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ENDPOINT MAPPING PROPERTIES OF THE LITTLEWOOD–PALEY SQUARE FUNCTION

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Abstract. We give an alternative proof of a theorem due to Bourgain concerning the growth of the constant in the Littlewood–Paley inequality on $\mathbb{T}$ as $p \to 1^+$. Our argument is based on the endpoint mapping properties of Marcinkiewicz multiplier operators, obtained by Tao and Wright, and on Tao's converse extrapolation theorem. Our method also establishes the growth of the constant in the Littlewood–Paley inequality on $\mathbb{T}^n$ as $p \to 1^+$. Furthermore, we obtain sharp weak-type inequalities for the Littlewood–Paley square function on $\mathbb{T}^n$, but when $n \geq 2$, the weak-type endpoint estimate on the product Hardy space over the n-torus fails, in contrast to what happens when $n = 1$.

1. Introduction. If f is a trigonometric polynomial on the torus $\mathbb{T}$, then the Littlewood–Paley square function $S(f)$ of f is given by

$$S(f)(x) = \Big(\sum_{k\in\mathbb{Z}} |\Delta_k(f)(x)|^2\Big)^{1/2},$$

where for $k \in \mathbb{N}$ one defines

$$\Delta_k(f)(x) = \sum_{n=2^{k-1}}^{2^k-1} \widehat{f}(n)e^{i2\pi nx} \quad \text{and} \quad \Delta_{-k}(f)(x) = \sum_{n=-2^k+1}^{-2^{k-1}} \widehat{f}(n)e^{i2\pi nx}$$

and $\Delta_0(f)(x) = \widehat{f}(0)$ for $x \in \mathbb{T}$.

The Littlewood–Paley square function S can be extended as a bounded operator on $L^p(\mathbb{T})$ for all $1 < p < \infty$, namely for each $1 < p < \infty$ there is a constant $C(p)$ such that

$$\|S(f)\|_{L^p(\mathbb{T})} \leq C(p)\|f\|_{L^p(\mathbb{T})}. \tag{1.1}$$

In [3, Theorem 1], Bourgain determined the behaviour of $C(p)$ in (1.1) as $p \to 1^+$. In particular, Bourgain showed that there exist absolute constants $c_1, c_2 > 0$ such that

$$c_1(p-1)^{-3/2} < C(p) < c_2(p-1)^{-3/2} \tag{1.2}$$

for every $1 < p \leq 2$.

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In Section 3 we give a simple proof of the upper estimate in (1.2) based on results of Tao and Wright [12] and Tao [11]. More precisely, using the observation that Marcinkiewicz multipliers locally map $L\log^{1/2} L$ to $L^{1,\infty}$ [12, Theorem 1.2], together with interpolation and Tao's converse extrapolation [11], one deduces that $\| \sum_{k\in\mathbb{Z}} \pm\Delta_k\|_{L^p(\mathbb{T})\to L^p(\mathbb{T})} \lesssim (p-1)^{-3/2}$, which is essentially the upper estimate in (1.2). Furthermore, we extend (1.2) to higher dimensions. Indeed, by using $\| \sum_{k\in\mathbb{Z}} \pm\Delta_k\|_{L^p(\mathbb{T})\to L^p(\mathbb{T})} \lesssim (p-1)^{-3/2}$ and iteration, we obtain higher-dimensional extensions of (1.2) in Section 4. In Section 5 we prove sharp weak-type inequalities for the multi-parameter Littlewood–Paley square function on $\mathbb{T}^n$, and in Section 6 we establish the corresponding weak-type endpoint results on $\mathbb{R}^n$. It is well-known that the Littlewood–Paley square function maps $H^1(\mathbb{T})$ to $L^{1,\infty}(\mathbb{T})$. Motivated by this fact, a natural question is whether the two-parameter Littlewood–Paley square function maps the product real Hardy space $H^1(\mathbb{T}\times\mathbb{T})$ to $L^{1,\infty}(\mathbb{T}^2)$. In Section 7 we show that this is not the case.

2. Notation and background

2.1. Notation and useful facts. If X and Y are positive quantities, the notation $X \lesssim Y$ (or $Y \gtrsim X$) means that there is a positive constant $C>0$ such that $X \le CY$. To specify the dependence of the constant C on some given parameters $r_1,\dots,r_n$, we shall write $X \lesssim_{r_1,\dots,r_n} Y$. If $X \lesssim Y$ and $Y \lesssim X$, we write $X \sim Y$.

Let (X,μ) be a measure space with $\mu(X)=1$ and let $r>0$. We set

$$\|f\|_{L\log^r L(X)} = \int_0^1 f^*(t)\log^r(e/t)\,dt,$$

where $f^*(t) = \inf\{\lambda>0 : \mu(\{x\in X : |f(x)|>\lambda\}) \le t\}$ is the decreasing rearrangement of f. It is well-known that $\|f\|_{L\log^r L(X)} < \infty$ if and only if $\int_X |f(x)|\log^r(1+|f(x)|)\,d\mu(x) < \infty$, and moreover one has (see e.g. [1, proof of Lemma 10.1])

$$\|f\|_{L\log^r L(X)} \lesssim_r 1 + \int_X |f|\log^r(1+|f|)\,d\mu \tag{2.1}$$

and

$$\int_X |f|\log^r(1+|f|)\,d\mu \lesssim_r 1+\|f\|_{L\log^r L(X)} + \|f\|_{L^1(X)}\log^r(\|f\|_{L^1(X)}). \tag{2.2}$$

For more details we refer the reader to [1] and [2].

In the present note, we identify functions on $\mathbb{T}$ with functions on $[0,1)$.

If K_n is the Fejér kernel on $\mathbb{T}$ of order n, then $V_n = 2K_{2n+1} - K_n$ is the de la Vallée Poussin kernel of order n. Since $\|K_n\|_{L^1(\mathbb{T})} = 1$ and $\|K_n\|_{L^\infty(\mathbb{T})} \lesssim n$ for every $n\in\mathbb{N}$, we deduce that $\int_{\mathbb{T}} |V_n(x)|\log^r(1+|V_n(x)|)\,dx \lesssim \log^r(n+1)$

for $r \geq 0$. Moreover, $\widehat{V_n}(j) = 1$ for all $|j| \leq n + 1$ and thus it follows that $\|\Delta_k(V_{2^N})\|_{L^1(\mathbb{T})} \gtrsim k$ for each $k \in \mathbb{N}$ with $k \leq N$.

Let (X, μ) be a given measure space. One has (see [7, p. 485])

$$\|g\|_{L^{1,\infty}(X)} \sim \sup_{\substack{E \subset X: \\ 0<\mu(E)<\infty}} \mu(E)^{-1} \|g\|_{L^{1/2}(E)}. \tag{2.3}$$

2.2. Hardy spaces. We define the real Hardy space $H^1(\mathbb{R})$ to be the space of all integrable functions whose Hilbert transform is also integrable.

The product real Hardy space $H^1(\mathbb{R} \times \mathbb{R})$ is the set of all functions $f \in L^1(\mathbb{R}^2)$ such that $H_1(f), H_2(f), H_1 \otimes H_2(f) \in L^1(\mathbb{R}^2)$, where H_i denotes the Hilbert transform with respect to the ith variable.

Let $R = I \times J$ be a dyadic rectangle in $\mathbb{R}^2$. Following [5], we say that a function a_R is a *rectangle atom* associated to R, if a_R is supported in R, $\|a_R\|_{L^2(\mathbb{R}^2)} \leq |R|^{-1/2}$, $\int_I a_R(x', y)\, dx' = 0$ for every $y \in J$ and $\int_J a_R(x, y')\, dy' = 0$ for every $x \in I$. We define $H^1_{\mathrm{rect}}(\mathbb{R} \times \mathbb{R})$ to be the space spanned by the class of all rectangle atoms, namely

$$H^1_{\mathrm{rect}}(\mathbb{R} \times \mathbb{R}) = \Big\{ \sum_R \lambda_R a_R : a_R \text{ is a rectangle atom and } \sum_R |\lambda_R| < \infty \Big\}.$$

A counterexample of Carleson [4] shows that $H^1_{\mathrm{rect}}(\mathbb{R} \times \mathbb{R})$ is a proper subspace of $H^1(\mathbb{R} \times \mathbb{R})$.

Similarly, by using the periodic Hilbert transform, one defines the real Hardy space $H^1(\mathbb{T})$ and the product real Hardy space $H^1(\mathbb{T} \times \mathbb{T})$. One defines rectangle atoms a_R associated to dyadic rectangles $R \subset \mathbb{T}^2$ as in the euclidean case. In the periodic setting, one also needs to consider constant functions on $\mathbb{T}^2$ and "essentially one-dimensional atoms", that is, functions defined on $\mathbb{T}^2$ that are constant in one variable $a(x, y) = a_I(x)$ (or $a(x, y) = a_I(y)$), a_I is supported in a dyadic interval $I \subset \mathbb{T}$, has mean zero and $\|a_I\|_{L^2(\mathbb{T})} \leq |I|^{-1/2}$. We define $H^1_{\mathrm{rect}}(\mathbb{T} \times \mathbb{T})$ to be the space spanned by rectangle atoms associated to dyadic rectangles in $\mathbb{T}^2$, constant functions on $\mathbb{T}^2$ and "essentially one-dimensional atoms" on $\mathbb{T}^2$. Furthermore, as in the euclidean case, $H^1_{\mathrm{rect}}(\mathbb{T} \times \mathbb{T})$ is a proper subspace of $H^1(\mathbb{T} \times \mathbb{T})$.

For more details on Hardy spaces, we refer the reader to [5] and [6].

3. A new proof of the upper estimate in (1.2). In [12], Tao and Wright proved that if T_m is a Marcinkiewicz multiplier operator acting on functions defined over $\mathbb{R}$ (namely, the corresponding symbol m of T_m is a bounded function on $\mathbb{R}$ and m is of uniform bounded variation over intervals of the form $\pm[2^k, 2^{k+1})$, $k \in \mathbb{Z}$), then it locally maps $L \log^{1/2} L$ to $L^{1,\infty}$. In particular, for every compact set $K \subset \mathbb{R}$ there exists a constant $C_{K,m} > 0$, depending on K and on $A_m = \|m\|_{L^\infty(\mathbb{R})} + \sup_{k\in\mathbb{Z}} \int_{\pm[2^k,2^{k+1})} |dm|$, such that

$$\|T_m(f)\|_{L^{1,\infty}(K)} \le C_{K,m}\|f\|_{L\log^{1/2}L(K)} \tag{3.1}$$

for all measurable functions f supported in K.

By adapting the proof of Tao and Wright to functions defined on the torus, one can show that for every $\omega \in [0,1]$ the prototypical Marcinkiewicz multiplier operator

$$T_\omega = \sum_{k\in\mathbb{Z}} r_k(\omega)\Delta_k$$

acting on functions defined on $\mathbb{T}$ maps $L\log^{1/2}L(\mathbb{T})$ to $L^{1,\infty}(\mathbb{T})$, where $(r_k)_{k\in\mathbb{Z}}$ denotes the set of Rademacher functions indexed by $\mathbb{Z}$. In particular,

$$\|T_\omega(f)\|_{L^{1,\infty}(\mathbb{T})} \le C\|f\|_{L\log^{1/2}L(\mathbb{T})}, \tag{3.2}$$

where $C > 0$ is an absolute constant independent of ω.

Using (3.2) and the fact that T_ω is bounded on $L^2(\mathbb{T})$ with operator norm equal to 1, one can easily show, by using a Marcinkiewicz-type interpolation argument (see Subsection 3.1), that T_ω is bounded from $L\log^{3/2}L(\mathbb{T})$ to $L^1(\mathbb{T})$. In particular, we obtain

$$\|T_\omega(f)\|_{L^1(\mathbb{T})} \lesssim \|f\|_{L\log^{3/2}L(\mathbb{T})},$$

where the implied constant is independent of ω. By Tao's converse extrapolation theorem [11], it follows that

$$\|T_\omega\|_{L^p(\mathbb{T})\to L^p(\mathbb{T})} \le \frac{A}{(p-1)^{3/2}} \quad (\text{as } p\to 1^+) \tag{3.3}$$

where $A > 0$ is a positive constant independent of ω. To complete the proof of the upper estimate in (1.2), we use (3.3) and Khinchin's inequality. More precisely, let $p > 1$ be close to 1 and let f be a trigonometric polynomial. Then, by Khinchin's inequality, we have for every $x \in \mathbb{T}$,

$$S(f)(x) \lesssim \int_{[0,1]} |T_\omega(f)(x)|\,d\omega,$$

where the implied constant is independent of $x \in \mathbb{T}$ and f. Therefore, by integrating over $\mathbb{T}$ and using Minkowski's inequality, we get

$$\begin{aligned}\|S(f)\|_{L^p(\mathbb{T})} &\lesssim \Big(\int_{\mathbb{T}}\Big[\int_{[0,1]} |T_\omega(f)(x)|\,d\omega\Big]^p dx\Big)^{1/p} \le \int_{[0,1]} \|T_\omega(f)\|_{L^p(\mathbb{T})}\,d\omega \\ &\lesssim \int_{[0,1]} \frac{1}{(p-1)^{3/2}}\|f\|_{L^p(\mathbb{T})}\,d\omega = \frac{1}{(p-1)^{3/2}}\|f\|_{L^p(\mathbb{T})},\end{aligned}$$

which is the upper estimate in (1.2).

Remark 3.1. Recently, another proof of Bourgain's result was obtained by Lerner [8].

3.1. A variant of Marcinkiewicz-type interpolation. In this subsection we present the following variant of Marcinkiewicz-type interpolation for $L\log^r L$ spaces that is used in several parts of the paper.

LEMMA 3.2. *Let (X,μ) be a measure space with $\mu(X)=1$ and let $r_0\geq 0$. If T is a sublinear operator that is bounded from $L\log^{r_0}L(X)$ to $L^{1,\infty}(X)$ and bounded on $L^2(X)$, then for every $r\geq 0$ one has*

$$\int_X |T(f)|\log^r(1+|T(f)|)\,d\mu \lesssim_{r,r_0} 1+\int_X |f|\log^{r_0+r+1}(1+|f|)\,d\mu. \tag{3.4}$$

Proof. The proof is a straightforward adaptation of the argument establishing the classical Marcinkiewicz interpolation theorem, as presented e.g. in [10, Chapter 1], and we include it here for the convenience of the reader.

For the case $r_0=0$, see e.g. [10, Section 5.2 in Chapter 1]. Consider now the case where $r_0>0$ and $r>0$; the other one (namely $r_0>0$ and $r=0$) is treated similarly. To prove (3.4) for $r_0,r>0$, note that

$$\int_X |T(f)|\log^r(1+|T(f)|)\,d\mu \lesssim_r 1+\int_1^\infty \mu(\{x\in X:|T(f)(x)|>\lambda\})\Big[\log^r(1+\lambda)+r\frac{\lambda}{\lambda+1}\log^{r-1}(1+\lambda)\Big]\,d\lambda$$

and so, by using the sublinearity of T, one has

$$\int_1^\infty \mu(\{x\in X:|T(f)(x)|>\lambda\})\Big[\log^r(1+\lambda)+r\frac{\lambda}{\lambda+1}\log^{r-1}(1+\lambda)\Big]\,d\lambda \leq I_1+I_2,$$

where

$$I_1=\int_1^\infty \mu(\{x\in X:|T(f^\lambda)(x)|>\lambda/2\})\Big[\log^r(1+\lambda)+r\frac{\lambda}{\lambda+1}\log^{r-1}(1+\lambda)\Big]\,d\lambda,$$

$$I_2=\int_1^\infty \mu(\{x\in X:|T(f_\lambda)(x)|>\lambda/2\})\Big[\log^r(1+\lambda)+r\frac{\lambda}{\lambda+1}\log^{r-1}(1+\lambda)\Big]\,d\lambda$$

and $f^\lambda=f\chi_{\{|f|>\lambda\}}$, $f_\lambda=f\chi_{\{|f|\leq\lambda\}}$.

To handle I_1, we use the assumption that T is bounded from $L\log^{r_0}L(X)$ to $L^{1,\infty}(X)$, (2.1), and then Fubini's theorem as follows:

$$\begin{aligned}I_1 &\lesssim_{r_0} \int_1^\infty\Big(\int_X |f^\lambda|\log^{r_0}(1+|f^\lambda|)\,d\mu\Big)\Big(\frac{\log^r(1+\lambda)}{\lambda}+r\frac{1}{\lambda+1}\log^{r-1}(1+\lambda)\Big)\,d\lambda\\ &\leq \int_X |f(x)|\log^{r_0}(1+|f(x)|)\Big[\int_1^{|f(x)|}\Big(\frac{\log^r(1+\lambda)}{\lambda}+\frac{r\log^{r-1}(1+\lambda)}{\lambda+1}\Big)\,d\lambda\Big]\,d\mu(x),\end{aligned}$$

and thus we deduce that $I_1\lesssim_{r,r_0} 1+\int_X |f|\log^{r_0+r+1}(1+|f|)\,d\mu$.

To get an appropriate bound for I_2, we use the $L^2(X)$-boundedness of T as well as the fact that the function $m : [e^{r_0+r+1}, \infty) \to \mathbb{R}$ given by $m(x) = x(\log^{r_0+r+1} x)^{-1}$ is increasing on $[e^{r_0+r+1}, \infty)$:

$$
\begin{aligned}
I_2 &\lesssim_{r,r_0} 1 + \int_{e^{r_0+r+1}}^{\infty} \left(\frac{\log^r \lambda}{\lambda^2} + \frac{\log^{r-1} \lambda}{\lambda^2} \right) \Big(\int_{e^{r_0+r+1} \le |f| \le \lambda} |f|^2 \, d\mu \Big) \, d\lambda \\
&\le 1 + \int_{e^{r_0+r+1}}^{\infty} \left(\frac{\log^r \lambda}{\lambda^2} + \frac{\log^{r-1} \lambda}{\lambda^2} \right) \Big(\int_{e^{r_0+r+1} \le |f| \le \lambda} |f| \lambda \frac{\log^{r_0+r+1} |f|}{\log^{r_0+r+1} \lambda} \, d\mu \Big) \, d\lambda \\
&\le 1 + \Big[\int_{e^{r_0+r+1}}^{\infty} \left(\frac{1}{\lambda \log^{r_0+1} \lambda} + \frac{1}{\lambda \log^{r_0+2} \lambda} \right) d\lambda \Big] \Big(\int_X |f| \log^{r_0+r+1}(1+|f|) \, d\mu \Big) \\
&\lesssim_{r,r_0} 1 + \int_X |f| \log^{r_0+r+1}(1+|f|) \, d\mu.
\end{aligned}
$$

Therefore, the proof of the lemma is complete. ■

Remark 3.3. If T satisfies the assumptions of Lemma 3.2, then it easily follows from (3.4) combined with (2.1) and (2.2) that

$$
\|T(f)\|_{L \log^r L(X)} \lesssim_{r,r_0} \|f\|_{L \log^{r_0+r+1} L(X)}. \tag{3.5}
$$

Indeed, by using (3.4), (2.1) and (2.2), we have

$$
\begin{aligned}
\|T(f)\|_{L \log^r L(X)} \lesssim_{r,r_0} 1 &+ \|f\|_{L \log^{r_0+r+1} L(X)} \\
&+ \|f\|_{L^1(X)} \log^{r_0+r+1}(\|f\|_{L^1(X)}),
\end{aligned}
$$

and so we deduce that if $\|f\|_{L \log^{r_0+r+1} L(X)} = 1$ then $\|T(f)\|_{L \log^r L(X)} \lesssim_{r,r_0} 1$. By the last implication and scaling, (3.5) follows.

4. Higher-dimensional extension of (1.2). For $n \in \mathbb{N}$ with $n \ge 2$, let S_n denote the n-parameter Littlewood–Paley square function on $\mathbb{T}^n$ initially defined on trigonometric polynomials on $\mathbb{T}^n$ by

$$
S_n(f)(x) = \Big(\sum_{k_1,\dots,k_n \in \mathbb{Z}} |\varDelta_{k_1,\dots,k_n}(f)(x)|^2 \Big)^{1/2},
$$

where $\varDelta_{k_1,\dots,k_n} = \varDelta_{k_1} \otimes \cdots \otimes \varDelta_{k_n}$. The corresponding n-parameter Littlewood–Paley inequality is

$$
\|S_n(f)\|_{L^p(\mathbb{T}^n)} \le C_p(n) \|f\|_{L^p(\mathbb{T}^n)}. \tag{4.1}
$$

Our goal in this section is to show that $C_p(n) \sim (p-1)^{-3n/2}$. As mentioned in the introduction, this can be done quite easily by iteration thanks to the fact that $\|T_\omega\|_{L^p(\mathbb{T}) \to L^p(\mathbb{T})} \lesssim (p-1)^{-3/2}$.

PROPOSITION 4.1. *There exist positive constants $c_1(n), c_2(n)$, depending only on the dimension n, such that*

$$\frac{c_1(n)}{(p-1)^{3n/2}} < C_p(n) < \frac{c_2(n)}{(p-1)^{3n/2}}, \tag{4.2}$$

where $C_p(n)$ is the constant in (4.1).

Proof. To obtain the upper estimate in (4.2), let $\omega_1, \ldots, \omega_n$ be arbitrary numbers in $[0,1]$. Then, by using (3.3) and iteration, we deduce that

$$\|T_{\omega_1} \otimes \cdots \otimes T_{\omega_n}\|_{L^p(\mathbb{T}^n)\to L^p(\mathbb{T}^n)} \le \frac{A^n}{(p-1)^{3n/2}}.$$

As in the one-dimensional case, by using multi-dimensional Khinchin's inequality (see e.g. [10, Appendix D]) and Minkowski's inequality, we obtain

$$\|S_n(f)\|_{L^p(\mathbb{T}^n)} \le \frac{c_2(n)}{(p-1)^{3n/2}} \|f\|_{L^p(\mathbb{T}^n)},$$

where $c_2(n)$ is a constant that depends only on $n \in \mathbb{N}$.

To prove the lower estimate, we use the corresponding argument of Bourgain that shows the lower estimate in (1.2). As in [3], given $p > 1$, take $N \in \mathbb{N}$ to be such that $\log N \sim (p-1)^{-1}$ and set $f = V_N$. Hence $\|f\|_{L^p(\mathbb{T})} \sim 1$, and since $\|S(V_N)\|_{L^p(\mathbb{T})} \gtrsim (p-1)^{-3/2}$, we have

$$\|S_n(V_N \otimes \cdots \otimes V_N)\|_{L^p(\mathbb{T}^n)} = \|S(V_N)\|_{L^p(\mathbb{T})} \cdots \|S(V_N)\|_{L^p(\mathbb{T})} \gtrsim_n (p-1)^{-3n/2},$$

as desired. ∎

It is worth noting that by adapting the method presented in Section 3 one can give an alternative proof of the upper estimate in (4.2). In particular, one can first study the endpoint mapping properties of n-dimensional Marcinkiewicz multiplier operators of the form $T_{\omega_1} \otimes \cdots \otimes T_{\omega_n}$, and then use converse extrapolation to deduce the growth of $C_p(n)$ as $p \to 1^+$ (see also Remark 5.4). The advantage of this indirect approach is that it motivates the study of sharp weak-type inequalities for S_n, which can be regarded as a rudimentary prototype of general Marcinkiewicz multipliers in higher dimensions. This is a problem interesting in its own right.

5. Sharp weak-type estimates for the Littlewood–Paley square function on $\mathbb{T}^n$

5.1. The one-dimensional case. Assume that for some given $r \ge 0$ the Littlewood–Paley square function S satisfies a weak-type inequality of the form

$$\|S(f)\|_{L^{1,\infty}(\mathbb{T})} \le C\|f\|_{L\log^r L(\mathbb{T})}$$

for all trigonometric polynomials f on $\mathbb{T}$, where $C > 0$ is some absolute constant. We shall prove that necessarily $r \ge 1/2$. For this, note that by using the above inequality and the fact that S is bounded on $L^2(\mathbb{T})$, it follows from Lemma 3.2 that

$$\|S(f)\|_{L^1(\mathbb{T})} \lesssim \|f\|_{L\log^{r+1}L(\mathbb{T})}$$

for all trigonometric polynomials f on $\mathbb{T}$. However, if we take $f = V_{2^N}$, then we have

$$\|f\|_{L\log^{r+1}L(\mathbb{T})} \lesssim 1 + \int_{\mathbb{T}} |f| \log^{r+1}(1+|f|) \lesssim N^{r+1},$$

and moreover, by Minkowski's inequality,

$$\begin{aligned}\|S(f)\|_{L^1(\mathbb{T})} \geq \Big\| \Big(\sum_{k=1}^{N} |\Delta_k(V_{2^N})|^2 \Big)^{1/2} \Big\|_{L^1(\mathbb{T})} &\geq \Big(\sum_{k=1}^{N} \|\Delta_k(V_{2^N})\|_{L^1(\mathbb{T})}^2 \Big)^{1/2} \\ &\gtrsim \Big(\sum_{k=1}^{N} k^2 \Big)^{1/2} \gtrsim N^{3/2}.\end{aligned}$$

We thus get $N^{3/2} \lesssim N^{r+1}$ and hence, by letting $N \to \infty$, it follows that the best we can expect is $r \geq 1/2$.

Proposition 5.1. *The Littlewood–Paley square function S satisfies the weak-type inequality*

$$\|S(f)\|_{L^{1,\infty}(\mathbb{T})} \leq C \|f\|_{L\log^{1/2}L(\mathbb{T})} \tag{5.1}$$

for all trigonometric polynomials f on $\mathbb{T}$, where $C > 0$ is an absolute constant.

Proof. This follows immediately from the work of Tao and Wright [12, Theorem 1.2]. In particular, (5.1) can be regarded as a vector-valued version of (3.2).

More precisely, to prove (5.1), let f be a fixed trigonometric polynomial on $\mathbb{T}$. Note that for every measurable subset E of $\mathbb{T}$ with $|E| > 0$, by Khinchin's inequality and Fubini's theorem, there is a choice of $\omega' \in [0,1]$, depending on f and E, such that $\|T_{\omega'}(f)\|_{L^{1/2}(E)} \gtrsim \|S(f)\|_{L^{1/2}(E)}$. Hence, (5.1) follows from (2.3) and (3.2). ∎

5.2. The higher-dimensional case. In this subsection we extend (5.1) to higher dimensions, namely we obtain weak-type estimates for the n-parameter Littlewood–Paley square function S_n. To do this, as in the one-dimensional case, we reduce the problem to the study of the corresponding mapping properties of certain randomised analogues of S_n, namely we study first the mapping properties of Marcinkiewicz multiplier operators of the form $T_{\omega_1} \otimes \cdots \otimes T_{\omega_n}$ on $\mathbb{T}^n$, where $\omega_i \in [0,1]$. Towards this end, note that by using Lemma 3.2 for $r_0 = 1/2$ and induction, one can easily establish sharp weak-type estimates for Marcinkiewicz multiplier operators of the form $T_{\omega_1} \otimes \cdots \otimes T_{\omega_n}$ on $\mathbb{T}^n$, $\omega_i \in [0,1]$.

LEMMA 5.2. *Let $n \in \mathbb{N}$ be a given dimension. For $\omega_1, \dots, \omega_n \in [0,1]$ the n-dimensional Marcinkiewicz multiplier operator $T_{\omega_1} \otimes \dots \otimes T_{\omega_n}$, where T_{ω_i} is as in Section 3, maps $L \log^{a_n} L(\mathbb{T}^n)$ to $L^{1,\infty}(\mathbb{T}^n)$, where $a_n = 1/2 + 3(n-1)/2$, and in particular*

$$\|T_{\omega_1} \otimes \dots \otimes T_{\omega_n}(f)\|_{L^{1,\infty}(\mathbb{T}^n)} \lesssim_n 1 + \int_{\mathbb{T}^n} |f| \log^{a_n}(1+|f|), \tag{5.2}$$

where the implied constant depends only on n and not on $\omega_1, \dots, \omega_n \in [0,1]$.

Proof. We proceed by induction on $n \in \mathbb{N}$. The case $n = 1$ corresponds to (3.2).

Assume now that for some integer $n > 1$ the desired inequality (5.2) holds. To obtain the $(n+1)$-dimensional case, fix an arbitrary $\lambda > 0$ and an f in $L \log^{a_{n+1}} L(\mathbb{T}^{n+1})$. Then, by using Fubini's theorem, we may write

$$|\{(x_1, \dots, x_{n+1}) \in \mathbb{T}^{n+1} : |T_{\omega_1} \otimes \dots \otimes T_{\omega_{n+1}}(f)(x_1, \dots, x_{n+1})| > \lambda\}| = \int_{\mathbb{T}} |\{(x_1, \dots x_n) \in \mathbb{T}^n : |T_{\omega_1} \otimes \dots \otimes T_{\omega_n}(T_{\omega_{n+1}}(f))(x_1, \dots, x_{n+1})| > \lambda\}| \, dx_{n+1}$$

and so, by our inductive hypothesis and Fubini's theorem,

$$\begin{aligned} &\lambda|\{(x_1, \dots, x_{n+1}) \in \mathbb{T}^{n+1} : |T_{\omega_1} \otimes \dots \otimes T_{\omega_{n+1}}(f)(x_1, \dots, x_{n+1})| > \lambda\}| \\ &\qquad \lesssim_n 1 + \int_{\mathbb{T}^n} \Big[\int_{\mathbb{T}} |T_{\omega_{n+1}}(f)| \log^{a_n}(1 + |T_{\omega_{n+1}}(f)|) \, dx_{n+1} \Big] \, dx_1 \cdots dx_n. \end{aligned}$$

Hence, by using (3.4) for $r_0 = 1/2$ and $r = a_n$ together with Fubini's theorem, we get

$$\begin{aligned} \lambda|\{(x_1, \dots, x_{n+1}) \in \mathbb{T}^{n+1} &: |T_{\omega_1} \otimes \dots \otimes T_{\omega_{n+1}}(f)(x_1, \dots, x_{n+1})| > \lambda\}| \\ &\lesssim_n 1 + \int_{\mathbb{T}^n} \Big[\int_{\mathbb{T}} |f| \log^{a_n + 3/2}(1+|f|) \, dx_{n+1} \Big] \, dx_1 \cdots dx_n \\ &= 1 + \int_{\mathbb{T}^{n+1}} |f| \log^{a_n+3/2}(1+|f|) \, dx_1 \cdots dx_{n+1}. \end{aligned}$$

Since $a_{n+1} = a_n + 3/2$, the proof of the lemma is complete. ■

Now, an adaptation of the argument used in the one-dimensional case gives the main result of this subsection.

PROPOSITION 5.3. *For any given $n \in \mathbb{N}$, there is a constant $C_n > 0$ such that the n-parameter Littlewood–Paley square function satisfies the weak-type inequality*

$$\|S_n(f)\|_{L^{1,\infty}(\mathbb{T}^n)} \le C_n \Big[1 + \int_{\mathbb{T}^n} |f| \log^{a_n}(1+|f|) \Big] \tag{5.3}$$

for all trigonometric polynomials f on $\mathbb{T}^n$, where $a_n = 1/2 + 3(n-1)/2$. Moreover, the exponent a_n in (5.3) is sharp.

Proof. As in the one-dimensional case, we use Khinchin's inequality and (2.3) to show that there exists a choice of $\omega_1',\dots,\omega_n' \in [0,1]$ such that

$$\|S_n(f)\|_{L^{1,\infty}(\mathbb{T}^n)} \lesssim_n \|T_{\omega_1'}\otimes\cdots\otimes T_{\omega_n'}(f)\|_{L^{1,\infty}(\mathbb{T}^n)}.$$

Hence, by using (5.2), we deduce that S_n satisfies the desired weak-type inequality (5.3). Note that, arguing as in Remark 3.3, one may replace the right-hand side of (5.3) by $C_n'\|f\|_{L\log^{a_n}L(\mathbb{T}^n)}$.

To prove that the exponent a_n in (5.3) cannot be improved, assume that the inequality holds for some $r \ge 0$. Since S_n is bounded on $L^2(\mathbb{T}^n)$, it follows from Lemma 3.2 that

$$\|S_n(f)\|_{L^1(\mathbb{T}^n)} \lesssim 1 + \int_{\mathbb{T}^n} |f(x_1,\dots,x_n)|\log^{r+1}(1+|f(x_1,\dots,x_n)|)\,dx_1\cdots dx_n.$$

If we take f to be $V_{2^N}\otimes\cdots\otimes V_{2^N}$, then

$$\|S_n(f)\|_{L^1(\mathbb{T}^n)} = \|S(V_{2^N})\|_{L^1(\mathbb{T})}\cdots\|S(V_{2^N})\|_{L^1(\mathbb{T})} \gtrsim N^{3n/2}$$

but $\int_{\mathbb{T}^n}|f|\log^{r+1}(1+|f|) \lesssim N^{r+1}$. Hence, by letting $N\to\infty$, we see that we must have $r \ge -1+3n/2 = a_n$. ∎

REMARK 5.4. As mentioned in Section 4, by using Lemma 5.2, Lemma 3.2, and Tao's converse extrapolation (as in the one-dimensional case), one can give an alternative proof of Proposition 4.1.

6. Endpoint mapping properties of the multi-parameter rough Littlewood–Paley square function in the euclidean case. If f is a Schwartz function on $\mathbb{R}$, we define its rough Littlewood–Paley square function $S_{\mathbb{R}}(f)$ by

$$S_{\mathbb{R}}(f)(x) = \Big(\sum_{k\in\mathbb{Z}} |P_k(f)(x)|^2\Big)^{1/2},$$

where $(P_k(f))^\wedge(\xi) = \chi_{[2^k,2^{k+1})}(\xi)\widehat{f}(\xi) + \chi_{(-2^{k+1},-2^k]}(\xi)\widehat{f}(\xi)$ is the rough Littlewood–Paley projection at frequencies $|\xi|\sim 2^k$, $k\in\mathbb{Z}$. For $n\in\mathbb{N}$, the n-parameter rough Littlewood–Paley square function is given by

$$S_{\mathbb{R}^n}(f)(x) = \Big(\sum_{k_1,\dots,k_n\in\mathbb{Z}} |P_{k_1}\otimes\cdots\otimes P_{k_n}(f)(x)|^2\Big)^{1/2}$$

for f initially belonging to the class of Schwartz functions on $\mathbb{R}^n$.

In the following proposition we show that the n-parameter rough Littlewood–Paley square function on $\mathbb{R}^n$ satisfies weak-type inequalities analogous to the ones obtained in the previous section, if we restrict ourselves to compact subsets of $\mathbb{R}^n$.

PROPOSITION 6.1. *For any given $n\in\mathbb{N}$ and each compact set K in $\mathbb{R}^n$, there is a constant $C_{K,n}>0$ such that the n-parameter Littlewood–Paley* *square function satisfies the weak-type inequality*

$$\|S_{\mathbb{R}^n}(f)\|_{L^{1,\infty}(K)} \leq C_{K,n}\Big[1 + \int_K |f| \log^{a_n}(1+|f|)\Big] \tag{6.1}$$

for each measurable function f supported in K, where $a_n = 1/2+3(n-1)/2$. Moreover, the exponent a_n in (6.1) *is sharp.*

Proof. The argument that establishes (6.1) is similar to the one given in the previous section, where one uses (3.1) instead of (3.2).

It remains to prove sharpness. Consider the one-dimensional case first. For this, assume that for some $r \geq 0$ one has

$$\|S_{\mathbb{R}}(f)\|_{L^{1,\infty}([-1,1])} \lesssim 1 + \int_{[-1,1]} |f| \log^{r}(1+|f|)$$

for every measurable function f supported in $[-1,1]$. Arguing as in Subsection 3.1, we deduce that

$$\|S_{\mathbb{R}}(f)\|_{L^{1}([-1,1])} \lesssim 1 + \int_{[-1,1]} |f| \log^{r+1}(1+|f|) \tag{6.2}$$

for all measurable functions f with $\mathrm{supp}(f) \subset [-1,1]$. To show that $r \geq a_1 = 1/2$, let N be a large positive integer to be chosen later and let ϕ be a fixed Schwartz function such that $\mathrm{supp}(\phi) \subset [-2,2]$ and $\phi|_{[-1,1]} \equiv 1$. Define $g(x) = 2^N \check{\phi}(2^N x)$, $x \in \mathbb{R}$. Then g is a Schwartz function satisfying $\|g\|_{L^1(\mathbb{R})} \sim 1$, $\|g\|_{L^\infty(\mathbb{R})} \lesssim 2^N$, where the implied constants depend only on ϕ and not on N. Hence,

$$\int_{[-1,1]} |g| \log^{r+1}(1+|g|) \lesssim N^{r+1}. \tag{6.3}$$

Using Minkowski's inequality and the fact that $\widehat{g}|_{[-2^N,2^N]} \equiv 1$ we get

$$\begin{aligned}\|S_{\mathbb{R}}(g)\|_{L^1([-1,1])} \geq \Big(\sum_{k\in\mathbb{Z}} \|P_k(g)\|^2_{L^1([-1,1])}\Big)^{1/2} &\geq \Big(\sum_{k=1}^{N} \|P_k(g)\|^2_{L^1([-1,1])}\Big)^{1/2} \\ &\gtrsim N^{3/2}.\end{aligned}$$

Define $f = g\chi_{[-1,1]}$ and $e = g - f$. One can easily check that, by the construction of g, the "error" satisfies $\|e\|_{L^2(\mathbb{R})} \lesssim 1$. Moreover, f is supported in $[-1,1]$ and $\|S_{\mathbb{R}}(g)\|_{L^1([-1,1])} \leq \|S_{\mathbb{R}}(f)\|_{L^1([-1,1])} + \|S_{\mathbb{R}}(e)\|_{L^1([-1,1])}$. By using the Cauchy–Schwarz inequality, $\|S_{\mathbb{R}}(e)\|_{L^1([-1,1])} \leq \sqrt{2}\,\|S_{\mathbb{R}}(e)\|_{L^2([-1,1])} \leq \sqrt{2}\,\|S_{\mathbb{R}}(e)\|_{L^2(\mathbb{R})}$, and since $\|S_{\mathbb{R}}(e)\|_{L^2(\mathbb{R})} = \|e\|_{L^2(\mathbb{R})} \lesssim 1$, we deduce that

$$\|S_{\mathbb{R}}(f)\|_{L^1([-1,1])} \gtrsim N^{3/2}. \tag{6.4}$$

Since $|f| \leq |g|$, (6.3) implies that

$$\int_{[-1,1]} |f| \log^{r+1}(1+|f|) \lesssim N^{r+1}. \tag{6.5}$$

Combining (6.2), (6.4) and (6.5), we get $N^{3/2} \lesssim N^{r+1}$. Letting $N \to \infty$ shows that $r \geq a_1 = 1/2$, as desired.

To prove sharpness in the higher-dimensional case, fix a dimension $n \in \mathbb{N}$ and assume that (6.1) holds for some $r \geq 0$. For f being as above, take $h = f \otimes \cdots \otimes f$. Then h is supported in $[-1,1]^n$, $\int_{[-1,1]^n} |h| \log^{r+1}(1+|h|) \lesssim N^{r+1}$ and

$$\|S_{\mathbb{R}^n}(h)\|_{L^1([-1,1]^n)} = \|S_{\mathbb{R}}(f)\|_{L^1([-1,1])} \cdots \|S_{\mathbb{R}}(f)\|_{L^1([-1,1])} \gtrsim N^{3n/2}.$$

Therefore, we must have $r \geq a_n = 1/2 + 3(n-1)/2$. ■

7. Negative results. It is well-known that $S_{\mathbb{R}}$ maps $H^1(\mathbb{R})$ to $L^{1,\infty}(\mathbb{R})$. Indeed, one may write

$$S_{\mathbb{R}}(f)(x) = \Big(\sum_{k \in \mathbb{Z}} |P_k(f_k)(x)|^2\Big)^{1/2}, \tag{7.1}$$

where $f_k = \widetilde{P}_k(f)$ and $\widetilde{P}_k$ denotes the multiplier operator whose corresponding symbol is $\eta(2^{-k}\cdot)$, η being an even Schwartz function supported in $\pm[1/4,4]$ with $\eta|_{[1,2]} \equiv 1$. By [7, Corollary 2.13, p. 488], one has

$$\Big\|\Big(\sum_{k \in \mathbb{Z}} |P_k(f_k)|^2\Big)^{1/2}\Big\|_{L^{1,\infty}(\mathbb{R})} \leq C \Big\|\Big(\sum_{k \in \mathbb{Z}} |f_k|^2\Big)^{1/2}\Big\|_{L^1(\mathbb{R})}. \tag{7.2}$$

Hence, the estimate $\|S_{\mathbb{R}}(f)\|_{L^{1,\infty}(\mathbb{R})} \lesssim \|f\|_{H^1(\mathbb{R})}$ follows from (7.1) and the fact that the right-hand side of (7.2) is majorised by $A\|f\|_{H^1(\mathbb{R})}$, where $A > 0$ is a constant that depends only on the choice of η (see [9]). Similarly, the Littlewood–Paley square function S maps $H^1(\mathbb{T})$ to $L^{1,\infty}(\mathbb{T})$.

A natural question is whether an analogous weak-type estimate holds for the two-parameter rough Littlewood–Paley square function. In the two-parameter setting, a candidate endpoint function space is the product Hardy space $H^1(\mathbb{R} \times \mathbb{R})$. Our next result shows that such an estimate is not possible in the product setting, as $S_{\mathbb{R}^2}$ does not even locally map $H^1_{\mathrm{rect}}(\mathbb{R} \times \mathbb{R})$ to $L^{1,\infty}(\mathbb{R}^2)$.

PROPOSITION 7.1. *The two-parameter rough Littlewood–Paley square function does not locally map $H^1_{\mathrm{rect}}(\mathbb{R} \times \mathbb{R})$ to $L^{1,\infty}(\mathbb{R}^2)$.*

Proof. Let $N \geq 5$ be a large positive integer to be chosen later. Consider the function $a_N(x) = 2^{N-1} e^{i2\pi 2^{N-1} x} \chi_{[0,2^{-(N-1)})}(x)$. Note that for $x \neq 0$ the kernel of P_{N-1} is given by

$$\int_{2^{N-1}}^{2^N} e^{i2\pi\xi x}\,d\xi + \int_{-2^N}^{-2^{N-1}} e^{i2\pi\xi x}\,d\xi = \frac{e^{i2\pi 2^N x} - e^{i2\pi 2^{N-1}x}}{i2\pi x} + \frac{e^{-i2\pi 2^{N-1}x} - e^{-i2\pi 2^N x}}{i2\pi x}$$

and hence for $8 \cdot 2^{-(N-1)} \le x \le 1$ one has

$$\begin{aligned}
P_{N-1}(a_N)(x) = & \int_{[0,2^{-(N-1)})} 2^{N-1} e^{i2\pi 2^{N-1}y} \frac{e^{i2\pi 2^N(x-y)} - e^{i2\pi 2^{N-1}(x-y)}}{2\pi i(x-y)}\,dy \\
& + \int_{[0,2^{-(N-1)})} 2^{N-1} e^{i2\pi 2^{N-1}y} \frac{e^{-i2\pi 2^{N-1}(x-y)} - e^{-i2\pi 2^N(x-y)}}{2\pi i(x-y)}\,dy \\
= & -\frac{2^{N-1}e^{i2\pi 2^{N-1}x}}{i2\pi} \int_{[0,2^{-(N-1)})} \frac{1}{x-y}\,dy \\
& + \frac{2^{N-1}e^{i2\pi 2^N x}}{i2\pi} \int_{[0,2^{-(N-1)})} \frac{e^{-i2\pi 2^{N-1}y}}{x-y}\,dy \\
& + \frac{2^{N-1}e^{-i2\pi 2^{N-1}x}}{i2\pi} \int_{[0,2^{-(N-1)})} \frac{e^{i2\pi 2^N y}}{x-y}\,dy \\
& - \frac{2^{N-1}e^{-i2\pi 2^N x}}{i2\pi} \int_{[0,2^{-(N-1)})} \frac{e^{i6\pi 2^{N-1}y}}{x-y}\,dy \\
= & I_1^{(N)}(x) + I_2^{(N)}(x) + I_3^{(N)}(x) + I_4^{(N)}(x).
\end{aligned}$$

Note that for each $8 \cdot 2^{-(N-1)} \le x \le 1$ one has

$$|I_1^{(N)}(x)| = \frac{2^{N-1}}{2\pi} \int_{[0,2^{-(N-1)})} \frac{1}{x-y}\,dy \ge \frac{1}{2\pi x}.$$

We shall bound $|I_2^{(N)}(x)|$, $|I_3^{(N)}(x)|$ and $|I_4^{(N)}(x)|$ from above. To bound $|I_2^{(N)}(x)|$, we make use of the cancellation of $e^{-i2\pi 2^{N-1}y}$ over $[0, 2^{-(N-1)})$,

$$\begin{aligned}
|I_2^{(N)}(x)| = & \frac{2^{N-1}}{2\pi} \left| \int_{[0,2^{-(N-1)})} e^{-i2\pi 2^{N-1}y} \left(\frac{1}{x-y} - \frac{1}{x - 2^{-1}\cdot 2^{-(N-1)}} \right) dy \right| \\
\le & \frac{2^{N-1}}{2\pi} \int_{[0,2^{-(N-1)})} \left| \frac{2^{-1}\cdot 2^{-(N-1)} - y}{(x-y)(x - 2^{-1}\cdot 2^{-(N-1)})} \right| dy \le \frac{2}{15\pi x},
\end{aligned}$$

since $x - y \ge x/2$ for all $y \in [0, 2^{-(N-1)})$ and $x - 2^{-1}\cdot 2^{-(N-1)} \ge 15x/16$.

Similarly, $|I_3^{(N)}(x)| \le 2/(15\pi x)$ and $|I_4^{(N)}(x)| \le 2/(15\pi x)$. Therefore,

$$|P_{N-1}(a_N)(x)| \ge |I_1^{(N)}(x)| - |I_2^{(N)}(x)| - |I_3^{(N)}(x)| - |I_4^{(N)}(x)| \ge \frac{1}{10\pi x}$$

for all $8 \cdot 2^{-(N-1)} \le x \le 1$, and hence

$$S_{\mathbb{R}^2}(a_N \otimes a_N)(x,y) \ge |(P_{N-1} \otimes P_{N-1})(a_N \otimes a_N)(x,y)| \ge \frac{1}{100\pi^2 xy}$$

for $(x,y) \in [8 \cdot 2^{-(N-1)}, 1]^2$. It thus follows that

$$\|S_{\mathbb{R}^2}(a_N \otimes a_N)\|_{L^{1,\infty}([0,1]^2)} \gtrsim N.$$

Since $a_N \otimes a_N$ is a rectangle atom, by letting $N \to \infty$ one deduces that $S_{\mathbb{R}^2}$ does not locally map $H^1_{\mathrm{rect}}(\mathbb{R} \times \mathbb{R})$ to $L^{1,\infty}(\mathbb{R}^2)$. ∎

By adapting the proof of the previous proposition we obtain a corresponding negative result in the periodic setting.

Proposition 7.2. *The two-parameter Littlewood–Paley square function S_2 does not map $H^1_{\mathrm{rect}}(\mathbb{T} \times \mathbb{T})$ to $L^{1,\infty}(\mathbb{T}^2)$.*

Proof. Let $N \ge 9$ be an integer to be chosen later. We decompose the kernel of Δ_N as

$$\sum_{n=2^{N-1}}^{2^N-1} e^{i2\pi nx} = \frac{e^{i2\pi 2^N x} - e^{i2\pi 2^{N-1}x}}{e^{i2\pi x} - 1} = \beta_N(x) + \gamma_N(x),$$

where for $x \in (0,1)$ one has

$$\beta_N(x) = (e^{i2\pi 2^N x} - e^{i2\pi 2^{N-1}x})\left(\frac{1}{e^{i2\pi x} - 1} - \frac{1}{i2\pi x}\right),$$

$$\gamma_N(x) = \frac{e^{i2\pi 2^N x} - e^{i2\pi 2^{N-1}x}}{i2\pi x}$$

and $\beta_N(0) = 0$, $\gamma_N(0) = 2^{N-1}$. Define $a_N(x) = 2^{N-1} e^{i2\pi 2^{N-1}x} \chi_{[0,2^{-(N-1)})}(x)$ for $x \in [0,1)$. Arguing as in the proof of Proposition 7.1, one shows that

$$|\gamma_N * a_N(x)| \ge \frac{11}{30\pi x}$$

for all $8 \cdot 2^{-(N-1)} \le x < 1$. Using the series expansion of $e^{i2\pi x}$ and the fact that $\sin(2\pi x) \ge 4x$ for every $0 \le x \le 2^{-2}$, one obtains $|\beta_N(x)| \le \pi e^{\pi/2}$ for all $0 \le x \le 2^{-2}$. Since $\|a_N\|_{L^1(\mathbb{T})} = 1$, it follows that $|\beta_N * a_N(x)| \le \pi e^{\pi/2}$ for every $2^{-(N-1)} \le x \le 2^{-2}$. Therefore, for each $8 \cdot 2^{-(N-1)} \le x \le 2^{-8}$ one has

$$\begin{aligned} |\Delta_N(a_N)(x)| &\ge |\gamma_N * a_N(x)| - |\beta_N * a_N(x)| \\ &\ge \frac{11}{30\pi x} - \pi e^{\pi/2} \ge \frac{11}{30\pi x} - \frac{1}{16x} \sim \frac{1}{x}. \end{aligned}$$

Since we may regard $a_N \otimes a_N$ as an atom of $H^1_{\mathrm{rect}}(\mathbb{T}\times\mathbb{T})$, by letting $N \to \infty$ we deduce that S_2 does not map $H^1_{\mathrm{rect}}(\mathbb{T}\times\mathbb{T})$ to $L^{1,\infty}(\mathbb{T}^2)$. ∎

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