 be the union of the compact irreducible components of $\alpha^{-1}(0)$. Then, by a well known criterion for exceptional curves on surfaces, it is easy to see that $[D_1]|_{D_1\cap D_2}$ is positive. Let D be any irreducible component of D_2. Then $D \setminus D_1$ is a modification of an affine algebraic surface. Therefore, by blowing up along the exceptional set in $D \setminus D_1$ for each D, if necessary, we may assume that $[D_2]$ has a singular fiber metric whose curvature current is strictly negative, by using the existence of a strictly plurisubharmonic function on $X \setminus A$. Hence the conclusion follows from Theorem 3.1, by extending holomorphic functions with sufficiently high order of zeros along D_2. ■

5. Proof of Theorem 1.2. The proof will be given under the assumption that X is connected. By Proposition 3.3, it suffices to prove the assertion when the fibers of f have finitely many irreducible components. Let $w \in \mathbb{C}$. Since $f^{-1}(w)$ is two-dimensional and holomorphically convex, its irreducible components are either compact, 1-convex, or properly fibered over a Stein space of dimension one. The conclusion is obvious if some fiber of f is compact. So we assume that no fiber of f is compact. Let Y_w be the union of the irreducible components which are fibered over Stein spaces of dimension one. Then Y_w is obviously holomorphically convex and it is easy to see that Y_w has a weakly 1-complete neighborhood system in X.

Let $\gamma_w : Y_w \to V_w$ be the Remmert reduction. Since K_X is seminegative, the fibers of γ_w are either rational or irreducible elliptic curves. We put $Y = \bigcup_{w\in\mathbb{C}} Y_w$. It is clear that Y is a complex analytic subset of X.

If $\dim Y = 3$, it follows that $X = Y$. Hence the Remmert reduction of X is two-dimensional, weakly 1-complete and admits a nonconstant holomorphic function. Hence it is holomorphically convex by Takegoshi's theorem. Therefore X is holomorphically convex in this case.

If a fiber of γ_w is elliptic for some w, it is easy to see from the seminegativity of K_X that $\dim Y = 3$, so that there is nothing left to prove.

If $\dim Y = 2$ and the fibers of γ_w are rational, then the bundle $[Y]|_{\gamma_w}$ must be negative for all w, because there would be an analytic family of rational curves filling a neigborhood of Y otherwise (cf. [Kd]), which contradicts the definition of Y. (See also [N], [F-N], [F] and [Oh-1].) Hence it follows in this specific case that Y is contractible to its Remmert reduction in X. Hence, combining this observation with Theorem 4.1, we conclude that f induces a locally Stein holomorphic map from the Remmert reduction of X to $\mathbb{C}$. (Note that f can be approximated on compact sets by 1-convex maps in this situation.) Therefore, the conclusion follows from Proposition 3.4. ■

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