

THE ZIEGLER SPECTRUM OF THE  $D$ -INFINITY PLANE  
SINGULARITY

BY

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**Abstract.** We describe the Cohen–Macaulay part of the Ziegler spectrum of the  $D_\infty$  plane singularity  $S$  and calculate the nilpotency index of the radical of the category of finitely generated Cohen–Macaulay  $S$ -modules.

**1. Introduction.** A local commutative noetherian ring is said to be of *countable* (or *discrete*) *Cohen–Macaulay representation type* if it has only countably (infinitely) many nonisomorphic finitely generated maximal Cohen–Macaulay modules. For complete 1-dimensional surfaces the classification of [3] shows that there are essentially two possibilities:  $R = F[[x, y]]/(x^2)$ , the  $A_\infty$  plane singularity; and  $S = F[[x, y]]/(x^2y)$ , the  $D_\infty$  plane singularity.

The model theory of Cohen–Macaulay  $R$ -modules was investigated in [15]. In particular, the Cohen–Macaulay part of the Ziegler spectrum of  $R$ ,  $ZCM_R$ , was described, and the nilpotency index of the radical of the category of finitely generated Cohen–Macaulay modules was calculated.

In this paper we address similar questions for the ring  $S$ . By factoring out  $x^2$  we obtain a natural surjection  $S \rightarrow R$ , hence an embedding from  $ZCM_R$  onto a closed subset of  $ZCM_S$ . Our initial feeling was that the Cantor–Bendixson rank of points in the former space will jump to higher values via this embedding (see [10] or [16] for this effect). However, this is not the case: we will show that the points preserve their ranks in the ambient space; and the Cantor–Bendixson rank of  $ZCM_S$  equals 2.

From the point of view of the properties we investigate in this paper, the category of Cohen–Macaulay modules over  $S$  is a ‘double’ of the corresponding category of  $R$ -modules. For instance, the Krull–Gabriel dimension of the definable category generated by finitely generated Cohen–Macaulay

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modules equals 2 for  $R$  and for  $S$ . Further, similarly to the  $A_\infty$ -case, from the Auslander–Reiten quiver of the category of finitely generated Cohen–Macaulay  $S$ -modules one can glue the Ringel quilt, which is the Möbius stripe.

This gives a global geometric structure to the above category, including objects, but also morphisms—a part which is so often neglected. Using this realization we will show that the nilpotency index of the radical of this category equals  $\omega \cdot 2$ . Note that, by Schröer [21, Prop. 6.1], for finite-dimensional algebras this index cannot be a limit ordinal.

To classify points of the space  $\text{ZCM}_S$ , i.e. indecomposable pure injective Cohen–Macaulay modules, we will employ the so-called ‘interval method’ which is based on Ziegler’s results [23]. Namely, each point in this space is uniquely determined by a nontrivial cut it creates on a given interval in the lattice of positive primitive formulae. Taking intervals means to take open sets covering the category of finitely generated Cohen–Macaulay modules considered as a surface. We will show that two basic open sets are sufficient to detect all points of this space and calculate their bases of open sets in the Ziegler topology.

Due to many technicalities involved in the proofs, we will be quite concise when executing them in detail. Say, the description of irreducible morphisms between finitely generated Cohen–Macaulay modules over  $S$  is known to experts in this area, but no written account seems to exist. Further, because the model-theoretic part of this paper is similar to the  $A_\infty$ -case, which was carefully explained in [15], we will allow ourselves to be quite sketchy in some instances.

The point of this paper is to propel the method rather than calculations. It proved to be successful for some classes of finite-dimensional algebras and some Cohen–Macaulay singularities. The reason for this is obvious: both finite-dimensional modules and finitely generated Cohen–Macaulay modules over complete Cohen–Macaulay rings are pure injective.

**2. The ring.** In this part we will introduce the object of our interest, the ring  $S$ , and mention its basic properties which will be used in this paper. Let  $F$  be an algebraically closed field of characteristic different from 2, and let  $S = F[[x, y]]/(x^2y)$  be the quotient of the ring of power series  $F[[x, y]]$  by the ideal generated by  $x^2y$ , the so-called  $D_\infty$  *plane singularity*. Many properties of  $S$  can be extracted from fairly general commutative algebra.

In particular,  $S$  is a local complete 1-dimensional commutative noetherian ring with zero divisors. Namely, if  $P$  is a prime ideal of  $S$  then  $x^2y = 0$  yields  $x \in P$  or  $y \in P$  (or both). It follows easily that  $P = xS$ ,  $P = yS$  or  $P = \mathfrak{m} = (x, y)$ , the maximal ideal of  $S$ .

Note that there is a natural isomorphism  $S/x^2S \cong R = F[[x, y]]/(x^2)$ , therefore every  $R$ -module is an  $S$ -module. Further,  $x^2S \cong S/yS \cong F[[x]]$ , hence writing  $F[[x]]$  or  $F((x))$ , the field of Laurent power series, we mean the corresponding  $S$ -module. Also  $xyS \cong S/xS \cong F[[y]]$ , hence the same applies to  $F[[y]]$  and  $F((y))$ .

The ring-theoretical properties of  $R$  and its modules can be found in [15]; we will use them freely. For instance, the quotient ring  $Q_R$  of  $R$  is a uniform module, and the integral closure  $\tilde{R}$  of  $R$  in  $Q_R$  is a nonnoetherian valuation ring of Krull dimension 1. Further,  $\tilde{R}$  can be identified with  $R + xQ_R$ .

Because  $S$  is 1-dimensional, it suffices to invert any nonzero divisor, say  $x + y$ , to obtain the ring of quotients  $Q_S$ , for instance  $Q_S = S[(x + y)^{-1}]$ . Namely,  $T = (x + y)^{-k}$ ,  $k = 1, 2, \dots$ , is a multiplicatively closed set, and the localization  $ST^{-1}$  has only two prime ideals  $xST^{-1}$  and  $yST^{-1}$ , both consisting of zero divisors. It follows that each nonzero divisor in this ring is invertible, hence  $ST^{-1} = Q_S$ .

For basic facts in the theory of Cohen–Macaulay rings and modules see [11] or [22]. We will use the homological definition of Cohen–Macaulayness. Because  $S$  is 1-dimensional, a finitely generated  $S$ -module  $M$  is said to be *Cohen–Macaulay*, CM, if  $\text{Hom}(F, M) = 0$ . Clearly this is equivalent to  $\text{Soc}(M) = 0$ , i.e.  $mx = my = 0$  for  $m \in M$  implies  $m = 0$ . For instance, the ring  $S$  itself is Cohen–Macaulay.

Furthermore,  $S$  is *Gorenstein*, that is, of injective dimension 1. Namely,  $x + y$  is a nonzero divisor, and the ring  $S/(x + y)S \cong F[[x]]/(x^3)$  has simple socle, hence is zero-dimensional Gorenstein (see [6, Prop. 21.5]). Then  $S$  is Gorenstein by [2, Prop. 3.1.19(b)].

Also  $S$  has uniform dimension 2, therefore its injective envelope  $I(S)$  is a decomposable module. In fact, because  $S$  is noetherian, by [2, Prop. 2.9] we have the injective resolution

$$0 \rightarrow S \rightarrow I(S/xS) \oplus I(S/yS) \rightarrow I(S/\mathfrak{m}) \rightarrow 0.$$

From  $S/yS \cong F[[x]]$  and  $x^2y = 0$  it follows that  $I(S/yS)$  is annihilated by  $y$ , hence  $I(S/yS) \cong F((x))$ . Similarly  $S/xS \cong F[[y]]$  implies that  $x^2$  annihilates  $I(S/xS)$ , therefore  $I(S/xS) \cong Q_R$ .

Comparing this with the injective resolution  $0 \rightarrow S \rightarrow Q_S \rightarrow Q_S/S \rightarrow 0$  (see Matlis [12, Thm. 13.1]) we conclude that  $Q_S \cong F((x)) \oplus Q_R$  and  $Q_S/S \cong I(S/\mathfrak{m})$ , the minimal injective cogenerator in the category of  $S$ -modules.

We denote by  $\tilde{S}$  the integral closure of  $S$  in  $Q_S$ . If  $u = x(x + y)^{-1}$  then  $u^2 = u^3$ , hence  $u \in \tilde{S}$ , and  $e = u^2$  is an idempotent in  $\tilde{S}$ . Now we can see the above decomposition of  $Q_S$  more clearly.

LEMMA 2.1.  $Q_S = x^2Q_S \oplus yQ_S$ . Further,  $x^2Q_S = eQ_S \cong F((x))$  and  $yQ_S = (1 - e)Q_S \cong Q_R$ .

*Proof.* Since  $e \in Q_S$  is an idempotent, we have  $Q_S = eQ_S \oplus (1 - e)Q_S$ . Clearly  $eQ_S = x^2Q_S \cong F((x))$ . Further,  $1 - e = y(2x + y)(x + y)^{-2} \in yQ_S$ , therefore  $(1 - e)Q_S \subseteq yQ_S$ . Also  $ye = 0$  implies  $y \in (1 - e)Q_S$ , from which the inverse inclusion follows.

Finally,  $yQ_S \cong Q_R$  by sending  $y$  to 1. ■

We will consider  $Q_S$  as the *universal domain* which contains all objects of our interest. In particular, we identify  $R_S$  with  $yS \subset Q_S$ ; and  $Q_R$ , considered as an  $S$ -module, with  $yQ_S$ .

Recall that an *overring* of  $S$  is any subring of the integral closure  $\tilde{S}$  containing  $S$ . Since the ideal  $\mathbf{m}^2$  requires three generators, it follows that  $S$  is not *Bass*, i.e. there is an overring of  $S$  which is not Gorenstein (see [11, Thm. 4.18]). We will pinpoint such an overring.

LEMMA 2.2. *There is the least proper overring  $S' = \text{End}_S(\mathbf{m})$  of  $S$  generated over  $S$  by  $z = x^2(x + y)^{-1}$ . Further,  $S'$  is a local non-Gorenstein ring which coincides with the endomorphism ring of  $S'$  considered as an  $S$ -module.*

*Proof.* Because  $S$  is Gorenstein, it follows (this fact is called the *rejection lemma* by the Kiev school) that  $S$  has the least proper overring  $S'$ . It can be identified with the endomorphism ring of  $\mathbf{m}$ :  $S' = S[z]$  where  $z = x^2(x + y)^{-1} \in \tilde{S}$ . For instance, multiplication by  $z$  defines an endomorphism of  $\mathbf{m}$  which sends  $x$  to  $x^2$  and  $y$  to zero.

In fact (see [11, Example 14.23]),  $S'$  is isomorphic to the ring obtained from  $F[[x, y, z]]$  by imposing the relations  $z^2 = zx = x^2$  and  $yz = 0$ , the so-called  $E_\infty$ -singularity. Clearly  $S'$  is a local ring whose Jacobson radical is generated by  $x, y, z$ .

Further,  $S'$  is no longer Gorenstein. Namely,  $x + y$  is a nonzero divisor in  $S'$ , and  $S'/(x + y)S'$  is isomorphic to the ring obtained from  $F[[x, z]]$  by imposing the relations  $z^2 = zx = x^2 = 0$ . This ring has a 2-dimensional socle, hence it is not Gorenstein, therefore the same holds true for  $S'$ .

Since the image of each  $f \in \text{End}_S(S')$  is uniquely determined by  $f(1) \in S'$  it follows that  $\text{End}_S(S')$ , coincides with  $S'$ , in particular  $S'_S$  is an indecomposable module. ■

Next we will describe the integral closure of  $S$ . Recall that  $u = x(x + y)^{-1} \in \tilde{S}$  and  $e = u^2 = x^2(x + y)^{-2}$  is an idempotent.

LEMMA 2.3.  $\tilde{S} = S[u] + xyQ_S$ .

*Proof.* Clearly  $xyQ_S \subseteq \tilde{S}$ , because this ideal is nilpotent, therefore  $S[u] + xyQ_S \subseteq \tilde{S}$ .

To prove the converse inclusion note that  $u = x(x + y)^{-1} \in \tilde{S}$  implies  $x^k(x + y)^{-k} \in \tilde{S}$ ; and similarly  $y^k(x + y)^{-k} \in \tilde{S}$ . Further, clearly

$x(x+y)^{-2} \notin S$ . By multiplying this by  $e$  we conclude that  $x^k(x+y)^{-l}$  is not in  $\tilde{S}$  as soon as  $1 \leq k < l$ , and the same holds true for  $y^k(x+y)^{-l}$ . It remains to show that no nonzero element  $(\sum_{i < k} \alpha_i x^i + \sum_{j < k} \beta_j y^j)(x+y)^{-k}$  is in  $\tilde{S}$  if  $k \geq 1$ ; but this is easily checked. ■

From this we obtain the following decomposition of  $\tilde{S}_S$ .

LEMMA 2.4.  $\tilde{S}_S \cong e\tilde{S} \oplus (1-e)\tilde{S}$ , where  $e\tilde{S} = eS$  is isomorphic to  $x^2S = F[[x]]$  and  $(1-e)\tilde{S} = y^2(x+y)^{-1}S + xyQ_S$  is isomorphic to  $\tilde{R}$  (as  $S$ -modules).

*Proof.* Since  $exy = 0$  and  $eu = e$ , by Lemma 2.3 we conclude that  $e\tilde{S} = eS$ . Also  $eS = x^2(x+y)^{-2}S$  is isomorphic to  $x^2S$  as an  $S$ -module.

Further,  $1-e = y(2x+y)(x+y)^{-2}$  equals  $z' = y^2(x+y)^{-2}$  modulo  $xyQ_S$ . Since  $z'u \in xyQ_S$ , we obtain  $(1-e)\tilde{S} = z'S + xyQ_S$ . It is easily checked that the map  $z' \mapsto 1$  extends uniquely to an isomorphism from this submodule onto  $\tilde{R}_S$ . ■

It will be easier to deal with another copy of  $\tilde{R}$  within  $Q_S$ . Namely, it is easily shown that the submodule  $yS + xyQ_S$  is isomorphic to  $\tilde{R}_S$  via the map  $y \mapsto 1$ . We will identify  $\tilde{R}_S$  with this submodule.

We extend the above definition of finitely generated Cohen–Macaulay modules to arbitrary modules. Namely, an  $S$ -module  $M$  is said to be *Cohen–Macaulay* if  $\text{Hom}(F, M) = 0$ , i.e.  $\text{Soc}(M) = 0$ .

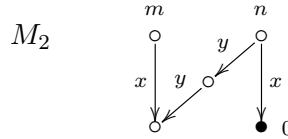
Clearly CM-modules over  $S$  form a *definable class (category)*, i.e. (see [5] or [9]) this class is closed with respect to direct products, direct limits and pure submodules. It is defined by the first order sentence  $\forall x (vx = 0 \wedge vy = 0 \rightarrow v = 0)$ , where  $v$  denotes a variable; therefore we obtain the *theory*  $T_{\text{CM}}$  whose models are CM-modules.

Recall (see [13, Sect. 3.3.4]) that there is a natural one-to-one correspondence between definable categories of modules over any ring  $U$  and first order theories of  $U$ -modules whose models are closed with respect to direct products. We will use these terms synonymously.

**3. Finitely generated CM-modules.** In this section we recall the classification of indecomposable finitely generated CM-modules over  $S$ . It can be deduced from [22, 20, 11]; we will use the notation of [22, p. 78]. In particular, each indecomposable finitely generated CM-module is isomorphic to either an ideal of  $S$ , or a submodule of  $S^2$ . We will mention this realization, but prefer to define such modules using generators and relations.

1)  $S$ ,  $A = x^2S$ ,  $B = yS$ ,  $C = xyS$  and  $D = xS$ . Note that  $Bx^2 = Cx^2 = 0$ , hence  $B$  and  $C$  are  $R$ -modules.

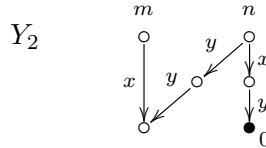
**2)** For each  $k \geq 1$  the module  $M_k$  is given by generators  $m, n$  and relations  $mx = ny^k$ ,  $nx = 0$ ; for instance, here is a diagram for  $M_2$ :



It is easily seen that the map  $m \mapsto y^{k+1}$  and  $n \mapsto xy$  gives an isomorphism from  $M_k$  onto the ideal  $(y^{k+1}, xy)$  of  $S$ . Further,  $M$  is annihilated by  $x^2$ , hence is a CM-module over  $R$ .

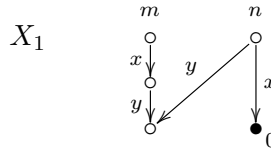
Also  $M_k$  is isomorphic to the submodule of  $Q_S$  generated by  $m = y$  and  $n = xy(x + y)^{-k}$ .

**3)** For each  $k \geq 1$  the module  $Y_k$  is given by generators  $m, n$  and relations  $mx = ny^k$ ,  $nx = 0$ ; here is a diagram for  $Y_2$ :



$Y_k$  is isomorphic to the ideal  $(y^k, x)$  of  $S$  via the map  $m \mapsto y^k$  and  $n \mapsto x$ , in particular  $Y_1 \cong \mathfrak{m}$ .

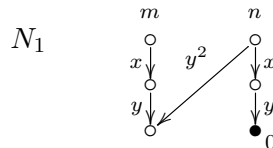
**4)** The module  $X_k$ ,  $k \geq 1$ , is generated by  $m, n$  with relations  $mx = ny^k$  and  $nx = 0$ ; here is a diagram for  $X_1$ :



$X_1$  is isomorphic to  $S'_S$  via the map  $m \mapsto 1$  and  $n \mapsto x - z$ .

Further,  $X_k$  is isomorphic to the submodule of  $S^2$  generated by  $m = (y^k, x)$  and  $n = (xy, 0)$ ; and can be identified with the submodule of  $Q_S$  generated by  $m = 1$  and  $n = xy(x + y)^{-k}$ .

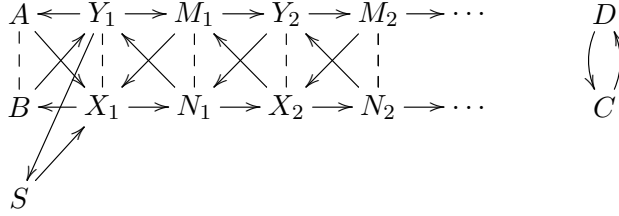
**5)** The module  $N_k$ ,  $k \geq 1$ , is generated by  $m, n$  with relations  $mx = ny^{k+1}$  and  $nx = 0$ ; here is a diagram for  $N_1$ :



$N_k$  is isomorphic to the submodule of  $S^2$  generated by  $m = (y^k, x)$  and  $n = (x, 0)$ .

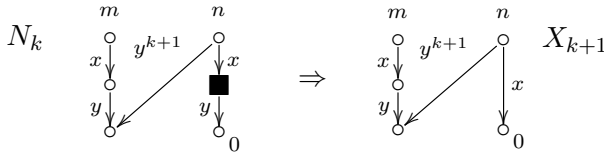
The following is the AR-quiver of the category of finitely generated CM-modules over  $S$  extrapolated from Yoshino [22, p. 78] and [20, p. 25],

ignoring the precise form of irreducible maps:



The irreducible morphisms in this diagram resemble irreducible morphisms between string modules (see [4]) given by graph maps: first factor, then embed. They could be guessed and checked. Here is a complete list:

- 1)  $Y_k \rightarrow M_k$  is given by factoring out  $nx$ .
- 2)  $Y_k \rightarrow N_{k-1}$ ,  $k \geq 2$ , sends  $m \mapsto my$  and  $n \mapsto n$ .
- 3)  $M_k \rightarrow Y_{k+1}$  sends  $m \mapsto m$  and  $n \mapsto ny$ .
- 4)  $M_k \rightarrow X_k$  is given by  $m \mapsto my$  and  $n \mapsto n$ .
- 5)  $X_k \rightarrow N_k$  is provided by  $m \mapsto m$  and  $n \mapsto ny$ .
- 6)  $X_k \rightarrow M_{k-1}$ ,  $k \geq 2$ , is given by annihilating  $mx - ny^{k-1}$ .
- 7)  $N_k \rightarrow Y_k$  is provided by gluing  $mx$  with  $ny^k$ .
- 8)  $N_k \rightarrow X_{k+1}$  is given by annihilating  $nx$ .



Here the black square indicates the element of the first module which is sent to zero by this morphism.

- 9)  $B \rightarrow Y_1$  sends  $y \mapsto m$ , hence we have the inclusion  $yS \subset \mathbf{m}$ .
- 10)  $Y_1 \rightarrow A$  is given by  $n \mapsto x^2$  and  $m \mapsto 0$ . If we identify  $Y_1$  with  $\mathbf{m}$  by setting  $m = y$  and  $n = x$ , then this map is given by multiplication by  $z$ .
- 11)  $X_1 \rightarrow B$  is given by gluing  $mx$  and  $n$ , hence by factoring out  $z$  in  $S' = X_1$ .
- 12)  $S \rightarrow X_1$  sends  $1 \mapsto m$ , hence we have the inclusion  $S \subset S'$ .
- 13)  $Y_1 \rightarrow S$  sends  $m \mapsto y$  and  $n \mapsto x$ , hence we have the inclusion  $\mathbf{m} \subset S$ .

14)  $A \rightarrow X_1$  is given by sending  $x^2 \mapsto mx - n$ .

15)  $C \rightarrow D$  is the inclusion and  $D \rightarrow C$  is given by multiplication by  $y$ .

In Figure 1 we show the unbridged version of the AR-quiver, which includes the relations between irreducible morphisms. We mark the finitely generated CM-modules over  $R$  by solid circles; they form roughly a quarter of the category of finitely generated CM-modules over  $S$ .

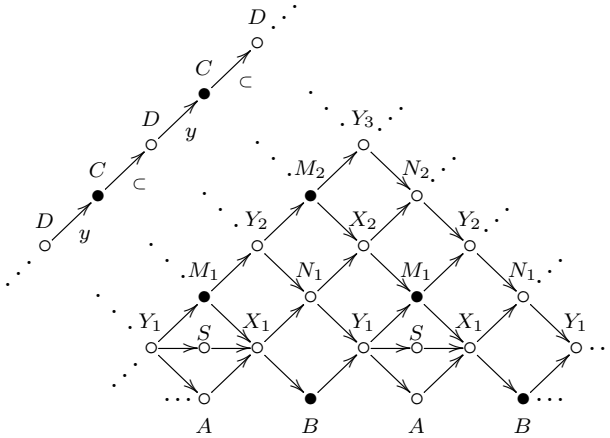


Fig. 1

For instance, we have the following almost split sequences in the category of finitely generated CM-modules:

$$\begin{aligned} 0 \rightarrow M_k \rightarrow Y_{k+1} \oplus X_k \rightarrow N_k \rightarrow 0, \quad 0 \rightarrow N_k \rightarrow Y_k \oplus X_{k+1} \rightarrow M_k \rightarrow 0, \\ 0 \rightarrow Y_k \rightarrow M_k \oplus N_{k-1} \rightarrow X_k \rightarrow 0, \quad k \geq 2, \\ 0 \rightarrow Y_1 \rightarrow M_1 \oplus S \oplus A \rightarrow X_1 \rightarrow 0. \end{aligned}$$

Symmetrically we obtain the following AR-sequences:

$$\begin{aligned} 0 \rightarrow X_k \rightarrow M_{k-1} \oplus N_k \rightarrow Y_k \rightarrow 0, \quad k \geq 2, \\ 0 \rightarrow X_1 \rightarrow N_1 \oplus B \rightarrow Y_1 \rightarrow 0, \\ 0 \rightarrow A \rightarrow X_1 \rightarrow B \rightarrow 0, \quad 0 \rightarrow B \rightarrow Y_1 \rightarrow A \rightarrow 0. \end{aligned}$$

Note that  $S$  is projective and Ext-injective in the category of finitely generated CM-modules, hence no almost split sequence starts or ends with  $S$ . Also neither  $C$  nor  $D$  is a source or sink of an AR-sequence in this category.

Thus the AR-quiver consists of two components: the top one consisting of the modules  $C$  and  $D$ , and the bottom one. To explain why  $C$  or  $D$  are located as shown in Figure 1, note that the composite morphism  $Y_{k+1} \rightarrow N_k \rightarrow Y_k$  is just the inclusion of ideals  $(y^{k+1}, x) \subset (y^k, x)$ . The inverse limit (i.e. the intersection) along the coray of morphisms going northwestward through the  $Y_k$  equals  $xS = D$ . This is why  $D$  is positioned at the end of this coray.

Similarly the composite  $M_k \rightarrow X_k \rightarrow M_{k-1}$  is the inclusion  $(y^{k+1}, xy) \subset (y^k, xy)$ , therefore  $C = xyS$  is the inverse limit of the coray of morphisms going northwestward through the  $M_k$ .

Note that there is a duality of the category of finitely generated CM-modules over  $S$  given by the functor  $\text{Hom}(-, S)$  (see [6, Thm. 21.21] for a



general statement). It is easily checked that this duality interchanges  $X_k$  with  $Y_k$ , but fixes the  $M_k, N_k$  and  $A, B, C, D, S$ .

**4. Some basics in model theory.** In this section we recall a few notions from the model theory of modules which are relevant to this paper. We use Prest's book [13] as the main reference.

Suppose that  $U$  is a commutative ring. A *positive primitive formula*  $\varphi(v)$  in one free variable  $v$  is an existential formula  $\exists \bar{w} (\bar{w}A = v\bar{b})$  in the language of  $U$ -modules, where  $\bar{w} = (w_1, \dots, w_k)$  is a tuple of bound variables,  $A$  is a  $k \times l$  matrix over  $U$ , and  $\bar{b}$  is a column of height  $l$  with entries in  $U$ . If  $M$  is an  $U$ -module and  $m \in M$  then we say that  $M$  *satisfies*  $\varphi(m)$ , written  $M \models \varphi(m)$ , if there exists a tuple  $\bar{m} = (m_1, \dots, m_k)$  of elements in  $M$  such that  $\bar{m}A = m\bar{b}$ .

For instance, if  $r \in U$ , then the *divisibility formula*  $r|v \doteq \exists w (wr = v)$  is satisfied by an element  $m \in M$  iff  $m \in Mr$ . Similarly, the *annihilator formula*  $vs = 0$  holds on  $m \in M$  iff  $ms = 0$ . Let  $\varphi$  and  $\psi$  be pp-formulae. We say that  $\varphi$  *implies*  $\psi$ , written  $\varphi \rightarrow \psi$ , if, for each pair  $m \in M$ , from  $M \models \varphi(m)$  it follows that  $M \models \psi(m)$ . For instance, the divisibility formula  $rs|v$  implies  $s|v$ .

The implication relation defines a partial ordering  $\leq$  on the set of pp-formulas. We say that the pp-formulae  $\varphi$  and  $\psi$  are *equivalent* if both implications  $\varphi \rightarrow \psi$  and  $\psi \rightarrow \varphi$  hold. If we identify equivalent pp-formulas, the resulting poset is a modular lattice  $L$  called the *lattice of pp-formulae* (in one variable) over  $U$ . For instance, the zero formula  $v = 0$  is the least element in this lattice, and the trivial formula  $v = v$  is the largest element of  $L$ . Further, the meet in this lattice is the conjunction of pp-formulae, and the join is the sum of pp-formulae:  $(\varphi + \psi)(v) \doteq \exists w (\varphi(w) \wedge \psi(v - w))$ .

If  $\varphi$  is a pp-formula and  $M$  is an  $U$ -module, then  $\varphi(M) = \{m \in M \mid M \models \varphi(m)\}$  is the *definable submodule* of  $M$ . For instance, the divisibility formula  $r|v$  defines the submodule  $Mr$ , and the annihilator formula  $vs = 0$  defines the submodule consisting of  $m \in M$  such that  $ms = 0$ . This way we obtain the *lattice of pp-definable submodules of  $M$* ,  $L(M)$ , which is the factor of the lattice  $L$  of pp-formulae by a certain congruence relation.

The correspondence  $M \mapsto \varphi(M)$  defines a finitely generated subfunctor of the functor  $\text{Hom}(U, -)$  from the category of finitely presented  $U$ -modules to the category of abelian groups; in fact, pp-formulae can be introduced as such additive functors.

There are 'minimal' realizations of pp-formulae, called free realizations. Namely, we say that the pointed module  $(M, m)$  is a *free realization of a pp-formula  $\varphi$*  if 1)  $M \models \varphi(m)$  and 2) if  $M \models \psi(m)$  for some pp-formula  $\psi$ , then  $\varphi$  implies  $\psi$ . Though a minimal realization is never unique, each pp-formula has a free realization  $(M, m)$  such that  $M$  is a finitely presented

module. For instance,  $(U, r)$  is a free realization of the divisibility formula  $r|x$ , and  $\bar{1} \in U/sU$  freely realizes the annihilator formula  $vs = 0$ . We will often write  $m \in M$  to denote the pointed module  $(M, m)$ . Further, each finitely presented pointed module  $m \in M$  is a free realization of some pp-formula  $\varphi$  (which generates the pp-type of  $m$  in  $M$ , see definitions below).

If  $M$  is finitely presented, then the above condition 2) means that for each pointed module  $(N, n)$  satisfying  $\varphi$  there is a *pointed morphism* from  $(M, m)$  to  $(N, n)$ , i.e. a morphism  $f : M \rightarrow N$  such that  $f(m) = n$ .

All this can be relativized to any definable class of modules. For instance, let us consider the class of Cohen–Macaulay modules over  $S$ . Some nonequivalent pp-formulae in  $L$  get identified when restricted to CM-modules, hence we obtain the *lattice* of CM-formulae,  $L_{\text{CM}}$ , as a proper factor of  $L$ . The following arguments show that this definable class is *covariantly finite* (see [8]), i.e. admits finitely generated free realizations.

Take any pp-formula  $\varphi$  over  $S$  and choose a free realization  $m \in M$  for this formula, where  $M$  is a finitely generated  $S$ -module. Factor out the socle of  $M$  obtaining  $M' = M/\text{Soc}(M)$ , then kick off the socle of  $M'$  and so on. Since  $M$  is noetherian, in finitely many steps we will obtain a CM-module  $\bar{M}$  over  $S$ . Let  $\bar{m}$  denote the canonical image of  $m$  in this module and let a pp-formula  $\varphi_{\text{CM}}$  generate the pp-type of  $\bar{m}$  in  $\bar{M}$ . Directly from the definition we obtain the following:

- 1)  $\bar{\varphi}$  implies  $\varphi$  in the theory of all  $S$ -modules.
- 2)  $\bar{\varphi}$  is equivalent to  $\varphi$  in the theory of CM-modules over  $S$ .

We will call formulae of the form  $\varphi_{\text{CM}}$  *CM-formulae* and use pointed modules  $\bar{m} \in \bar{M}$  to denote them. Thus the lattice  $L_{\text{CM}}$  can be described as follows. It consists of the equivalence classes of pointed finitely generated CM-modules over  $S$ , where  $(M, m) \leq (N, n)$  if there exists a pointed morphism from the latter module to the former. For instance, the formula  $x^2 \in S$  implies  $x \in S$ , but these formulae are not equivalent.

Further, the sum of CM-formulas  $m \in M$  and  $n \in N$  is given by the pointed module  $(m, n) \in M \oplus N$ , and their conjunction is defined using the following pushout:

$$\begin{array}{ccc} (R, 1) & \longrightarrow & (M, m) \\ \downarrow & & \downarrow \\ (N, n) & \twoheadrightarrow & (K, k) \xrightarrow{\pi} (\bar{K}, \bar{k}) \end{array}$$

**5. The first interval.** In this section we will create the first cut along the category  $\text{CM}_{\text{fg}}$  of finitely generated CM-modules over  $S$ , i.e. we calculate a particular interval in the lattice of CM-formulae, which later will

be used as a net to catch indecomposable pure injective modules. For this we use the following notion borrowed (with some adjustments) from Ringel [17, Sect. 3], where he calculates the hammock functors.

Suppose that  $(M, m)$  is a pointed module, where  $M$  is an indecomposable finitely generated CM-module over  $S$ . Consider the set  $P(M)$  of all pointed modules  $(N, n)$  such that  $N$  is an indecomposable finitely generated CM-module, and there is a pointed morphism from  $(M, m)$  to  $(N, n)$ . In particular, if  $(M, m)$  realizes a pp-formula  $\varphi$  freely and  $(N, m)$  is a free realization of  $\psi$ , then  $\psi$  implies  $\varphi$ .

We introduce a partial order on  $P$  by setting  $(N, n) \geq (K, k)$  if there is a pointed morphism from the former module to the latter. Finally, we factor this collection of pointed modules by the corresponding equivalence relation ( $\leq$  and  $\geq$ ) to obtain the *pattern* of  $M$ . Thus  $P(M)$  is a partially ordered set with the least element (corresponding to the zero morphism) and the largest element  $1_M$ ; however, it is not clear that this poset is a lattice.

We keep in mind the interpretation of this poset in terms of pp-formulae: if  $(M, m)$  is a free realization of all  $\varphi$ , then  $P(M)$  consists of all formulae in the interval  $[v = 0, \varphi]_{\text{CM}}$  with indecomposable free realizations, i.e. of formulae which are *+irreducible*. Thus the pattern is the ‘frame’ of the corresponding interval in the lattice of CM-formulae.

Because we know indecomposable finitely generated CM-modules (see Section 3), the calculation of patterns is straightforward. Here is an important example.

**PROPOSITION 5.1.** *The diagram in Figure 2 shows the pattern of the pointed module  $(S, x^2)$ .*

*Proof.* To calculate this pattern we should look at the image of  $x^2$  under morphisms  $f$  from  $S$  to indecomposable finitely generated CM-modules  $M$ . Further, we will identify morphisms  $f$  and  $g$  such that  $f(x^2)$  and  $g(x^2)$  are free realizations of equivalent pp-formulas. For example, if  $f(x^2) = 0$  then  $f$  represents the zero element in the pattern. Also the irreducible morphism  $S \rightarrow X_1$  which sends  $1 \mapsto m$  shows that  $(S, x^2)$  covers  $(X_1, mx^2)$  in the pattern.

Note that there is an almost split morphism  $f : X_1 \rightarrow N_1 \oplus B$ , and the image of  $mx^2$  via this map is  $(mx^2, 0)$ ; hence  $f$  is equivalent to the irreducible morphism  $X_1 \rightarrow N$  given by the first component of  $f$ . It follows that  $mx^2 \in N_1$  is the largest formula strictly below  $mx^2 \in X_1$ . The rest is by inspection. ■

We have already noticed that Figure 2 represents the structure of the interval below the formula  $x^2 \mid v$  in the lattice of CM-formulae. We will re-

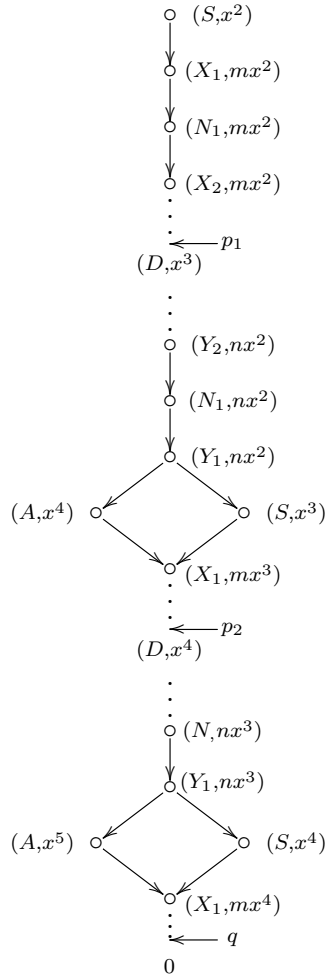


Fig. 2

cover the whole structure of this interval. First, we will prove an auxiliary lemma.

LEMMA 5.2. *The interval  $[v = 0, x^2 \mid v]$  in the lattice  $L_{\text{CM}}$  of all CM-formulae over  $S$  is distributive.*

*Proof.* Recall that a module is said to be *uniserial* if its lattice of submodules is a chain.

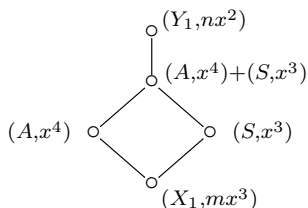
By [14, Prop. 4.4] it suffices to prove that the formula  $x^2 \mid v$ , when evaluated on an indecomposable finitely generated CM-module  $M$  over  $S$ , defines a submodule which is uniserial as a module over the endomorphism ring of  $M$ . For instance, there is nothing to prove for  $M_k$ , since  $M_k x^2 = 0$ .

The most difficult case is  $N_k$ ; let  $V = \text{End}(N_k)$ . Note that there exists an endomorphism of  $N_k$  sending  $m \mapsto n$ ,  $n \mapsto 0$ ; and there is an endomorphism such that  $m \mapsto my - n$ ,  $n \mapsto mx$ . Using this we conclude that nonzero endosubmodules of  $N_k$  contained in  $N_k x^2$  form the following chain:  $Vmx^2 \supset Vnx^2 \supset Vmx^3 \supset \dots$ . ■

Now we are in a position to describe the lattice of CM-formulae in the above interval. Since every finitely generated CM-module is a direct sum of indecomposable modules, each formula in this interval is equivalent to a sum  $\varphi_1 + \dots + \varphi_n$  of formulae with free realization given by modules in Figure 2. Of course, if  $\varphi_i$  implies  $\varphi_j$  in this sum, the former formula can be skipped. Because most formulas in the diagram are comparable, it remains to consider what happens if we add, say,  $x^4 \in A$  to  $x^3 \in S$ .

This situation admits a very general analysis. By using distributivity (Lemma 5.2) and the fact that the endomorphism ring of each indecomposable CM-module in the diagram is local, as in [15, Prop. 4.3] we derive the following. If  $\varphi = \sum_i \varphi_i$  and  $\psi = \sum_j \psi_j$ , where  $\varphi_i$  and  $\psi_j$  are pp-formulae in the diagram, then  $\varphi$  implies  $\psi$  only for the trivial reason: if for each  $i$  there exists  $j$  such that  $\varphi_i \leq \psi_j$  in the pattern.

Thus to get the interval  $[v = 0, x^2 \mid v]$  from the pattern we need only insert additional formulas  $(A, x^{k+1}) + (S, x^k)$ ,  $k \geq 3$ , above each summand and strictly below the formula  $(Y_1, nx^{k-1})$ . For instance, we will obtain the following diagram below  $(Y_1, nx^2)$ , where the top interval is simple. The diagram corresponds to the projection  $Y_1 \rightarrow A \oplus S$  onto the first two coordinates of the left almost split morphism  $Y_1 \rightarrow A \oplus S \oplus M_1$ .



It also follows that conjunction of pp-formulas in this interval coincides with intersection in the pattern. Thus the pattern consists of  $+$ -irreducible elements of the interval, and is a lattice itself, but the sums in these lattices may be different.

**6. The Ziegler spectrum.** In this section we classify points of the Cohen–Macaulay part of the Ziegler spectrum of  $S$ .

Let  $U$  be a commutative ring. An inclusion  $M \subseteq N$  of  $U$ -modules is said to be *pure* if, for each pp-formula  $\varphi$  and any  $m \in M$ , from  $N \models \varphi(m)$  it follows that  $M \models \varphi(m)$ . For instance, each injective module is pure in any module containing it. This definition obviously extends to embeddings of modules.

A module  $M$  is said to be *pure injective* if it is injective with respect to pure embeddings. For instance, each injective module is pure injective, as is any module which is linearly compact over its endomorphism ring (see [18]). Since  $U$  is commutative, each linearly compact module is pure injective.

The *Ziegler spectrum of  $U$* ,  $\text{Zg}_U$ , is a topological space whose points are indecomposable pure injective modules. The topology is given by *basic open sets*  $(\varphi/\psi)$ , where  $\varphi$  and  $\psi$  are pp-formulae. Here  $(\varphi/\psi)$  consists of points  $M \in \text{Zg}_S$  such that  $\varphi(M)$  is not contained in  $\psi(M)$ . For instance, if  $r \in U$ , then the pair  $vr = 0$  over  $v = 0$  defines the open set consisting of all indecomposable pure injective modules with  $r$ -torsion. Each basic open set is compact, in particular  $\text{Zg}_U$  is a quasi-compact space which is usually non-Hausdorff.

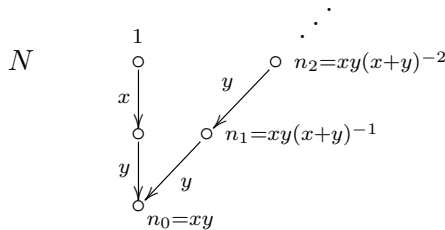
Definable classes (categories)  $\mathcal{D}$  of  $U$ -modules can be identified with closed subsets of the Ziegler spectrum whose points are indecomposable pure injective modules in  $\mathcal{D}$ , and the topology is induced from the whole spectrum. This way we obtain the Ziegler spectrum of this category. For instance, by considering the category of Cohen–Macaulay  $S$ -modules we will obtain the *Cohen–Macaulay part of the Ziegler spectrum of  $S$* , written  $\text{ZCM}_S$ . To investigate this topological space is the main objective of this paper.

First we show there is a sufficient supply of points in this space.

LEMMA 6.1. *Each indecomposable finitely generated CM-module over  $S$  is pure injective, hence is the point of  $\text{ZCM}_S$ . The same holds true for  $Q_R, \tilde{R}$ , and the generic modules  $G_y = F((y)), G_x = F((x))$ .*

*Proof.* It is not difficult to check that each module in the above list is linearly compact, which implies pure injectivity. For instance, for  $R$ -modules in this list, hence for  $B, C, M_k$  and  $Q_R, \tilde{R}, G_y$ , this follows from [15]. ■

We need to add one more module to this list. If we identify the module  $X_k$  with the submodule of  $Q_S$  generated by 1 and  $n_k = xy(x+y)^{-k}$ ,  $k = 1, 2, \dots$ , then these modules form the natural ray of inclusions, via  $n_k \mapsto n_{k+1}y$ , going in the northeastern direction in Figure 1. The direct limit along this ray will give us the CM-module  $N$ , which is the submodule of  $Q_S$  generated by 1 and the  $n_k$ .



LEMMA 6.2.  $N$  is indecomposable and pure injective as an  $S$ -module, hence is a point in  $\text{ZCM}_S$ .

*Proof.* Let  $N'$  be the submodule of  $N$  generated by the  $n_k$ , hence  $N' = xyQ_S$ . Clearly  $N'$  is isomorphic to  $G_y$ , hence uniserial. Further,  $N'$  is a linearly compact module, so  $N/N' \cong D = xS$ . It follows that  $N$  is linearly compact, hence pure injective.

It is straightforward to check that the endomorphism ring of  $N_S$  is  $N$  itself, which is a local ring. Thus  $N_S$  is an indecomposable module, hence is a point in  $\text{ZCM}_S$ . ■

The following property is similar to what is known for finite-dimensional algebras (see [13, Cor. 5.3.37]).

LEMMA 6.3. *Finitely generated points are dense in the CM-part of the Ziegler spectrum of  $S$ .*

*Proof.* Clearly each CM-module over  $S$  is a direct limit of its finitely generated submodules, which are Cohen–Macaulay, hence the result follows. ■

To approach the classification of points in  $\text{ZCM}_S$ , we need more explanations. Let  $M$  be a module over a commutative ring  $U$ , and let  $m$  be a (usually nonzero) element of  $M$ . The *positive primitive type*  $p = \text{pp}_M(m)$  consists of all pp-formulae which are satisfied by  $m$  in  $M$ . Clearly this set is a *filter*, i.e. it is closed with respect to finite conjunctions and implications. The converse is also true: for each filter  $p$  in the lattice of pp-formulae there exists a module  $M$  and an  $m \in M$  such that  $p = \text{pp}_M(m)$ . There always exists a minimal pure injective module  $M$  realizing  $p$ , which is unique up to an isomorphism over the realization. This module is called the *pure injective envelope* of  $p$ , written  $\text{PE}(p)$ .

We say that a pp-type  $p$  is *indecomposable* if its pure injective envelope is an indecomposable module. There is an important syntactical criterion which recognizes the indecomposability of a type. We will denote by  $p^-$  the complement of  $p$ , i.e. the collection of pp-formulae which are not in  $p$ . For an indecomposable pp-type this collection is almost a cofilter.

FACT 6.4 (see [23, Thm. 4.4]). *A pp-type  $p$  is indecomposable iff for each  $\varphi_1, \varphi_2 \in p^-$  there exists  $\varphi \in p$  such that  $(\varphi_1 \wedge \varphi) + (\varphi_2 \wedge \varphi) \in p^-$ .*

Suppose that  $\varphi$  and  $\psi$  are pp-formulae such that  $\varphi$  does not imply  $\psi$ , hence we have the nontrivial interval  $[\psi \wedge \varphi, \varphi]$  in the lattice of pp-formulae. Suppose that  $p$  is a pp-type such that  $\varphi \in p$  and  $\psi \in p^-$ . Then  $p$  defines, by restriction, a *cut* on this interval, whose upper part (i.e. the formulae in  $p$  from this interval) is a filter. The following fact shows that (like analytic functions) the pure injective envelope of  $p$  can be recovered from any cut, no matter how small.

FACT 6.5 (see [23, Cor. 4.16]). *Let  $p$  be an indecomposable pp-type with the pure injective envelope  $M$ . If  $\varphi \in p$  and  $\psi$  in  $p^-$  are pp-formulae, then the isomorphism type of  $M$  is uniquely determined by the cut defined by  $p$  on the interval  $[\varphi \wedge \psi, \varphi]$ . Further, the open sets  $(\varphi' / \psi')$ , where  $\psi \leq \psi' < \varphi' \leq \varphi$  and  $\varphi' \in p$ ,  $\psi' \in p^-$ , form a basis of open sets for  $M$  in the Ziegler topology.*

For instance, the irreducibility criterion from Fact 6.4 can be used locally. Namely, suppose that  $p$  is predefined by a filter in a given interval, and is indecomposable there, i.e. satisfies the above conditions with the  $\varphi_i$  and  $\varphi$  within the interval. Then  $p$  can be extended to an indecomposable pp-type, and all such extensions (no matter how different) lead to isomorphic indecomposable pure injective modules.

All this relativizes to any definable class of modules, with obvious changes. From this we derive the general approach to a classification of CM-modules over  $S$ . First choose a ‘nice’ interval whose lattice structure is known. Then using Ziegler’s criterion describe indecomposable pp-types in this interval (or rather cuts of such types). Finally, find indecomposable pure injective modules which realize these pp-types locally. This will complete the classification.

Now we are in a position to classify points of  $\text{ZCM}_S$ .

THEOREM 6.6. *Let  $M$  be an indecomposable pure injective Cohen–Macaulay module over  $S$ . Then either it is finitely generated, or  $M$  is isomorphic to one of the modules  $Q_R$ ,  $\tilde{R}$ ,  $N$ ,  $G_y = F((y))$  or  $G_x = F((x))$ .*

*Proof.* If  $M$  is annihilated by  $x^2$ , then it is pure injective indecomposable CM-module over  $R$ . It follows from [15] that  $M$  is one of the finitely generated modules  $B, C, M_k$ , or one of the modules  $Q_R, \tilde{R}$  or  $G_y$ .

Otherwise there exists a nonzero  $m \in Mx^2$ . Thus if  $p = \text{pp}_M(m)$  then  $x^2 \mid v \in p$  and  $v = 0 \in p^-$ , hence this type (therefore the module) is visible within the interval in Figure 2. Furthermore, to apply the Ziegler criterion for indecomposability (Fact 6.4) note that this interval is a distributive lattice. Thus a pp-type  $p$  within this interval is indecomposable iff  $p^-$  is a cofilter, i.e.  $\varphi_1 + \varphi_2 \in p^-$  for all pp-formulas  $\varphi_i$  in the negative part of  $p$ .

It easily follows that any cut on the frame gives an indecomposable pp-type. The *principal cut* whose upper part is generated by one pp-formula is realized in the corresponding indecomposable finitely generated CM-module. For instance, the cut whose positive part is generated by  $(D, x^3)$  corresponds to the module  $D$ .

What remains is to identify the modules which are produced by the infinitely generated cuts  $p_i$ ,  $i \geq 1$ , and by  $q$  (see Figure 2). The latter case



is easy to resolve. Namely, the element  $1 \in G_x = F((x))$  is divisible by any power of  $x$ , hence defines the required cut. Thus  $\text{PE}(q) \cong G_x$ .

Now it is easily checked that the element  $x^2 \in N$  realizes the pp-type whose restriction to our interval is  $p_1$ ; furthermore,  $x^k$ ,  $k > 2$ , realizes in  $N$  the pp-type whose restriction coincides with  $p_{k-1}$ . Thus we have filled in all vacant positions for points. ■

Recall that a pair  $(\varphi/\psi)$  of pp-formulae is said to be *minimal* in a theory  $T$  if  $\psi < \varphi$  and there is no pp-formula strictly between  $\psi$  and  $\varphi$  in the lattice of pp-formulas relativized to  $T$ .

From Fact 6.5 it is easy to construct a basis of open sets for each point visible in Figure 2. For instance, we immediately obtain the following.

LEMMA 6.7. *Each finitely generated point which occurs in Figure 2, except  $D$ , is isolated by a minimal pair in the theory  $T_{\text{CM}}$  of CM-modules over  $S$ .*

This follows from Fact 6.5.

For example,  $X_1$  is isolated by the pair  $mx^2 \in X_1$  over  $mx^2 \in N_1$ , which corresponds to the irreducible map  $X_1 \rightarrow N_1$ .

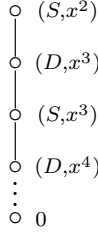
Now we proceed with calculating the Cantor–Bendixson rank of points. The *Cantor–Bendixson analysis* of a closed subset  $T$  of the Ziegler spectrum runs as follows. The first derivative  $T'$  is obtained from  $T$  by deleting all isolated points. Those points are assigned CB-rank 0. Further,  $T''$  is obtained from  $T'$  by deleting its isolated points, which are assigned CB-rank 1, and so on. If  $T'''$  is an empty set, then we say that the CB-rank of  $T$  equals 2.

There is a parallel analysis of the corresponding definable category which leads to the notion of *Krull–Gabriel dimension* (see [9, Ch. 7]). Suppose that the *isolation condition* holds true, i.e. (see [13, Sect. 5.3.2]) on each stage of the CB-analysis, each point is isolated by a minimal pair. Then both procedures give the same value. This means that that lattice of pp-formulae of the next derivative of  $T$  is obtained from the previous one by collapsing each interval of finite length. The corresponding dimension of the lattice is called the *m-dimension* in [13, Sect. 7.7.2], and coincides with the Krull–Gabriel dimension of the definable category.

We apply this analysis to the theory  $T_{\text{CM}}$  of CM-modules over  $S$ . We will see below that each isolated point in  $T_{\text{CM}}$  is isolated by a minimal pair. It follows that the lattice of pp-formulae  $L_1$  of  $T'_{\text{CM}}$  is obtained from the lattice  $L$  of CM-formulae over  $S$  by collapsing intervals of finite length.

All this is applicable to any given interval, say, to the one in Figure 2. We conclude that the interval  $[v = 0, x^2 \mid v]$  in  $T'_{\text{CM}}$  is the following chain of

order type  $1 + \omega^*$ :



Now we are in a position to calculate the rank of  $D$ .

LEMMA 6.8.  $D$  is isolated in the first derivative  $T'_{\text{CM}}$  by the minimal pair  $(D, x^3)$  over  $(S, x^3)$ , in particular the Cantor–Bendixson rank of  $D$  is 1.

*Proof.* From Figure 2 we conclude that the basis of open sets for  $D$  in the Ziegler topology is given by pairs of formulae  $x^3 \in D$  over  $nx^k \in Y_k$ ,  $k = 1, 2, \dots$ , in particular  $D$  is not isolated. It remains to look at the above diagram. ■

To include the module  $N$  in our considerations, we need one more definition. A point  $M$  in a closed set  $T$  is said to be *neg-isolated* (relative to  $T$ ) if there is a nonzero  $m \in M$  such that the negative part of the pp-type of  $m$  contains the largest formula. By [13, Prop. 5.3.46] this notion does not depend on the choice of  $m$ .

LEMMA 6.9.  $N$  is isolated in  $T'_{\text{CM}}$  by the minimal pair  $x^2 \in S$  over  $x^3 \in D$ , in particular  $N$  has CB-rank 1. Further,  $N$  is neg-isolated in  $T_{\text{CM}}$ .

*Proof.* The basis of open sets for  $N$  is given by pairs  $mx^k \in X_k$ ,  $k = 1, 2, \dots$ , over  $x^3 \in D$ , in particular this point is neg-isolated and not isolated. It remains to look at the above diagram. ■

Thus the only remaining point in the above interval is  $G_x = \text{PE}(q)$ .

LEMMA 6.10.  $G_x$  has Cantor–Bendixson rank 2.

*Proof.* From Figure 2 we see that the basis of open sets for  $G_x$  is given by pairs  $x^k \in S$ ,  $k = 1, 2, \dots$ , over 0, hence  $q$  is a pp-type maximal over 0 (usually called *critical*). It easily follows that  $G_x$  is not isolated on levels zero and one of the CB-analysis.

By applying the  $m$ -dimension analysis to the interval from the above diagram, we obtain the two-point interval. It follows that  $G_x$  is the only point in the interval  $x^2 \in S$  over 0 in  $T''_{\text{CM}}$ . ■

**7. The second interval.** To complete the description of the topology of  $\text{ZCM}_S$  we need to find the bases of open sets for points which are not visible in Figure 2, hence for indecomposable pure injective CM-modules over  $R$ . Here is a complete list of those points:

- 1)  $B$  and  $M_k$ ,  $k \geq 1$ ;
- 2)  $C$  and  $\tilde{R}$ ;
- 3)  $G_y = F((y))$  and  $Q_R$ .

From [15] we know that the CB-ranks of these points in the relative topology of  $ZCM_R$  are 0, 1 and 2, but we have to recalculate these ranks in the ambient space  $ZCM_S$ . For this we construct another interval in the lattice  $L$  of CM-formulae, which captures the points in question.

PROPOSITION 7.1. *The diagram in Figure 3 represents the pattern of the pointed module  $(C, xy)$ .*

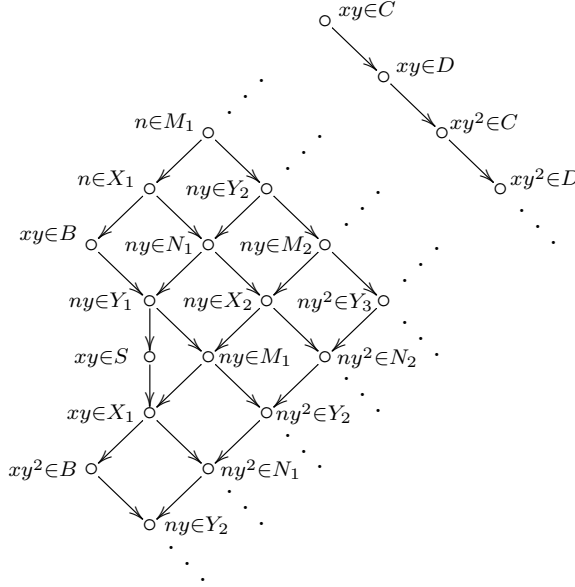


Fig. 3

*Proof.* By inspection. A very similar proof (with a similar diagram) is given in [15, Prop. 4.3]. ■

Note that the top pair  $xy \in C$  is a free realization of the formula  $vx = 0$ . It follows that the interval  $[v = 0, vx = 0]$  in  $L$  is generated by pp-formulae in the figure with respect to free sums, for instance it is distributive. This diagram allows us to calculate the ranks of the remaining points.

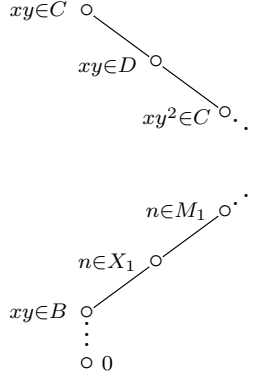
LEMMA 7.2. *Each finitely generated point  $M_k$  is isolated in  $ZCM_S$  by a minimal pair, and the same holds true for  $B$ .*

*Proof.* This follows directly from the diagram, because the  $M_k$  and  $B$  are sources of almost split sequences in the category of finitely generated CM-modules over  $S$ .

For instance, the module  $M_1$  is isolated by the pair  $n \in M_1$  over  $(X_1, n) + (Y_2, ny)$  which corresponds to the left almost split map  $M_1 \rightarrow X_1 \oplus Y_2$ . ■

It also follows that none of the remaining points is isolated in  $T_{\text{CM}} = \text{ZCM}_S$ , in particular each isolated point is isolated by a minimal pair. For instance, the lattice  $L_1$  of pp-formulae in  $T'_{\text{CM}}$  coincides with the lattice obtained from the lattice  $L$  of CM-formulae by collapsing intervals of finite length. In particular, this holds true for each interval.

LEMMA 7.3. *The interval  $xy \in C$  over zero in  $L_1$  is the following chain of order type  $1 + \mathbb{Z} + \omega^*$ :*



*Proof.* By direct calculations. For instance, the interval  $n \in X_1$  over  $ny \in N_1$  includes only one additional formula  $(B, xy) + (N_1, ny)$ , hence has length 2, and therefore becomes trivial in  $L_1$ . Similarly each consecutive interval along the southeastward ray  $(X_1, n) \rightarrow (N_1, ny) \rightarrow (X_2, ny) \rightarrow \dots$  has finite length in  $L$ , hence gets collapsed in  $L_1$ .

On the other hand, the interval  $(X_1, n)$  over  $(B, xy)$  contains the descending chain  $(B, xy) + (X_{k+1}, ny^k)$ ,  $k = 1, 2, \dots$ , hence produces a nontrivial (in fact simple) interval in  $L_1$ . ■

Recall that  $C$  and  $\tilde{R}$  have CB-rank 1 in  $\text{ZCM}_R$ , therefore their ranks in  $\text{ZCM}_S$  cannot be smaller. To our surprise, these ranks are preserved in the ambient space.

LEMMA 7.4.  *$C = xyS$  is isolated in  $T'_{\text{CM}}$  by the minimal pair  $(C, xy)$  over  $(D, xy)$ , in particular  $\text{CB}(C) = 1$ .*

*Proof.* From Figure 3 we see that the basis of open sets for  $C$  is given by pairs  $xy \in C$  over  $(D, xy) + (M_k, n)$ . By evaluating, it is easily seen that any such pair is open on  $M_{k+1}$ , therefore  $C$  is not isolated.

On the other hand, in  $L_1$  the interval  $(C, xy)$  over  $(D, xy)$  is simple, hence isolates  $C$ . ■

Now we proceed with  $\tilde{R}$ .

LEMMA 7.5.  *$\tilde{R}$  is isolated in  $T'_{\text{CM}}$  by the minimal pair  $(M_1, m)$  over  $(X_1, n)$ , in particular  $\text{CB}(\tilde{R}) = 1$ . Further,  $\tilde{R}$  is neg-isolated.*

*Proof.* From Figure 3 we see that the basis for the pointed module  $(\tilde{R}, xy)$  is given by the ray heading from  $(M_1, n)$  southeast, hence by the pairs  $ny^{k-1} \in M_k$  over  $n \in X_1$ . In particular, the latter formula neg-isolates the pp-type of  $xy$  in  $\tilde{R}$ . Since each such pair opens  $M_{k+1}$ , this point is not isolated. It remains to look at the above diagram. ■

It remains to consider the points  $Q_R$  and  $G_y$  whose CB-rank is at least 2. From the above analysis it follows that the lattice of pp-formulae of  $T''_{\text{CM}}$  coincides with the lattice  $L_2$  obtained from  $L_1$  by collapsing all finite length intervals. From the above diagram we conclude that the interval  $xy \in C$  over zero in  $L_2$  is the following chain of length 2:

$$\begin{array}{c} \circ \quad xy \in C \\ | \\ \circ \quad xy \in B \\ | \\ \circ \quad 0 \end{array}$$

The simple intervals in this chain will capture the remaining points of  $\text{ZCM}_S$ .

LEMMA 7.6.  *$Q_R$  is isolated in  $T''_{\text{CM}}$  by the minimal pair  $(B, xy)$  over 0, hence  $\text{CB}(Q_R) = 2$ . Further,  $G_y$  is isolated in  $T''_{\text{CM}}$  by the minimal pair  $(C, xy)$  over  $(B, xy)$ , hence  $\text{CB}(G_y) = 2$ .*

Now we deduce the global value of the CB-rank.

THEOREM 7.7. *The Cantor–Bendixson rank of the Cohen–Macaulay part of the Ziegler spectrum of  $S$  equals 2. The same holds true for the Krull–Gabriel dimension of the (definable) category of Cohen–Macaulay  $S$ -modules.*

*Proof.* The first assertion follows from the classification of points in  $\text{ZCM}_S$  (Theorem 6.6), and the values of their ranks calculated in this section and the previous one. Further, at each stage of the CB-analysis, each isolated point is isolated by a minimal pair. It follows that the Krull–Gabriel dimension of the category of CM-modules coincides with the  $m$ -dimension of the lattice  $L$ , and with the CB-rank of the space  $\text{ZCM}_S$ , which have already been calculated. ■

Note that the basic open set  $(vx = 0/v = 0)$  contains all points in  $\text{ZCM}_S$ , but  $A$  and  $G_x$ .

**8. Auslander–Reiten sequences.** It follows from the above analysis that almost split sequences we mentioned in Section 3 preserve their defining properties in the category of all CM-modules (finitely generated or not).

In this section we will produce a few more AR-sequences that will be used in the next section to glue a surface from the category of finitely generated CM-modules over  $S$ . Though no direct reference is possible, the idea goes back to Auslander [1] who constructed almost split sequences in the category

of all  $U$ -modules (over any ring  $U$ ) targeting a finitely presented module with local endomorphism ring. As follows from Herzog's [7] interpretation of this result, the source of this AR-sequence is an indecomposable pure injective module which is neg-isolated.

That is how we guessed almost split sequences in the category of CM-modules over  $S$  with infinitely generated terms. Namely, there are few indecomposable infinitely generated pure injective CM-modules which are neg-isolated, hence few candidates for sources of AR-sequences. The guess was difficult, but it is surprisingly easy to check the required properties.

Recall that  $N$  is identified with the submodule of  $Q_S$  generated by 1 and the  $n_k = xy(x+y)^{-k}$ . Further,  $N'$  is the submodule of  $N$  generated by the  $n_k$ , hence  $N' = xyQ_S$ . Then the factor  $N/N'$  is isomorphic to  $D = xS$  via the map which sends  $\bar{1}$  to  $x$ . We will also identify  $\tilde{R}$  with the submodule  $yS + N'$  of  $Q_S$ . In particular,  $\tilde{R}x = Sxy$  and  $\tilde{R}/\tilde{R}x \cong yS/xyS$ , which is isomorphic to  $S/xS \cong C = xyS$ .

With this identification in mind, we will produce the first AR-sequence with infinitely generated terms.

LEMMA 8.1. *The following sequence is almost split in the category of CM-modules over  $S$ :*

$$0 \rightarrow \tilde{R} \xrightarrow{f=(\iota_1, \pi_1)} N \oplus C \xrightarrow{u=(\pi'_1, -\iota'_1)} D \rightarrow 0.$$

Here  $\iota_1$  denotes the inclusion  $\tilde{R} = yS + N' \subseteq N$ , and  $\pi_1$  is the above epimorphism  $\tilde{R} \rightarrow \tilde{R}/\tilde{R}x \cong C$  which sends  $y \in \tilde{R}$  to  $xy \in C$ , hence is given by multiplication by  $x$ . Further,  $\pi'_1$  is the composition of the epimorphism  $N \rightarrow N/N'$  with the isomorphism  $N/N' \cong D = xS$  such that  $1 \mapsto x$ , hence is given by multiplication by  $x$ ; and  $\iota'_1$  is the inclusion  $C \subset D$ .

*Proof.* It is not difficult to check that this sequence is exact. For instance,  $uf(y) = u(y, xy) = xy - xy = 0$ . Further, the cokernel of  $f$  is obtained by identifying  $y \in N$  with  $-xy \in C$ , hence kicks off  $N'$ , and the resulting module is isomorphic to  $N/N' = D$ .

Since the endomorphism ring of the target  $D$  is local, it follows from [1, Prop. 4.4] that it suffices to check that  $f$  is left almost split. Suppose that  $K$  is a CM-module over  $S$ , and  $g : \tilde{R} \rightarrow K$  is a morphism which is not a split monomorphism. We need to find  $h : N \oplus C \rightarrow K$  completing the diagram

$$\begin{array}{ccc} \tilde{R} & \xrightarrow{f} & N \oplus C \\ g \downarrow & \nearrow h & \\ K & & \end{array}$$

Note that  $f(xy) = (xy, 0)$  belongs to  $N$ .

Assume first that  $g(xy) \neq 0$  and let  $q$  be the pp-type of  $g(xy)$  in  $K$ . Since  $g$  is not a split monomorphism, it increases the pp-type  $p$  of  $xy \in \tilde{R}$ , which (see Figure 3) is neg-isolated by the formula  $n \in X_1$ , hence this formula is in  $q$ . Since  $q$  is a filter, by taking conjunctions we find that each formula in the northeastward ray starting from  $n \in X_1$  belongs to  $p$ . We conclude that  $q$  includes each formula in the pp-type of  $xy \in N$ . It follows that there exists a morphism  $h : N \rightarrow K$  (considered as a morphism  $N \oplus C \rightarrow K$ ) such that  $g(xy) = hf(xy)$ .

Thus by considering  $g' = g - hf : \tilde{R} \rightarrow N \oplus K$  we may assume that  $g(xy) = 0$ . Since  $K$  is Cohen–Macaulay, it follows that  $g(xyQ_S) = 0$ , therefore  $g$  factors through  $\tilde{R}/xyQ_S \cong C$ , hence through  $\pi_1$ . ■

By swapping  $C$  with  $D$ , and  $\tilde{R}$  with  $N$ , we obtain a ‘dual’ almost split sequence.

LEMMA 8.2. *The following sequence is almost split in the category of CM-modules over  $S$ :*

$$0 \rightarrow N \xrightarrow{f=(\iota_2, \pi'_1)} \tilde{R} \oplus D \xrightarrow{u=(\pi_1, -\iota'_2)} C \rightarrow 0.$$

Here  $\iota_2 : N \rightarrow \tilde{R}$  and  $\iota'_2 : D \rightarrow C$  are given by multiplication by  $y$ . The epimorphisms  $\pi_1$  and  $\pi'_1$  are defined in Lemma 8.1.

*Proof.* We will give only the beginning of the proof. The remaining part is similar to that of the previous lemma.

Because  $uf(1) = u(y, x) = xy - xy = 0$ , the composite map is zero. Further, because  $y \in \tilde{R}$  is identified with  $-x \in D$ , we conclude that  $xy^2 = 0$  in the cokernel. Because this element equals  $xy(x + y)$ , we derive  $xy = 0$ , hence the submodule  $xyQ_S$  of  $\tilde{R}$  is annihilated. ■

By projecting the above almost split sequences we produce the following irreducible maps.

COROLLARY 8.3. *The following morphisms are irreducible in the definable category of CM-modules over  $S$ :*

- 1) The epimorphisms  $\tilde{R} \xrightarrow{x} C$ ,  $N \xrightarrow{x} D$ ,  $D \xrightarrow{y} C$  and  $N \xrightarrow{y} \tilde{R}$ .
- 2) The monomorphisms  $\tilde{R} \subset N$  and  $C \subset D$ .

**9. The Ringel quilt.** In this section we will add indecomposable infinitely generated pure injective CM-modules over  $S$  to the AR-quiver of the category  $\text{CM}_{\text{fg}}$  of finitely generated CM-modules to create the *Ringel quilt* of this category. The term is borrowed from Ringel [19], where it is called the *Auslander–Reiten quilt*.

We start with the unbridged version of the AR-quiver of the category  $\text{CM}_{\text{fg}}$  (see Figure 1). Now we compactify (add the boundary to) this diagram

taking into account the direct limits of rays, and inverse limits of corays (see Figure 4).

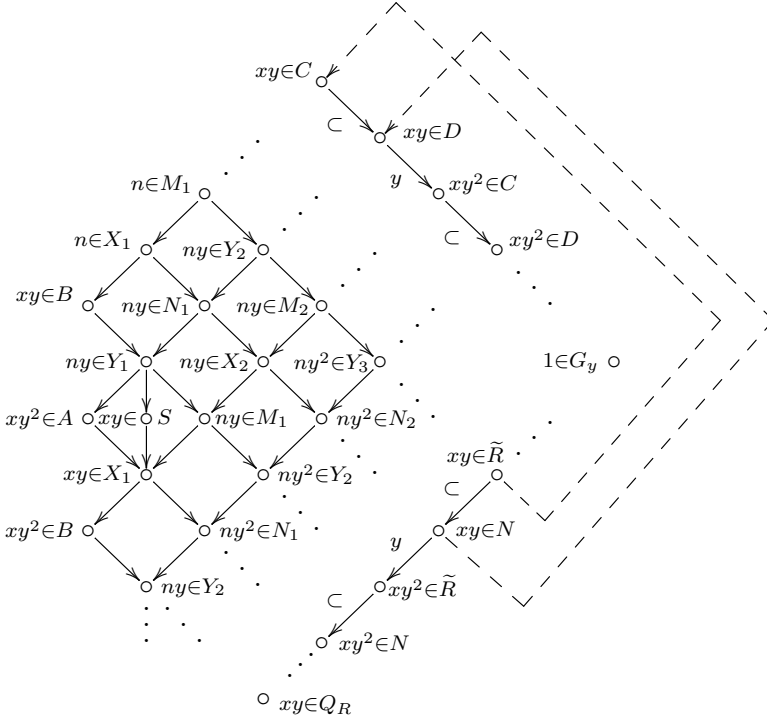


Fig. 4

Namely, if we identify  $M_k$  with the submodule of  $Q_S$  generated by  $y$  and  $n_k = xy(x+y)^{-k}$  then the composite morphism  $M_k \rightarrow M_{k+1}$  in the diagram is the inclusion. Further, the pointed module  $xy \in \tilde{R}$  is the direct limit of this ray; we add this pointed module to its end.

Going northeast along the parallel ray  $(X_1, n) \rightarrow (N_1, ny) \rightarrow \dots$ , we obtain the pointed module  $(N, xy)$  as its direct limit. Add this module to the end of this ray and attach the irreducible map  $\tilde{R} \subset N$  of these pointed modules. Now repeat this procedure with all parallel northeastward rays, for instance add irreducible maps  $N \xrightarrow{y} \tilde{R}$ .

Further, attach the point  $1 \in G_y$  to the end of the ray  $(C, xy) \rightarrow (D, xy) \rightarrow (C, xy^2) \rightarrow \dots$ , because this pointed module is the corresponding direct limit. On the other hand, we attach the same module to the beginning of the coray  $\dots \rightarrow (\tilde{R}, xy) \rightarrow (N, xy)$ , because the pp-type of  $1 \in G_y$  is the same as the pp-type of the inverse limit of pointed modules in this coray.

Now add the pointed module  $xy \in Q_R$  to the end of the ray  $(\tilde{R}, xy) \rightarrow (N, xy) \rightarrow (\tilde{R}, xy^2) \rightarrow \dots$  heading southwest. We will also connect  $\tilde{R}$  to the



module  $C$  by the irreducible map  $\tilde{R} \xrightarrow{x} C$ ; and connect the next module  $N$  in the ray to the next module  $D$  by the irreducible map  $N \xrightarrow{x} D$ .

These four modules form the following commutative square whose edges are irreducible maps:

$$\begin{array}{ccc} \tilde{R} & \xrightarrow{C} & N \\ x \downarrow & & \downarrow x \\ C & \xrightarrow{C} & D \end{array}$$

Further, the next morphism  $N \xrightarrow{y} \tilde{R}$  along the southeastward ray leads to a similar commutative square:

$$\begin{array}{ccc} N & \xrightarrow{y} & \tilde{R} \\ x \downarrow & & \downarrow x \\ D & \xrightarrow{y} & C \end{array}$$

Note that the sides of these squares change direction in Figure 4—from going southeast for the top maps to heading southwest for bottom morphisms.

Gluing the boarder lines of this surface using the irreducible morphisms just discussed we obtain the Möbius stripe. For instance (see Figure 5),

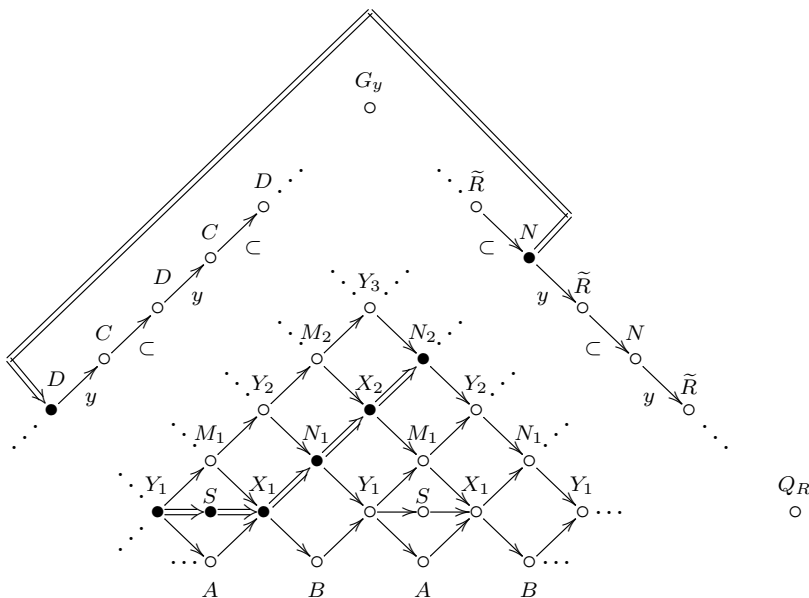


Fig. 5

starting from  $S$  we may go through  $X_1 \rightarrow N_1 \rightarrow \dots$  along the ray heading northeast, leave the first component of the AR-quiver when reaching  $N$ ,

and then enter the second component at point  $D$ . Now we could reenter the first component through a coray towards  $Y_1$ , hence complete the whole revolution by entering  $S$ . It is easily checked that the resulting map amounts to multiplication by  $x$ .

There is one misleading point in this description. The initial choice of the point  $D$  to enter (after  $N$ ) the second component of the AR-quiver was arbitrary, hence in general we will not get a closed path. Furthermore, the module  $G_x$  is missing from our description. It seems that a better way to give the global picture to the category  $\text{CM}_{\text{fg}}$  is shown in Figure 6. Thus this category is the quotient of the vertical strip by the action given by the glide reflection shifting upwards.

This way we could add the point  $G_x$  to the top of this diagram, because it is the direct limit of all rays heading upwards. We will also mark in this figure the points of  $\text{ZCM}_S$  included in the basic open set  $(x^2 | v/v = 0)$  corresponding to the first interval (see Section 5).

**10. The infinite radical.** In this section we will calculate the nilpotency index of the radical of the category  $\text{CM}_{\text{fg}}$  of finitely generated CM-modules over  $S$ . Recall that the *radical*,  $\text{rad}$ , of this category is defined to consist of all morphisms  $f : M \rightarrow N$  between modules in this category such that for each indecomposable  $K \in \text{CM}_{\text{fg}}$ , no composite  $K \rightarrow M \xrightarrow{f} N \rightarrow K$  is invertible. It follows easily that each irreducible morphism is in the radical, in fact  $\text{rad}$  is generated by irreducible morphisms.

The *transfinite powers* of the radical,  $\text{rad}^\lambda$ , are defined as follows. Set  $\text{rad}^1 = \text{rad}$ . If  $\lambda$  is a limit ordinal, then define  $\text{rad}^\lambda = \bigcap_{\mu < \lambda} \text{rad}^\mu$ . Otherwise  $\lambda = \nu + k$  for a limit ordinal  $\nu$ , hence set  $\text{rad}^\lambda = (\text{rad}^\nu)^{k+1}$ . For instance,  $\text{rad}^\omega$  is known as the *infinite radical*.

The least ordinal  $\lambda$  such that  $\text{rad}^\lambda = 0$  (if it exists) is called the *nilpotency index of the radical*,  $\text{nil}(\text{rad})$ . If it is defined, each morphism  $f$  in  $\text{CM}_{\text{fg}}$  is assigned its *nilpotency index*,  $\text{nil}(f)$ , as the least  $\mu$  such that  $f \in \text{rad}^\mu \setminus \text{rad}^{\mu+1}$ . Thus the nilpotency index of the radical is the supremum of the indices of morphisms between indecomposable modules.

Here is the main result of this section.

**THEOREM 10.1.** *The nilpotency index of the radical of the category of finitely generated Cohen–Macaulay  $S$ -modules equals  $\omega \cdot 2$ .*

Recall that each ordinal can be uniquely written in the form  $\omega^{\lambda_1} n_1 + \dots + \omega^{\lambda_k} n_k$  for some ordinals  $\lambda_1 > \dots > \lambda_k$ , and some natural numbers  $n_i$ . Thus  $\omega \cdot 2$  is the standard form for  $\omega + \omega$ .

One implication in this theorem is easy. Namely, the ‘one revolution’ map  $f : S \xrightarrow{x} S$  is in the infinite radical, hence its power  $f^{k+1}$  is a nonzero map in  $\text{rad}^{\omega+k}$ , as desired.

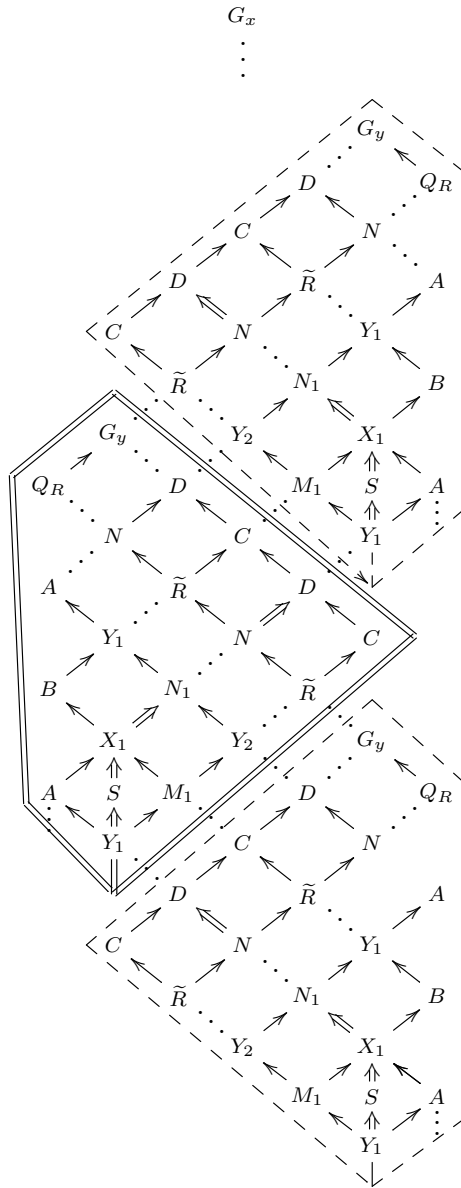


Fig. 6

To prove the converse let us make the following remark. Suppose that  $f : M \rightarrow N$  is a morphism between indecomposable modules in  $\text{CM}_{\text{fg}}$  which belongs to the infinite radical. If  $M$  lies in the top component of the AR-quiver in Figure 4 (i.e.  $M$  is  $C$  or  $D$ ), then factoring through irreducible maps, we conclude that  $f$  factors through a module in the second compo-

ment. Similarly, if  $M$  is in the second component, then  $f$  factors through multiplication by  $x$ .

It follows that if  $f \in \text{rad}^{\omega-2}$  then for each  $m \in M$  the image  $f(m)$  is divisible by any power of  $x$ . Looking at Figure 2 we conclude that  $f(m) = 0$ , hence  $f = 0$ . From this the result easily follows.

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