

MONOID EMBEDDINGS OF SYMMETRIC VARIETIES

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Abstract. We determine when an antiinvolution on an adjoint semisimple linear algebraic group extends to an antiinvolution on a J -irreducible monoid. Using this information, we study a special class of compactifications of symmetric varieties. Extending the work of Springer on involutions, we describe the parametrizing sets of Borel orbits in these special embeddings.

1. Introduction. Let G be a complex reductive algebraic group and let $\theta : G \rightarrow G$ be an algebraic group automorphism such that $\theta^2 = \text{id}$. The fixed subgroup $H := \{g \in G : \theta(g) = g\}$ is called the *symmetric subgroup* associated with θ , and the corresponding quotient G/H is called a *symmetric variety*. Let B be a Borel subgroup of G . From the works of Richardson and Springer [16] and Helminck [7, 8], we know that there is a close relationship between the set of B -orbits in G/H and the set of involutions in the Weyl group of G . In particular, we know that the number of B -orbits in G/H is finite [10, 17].

A purpose of our paper is to show that the sets of B -orbits in certain “monoid embeddings” of the symmetric varieties are closely related to the sets of involutions in certain finite inverse semigroups. We proceed to explain what we mean by a monoid embedding.

A *reductive monoid* is a linear algebraic monoid whose group of units is a reductive algebraic group. Let M be a reductive monoid and let θ_{an} be an antiinvolution, that is, $\theta_{\text{an}} : M \rightarrow M$ is an automorphism of M such that $\theta_{\text{an}}^2 = \text{id}$ and for every $m_1, m_2 \in M$ we have $\theta_{\text{an}}(m_1 m_2) = \theta_{\text{an}}(m_2) \theta_{\text{an}}(m_1)$. If G denotes the group of units of M , then we denote the restriction of θ_{an} to G by the same notation. Then the morphism $\theta : G \rightarrow G$ defined by $\theta(g) := \theta_{\text{an}}(g)^{-1}$ ($g \in G$) is an involutory algebraic group automorphism. As before, let us denote by H the fixed subgroup of θ . The morphic map

$$(1) \quad \tau : M \rightarrow M, \quad m \mapsto m\theta_{\text{an}}(m),$$

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restricts to give a morphism on G and we denote this restriction by τ as well. The image of τ on G is denoted by P . Note that P is a closed subvariety of G , and furthermore it is isomorphic to G/H as a variety (see [15, Lemma 2.4]).

The *twisted (conjugation) action* of G on M , denoted by $*$, is defined by

$$g * m = gm\theta_{\text{an}}(g) = gm\theta(g)^{-1} \quad (g \in G, m \in M).$$

It is easy to check that P is stable under the twisted action. In fact, $P = G * 1_G$. It is not hard to see also that the set of $B*$ -orbits in P is in bijection with the set of B -orbits in G/H . We call the Zariski closure of P in M the *monoid embedding* of P . Since P is isomorphic to G/H and since \bar{P} is $G*$ -stable, we view the monoid embedding of P as an equivariant embedding of G/H in the reductive monoid M .

An important auxiliary variety for our purposes is the fixed subvariety Q defined by

$$Q := \{g \in G : \theta_{\text{an}}(g) = g\}.$$

It is easy to check that Q is closed in G , $P \subset Q$, and that Q is $G*$ -stable. We know from Springer's work [17] that if B is a θ -stable Borel subgroup of G , then Q has only finitely many $B*$ -orbits. As in the case of P , the parametrizing set of $B*$ -orbits in Q is closely related to the set of involutions in the Weyl group $W := N_G(T)/T$, where T is a maximal torus contained in B and $N_G(T)$ is the normalizer of T in G . (Here, by an involution in W we mean an element $\sigma \in W$ such that $\sigma^2 = \text{id}$.)

Let M_Q denote the following (closed) subvariety of M :

$$(2) \quad M_Q := \{m \in M : \theta_{\text{an}}(m) = m\}.$$

Clearly, $\bar{P} \subseteq \bar{Q} \subseteq M_Q$ and G acts on the sets P, Q, \bar{P}, \bar{Q} , and on M_Q by the same formula $g * m = gm\theta_{\text{an}}(g)$. Our first main result is about the parametrizing sets of $B*$ -orbits in the embeddings of P and Q in M .

THEOREM 1.1. *Let M be a normal reductive monoid with unit group G , θ_{an} be an antiinvolution on M , and let θ denote the involutive automorphism on G that is defined by $\theta(g) = \theta_{\text{an}}(g)^{-1}$ for $g \in G$. We fix a pair (T, B) of a θ -stable maximal torus and a Borel subgroup in G , and we let \bar{N} denote the Zariski closure in M of the normalizer of T in G . In this case, the following sets are finite and they are in bijection with each other:*

- (1) $B*$ -orbits in \bar{Q} (respectively, $B*$ -orbits in \bar{P}),
- (2) $T \times H$ -orbits in $\tau^{-1}(\bar{N} \cap \bar{Q})$ (respectively, $T \times H$ -orbits in $\tau^{-1}(\bar{N} \cap \bar{P})$).

The *Renner monoid* of M , defined by $R = \bar{N}/T$, is a generalization of the Weyl group of G (see [13]). It is a finite inverse semigroup and W is its group of invertible elements. As a consequence of Theorem 1.1, we will show that a certain subset of R can be used for studying $B*$ -orbits in \bar{P} . In

some special cases this subset of R gives a parametrization of the full set of B^* -orbits; see the examples in Section 4.

With hindsight, our first main result raises the question of finding antiinvolutions on reductive monoids. To answer this question, in our second main result, we focus on a particular subclass of reductive monoids. A *semisimple monoid* is a reductive monoid which is normal, has a one-dimensional center and a zero element. Interesting examples of such monoids include the cones over certain representations of semisimple groups.

Let G_0 be a semisimple algebraic group of adjoint type and let $\rho : G \rightarrow \mathrm{GL}(V)$ be a finite-dimensional irreducible rational representation of G_0 . Let Z_V denote the cone over $\rho(G)$ in $\mathrm{End}(V)$. Then Z_V has the structure of a reductive monoid. Let us mention that Z_V is known to be normal as an algebraic variety if the representation (ρ, V) is a minuscule representation in the sense of [3, Theorem 3.1]. (We will review De Concini's theorem in the preliminaries section.) In the following result we denote by G the reductive group of units in Z_V .

THEOREM 1.2. *Let G_0 be a complex semisimple algebraic group of adjoint type and let $\theta_0 \in \mathrm{Aut}(G_0)$ be an involutory algebraic group automorphism. If (ρ, V) is a minuscule representation of G_0 with the highest weight ω such that $\theta_0^*\omega = -\omega$ and Z_V is normal, then there exists a unique morphism $\theta_{\mathrm{an}} : Z_V \rightarrow Z_V$ such that*

- (1) $\theta_{\mathrm{an}}(xy) = \theta_{\mathrm{an}}(y)\theta_{\mathrm{an}}(x)$ for all $x, y \in Z_V$;
- (2) θ_{an}^2 is the identity map on Z_V ;
- (3) $\theta_{\mathrm{an}}(g) = \theta(g)^{-1}$ for all $g \in G$, where θ is the unique extension of θ_0 to G .

We conclude our introduction by giving a brief overview of our article. In Section 2, we set our notation and review some facts from the theory of linear algebraic monoids and the representation theory of reductive algebraic groups. In Section 3, we prove our second main result, Theorem 1.2. In Section 4, we characterize the parametrizing sets of Borel orbits in \bar{P} . In particular we prove our Theorem 1.1 in Section 4. Finally, we close our paper with some remarks in Section 5.

2. Preliminaries. Unless otherwise mentioned, all reductive groups are assumed to be connected and all semigroups are defined over \mathbb{C} . The representations we consider here are all rational and finite-dimensional.

The general linear group of invertible $n \times n$ matrices is denoted by GL_n , and the monoid of $n \times n$ matrices by Mat_n . The Lie algebra of a linear algebraic group G is denoted by $\mathrm{Lie}(G)$.

Let G be a reductive algebraic group, T a maximal torus in G , and B a Borel subgroup containing T . We use $X(T)$ to denote the character group

of T , and E to denote the real vector space $X(T) \otimes_{\mathbb{Z}} \mathbb{R}$. In addition, we fix the following notation:

$\Phi \subset E$: the set of weights of the adjoint representation;

$\Delta \subset \Phi$: the set of simple roots determined by (B, T) ;

$\Phi^+ \subset \Phi$: the set of positive roots determined by Δ ;

$A_r \subset X(T)$: the root lattice generated by Δ .

If α is a root from Φ , then the associated coroot, $2\alpha/(\alpha, \alpha)$, is denoted by $\check{\alpha}$. Suppose that $\alpha_1, \dots, \alpha_n$ is the list of simple roots from Δ . The set of *fundamental weights*, $\{\omega_1, \dots, \omega_n\}$, is the dual of the coroot basis $\{\check{\alpha}_1, \dots, \check{\alpha}_n\}$ for the dual vector space $\text{Lie}(T)^*$.

Irreducible representations of G are parametrized by the semigroup of dominant weights (with respect to T). A dominant weight λ is called *minuscule* if $\langle \lambda, \check{\alpha} \rangle \leq 1$ for all positive coroots $\check{\alpha}$.

The dominance partial order on weights is defined by $\mu \preceq \lambda$ if and only if $\lambda - \mu$ is a positive linear combination of positive roots.

If λ is a dominant weight, then we denote by $\Sigma(\lambda)$ the set of dominant weights μ such that $\mu \preceq \lambda$. The set $\Sigma(\lambda)$ is finite and it is called the *saturation* of λ .

2.1. Reductive monoids. The purpose of this section is to introduce the notion of a reductive monoid. For details, see [14, 11]. For a more recent exposition of the basic ideas behind algebraic monoids we recommend Brion's article [1], and for the combinatorics of Renner monoids we recommend [9].

Let M be a linear algebraic monoid with the group of invertible elements G . The set of idempotents in M is denoted by $E(M)$. If G is a reductive group and M is an irreducible algebraic variety, then M is called a *reductive monoid*. Note that there is no normality assumption on M .

Let T be a maximal torus in G and let B be a Borel subgroup containing T . Clearly, \overline{T} is a reductive and commutative submonoid of M . As before, we denote by R (resp. by W), the *Renner monoid* $\overline{N_G(T)}/T$ (resp. the *Weyl group* $N_G(T)/T$) of M (resp. of G).

The “generalized” *Bruhat–Chevalley order* on R is defined by

$$(3) \quad \sigma \preceq \tau \quad \text{if and only if} \quad B\sigma B \subseteq \overline{B\tau B},$$

where τ and σ are from R and the bar stands for the Zariski closure in M .

There is a canonical partial order \leq on the set $E(\overline{T})$ of idempotents of \overline{T} defined by

$$(4) \quad e \leq f \quad \text{if and only if} \quad ef = e = fe.$$

Notice that $E(\overline{T})$ is invariant under the conjugation action of the Weyl group W . A subset $A \subseteq E(\overline{T})$ is called a *cross-section lattice* (or a *Putcha*

lattice) if Λ is a set of representatives for the W -orbits on $E(\overline{T})$ and the bijection $\Lambda \rightarrow G \backslash M / G$ defined by $e \mapsto GeG$ is order preserving. There is a close relationship between cross-section lattices and Borel subgroups. The right centralizer of Λ in G is the subgroup

$$C_G^r(\Lambda) = \{g \in G : ge = ege \text{ for all } e \in \Lambda\}.$$

If M has a zero, then for all Borel subgroups of G containing T the set $\Lambda(B) = \{e \in E(\overline{T}) : Be = eBe\}$ is a cross-section lattice with $B = C_G^r(\Lambda)$, and for any cross-section lattice Λ , the right centralizer $C_G^r(\Lambda)$ is a Borel subgroup containing T with $\Lambda = \Lambda(C_G^r(\Lambda))$. See [11, Theorem 9.10].

The decomposition $M = \bigsqcup_{e \in \Lambda} GeG$ into $G \times G$ orbits has a finite counterpart: $R = \bigsqcup_{e \in \Lambda} WeW$. Moreover, the partial order (4) on Λ agrees with the order induced from the Bruhat–Chevalley order (3).

$E(\overline{T})$ is a relatively complemented lattice, antiisomorphic to a face lattice of a convex polytope. For \overline{T} contained in a J -irreducible monoid, the associated polytope is described explicitly in Section 2.2. Let Λ be a cross-section lattice in $E(\overline{T})$. The Weyl group of T (relative to $B = C_G^r(\Lambda)$) acts on $E(\overline{T})$, and furthermore

$$E(\overline{T}) = \bigsqcup_{w \in W} w\Lambda w^{-1}.$$

Let S be a semigroup and let $M = S^1$ be the monoid obtained from S by adding a unit element if it is not already present. Let $a, b \in M$. The following are four of the five equivalence relations which are collectively known as *Green’s relations*, of utmost importance for semigroup theory:

- $a \mathcal{L} b$ if $Ma = Mb$;
- $a \mathcal{R} b$ if $aM = bM$;
- $a \mathcal{J} b$ if $MaM = MbM$;
- $a \mathcal{H} b$ if $a \mathcal{L} b$ and $a \mathcal{R} b$.

It turns out that the unit group G of a reductive monoid M is big in the sense that $a \mathcal{L} b$ if $Ga = Gb$, $a \mathcal{R} b$ if $aG = bG$, and $a \mathcal{J} b$ if $GaG = GbG$ (see [11, Proposition 6.1]). Furthermore, a cross-section lattice is a representative for the set of \mathcal{J} -classes in M . A reductive monoid M is called *J -irreducible* if M has a unique, nonzero, minimal $G \times G$ -orbit.

We continue with the assumption that M is a reductive group with unit group G . Among the important submonoids of M are those of the form eMe ($e \in E(\overline{T})$). Let $C_G(e)$ denote the centralizer of e in G . If e is from the cross-section lattice Λ , then $eC_G(e)$ is the unit group of eMe . We will also need the following fact.

LEMMA 2.1. *Let $e \in E(\overline{T})$ be an idempotent and let H denote its \mathcal{H} -class, that is, $H = eC_G(e)$. Let B be a Borel subgroup of G containing T ,*

hence $e \in E(\overline{B})$. In this case, $C_B(e)$ and $eBe = eC_B(e)$, respectively, are Borel subgroups of $C_G(e)$ and H .

For the proofs of the facts stated in the paragraph before the lemma as well as for the proof of the lemma, see [11, Corollary 7.2].

2.2. J -irreducible monoids. Let G_0 denote a semisimple linear algebraic group of adjoint type, and T_0 be a maximal torus in G_0 . If (ρ_0, V) is a representation of G_0 , then the group $\mathbb{C}^* \cdot \rho_0(G_0)$, which we denote by G , is reductive. If ρ_0 is faithful, then up to isomorphism, T_0 and $\rho_0(T_0)$ differ by a finite set of central elements. In this case, when there is no danger of confusion, we will denote the image $\rho_0(T_0)$ by T_0 .

Let $T \subseteq G$ be a maximal torus containing T_0 , and let $\mathbf{T} \subset \mathrm{GL}(V)$ denote an n -dimensional maximal torus containing T . (Here, $n = \dim V$.) Accordingly, we have a nested sequence of Euclidean spaces:

$$E_0 = X(T_0) \otimes_{\mathbb{Z}} \mathbb{R} \subset E = X(T) \otimes_{\mathbb{Z}} \mathbb{R} \subset \mathbf{E} = X(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Note that $\dim E = \dim E_0 + 1$.

If ε_i (for $i = 1, \dots, n$) denotes the standard i th coordinate function on \mathbf{T} , then $\{\varepsilon_1, \dots, \varepsilon_n\}$ is a basis for \mathbf{E} , and E is spanned by the restrictions $\varepsilon_i|_T$, $i = 1, \dots, n$. Let $\chi \in X(T)$ denote the restriction of the character whose n th power is the determinant on $\mathrm{GL}(V)$. Stated differently in additive notation, χ is the restriction to T of the rational character $\frac{1}{n}(\varepsilon_1 + \dots + \varepsilon_n)$. We denote $\varepsilon_i|_T$ by χ_i and set

$$\tilde{\chi}_i := \chi_i - \chi \quad \text{for } i = 1, \dots, n.$$

If K is an arbitrary group, then the center of K is customarily denoted by $Z(K)$. In our case, since $Z(G) = \mathbb{C}^* \cdot Z(G_0)$, the character group of $Z(G)$ is generated by one element, which is χ . Thus, $E = \mathbb{R}\chi \oplus E_0$. In fact, χ vanishes on T_0 . It follows from these observations that

- $\{\tilde{\chi}_1, \dots, \tilde{\chi}_{n-1}, \chi\}$ spans E ;
- if $x \in T$ lies in $T_0 \subset T$, then $\tilde{\chi}_i|_{T_0}(x) = \chi_i(x)$;
- $\{\tilde{\chi}_1, \dots, \tilde{\chi}_{n-1}\}$ spans E_0 .

In this paper, we are interested in the J -irreducible monoids that come from a faithful representation.

DEFINITION 2.1. Let (ρ_0, V) be a faithful, irreducible representation of G_0 . The J -irreducible monoid associated with (ρ_0, V) is the affine variety $\overline{\mathbb{C}^* \cdot \rho_0(G_0)}$ together with its monoid structure induced from $\mathrm{End}(V)$.

REMARK 2.1. The definition of a J -irreducible monoid which is given at the end of Section 2.1 agrees with Definition 2.1 (see [14, Lemma 7.8]).

The representation (ρ_0, V) of G_0 gives a representation of G by

$$(5) \quad zg \cdot v = z\rho_0(g)(v) \in V,$$

where $z \in \mathbb{C}^*$, $g \in G_0$, and $v \in V$. Another such simple but useful observation is that, since $G_0 \xrightarrow{\rho_0} (G, G)$, the Weyl group W of (G_0, T_0) is isomorphic to that of the pair (G, T) .

LEMMA 2.2 ([12, Proposition 3.5]). *Let (ρ_0, V) be an irreducible representation of G_0 . If λ is the T_0 -highest weight of (ρ_0, V) and \mathcal{P} denotes the convex hull of $\{w \cdot (\chi + \lambda) : w \in W\} = \{\chi + w \cdot \lambda : w \in W\}$, then the set of weights of T with respect to (5) is contained in \mathcal{P} .*

Next, we briefly review a result of De Concini on the normality of J -irreducible monoids. Let λ be a dominant weight for G_0 and let (ρ_0, V) denote the corresponding irreducible representation of G_0 . We define (η, W_λ) as the following sum of irreducible representations of G_0 :

$$W_\lambda := \bigoplus_{\mu \in \Sigma(\lambda)} V(\mu).$$

Here $V(\mu)$ denotes the irreducible representation of G_0 with highest weight μ . Finally, we set

$$Z_\lambda := Z_{V(\lambda)} \quad \text{and} \quad \mathcal{Z}_\lambda := Z_{W_\lambda},$$

where Z_{W_λ} is the cone over $\eta(W_\lambda)$ in $\text{End}(W_\lambda)$.

THEOREM 2.1 (De Concini [3, Theorem 3.1]).

- (1) \mathcal{Z}_λ is a normal variety with rational singularities.
- (2) If V is a G_0 -module of highest weight λ , then \mathcal{Z}_λ is the normalization of Z_λ , and it is equal to Z_λ if and only if W_λ is a subrepresentation of V . In particular, $\mathcal{Z}_\lambda = Z_\lambda$ if and only if λ is minuscule.

3. A proof of Theorem 1.2. Let G_0 be a semisimple algebraic group of adjoint type, and θ_0 be an involutory linear algebraic group automorphism of G_0 . Let (T_0, B_0) be a θ_0 -stable pair of a maximal torus T_0 and a Borel subgroup B such that $T_0 \subset B_0$. The *isotropic subtorus* T'_0 and the *anisotropic subtorus* T'_1 are defined by

$$T'_0 = \{t \in T_0 : \theta(t) = t\}, \quad T'_1 = \{t \in T_0 : \theta(t) = t^{-1}\}.$$

The multiplication map $T'_1 \times T'_0 \rightarrow T_0$ is an *isogeny*, that is, a surjective homomorphism with a finite kernel.

Among all θ -stable maximal tori, we work with the one for which the dimension $l := \dim T'_1$ is maximal. The integer l is called the *rank* of the symmetric variety G_0/G_0^θ . (G_0^θ is the fixed subgroup of θ .)

Let Φ denote the set of roots of G_0 relative to T_0 . Passing to the Lie algebra setting by differentiation, we view Φ as a subset of the dual vector space $\text{Lie}(T_0)^*$ of the Lie algebra of T_0 . Since θ is an automorphism of T_0 , it

induces a linear map

$$\theta^* : \text{Lie}(T_0)^* \rightarrow \text{Lie}(T_0)^*,$$

which in turn induces an involution on Φ . Define

$$\Phi_0 = \{\alpha \in \Phi : \theta^*(\alpha) = \alpha\}, \quad \Phi_1 = \Phi - \Phi_0.$$

LEMMA 3.1 ([4, [Lemma 1.2]]). *There exists a system $\Phi^+ \subseteq \Phi$ of positive roots such that $\theta^*(\alpha) \in \Phi - \Phi^+$ for all $\alpha \in \Phi^+ \cap \Phi_1$.*

We fix a set Φ^+ of positive roots as in Lemma 3.1. Let Δ denote the associated set of simple roots and let

$$\Delta_0 = \Phi_0 \cap \Delta, \quad \Delta_1 = \Phi_1 \cap \Delta.$$

Observe that $|\Delta_1| \geq \dim T_1' = l$. It turns out that there exists an ordering $\{\alpha_1, \dots, \alpha_j\}$ of the elements of Δ_1 such that the differences $\alpha_i - \theta^*(\alpha_i)$ are mutually distinct for $i = 1, \dots, l$, and for each $i \in \{l+1, \dots, j\}$ there exists $s \in \{1, \dots, l\}$ such that $\alpha_i - \theta^*(\alpha_i) = \alpha_s - \theta^*(\alpha_s)$ (see [4, Section 1.4]). A *restricted simple root* $\bar{\alpha}$ is a weight of the form

$$\bar{\alpha} = \frac{\alpha_i - \theta^*(\alpha_i)}{2} \quad \text{for some } i \in \{1, \dots, l\}.$$

In this case, we denote $\bar{\alpha}$ by $\bar{\alpha}_i$, and denote by $\overline{\Delta_1} = \{\bar{\alpha}_1, \dots, \bar{\alpha}_l\}$ the set of all restricted simple roots. Suppose now that $\Delta_0 = \{\beta_1, \dots, \beta_k\}$. In accordance with the partitioning $\Delta = \Delta_0 \sqcup \Delta_1$, we divide the set of fundamental weights of Δ into two disjoint subsets $\{\omega_1, \dots, \omega_j\} \sqcup \{\zeta_1, \dots, \zeta_k\}$ so that for each $i \in \{1, \dots, j\}$,

$$(\omega_i, \beta_s^\vee) = 0 \quad \text{for } s = 1, \dots, k, \quad \text{and} \quad (\omega_i, \alpha_r^\vee) = \delta_{i,r} \quad \text{for } r = 1, \dots, j,$$

and similarly for ζ_i 's. As shown in [4, pp. 5, 6], θ^* induces an involution $\tilde{\theta}$ on the indices $\{1, \dots, j\}$ such that $\theta^*(\omega_i) = -\omega_{\tilde{\theta}(i)}$. Thus, we arrive at a crucial definition for our purposes:

DEFINITION 3.1. A dominant weight λ of G_0 is said to be *special* (or *θ -special*) if $\theta^*(\lambda) = -\lambda$. If (ρ, V) is an irreducible representation with a θ -special highest weight, then we call ρ a *θ -special representation* of G_0 .

Now, let $\theta_0 : G_0 \rightarrow G_0$ be an involutory automorphism on G_0 . We choose a θ_0 -stable maximal torus T_0 in G_0 . Let λ be a special, dominant weight with the corresponding irreducible representation (ρ_0, V) . Assume also that (ρ_0, V) is faithful. As before, we define the reductive group G by setting $G = \mathbb{C}^* \cdot \rho_0(G_0) \subset \text{GL}(V)$. We claim that there exists an “extension” $\theta : G \rightarrow G$ of θ_0 to G . To this end we define

$$(6) \quad \theta(c\rho_0(g)) = c^{-1}\rho_0(\theta_0(g)), \quad g \in G_0, c \in \mathbb{C}^*.$$

To prove θ is well defined, suppose $g, g' \in G_0$ and $c, c' \in \mathbb{C}^*$ are such that $c\rho_0(g) = c'\rho_0(g')$. Let $\alpha \in \mathbb{C}^*$ denote cc'^{-1} . Then $\rho_0(g^{-1}g') = \alpha 1_{\text{GL}(V)} \in \text{GL}(V)$.

But G_0 is of adjoint type, ρ_0 is faithful, and α is a central element in $\mathrm{GL}(V)$. Therefore, $\alpha = \mathrm{id}$, hence $g = g'$ and $c = c'$. Finally, note that

$$\begin{aligned} \theta(\theta(c\rho_0(g))) &= \theta(c^{-1}\rho_0(\theta_0(g))) = c\rho_0(\theta_0(\theta_0(g))) \\ &= c\rho_0(g) \quad \text{for all } c \in \mathbb{C}^* \text{ and } g \in G_0. \end{aligned}$$

The *antiinvolution* corresponding to θ is, by definition, the composition $\theta_{\mathrm{an}} := \theta \circ \iota$ of θ with the “inverting” morphism $\iota : g \mapsto g^{-1}$. The map induced by θ_{an} on the character group $X(T)$ is denoted θ_{an}^* . Then θ^* and θ_{an}^* are related by

$$\theta_{\mathrm{an}}^*(\chi) = -\theta^*(\chi) \quad \text{for } \chi \in X(T).$$

In particular, if $\theta^*(\lambda) = -\lambda$, then $\theta_{\mathrm{an}}^*(\lambda) = \lambda$.

We are ready to prove Theorem 1.2. Let us paraphrase it for completeness: If M is a normal J -irreducible monoid that is obtained from a θ -special minuscule representation of G , then there exists a unique morphism $\theta_{\mathrm{an}} : M \rightarrow M$ such that

- (1) $\theta_{\mathrm{an}}(xy) = \theta_{\mathrm{an}}(y)\theta_{\mathrm{an}}(x)$ for all $x, y \in M$;
- (2) θ_{an}^2 is the identity map on M ;
- (3) $\theta_{\mathrm{an}}(g) = \theta(g)^{-1}$ for all $g \in G$, where θ is the involution that is extended from θ_0 on G_0 .

Proof of Theorem 1.2. Since θ_{an} agrees with θ (after composing with ι , of course) on G , the uniqueness is clear. We are going to show that θ_{an} extends to the whole J -irreducible monoid $M := Z_\lambda$ associated to an irreducible representation (ρ_0, V) of G_0 with the highest weight λ . Let ρ denote the representation of G as defined in (5).

First, we note that, by Theorem 2.1, M is a normal reductive monoid. Let T_0 denote the maximal torus of G_0 such that $T = \mathbb{C}^* \cdot \rho_0(T_0)$, and let $\langle \Pi(\rho) \rangle$ denote the submonoid of $X(T)$ generated by the weights $\Pi(\rho)$ of T . The coordinate ring of the affine torus embedding \bar{T} is equal to the monoid-ring $R = \mathbb{C}[\langle \Pi(\rho) \rangle]$ (see [12, Lemma 3.2]). Therefore, $\bar{T} = \mathrm{Spec}(R)$. On the other hand, by Lemma 2.2, we know that $\Pi(\rho)$ is contained in the convex hull \mathcal{P} of $W \cdot (\lambda + \chi)$, where χ is the n th root of the determinant on $\mathrm{GL}(V)$.

Since λ is special, by [4, Lemma 1.6] there is a G -isomorphism $V^\theta \simeq V^*$, hence $\theta^*(\Pi(\rho)) = \Pi(\rho^*) = -\Pi(\rho)$ and it follows that $\theta_{\mathrm{an}}^*(\Pi(\rho)) = \Pi(\rho)$. In particular, it induces an antiinvolution θ_{an} on $\bar{T} = \mathrm{Spec}(R)$. Since $\theta \circ \iota = \theta_{\mathrm{an}}$ on T , by the “extension principle” (see [12, Corollary 4.5]) there exists a unique morphism $\theta_{\mathrm{an}} : M \rightarrow M$ which agrees with $\theta \circ \iota$ on G and agrees with θ_{an} on \bar{T} . Since $\theta_{\mathrm{an}}^2 = \mathrm{id}$ on G , and since G is dense on M , we see that $\theta_{\mathrm{an}}^2 = \mathrm{id}$ on M . Finally, since $(x, y) \mapsto \theta_{\mathrm{an}}(xy)$ and $(x, y) \mapsto \theta_{\mathrm{an}}(y)\theta_{\mathrm{an}}(x)$ are morphisms from $M \times M$ into M agreeing on the open dense set $G \times G$, they agree everywhere. ■

4. A proof of Theorem 1.1. We start with providing the details of some useful facts which we briefly mentioned earlier in Section 2.1.

LEMMA 4.1 (Generalized Bruhat–Chevalley decomposition, [13]). *Let M be a reductive monoid with the group of invertible elements G , and let $T \subseteq B$ be a maximal torus contained in a Borel subgroup. Let \overline{N} denote the closure in M of the normalizer $N = N_G(T)$ of T . If $m \in M$, then there exist $b_1, b_2 \in B$ and $\overline{n} \in \overline{N}$ such that $m = b_1 \overline{n} b_2$. This leads to the Bruhat–Chevalley decomposition*

$$(7) \quad M = \bigcup_{\overline{n} \in R} B \overline{n} B,$$

where the union is disjoint and $R = \overline{N}/T$ is the Renner monoid of M .

Fix $\overline{n} \in \overline{N}$, and let $V_{\overline{n}} \subseteq U$ denote the subgroup $\{u \in U : u \overline{n} B \subseteq \overline{n} B\}$. Then $V_{\overline{n}}$ is closed and T -stable under conjugation. Therefore, there exists a complementary subgroup

$$(8) \quad U_{\overline{n},1} = \prod_{U_\alpha \not\subseteq V_{\overline{n}}} U_\alpha.$$

Complementary in this context means that the product morphism $U_{\overline{n},1} \times V_{\overline{n}} \rightarrow U$ is an isomorphism of algebraic groups.

In a similar manner, let $Z_{\overline{n}} \subseteq U$ denote the closed subgroup $\{u \in U : \overline{n} T u = \overline{n} T\}$. Also in this case, $Z_{\overline{n}}$ is T -stable under conjugation; let

$$(9) \quad U_{\overline{n},2} = \prod_{U_\alpha \not\subseteq Z_{\overline{n}}} U_\alpha$$

denote its complementary subgroup. The precise structure of the orbit $B \overline{n} B$ is exhibited in the next result:

LEMMA 4.2 ([14, Lemma 13.1]). *The product morphism $U_{\overline{n},1} \times \overline{n} T \times U_{\overline{n},2} \rightarrow B \overline{n} B$ is an isomorphism of varieties.*

As a consequence of Lemmas 4.1 and 4.2 we have the following important observation:

UNIQUENESS CRITERION. *Given $m \in M$, there exist unique $u \in U_{\overline{n},1}$, $v \in U_{\overline{n},2}$, and $\overline{n} \in \overline{N}$ such that*

$$(10) \quad m = u \overline{n} v.$$

We continue under the assumption that M is a reductive monoid with an antiinvolution $\theta_{\text{an}} : M \rightarrow M$. Let $H \subseteq G$ denote, as usual, the fixed subgroup G^θ , where $\theta : G \rightarrow G$ is the involution $\iota \circ \theta_{\text{an}}$, and ι stands for the inverse map. Here, we are going to investigate the sets of B^* -orbits in the following varieties:

- the Zariski closure \overline{P} in M of $P = \{g\theta(g^{-1}) : g \in G\} \simeq G/H$;
- the Zariski closure \overline{Q} in M of $Q = \{g \in G : \theta(g) = g^{-1}\}$;
- the set $M_Q := \{x \in M : \theta_{\text{an}}(x) = x\}$.

Assume from now on that T is a θ -stable maximal torus of the θ -stable Borel subgroup $B \subseteq G$. Notice in this case that the corresponding unipotent subgroup $U \subset B$ has to be θ -stable as well. Moreover, since T is θ -stable, if n is an element from the normalizer $N = N_G(T)$, then $\theta(n)t\theta(n)^{-1} = \theta(n)\theta(t')\theta(n^{-1}) = \theta(nt'n^{-1}) \in T$ for some $t' \in T$. In other words, $\theta(N) = N$. It follows that the Zariski closure \overline{N} is θ_{an} -stable.

PROPOSITION 4.1. *Any B^* -orbit in M_Q contains an element of \overline{N} .*

Proof. For $m \in M_Q$, we know from the Uniqueness Criterion that there exist unique $u \in U_{\overline{n},1}$, $v \in U_{\overline{n},2}$, and $\overline{n} \in \overline{N}$ such that $m = u\overline{n}v$. Then

$$u\overline{n}v = m = \theta_{\text{an}}(m) = \theta_{\text{an}}(v)\theta_{\text{an}}(\overline{n})\theta_{\text{an}}(u) = \theta(v)^{-1}\theta_{\text{an}}(\overline{n})\theta(u)^{-1}.$$

Since the Bruhat–Chevalley decomposition (7) is a disjoint union, we see that $\theta_{\text{an}}(\overline{n}) \in \overline{n}T$. Let $t \in T$ be such that $\theta_{\text{an}}(\overline{n}) = \overline{n}t$.

Let a denote $v\theta(u)$. It is clear that $\overline{n}a$ lies in the B^* -orbit of m . Therefore, we have

$$\overline{n}a = \theta_{\text{an}}(\overline{n}a) = \theta_{\text{an}}(a)\theta_{\text{an}}(\overline{n}) = \theta(a)^{-1}\overline{n}t \quad \text{for some } t \in T,$$

or

$$(11) \quad \theta(a)\overline{n}a = \overline{n}t \quad \text{for some } t \in T.$$

Suppose $\overline{n} = en$ for some $e \in E(\overline{T})$ and $n \in N$. By (11), we see that

$$\theta(a)e = enta^{-1}n^{-1}.$$

In particular, $\theta(a)e = e\theta(a)e$. Since U is θ -stable, we know that $\theta(a) \in U$, therefore $\theta(a)e$ is an element of eUe . Notice that $nt = t'n$ for some $t' \in T$, so $\theta(a)e = et'na^{-1}n^{-1}$. At the same time, $na^{-1}n^{-1} \in U$. In other words, $\theta(a)e = et'u'$, where $t' \in T$, $u' := na^{-1}n^{-1} \in U$, and we know that $et'u' \in eUe$. Therefore, it is harmless to continue with $\theta(a)e = eu'$.

Since square roots exist in unipotent groups, we see that $(e\theta(a)e)^{1/2} = e\theta(a)^{1/2}e = (ena^{-1}n^{-1}e)^{1/2} = e(na^{-1}n^{-1})^{1/2}e = e(na^{-1/2}n^{-1})e$.

The unit group of eMe is $eC_G(e)$, and $eC_B(e) = eBe$ is a Borel subgroup of $eC_G(e)$ (see Lemma 2.1). Since $eUe \subseteq eC_B(e)$, we find that $e\theta(a)^{1/2}e = \theta(a)^{1/2}e$ and that $e(na^{-1/2}n^{-1})e = ena^{-1/2}n^{-1}$. Now, on the one hand, we have $\theta(a)^{1/2}e = ena^{-1/2}n^{-1}$, or equivalently $\theta(a)^{1/2}\overline{n}a^{1/2} = \overline{n}$. On the other hand, $\theta(a)^{1/2}\overline{n}a^{1/2} = \theta(a)^{1/2} * (\overline{n}a)$. Hence, \overline{n} is contained in the B^* -orbit of m . ■

REMARK 4.1. Recall that $\tau : M \rightarrow M$ is defined by $\tau(x) = x\theta_{\text{an}}(x)$. The image of τ is contained in M_Q : if $m \in M$, then

$$\theta_{\text{an}}(\tau(m)) = \theta_{\text{an}}(m\theta_{\text{an}}(m)) = \theta_{\text{an}}(\theta_{\text{an}}(m))\theta_{\text{an}}(m) = m\theta_{\text{an}}(m) = \tau(m).$$

LEMMA 4.3. $T \times H$ acts on $\tau^{-1}(\overline{N})$ by $(t, h) \cdot m = tmh^{-1}$.

Proof. It suffices to check that for all $t \in T$, $h \in H$, and $m \in \tau^{-1}(\overline{N})$, the image $\tau(tmh^{-1})$ is contained in \overline{N} . But

$$\tau(tmh^{-1}) = tm\theta_{\text{an}}(m)\theta(t)^{-1}.$$

Since $\tau(m) = m\theta_{\text{an}}(m) \in \overline{N}$ and since \overline{N} is T^* -stable, the proof is finished. ■

REMARK 4.2. Let (T, B) be a pair of a θ -stable maximal torus and a Borel subgroup such that $T \subseteq B$. Let \mathcal{V} denote the set of all $g \in G$ such that $\tau(g) \in N_G(T)$. It is easy to verify that $\mathcal{V} \subset G$ is closed under the action of $T \times H$,

$$(t, h) \cdot g = tgh^{-1} \quad \text{for } t \in T, h \in H, g \in G.$$

Let V denote the set of $T \times H$ -orbits in \mathcal{V} . For $v \in V$, let $x(v) \in \mathcal{V}$ denote a representative of the orbit v . The inclusion $\mathcal{V} \hookrightarrow G$ induces a bijection from V onto the set of $B \times H$ -orbits in G . In particular, G is the disjoint union of the double cosets $Bx(v)H$, $v \in V$ (see [8]).

THEOREM 4.1. *The following sets are in bijection with each other:*

- (1) B^* -orbits in M_Q ,
- (2) $T \times H$ -orbits in $\tau^{-1}(\overline{N}) \subset M$.

Proof. We start with an observation: under τ , the set of $B \times H$ -orbits in M is surjectively mapped onto the set of B^* -orbits in M_Q . To see this, first, we show that any $B \times H$ -orbit in M is mapped by τ onto a B^* -orbit in M_Q . Let \mathcal{O}_a be the $B \times H$ -orbit of an element a from M . Since

$$\tau(bah^{-1}) = bah^{-1}\theta_{\text{an}}(bah^{-1}) = ba\theta_{\text{an}}(a)\theta_{\text{an}}(b) = b * \tau(a),$$

we see that $\tau(\mathcal{O}_a) = B * \tau(a)$. Next, we will show that any B^* -orbit in M_Q comes from a $B \times H$ -orbit in M . Let $x \in M_Q$. By Proposition 4.1, we know that any B^* -orbit in M_Q intersects \overline{N} . In particular, $B * x \cap \overline{N} \neq \emptyset$. Let $\overline{n} \in \overline{N}$ be such that $b * x = \overline{n}$ for some $b \in B$. By Lemma 4.3, we know that $T \times H$ acts on $\tau^{-1}(\overline{N})$. Let $a \in \tau^{-1}(\overline{N})$ be such that $\tau(a) = \overline{n}$. Then the $B \times H$ -orbit \mathcal{O}_a of a is mapped to $B * \tau(a) = B * x$. Now we know that $B \times H$ -orbits in M are mapped onto B^* -orbits in M_Q .

Incidentally, the argument in the above paragraph shows the following: the assignment defined by

$$(12) \quad f : (T \times H)a \mapsto T * \tau(a) \mapsto B * \tau(a)$$

is a surjective map between the set of $T \times H$ -orbits in $\tau^{-1}(\overline{N})$ and the set of B^* -orbits in M_Q . We proceed to show that f is injective.

Let O be a B^* -orbit in M_Q and let $\overline{n}_1, \overline{n}_2 \in \overline{N} \cap O$. Then there exists $b \in B$ such that

$$\overline{n}_1 = b * \overline{n}_2 = b\overline{n}_2\theta_{\text{an}}(b).$$

Since the Bruhat–Chevalley decomposition $M = \bigcup_{r \in R} BrB$ is a disjoint union, and B is θ_{an} -stable, we see from the uniqueness criterion that $b \in T$

and $\bar{n}_2 \in \bar{n}_1 T$. In other words, there exists $t \in T$ such that $t * \bar{n}_2 = \bar{n}_1$. Let $a_1, a_2 \in \tau^{-1}(\bar{N})$ be such that $\tau(a_1) = \bar{n}_1$ and $\tau(a_2) = \bar{n}_2$. Then $t * \tau(a_2) = \tau(a_1)$. But $t * \tau(a_2) = \tau(ta_2)$, or equivalently $ta_2 \in \tau^{-1}(\bar{n}_1)$. Consequently, the intersection with \bar{N} of a B^* -orbit $O (= \tau(\mathcal{O}_{a_1}))$ is covered by a single $T \times H$ -orbit in $\tau^{-1}(\bar{N})$. In particular, the map (12) is one-to-one. ■

REMARK 4.3. An important corollary of the proof of Theorem 4.1 is that the number of B^* -orbits in M_Q is finite. Indeed, any B^* -orbit in M_Q intersects \bar{N} along a T^* -orbit and \bar{N}/T is a finite semigroup.

Now we are ready to prove our first main result, which states that the following sets are finite and they are in bijection with each other:

- (1) B^* -orbits in \bar{Q} (respectively, B^* -orbits in \bar{P}),
- (2) $T \times H$ -orbits in $\tau^{-1}(\bar{N} \cap \bar{Q})$ (respectively, $T \times H$ -orbits in $\tau^{-1}(\bar{N} \cap \bar{P})$).

Proof of Theorem 1.1. Since M_Q is closed, the inclusions $P \subseteq Q \subseteq M_Q$ imply that $\bar{P} \subseteq \bar{Q} \subseteq M_Q$. Moreover, we know that P and Q , and hence \bar{P} and \bar{Q} , are B^* -stable. By Remark 4.3 we know that M_Q comprises finitely many B^* -orbits. The rest of the proof follows from Theorem 4.1. ■

There is a well known classification, due to Cartan, of involutions on semisimple groups. For the classical groups, up to inner automorphisms there are seven types of involutions in total. For the exceptional groups there are in total nine involutions. See [6, Chapter X, Section 6] for a complete list. We finish this section by presenting some examples.

EXAMPLE 4.1. Let G_0 denote PSL_n , the projective special linear group of $n \times n$ matrices with determinant 1. Then $\theta_0 : G_0 \rightarrow G_0$ defined by $\theta_0(g) = (g^{-1})^\top$ ($g \in G_0$) is an involutory automorphism. Let T_0 denote the maximal torus of diagonal matrices in G_0 and let ω_1 denote the first fundamental weight. Let $(\rho_0, V) \cong (\mathrm{id}, \mathbb{C}^n)$ denote the corresponding irreducible (minuscule) representation. Then the J -irreducible monoid associated with ω_1 is nothing but the monoid of $n \times n$ matrices,

$$Z_{\omega_1} := \overline{\mathbb{C}^* \cdot \mathrm{PSL}_n} = \mathrm{Mat}_n,$$

which we denote by M . Then the unit group of M is $G = \mathrm{GL}_n$. Clearly, θ_0 extends to G by the same formula, $\theta(g) = (g^{-1})^\top$ ($g \in G$). The G^* -orbit of the identity is equal to the set of invertible symmetric $n \times n$ matrices,

$$G * 1_{\mathrm{GL}_n} = P = \{gg^\top : g \in \mathrm{GL}_n\}.$$

We observe that for our choices of θ and G_0 , the subvariety $Q := \{g \in G : \theta(g) = g^{-1}\}$ is equal to P . Therefore, in M , we have

$$\bar{Q} = \bar{P} = \mathrm{Sym}_n,$$

the affine variety of symmetric $n \times n$ matrices. Also, we notice that the unique antiinvolution on M that is extended from the involution θ on G is given by

$\theta_{\text{an}}(m) = m^\top$ for $m \in M$. Therefore, M_Q is equal to Sym_n as well. Finally, we know from [18] that B^* -orbits in Sym_n are parametrized by the $n \times n$ “partial involutions” in the “rook monoid” R_n . Here, the *rook monoid* is the Renner monoid of Mat_n ; it is the finite monoid which consists of $n \times n$ 0/1 matrices with at most one 1 in each row and column. A *partial involution* in R_n is an element $x \in R_n$ such that $x^\top = x$.

EXAMPLE 4.2. Let (ρ_0, V) denote the second fundamental representation $V = \bigwedge^2 \mathbb{C}^{2n}$ of $G_0 := \text{PSL}_{2n}$. As before, let T_0 denote the maximal torus consisting of all diagonal matrices in G_0 . We consider the involution $\theta_0(g) = -J(g^{-1})^\top J$ ($g \in G_0$), where J is the $2n \times 2n$ block diagonal matrix

$$J = \text{diag}(J_2, \dots, J_2) \quad \text{with} \quad J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

More explicitly, V is equal to the space of $2n \times 2n$ skew-symmetric matrices, and the action of G_0 on V is given by

$$g \cdot A = (g^{-1})^\top A g^{-1}.$$

For simplicity, denote by ϕ_g the operator $\rho_0(g)$ on V ($g \in G_0$). Note that for $g = J$ we have $\phi_J^2 = 1_{\text{GL}(V)}$. It is not difficult to show that ρ_0 is faithful, and that the extension of θ_0 to $G = \mathbb{C}^* \cdot \rho_0(G_0)$ is given by

$$\theta(c\phi_g) = c\phi_J\phi_{(g^{-1})^\top}\phi_J \quad \text{for all } g \in G_0 \text{ and } c \in \mathbb{C}^*.$$

Now, let $y \in Q$. If $y = c\phi_g$ for some $g \in G_0$, and $c \in \mathbb{C}^*$, then

$$c^{-1}\phi_g^{-1} = \theta(y) = c\phi_J\phi_{(g^{-1})^\top}\phi_J, \quad \text{or equivalently} \quad 1_V = c^2\phi_g J (g^{-1})^\top J.$$

Since ρ_0 is faithful, we have $c = 1$, and $gJ(g^{-1})^\top J = 1_{G_0}$, or $g^{-1} = J(g^{-1})^\top J$. In other words, Q is isomorphic to

$$Q_0 := \{g \in \text{PSL}_{2n} : \theta_0(g) = g^{-1}\}.$$

On the other hand, we know that Q_0 is equal to $P_0 := \{g\theta_0(g^{-1}) : g \in \text{PSL}_{2n}\}$ (see [5, Section 11.3.5]). Since the image of P_0 under ρ_0 equals P , we see that $P = Q$, so P is a closed subvariety of G .

Let θ_{an} be the unique antiinvolution extension of θ to the monoid M of (ρ_0, V) . Then

$$\overline{Q} = M_Q = \{y \in M : \theta_{\text{an}}(y) = y\}.$$

Next, we compute the parametrizing set of B^* -orbits in $M_Q = M_P$. To this end, we determine the normalizer of T in G . We claim that $N_G(T) = \mathbb{C}^* \rho_0(N_{G_0}(T_0))$. Indeed, let $x = c\rho_0(g) \in G$ be an element from the normalizer of T , and let $t \in T$. Since $t = d\rho_0(t')$ for some $t' \in T_0$ and $d \in \mathbb{C}^*$, we have $xtx^{-1} = d\rho_0(gt'g^{-1}) \in T$, or equivalently $gt'g^{-1} \in T_0$. Thus, t lies in $\mathbb{C}^* \rho_0(N_{G_0}(T_0))$. The converse inclusion is obvious.

Let us look at a typical element of $N_G(T)$. Assume that g is a *monomial matrix*, that is, every row and every column have exactly one nonzero entry. We will prove that, once a basis is fixed, $\rho_0(g) = \phi_g$ is a monomial matrix as well. Towards this end, we choose the following basis:

$$F_{i,j} = E_{i,j} - E_{j,i}, \quad 1 \leq i < j \leq 2n,$$

where $E_{i,j}$'s are the elementary matrices. Suppose that the inverse of $g \in G_0$ is the matrix $g^{-1} = (g_{k,l})_{k,l=1}^{2n}$. Obviously, g^{-1} is a monomial matrix as well. Since $\rho_0(g) \cdot E_{i,j} = (g^{-1})^\top E_{i,j} g^{-1} = (g_{i,k} g_{j,l})_{k,l=1}^{2n}$, we see that

$$(13) \quad g \cdot F_{i,j} = \rho_0(g) \cdot F_{i,j} = (g_{i,k} g_{j,l} - g_{j,k} g_{i,l})_{k,l=1}^{2n}.$$

We continue with a special case of our claim by assuming that g is a diagonal matrix. Then the (k, l) th entry (with $k < l$) of $g \cdot F_{i,j}$ is nonzero if and only if $i = k$ and $j = l$. In this case, $g \cdot F_{i,j} = g_{i,i} g_{j,j} F_{i,j}$. Thus, the matrix representing $\rho_0(g)$ is the $n(n-1) \times n(n-1)$ diagonal matrix $\text{diag}(s_{1,2}, s_{1,3}, \dots, s_{n-1,n})$ with $s_{i,j} = g_{i,i} g_{j,j}$. Now, more generally, assume that g is a monomial matrix. Then the (k, l) th entry (with $k < l$) $g_{i,k} g_{j,l} - g_{j,k} g_{i,l}$ of $g \cdot F_{i,j}$ is nonzero if and only if one of the following is true:

- (i) the entries of g^{-1} at its (i, k) th and the (j, l) th positions are nonzero at the same time, or
- (ii) the entries of g^{-1} at its (i, l) th and the (j, k) th positions are nonzero at the same time.

Observe that (i) and (ii) do not hold true at the same time. Observe also that for each $i < j$ there exists a unique pair (k, l) with $k < l$ such that either (i) is true, or (ii) is true. Therefore, if g^{-1} is a monomial matrix, then

$$(14) \quad g \cdot F_{i,j} = \begin{cases} g_{i,k} g_{j,l} F_{k,l} & \text{if } g_{i,k} g_{j,l} \neq 0, \\ g_{i,l} g_{j,k} F_{k,l} & \text{if } g_{i,l} g_{j,k} \neq 0. \end{cases}$$

It follows that if g is a monomial matrix, then so is the matrix of $\rho_0(g) = \phi_g$.

Now, let $x \in M$ be an element from $\overline{N_G(T)}$. Since the elements of $\overline{N_G(T)}$ are obtained from those of $N_G(T)$ by taking limits (in the algebraic sense), we see that $x \cdot F_{i,j}$ is either identically zero, or is a scalar multiple of $F_{k,l}$ for some k, l as in (14). In other words, x is obtained from the image of a monomial matrix in G_0 by replacing some of its entries by zeros.

It is well known that the invertible symmetric monomial matrices modulo the maximal torus of diagonal matrices represent the B^* -orbits in Q , and furthermore the finite set of orbit representatives is in bijection with the fixed point free involutions of the symmetric group S_{2n} (see [16]). Thus, in our case, the representing matrices are those that are obtained from the fixed point free monomial matrices by replacing some of the nonzero entries by

zeros. These are precisely the “partial fixed point free involutions,” introduced in [2].

5. Final remarks. Given a reductive monoid M with an antiinvolution θ_{an} , we now have the notion of a *symmetric submonoid*

$$(15) \quad M_{\text{an}} := \{m \in M : m\theta_{\text{an}}(m) = 1_M\}.$$

Observe that the identity element 1_M of M is the identity element 1_G of G . Therefore, $\theta_{\text{an}}(1_M) = \theta_{\text{an}}(1_G) = \theta_{\text{an}}(1_G)\theta_{\text{an}}(1_G)$, hence $\theta_{\text{an}}(1_M) = 1_M$. In other words, $1_M \in M_{\text{an}}$. Also, if $m_1, m_2 \in M_{\text{an}}$, then

$$m_1 m_2 \theta_{\text{an}}(m_1 m_2) = m_1 m_2 \theta_{\text{an}}(m_2) \theta_{\text{an}}(m_1) = m_1 \cdot 1_M \cdot \theta_{\text{an}}(m_1) = 1_M.$$

Therefore, $m_1 m_2 \in M_{\text{an}}$. Note that if $g \in G$ lies in M_{an} , then $1_G = g^{-1} \theta_{\text{an}}(g^{-1}) = g^{-1} \theta(g)$, hence $\theta(g) = g$. In other words, the group of invertible elements of M_{an} is the fixed subgroup $H = G^\theta$. The above argument provides us with an effective way of producing new linear algebraic monoids, one for each antiinvolution θ_{an} on M .

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