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# *A NOTE ON LATTICE POINTS AND OPTIMAL STRETCHING*

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**Abstract.** We study the classical lattice point problem and its variant, an optimal stretching problem, associated with a special class of high-dimensional finite type domains. Our proofs rely on an explicit bound of the Fourier transform of indicator functions of domains under consideration.

**1. Introduction.** In this paper we study the classical lattice point problem and one of its variants, the optimal stretching problem, associated with the following domain in $\mathbb{R}^d$ $(d \geq 3)$:

$$\mathcal{D} = \Big\{x \in \mathbb{R}^d : \sum_{p=0}^{n-1} \Big( \sum_{l=1+d_p}^{d_{p+1}} x_l^{\omega_l} \Big)^{m_{p+1}} \leq 1 \Big\}, \tag{1.1}$$

where $\omega_l \in 2\mathbb{N}$ with $1 \leq l \leq d$, and $n$, $m_{p+1}, d_{p+1} \in \mathbb{N}$ with $0 \leq p \leq n-1$ and $0 = d_0 < d_1 < \cdots < d_{n-1} < d_n = d$. Throughout this paper, $\mathcal{D}$ will denote this domain.

Let $\mathcal{B} \subset \mathbb{R}^d$ be a compact convex domain which contains the origin in its interior and has a smooth boundary $\partial\mathcal{B}$. The *lattice point problem* is about counting the number of lattice points of $\mathbb{Z}^d$ in the enlarged domain $t\mathcal{B}$ and the main issue is to study the remainder

$$R_{\mathcal{B}}(t) := \#(t\mathcal{B} \cap \mathbb{Z}^d) - \mathrm{vol}(\mathcal{B})t^d \quad \text{for } t \geq 1.$$

Gauss was the first to study this kind of problem (in the special and perhaps the most important case of the unit disk in $\mathbb{R}^2$). This problem is so far unsolved to a large extent, even though it has already been extensively studied for over one hundred years.

During the past few years there was a burst of research on the preceding problem in a different form, the so-called *optimal stretching problem*, and closely related "eigenvalue optimization among rectangles" problems in spectral geometry. The optimal stretching problem we are interested in

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can be formulated as follows. Consider a volume-preserving stretch of the domain $\mathcal{B}$ in the form

$$A\mathcal{B} := \{(a_1x_1, \ldots, a_dx_d) : (x_1, \ldots, x_d) \in \mathcal{B}\}, \tag{1.2}$$

where $a_1, \ldots, a_d$ are positive numbers and $A = \operatorname{diag}(a_1, \ldots, a_d)$ is a $d \times d$ matrix satisfying $\det A = 1$. We would like to know the limiting behaviour of $A$ (as $t$ goes to infinity) for those matrices $A$ such that ([1]) $\#(\mathbb{N}^d \cap tA\mathcal{B})$ attains the largest value. A similar question can be asked for matrices $A$ such that $\#(\mathbb{Z}_+^d \cap tA\mathcal{B})$ attains the smallest value.

Let us first discuss the classical lattice point problem. If the boundary $\partial\mathcal{B}$ has points of vanishing Gaussian curvature, the problem is rather poorly understood. The solution in high dimensions is still far from complete though a few partial results are known. In [4] we studied model domains of finite type ([2]) in $\mathbb{R}^d$ (defined by (1.5) below). In this paper we study the more general finite type domain $\mathcal{D}$ (defined by (1.1)).

Our study of such domains is motivated by some examples in the literature. To mention a few, the super spheres

$$\mathcal{B} = \{x \in \mathbb{R}^d : |x_1|^\omega + \cdots + |x_d|^\omega \leq 1\}$$

are considered in Randol [18] for even $\omega \geq 3$ and in Krätzel [8] for odd $\omega \geq 3$, and it is proved that

$$R_{\mathcal{B}}(t) = O(t^{(d-1)(1-1/\omega)} + t^{d-2+2/(d+1)}) \tag{1.3}$$

and this estimate is the best possible when $\omega \geq d+1$. For further results on super spheres (ellipsoids) see [9, 7] and the references contained therein. Krätzel [10] and Krätzel and Nowak [11, 12] studied a special class of convex domains in $\mathbb{R}^3$,

$$\mathcal{B} = \{x \in \mathbb{R}^3 : |x_1|^{mk} + (|x_2|^k + |x_3|^k)^m \leq 1\} \tag{1.4}$$

with certain assumptions on the reals $k$ and $m$ (for example, in [12], $k > 2$, $m > 1$, and $mk \geq 7/3$). The contribution of flat points was evaluated precisely and that of other boundary points was estimated. In [4] the first author extended the considerations to the domain

$$\mathcal{B} = \{x \in \mathbb{R}^d : x_1^{\omega_1} + \cdots + x_d^{\omega_d} \leq 1\} \tag{1.5}$$

for $\omega_l \in 2\mathbb{N}$ with $1 \leq l \leq d$.

The domain $\mathcal{D}$ is formally an extension of both (1.4) and (1.5) in high dimensions. To state our first theorem we need some notations. For any $1 \leq i \leq d$ there exists a unique $0 \leq p(i) \leq n-1$ such that $1 + d_{p(i)} \leq i \leq d_{p(i)+1}$. Namely the $x_i^{\omega_i}$ term shows up in the $(p(i)+1)$th parenthesis in the defining

([1]) In this paper we use $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, and $\mathbb{Z}_*^d = \mathbb{Z}^d \setminus \{0\}$.

([2]) That is, at each boundary point each tangent line has finite order of contact.

equation of the domain $\mathcal{D}$ (see (1.1)). For any $1 \le j \le d$, denote

$$m_{j,l} = \begin{cases} 1 & \text{for } 1 \le l \le d \text{ with } p(l) = p(j), \\ m_{p(l)+1} & \text{for } 1 \le l \le d \text{ with } p(l) \ne p(j). \end{cases} \tag{1.6}$$

We then have

THEOREM 1.1. *For the domain $\mathcal{D}$ defined by* (1.1), *we have*

$$\begin{aligned} R_{\mathcal{D}}(t) = {} & \sum_{j=1}^{d} O, \Omega\left(t^{d-1-\sum_{1\le l\le d,\, l\ne j} \frac{1}{m_{j,l}\omega_l}}\right) \\ & + \sum_{j=1}^{d}\sum_{i=2}^{d}\sum_{S\in P_i(\mathbb{N}_d),S\ni j} O\left(t^{d-1-\frac{i-1}{d+1}-\frac{2d}{d+1}\sum_{1\le l\le d,\, l\notin S}\frac{1}{m_{j,l}\omega_l}}\right), \end{aligned} \tag{1.7}$$

*where* $\mathbb{N}_d = \{1,\dots,d\}$ *and* $P_i(\mathbb{N}_d)$ *is the collection of all subsets of* $\mathbb{N}_d$ *having* $i$ *elements. If*

$$\omega = \max_{1\le j,l\le d} m_{j,l}\omega_l, \tag{1.8}$$

*then*

$$|R_{\mathcal{D}}(t)| \lesssim t^{(d-1)(1-1/\omega)} + t^{d-2+2/(d+1)}. \tag{1.9}$$

REMARK 1.2. By taking all $m_{p+1}$'s to be 1 we recover the result in [4]. If we take $d = 3$, $n = 2$, $m_1 = m_2 = m$, $\omega_1 = \omega_2 = \omega_3 = k$, and $d_1 = 1$, the domain $\mathcal{D}$ is in the special form of (1.4). Since we only consider domains with smooth boundary, we did not allow the exponents $m$ and $k$ to be real numbers, while Krätzel and Nowak [11, 12] do allow such general exponents. We mainly use harmonic analysis tools, while Krätzel and Nowak apply a "cut-into-slices" method to reduce a three-dimensional problem to a two-dimensional one and then work carefully on the latter problem.

REMARK 1.3. Here we have the same phenomenon as in [4, Remark 1]: in (1.7), the first sum is the contribution of the boundary points which lie on coordinate axes; the term for $i = d$ is $O(t^{d-2+2/(d+1)})$, due to the boundary points that are not on any coordinate plane; all other terms for $2 \le i \le d-1$ come from the boundary points lying on coordinate planes but not on axes.

REMARK 1.4. Many authors made efforts to study general domains (instead of special examples) in $\mathbb{R}^3$ under different curvature assumptions. Partial results were obtained by Krätzel, Popov, Peter, Nowak, etc. We refer the interested readers to two excellent survey articles [7, 17] and the references given there. For domains in high dimensions, satisfactory answers are still to be found.

REMARK 1.5. For convex domains of finite type in $\mathbb{R}^d$ Iosevich, Sawyer, and Seeger [6, Theorem 1.3] provides an estimate of the remainder. Their results work for high dimensions and the curvature assumption looks quite neat. Unfortunately, even for some model domains, their bound may not be sharp. For example Randol's bound (1.3) (namely, (1.9)) for super spheres is better when $\omega$ is not too large (say, of size $< 2d^2 + O(d)$).

Let us now turn to the optimal stretching problem. It was initiated by Antunes and Freitas, who considered in [1] the stretch of the unit disk in $\mathbb{R}^2$ and proved that among all ellipses of the same area, those that enclose the most lattice points in the first quadrant must be more and more "round" as the area goes to infinity. Laugesen and Liu [14] and Ariturk and Laugesen [2] extended Antunes and Freitas' results to very general planar domains including $p$-ellipses for $0 < p < \infty$, $p \neq 1$. The subtle case of triangles was investigated by Marshall and Steinerberger [16]. Further generalizations to convex domains in high dimensions with nowhere vanishing Gaussian curvature were studied by Marshall [15]. Recently in [5] the first author and Wang studied the finite type domains (1.5).

We remark that the optimal stretching problem originated from the shape optimization problem for eigenvalues in spectral geometry. For recent progress on the latter topic and related issues, see Larson [13] and the references given there.

In this paper we continue our previous study by considering the more general domain (1.1). For each $t \geq 1$ we let ([3])

$$A(t) = \mathrm{argmax}_A \, \#(\mathbb{N}^d \cap tA\mathcal{D}), \tag{1.10}$$

where the argmax ranges over all positive definite diagonal matrices $A$ of determinant 1. $A(t)$ means a stretching matrix $A$ that makes the set $tA\mathcal{D}$ contain the most positive lattice points. Note that in general such an $A$ is not unique, i.e. there could be a set of optimal stretching matrices. So in what follows, when we write $A(t) = \mathrm{diag}(a_1(t), \dots, a_d(t))$, we really mean that $A(t)$ is any stretching matrix that maximizes the lattice counting function $\#(\mathbb{N}^d \cap tA\mathcal{D})$. For each $1 \leq j \leq d$, we denote by $\mathcal{D}_j$ the intersection of $\mathcal{D}$ with the coordinate hyperplane $x_j = 0$. We then have

Theorem 1.6. *Let $\mathcal{D}$ be the domain defined by* (1.1)*. Then any maximal stretching matrix $A(t) = \mathrm{diag}(a_1(t), \dots, a_d(t))$ in* (1.10) *satisfies*

$$\left| a_j(t) - \frac{|\mathcal{D}_j|}{(|\mathcal{D}_1| \cdots |\mathcal{D}_d|)^{1/d}} \right| = O(t^{-\gamma}), \qquad 1 \leq j \leq d, \tag{1.11}$$

*where $|\mathcal{D}_j|$ is the $(d-1)$-dimensional measure of $\mathcal{D}_j$, $\omega$ is defined by* (1.8)*,*

([3]) The notation $\mathrm{argmax}_x f(x)$ means the values of $x$ for which $f(x)$ attains the function's largest value.

*and*

$$\gamma = \min\left\{\frac{d-1}{2\omega}, \frac{d-1}{2d+2}\right\}. \tag{1.12}$$

Similarly we consider the optimal stretching problem for nonnegative lattice points. For each $t \geq 1$ we let

$$\widetilde{A}(t) = \operatorname{argmin}_{\widetilde{A}} \#(\mathbb{Z}_+^d \cap t\widetilde{A}\mathcal{D}), \tag{1.13}$$

where the argmin ranges over all positive definite diagonal matrices $\widetilde{A}$ of determinant 1. When we write $\widetilde{A}(t) = \operatorname{diag}(\tilde{a}_1(t), \ldots, \tilde{a}_d(t))$, we mean that $\widetilde{A}(t)$ is any stretching matrix that minimizes the lattice counting function $\#(\mathbb{Z}_+^d \cap t\widetilde{A}\mathcal{D})$. Then we have

Theorem 1.7. *Let $\mathcal{D}$ be the domain defined by* (1.1). *Then any minimal stretching matrix $\widetilde{A}(t) = \operatorname{diag}(\tilde{a}_1(t), \ldots, \tilde{a}_d(t))$ in* (1.13) *satisfies*

$$\left|\tilde{a}_j(t) - \frac{|\mathcal{D}_j|}{(|\mathcal{D}_1| \cdots |\mathcal{D}_d|)^{1/d}}\right| = O(t^{-\gamma}), \qquad 1 \leq j \leq d,$$

*where $\gamma$ is given by* (1.12).

Remark 1.8. A domain in $\mathbb{R}^d$ is said to be *balanced* if the $(d-1)$-dimensional measures of the intersections of the domain with each coordinate hyperplane are equal. The results above mean that the optimal domain which contains the most positive (or least nonnegative) lattice points is asymptotically balanced. They are consistent with results in the literature, for example, [5, Theorems 1.1 and 1.2].

*Notations.* The Fourier transform of any function $f \in L^1(\mathbb{R}^d)$ is $\widehat{f}(\xi) = \int f(x) \exp(-2\pi i x \cdot \xi)\, dx$. For functions $f$ and $g$ with $g$ taking nonnegative real values, $f \lesssim g$ means $|f| \leq Cg$ for some constant $C$. If $f$ is nonnegative, $f \gtrsim g$ means $g \lesssim f$. The Landau notation $f = O(g)$ is equivalent to $f \lesssim g$. The notation $f \asymp g$ means that $f \lesssim g$ and $g \lesssim f$. For lower bounds, $f(t) = \Omega_+(g(t))$ means that $\limsup f(t)/g(t) > 0$ as $t \to \infty$, $f(t) = \Omega_-(g(t))$ stands for $-f(t) = \Omega_+(g(t))$, and $f(t) = \Omega(g(t))$ means that at least one of the previous two assertions is true.

## 2. The Fourier transform of the indicator function $\chi_{\mathcal{D}}$.

Let $\mathcal{D}$ be defined by (1.1). If $x \in \partial\mathcal{D}$ let $T_x$ be the affine tangent plane to $\partial\mathcal{D}$ at $x$. Bruna, Nagel, and Wainger [3] define a "ball"

$$\tilde{B}(x, \delta) = \{y \in \partial\mathcal{D} : \operatorname{dist}(y, T_x) < \delta\}$$

to be a cap near $x$ cut off from $\partial\mathcal{D}$ by a plane parallel to $T_x$ at distance $\delta$ from it. For nonzero $\xi \in \mathbb{R}^d$ let $x(\xi)$ be the unique point on $\partial\mathcal{D}$ where the unit exterior normal is $\xi/|\xi|$.

Our main work in this paper is to get the following size estimate of the surface measure of $\tilde{B}(x(\xi), |\xi|^{-1})$. This result generalizes [4, Lemma 2.2] and its computation is harder.

LEMMA 2.1. *Let* $0 < \varepsilon_0 \le 1$ *be a constant and* $1 \le j \le d$ *an integer. For any nonzero* $\xi \in \mathbb{R}^d$ *with* $|\xi_j|/|\xi| \ge \varepsilon_0$*, we have*

$$\sigma\big(\tilde{B}(x(\xi), |\xi|^{-1})\big) \lesssim \prod_{\substack{l=1 \\ l\neq j}}^{d} \min\big\{|\xi|^{-\frac{1}{m_{j,l}\omega_l}}, |\xi|^{-1/2}(|\xi_l|/|\xi|)^{-\frac{m_{j,l}\omega_l-2}{2(m_{j,l}\omega_l-1)}}\big\},$$

*where* $m_{j,l}$ *is defined by* (1.6) *and the implicit constant only depends on* $\varepsilon_0$ *and* $\mathcal{D}$*.*

*Proof.* For an arbitrarily fixed nonzero $\xi \in \mathbb{R}^d$ with $|\xi_j|/|\xi| \ge \varepsilon_0$, denote $x(\xi) = (a_1, \dots, a_d) \in \partial\mathcal{D}$. Due to the symmetry of $\partial\mathcal{D}$, we may assume that all $\xi_l$'s and $a_l$'s are nonnegative. We only treat the case $j = 1$, as all other cases are similar.

$\partial\mathcal{D}$ is described by the equation

$$F(x) = 0 \tag{2.1}$$

with $F$ explicitly determined by (1.1). Hence

$$\frac{\nabla F}{|\nabla F|}(x(\xi)) = \frac{\xi}{|\xi|}, \tag{2.2}$$

where $|\nabla F| \asymp 1$.

By definition the cap of interest is the one near $x(\xi)$ cut off from $\partial\mathcal{D}$ by the plane

$$\sum_{l=1}^{d} \xi_l(x_l - a_l) + 1 = 0. \tag{2.3}$$

After changing variables $X_l = x_l - a_l$, combining equations (2.1) and (2.3), and eliminating $X_1$, we get

$$\Big(\Big(a_1 - \xi_1^{-1} - \xi_1^{-1}\sum_{l=2}^{d} \xi_l X_l\Big)^{\omega_1} + \sum_{l=2}^{d_1}(a_l + X_l)^{\omega_l}\Big)^{m_1} + \sum_{p=1}^{n-1}\Big(\sum_{l=1+d_p}^{d_{p+1}} (a_l + X_l)^{\omega_l}\Big)^{m_{p+1}} - 1 = 0. \tag{2.4}$$

To estimate $\sigma(\tilde{B}(x(\xi), |\xi|^{-1}))$ it suffices to show that if $(X_2, \dots, X_d)$ satisfies (2.4) then for each $2 \le l \le d$,

$$\max|X_l| \lesssim \min\big\{|\xi|^{-\frac{1}{m_{1,l}\omega_l}}, |\xi|^{-1/2}(|\xi_l|/|\xi|)^{-\frac{m_{1,l}\omega_l-2}{2(m_{1,l}\omega_l-1)}}\big\}. \tag{2.5}$$

To prove (2.5) we discuss two cases: $2 \le l \le d_1$ or $1 + d_1 \le l \le d$.

CASE 1: $2 \le l \le d_1$. We may assume $l = 2$ as other cases can be handled similarly.

SUBCASE 1.1: $\xi_2/|\xi| = 0$. Then (2.4) implies

$$|X_2| \lesssim |\xi|^{-1/\omega_2}. \tag{2.6}$$

Indeed, in this case $a_2 = 0$ by (2.2). We apply Taylor's expansion of order two to $(a_1 - \xi_1^{-1} - \xi_1^{-1} \sum_{l=2}^{d} \xi_l X_l)^{\omega_1}$ at $a_1$ and to $(a_l + X_l)^{\omega_l}$ at $a_l$ (for $3 \le l \le d$) with nonnegative remainders (due to the evenness of $\omega_l$). For each $0 \le p \le n-1$ we then apply Taylor's expansion of order two to the $m_{p+1}$th powers in (2.4) at $\sum_{l=1+d_p}^{d_{p+1}} a_l^{\omega_l}$. After using the condition $x(\xi) \in \partial\mathcal{D}$ to cancel the constant term, (2.2) to eliminate linear terms, and dropping nonnegative remainder terms, we get

$$X_2^{\omega_2} \le \omega_1 a_1^{\omega_1 - 1} \xi_1^{-1}, \tag{2.7}$$

which implies (2.6) since $\xi_1 \asymp |\xi|$.

SUBCASE 1.2: $\xi_2/|\xi| \ne 0$. In this case $a_2 \ne 0$. Besides all the expansions used in Subcase 1.1, we also need

$$(a_2 + X_2)^{\omega_2} = a_2^{\omega_2} + \omega_2 a_2^{\omega_2 - 1} X_2 + a_2^{\omega_2 - 2} X_2^2 (\omega_2(\omega_2 - 1)/2 + \delta_1) + X_2^{\omega_2}$$

by the binomial formula, where

$$\delta_1 = C_{\omega_2}^3 X_2/a_2 + C_{\omega_2}^4 (X_2/a_2)^2 + \cdots + C_{\omega_2}^{\omega_2 - 1} (X_2/a_2)^{\omega_2 - 3}.$$

As in Subcase 1.1, we get

$$a_2^{\omega_2 - 2} X_2^2 (\omega_2(\omega_2 - 1)/2 + \delta_1) + X_2^{\omega_2} \le \omega_1 a_1^{\omega_1 - 1} \xi_1^{-1} \tag{2.8}$$

as a replacement of (2.7). Note that $a_1 \gtrsim 1$ since $|\xi_1|/|\xi| \ge \varepsilon_0$. Hence the second equation in (2.2) implies $a_2 \asymp (\xi_2/|\xi|)^{1/(\omega_2 - 1)}$.

If $X_2 > 0$, then $\delta_1 > 0$. (2.8) immediately implies the desired bound

$$\max_{X_2 > 0} |X_2| \lesssim \min\left\{ |\xi|^{-1/\omega_2}, |\xi|^{-1/2} (|\xi_2|/|\xi|)^{-\frac{\omega_2 - 2}{2(\omega_2 - 1)}} \right\}.$$

If $X_2 < 0$ and $\max_{X_2 < 0} |X_2| \le c_1 a_2$ for a sufficiently small constant $c_1$ (say, such that $\omega_2(\omega_2 - 1)/2 + \delta_1 > \omega_2(\omega_2 - 1)/4$), then (2.8) implies the desired bound for $\max_{X_2 < 0} |X_2|$.

If $X_2 < 0$ and $\max_{X_2 < 0} |X_2| > c_1 a_2$, by a compactness argument there exists a constant $C_1$ (depending only on $c_1$ and $\mathcal{D}$) such that the cap $\tilde{B}(x(\xi), C_1 |\xi|^{-1})$ intersects the plane $x_2 = -a_2$. It suffices to estimate the size of this larger cap. So we need to study (2.4) with $\xi$ replaced by $\xi/C_1$ and to estimate $\max_{X_2 < 0} |X_2|$ subject to $\max_{X_2 < 0} |X_2| > 2a_2$. Like (2.8) we get

$$a_2^{\omega_2 - 2} X_2^2 (\omega_2(\omega_2 - 1)/2 + \delta_1) + X_2^{\omega_2} \lesssim |\xi|^{-1}.$$

We also note that if $-X_2 > 2a_2$ then

$$a_2^{\omega_2-2}X_2^2(\omega_2(\omega_2-1)/2+\delta_1)+X_2^{\omega_2}$$
$$= (a_2+X_2)^{\omega_2} - a_2^{\omega_2} - \omega_2 a_2^{\omega_2-1}X_2 \geq (a_2+X_2)^{\omega_2} \geq X_2^{\omega_2}/2^{\omega_2}.$$

Combining the two inequalities above yields

$$\max_{X_2<0}|X_2| \lesssim |\xi|^{-1/\omega_2}. \tag{2.9}$$

Hence $a_2 \lesssim |\xi|^{-1/\omega_2}$, which implies

$$|\xi|^{-1/\omega_2} \lesssim |\xi|^{-1/2}(|\xi_2|/|\xi|)^{-\frac{\omega_2-2}{2(\omega_2-1)}}. \tag{2.10}$$

By (2.9) and (2.10) we again get the desired bound for $\max_{X_2<0}|X_2|$. This finishes Subcase 1.2, hence Case 1 as well.

CASE 2: $1+d_1 \leq l \leq d$. We may assume $l=d$, as other cases can be handled similarly.

SUBCASE 2.1: $\xi_d/|\xi| = 0$. In this case $a_d = 0$ by (2.2). This case is the same as Subcase 1.1 except that we need to treat $X_d$ (instead of $X_2$) separately. More precisely, we apply

$$\Big(\sum_{l=1+d_{n-1}}^{d}(a_l+X_l)^{\omega_l}\Big)^{m_n} \geq \Big(\sum_{l=1+d_{n-1}}^{d-1}(a_l+X_l)^{\omega_l}\Big)^{m_n} + X_d^{m_n\omega_d}$$

and then like (2.7) we get

$$X_d^{m_n\omega_d} \leq m_1\omega_1\Big(\sum_{l=1}^{d_1}a_l^{\omega_l}\Big)^{m_1-1}a_1^{\omega_1-1}\xi_1^{-1},$$

which implies

$$\max|X_d| \lesssim |\xi|^{-1/(m_n\omega_d)}.$$

SUBCASE 2.2: $\xi_d/|\xi| \neq 0$. Note that the last equation of the system (2.2) implies

$$\Big(\sum_{l=1+d_{n-1}}^{d}a_l^{\omega_l}\Big)^{m_n-1}a_d^{\omega_d-1} \asymp \xi_d/|\xi|, \tag{2.11}$$

hence

$$a_d \lesssim (\xi_d/|\xi|)^{1/(m_n\omega_d-1)}. \tag{2.12}$$

If $X_d > 0$, we apply the binomial formula to $(a_d + X_d)^{\omega_d}$ and use

$$\Big(\sum_{l=1+d_{n-1}}^{d} (a_l + X_l)^{\omega_l}\Big)^{m_n} \geq X_d^{m_n\omega_d} + \Big(\sum_{l=1+d_{n-1}}^{d-1} (a_l + X_l)^{\omega_l} + a_d^{\omega_d} + \omega_d a_d^{\omega_d-1} X_d + \frac{\omega_d(\omega_d-1)}{2} a_d^{\omega_d-2} X_d^2\Big)^{m_n}$$

to get a separated term $X_d^{m_n\omega_d}$. As in Subcase 1.1, we get

$$\frac{m_n\omega_d(\omega_d-1)}{2}\Big(\sum_{l=1+d_{n-1}}^{d} a_l^{\omega_l}\Big)^{m_n-1} a_d^{\omega_d-2} X_d^2 + X_d^{m_n\omega_d} \leq m_1\omega_1\Big(\sum_{l=1}^{d_1} a_l^{\omega_l}\Big)^{m_1-1} a_1^{\omega_1-1}\xi_1^{-1} \lesssim |\xi|^{-1}.$$

The inequality above, combined with (2.11) and (2.12), yields the desired bound

$$\max_{X_d>0} |X_d| \lesssim \min\big\{|\xi|^{-\frac{1}{m_n\omega_d}}, |\xi|^{-1/2}(\xi_d/|\xi|)^{-\frac{m_n\omega_d-2}{2(m_n\omega_d-1)}}\big\}.$$

If $X_d < 0$, we do not need to separate an $X_d^{m_n\omega_d}$ term. We mimic the computation used to derive (2.8) in Subcase 1.2 and get

$$m_n\Big(\sum_{l=1+d_{n-1}}^{d} a_l^{\omega_l}\Big)^{m_n-1}\big(a_d^{\omega_d-2}X_d^2(\omega_d(\omega_d-1)/2+\delta_2) + X_d^{\omega_d}\big) \leq m_1\omega_1\Big(\sum_{l=1}^{d_1} a_l^{\omega_l}\Big)^{m_1-1} a_1^{\omega_1-1}\xi_1^{-1}, \tag{2.13}$$

where

$$\delta_2 = C_{\omega_d}^3 X_d/a_d + C_{\omega_d}^4 (X_d/a_d)^2 + \cdots + C_{\omega_d}^{\omega_d-1}(X_d/a_d)^{\omega_3-3}.$$

If $\max_{X_d<0}|X_d| \leq c_2 a_d$ for a sufficiently small constant $c_2$, then (2.13) (with (2.11) and (2.12)) implies

$$\max_{X_d<0}|X_d| \lesssim |\xi|^{-1/2}(\xi_d/|\xi|)^{-\frac{m_n\omega_d-2}{2(m_n\omega_d-1)}} \tag{2.14}$$

and

$$a_d^{\omega_d(m_n-1)} X_d^{\omega_d} \lesssim |\xi|^{-1}. \tag{2.15}$$

Since $\max_{X_d<0}|X_d| \leq c_2 a_d$, (2.15) implies

$$\max_{X_d<0}|X_d| \lesssim |\xi|^{-1/(m_n\omega_d)}. \tag{2.16}$$

Then (2.14) and (2.16) give the desired bound for $\max_{X_d<0}|X_d|$ under the assumption that $\max_{X_d<0}|X_d| \le c_2 a_d$.

If $\max_{X_d<0}|X_d| > c_2 a_d$, by a compactness argument there is a constant $C_2 \ge 1$ (depending only on $c_2$ and $\mathcal{D}$) such that the cap $\tilde{B}(x(\xi), C_2|\xi|^{-1})$ intersects the plane $x_d = -a_d$. It suffices to estimate the size of that cap. Hence we need to study (2.4) with $\xi$ replaced by $\xi/C_2$ and to estimate $\max_{X_d<0}|X_d|$ subject to $\max_{X_d<0}|X_d| > 2a_d$. Like (2.13), we get

$$m_n\Big(\sum_{l=1+d_{n-1}}^{d} a_l^{\omega_l}\Big)^{m_n-1}\big(a_d^{\omega_d-2}X_d^2(\omega_d(\omega_d-1)/2+\delta_2)+X_d^{\omega_d}\big) \lesssim |\xi|^{-1}.$$

We also note that if $-X_d > 2a_d$ then

$$a_d^{\omega_d-2}X_d^2(\omega_d(\omega_d-1)/2+\delta_2)+X_d^{\omega_d} \ge X_d^{\omega_d}/2^{\omega_d}.$$

Combining the two inequalities above yields

$$\Big(\sum_{l=1+d_{n-1}}^{d} a_l^{\omega_l}\Big)^{m_n-1}\Big(\max_{X_d<0}|X_d|\Big)^{\omega_d} \lesssim |\xi|^{-1}. \tag{2.17}$$

It then follows from (2.17) and $\max_{X_d<0}|X_d| > c_2 a_d$ that

$$\Big(\sum_{l=1+d_{n-1}}^{d} a_l^{\omega_l}\Big)^{m_n-1} a_d^{\omega_d-2}\Big(\max_{X_d<0}|X_d|\Big)^{2} \lesssim |\xi|^{-1},$$

which (with (2.11) and (2.12)) implies (2.14).

It remains to prove (2.16). Since the cap $\tilde{B}(x(\xi), C_2|\xi|^{-1})$ intersects the coordinate plane $x_d = 0$, we can take a point $P$ from the intersection. By [3, Theorem A] there exists a constant $C_3$ (depending only on $\mathcal{D}$) such that $\tilde{B}(x(\xi), C_2|\xi|^{-1}) \subset \tilde{B}(P, C_3C_2|\xi|^{-1})$. Applying to $\tilde{B}(P, C_3C_2|\xi|^{-1})$ the result of Subcase 2.1 yields (2.16). This finishes the estimate of $\max_{X_d<0}|X_d|$ when $\max_{X_d<0}|X_d| > c_2 a_d$ and the proof of Subcase 2.2, hence the entire proof of the lemma. ∎

The Gauss–Green formula, [3, Theorem B], and Lemma 2.1 easily yield the following generalization of [18, II, Theorem 2] and [4, Theorem 2.1].

THEOREM 2.2. *Let $0 < \varepsilon_0 \le 1$ be a constant and $1 \le j \le d$ an integer. For any $\xi \in S^{d-1}$ with $|\xi_j| \ge \varepsilon_0$ and $t > 0$ we have*

$$|\widehat{\chi}_{\mathcal{D}}(t\xi)| \lesssim t^{-1}\prod_{\substack{l=1\\ l\ne j}}^{d}\min\big\{t^{-\frac{1}{m_{j,l}\omega_l}}, t^{-1/2}|\xi_l|^{-\frac{m_{j,l}\omega_l-2}{2(m_{j,l}\omega_l-1)}}\big\},$$

*where $m_{j,l}$ is defined by (1.6) and the implicit constant only depends on $\varepsilon_0$ and $\mathcal{D}$.*

**3. Lattice points counting.** In this section we prove several results concerning the number of lattice points. Theorem 3.1 provides both upper and lower bounds for the lattice counting for the dilated and stretched domain $tA\mathcal{D}$. We track the effects of both the dilation and stretching. As an immediate consequence, Theorem 1.1 follows by taking $A$ to be the identity matrix. Then we derive Corollary 3.2 and Theorem 3.3, whose proofs are analogous to those in [5, Section 3].

Throughout this section we use the following notations. $\mathcal{D}$ represents the domain defined by (1.1), $\mathcal{D}_j$ is the intersection of $\mathcal{D}$ with the coordinate hyperplane $x_j = 0$, $A = \operatorname{diag}(a_1, \dots, a_d)$ is any positive definite diagonal matrix with determinant 1, and

$$a_* = \|A^{-1}\|_\infty = \max\{a_1^{-1}, \dots, a_d^{-1}\}.$$

THEOREM 3.1. *If $a_*^{-1}t \geq 1$ we have*

$$R_{A\mathcal{D}}(t) = \sum_{j=1}^{d} O, \mathit{\Omega}_A\left(t^d (a_*^{-1}t)^{-1-\sum_{1\leq l\leq d,\, l\neq j} \frac{1}{m_{j,l}\omega_l}}\right) + \sum_{j=1}^{d}\sum_{i=2}^{d} \sum_{S\in P_i(\mathbb{N}_d),\, S\ni j} O\left(t^d (a_*^{-1}t)^{-1-\frac{i-1}{d+1}-\frac{2d}{d+1}\sum_{1\leq l\leq d,\, l\notin S}\frac{1}{m_{j,l}\omega_l}}\right), \tag{3.1}$$

*where $\mathbb{N}_d = \{1, \dots, d\}$ and $P_i(\mathbb{N}_d)$ is the collection of all subsets of $\mathbb{N}_d$ having $i$ elements. All implicit constants depend on the domain $\mathcal{D}$, while those implicit in the big-$\mathit{\Omega}$ notations also depend on the matrix $A$. If $\omega = \max_{1\leq j,l\leq d} m_{j,l}\omega_l$, then*

$$|R_{A\mathcal{D}}(t)| \lesssim a_*^{1+(d-1)/\omega} t^{(d-1)(1-1/\omega)} + a_*^{2-2/(d+1)} t^{d-2+2/(d+1)}. \tag{3.2}$$

*Proof.* We start with the standard inequality

$$\chi_{(t-a_*\epsilon)A\mathcal{D}} * \rho_\epsilon \leq \chi_{tA\mathcal{D}} \leq \chi_{(t+a_*\epsilon)A\mathcal{D}} * \rho_\epsilon, \tag{3.3}$$

where $0 \leq \rho \in C_0^\infty(\mathbb{R}^d)$ satisfies $\int \rho = 1$, and $\rho_\epsilon(x) = \epsilon^{-d}\rho(\epsilon^{-1}x)$ with $\epsilon > 0$. By summing (3.3) over $\mathbb{Z}^d$ and using the Poisson summation formula we get

$$\sum_{k\in\mathbb{Z}^d} \widehat{\chi}_{(t-a_*\epsilon)A\mathcal{D}}(k)\widehat{\rho}(\epsilon k) \leq \sum_{k\in\mathbb{Z}^d} \chi_{tA\mathcal{D}}(k) \leq \sum_{k\in\mathbb{Z}^d} \widehat{\chi}_{(t+a_*\epsilon)A\mathcal{D}}(k)\widehat{\rho}(\epsilon k). \tag{3.4}$$

Note that

$$\sum_{k\in\mathbb{Z}^d} \widehat{\chi}_{(t\pm a_*\epsilon)A\mathcal{D}}(k)\widehat{\rho}(\epsilon k) = \operatorname{vol}(\mathcal{D})t^d + O(t^{d-1}a_*\epsilon) + (t\pm a_*\epsilon)^d \sum_{k\in\mathbb{Z}_*^d} \widehat{\chi}_{\mathcal{D}}((t\pm a_*\epsilon)Ak)\widehat{\rho}(\epsilon k). \tag{3.5}$$

Hence we need to estimate $\sum_{k\in\mathbb{Z}^d_*}\widehat{\chi}_{\mathcal{D}}(tAk)\widehat{\rho}(\epsilon k)$ for any $t$. By using a partition of unity we have

$$\sum_{k\in\mathbb{Z}^d_*}\widehat{\chi}_{\mathcal{D}}(tAk)\widehat{\rho}(\epsilon k)=\sum_{j=1}^{d}\sum_{k\in\mathbb{Z}^d_*}\Omega_j(Ak)\widehat{\chi}_{\mathcal{D}}(tAk)\widehat{\rho}(\epsilon k)=:\sum_{j=1}^{d}S_j,$$

where $\Omega_j$ is supported in $\Gamma_j=\{x\in\mathbb{R}^d : |x_j|/|x|\geq(2d)^{-1/2}\}$ and smooth away from the origin. We then split $S_j$ as follows:

$$S_j=\sum_{i=1}^{d}\sum_{(i)}\Omega_j(Ak)\widehat{\chi}_{\mathcal{D}}(tAk)\widehat{\rho}(\epsilon k)=:\sum_{i=1}^{d}S_{i,j},$$

where the summation $\sum_{(i)}$ is over all $k\in\mathbb{Z}^d_*$ with precisely $i$ components nonzero.

Now we estimate $S_1$. The support of $\Omega_1$ requires $|a_1k_1|/|Ak|\geq(2d)^{-1/2}$. Theorem 2.2 implies that

$$|S_{1,1}|\lesssim\sum_{k_1\in\mathbb{Z}^1_*}|ta_1k_1|^{-1}\prod_{l=2}^{d}|ta_1k_1|^{-1/(m_{1,l}\omega_l)}\lesssim(a_*^{-1}t)^{-1-\sum_{l=2}^{d}1/(m_{1,l}\omega_l)}\tag{3.6}$$

and that $|S_{i,1}|$, for each $i\geq 2$, is bounded by a sum in which the exponent of $|Ak|$ is negative. Since $a_*^{-1}\leq a_l$ and $a_*^{-1}|k|\leq|Ak|$, we get

$$|S_{i,1}|\lesssim\sum_{S\in P_i(\mathbb{N}_d),\,S\ni 1}(a_*^{-1}t)^{-\frac{i+1}{2}-\sum_{l=1,\,l\notin S}^{d}\frac{1}{m_{1,l}\omega_l}}\left(1+\epsilon^{-\frac{i-1}{2}+\sum_{l=1,\,l\notin S}^{d}\frac{1}{m_{1,l}\omega_l}}\right).\tag{3.7}$$

Note that the first term of the right side above is less than the bound of $|S_{1,1}|$ in (3.6) if $a_*^{-1}t\geq 1$. We take $\epsilon=(a_*^{-1}t)^{-(d-1)/(d+1)}$. Then (3.6) and (3.7) give

$$|S_1|\lesssim(a_*^{-1}t)^{-1-\sum_{l=2}^{d}\frac{1}{m_{1,l}\omega_l}}+\sum_{i=2}^{d}\sum_{\substack{S\in P_i(\mathbb{N}_d)\\ S\ni 1}}(a_*^{-1}t)^{-1-\frac{i-1}{d+1}-\frac{2d}{d+1}\sum_{l=1,l\notin S}^{d}\frac{1}{m_{1,l}\omega_l}}.$$

The estimations of $S_j$ for $2\leq j\leq d$ are similar. So we obtain a bound of $\sum_{k\in\mathbb{Z}^d_*}\widehat{\chi}_{\mathcal{D}}(tAk)\widehat{\rho}(\epsilon k)$. Note that $t\pm a_*\epsilon\asymp t$ if $a_*\epsilon\leq t/2$, which is equivalent to $a_*^{-1}t\geq 2^{(d+1)/(2d)}$. Thus combining (3.4), (3.5) and the bound of $\sum_{k\in\mathbb{Z}^d_*}\widehat{\chi}_{\mathcal{D}}(tAk)\widehat{\rho}(\epsilon k)$ yields the desired upper bound in (3.1). If $1\leq a_*^{-1}t\leq 2^{(d+1)/(2d)}$, the desired upper bound in (3.1) is trivial since $\#(tA\mathcal{D}\cap\mathbb{Z}^d)\leq\#(2^{(d+1)/(2d)}tA\mathcal{D}\cap\mathbb{Z}^d)\lesssim t^d$. Finally, notice that (3.2) follows easily from (3.1).

We now turn to the proof of the lower bound in (1.7) (see also [6, pp. 167–168]). We may assume $j=1$, as other cases are similar.

For the domain $\mathcal{D}$, the asymptotic expansion in Schulz [19] gives

$$\widehat{n_1 d\sigma}(tk) = C_4 i \sin(-2\pi t k_1 + \pi\nu/2)(tk_1)^{-\nu} + O((tk_1)^{-\nu-1/\eta}),$$

where $n_1$ is the 1st component of the Gauss map of $\partial\mathcal{D}$, $d\sigma$ is the surface measure, $k = (k_1, 0, \ldots, 0)$ with $k_1 \in \mathbb{N}$, $C_4$ is a real number which depends on $\mathcal{D}$, $\nu = \sum_{l=2}^{d} 1/(m_{1,l}\omega_l)$, and $\eta$ is the least common multiple of $m_{1,2}\omega_2, \ldots, m_{1,d}\omega_d$. Then we apply the Gauss–Green formula to get

(3.8)

$$\widehat{\chi}_{\mathcal{D}}(tAk) = C_5 \sin(-2\pi t a_1 k_1 + \pi\nu/2)(t a_1 k_1)^{-1-\nu} + O((t a_1 k_1)^{-1-\nu-1/\eta}),$$

where $C_5$ is a real number.

We rewrite $S_{1,1}$ as

$$\sum_{\substack{k=(k_1,0,\ldots,0)\\ k_1\in\mathbb{Z}^1_*}} \widehat{\chi}_{\mathcal{D}}(tAk) + \sum_{\substack{k=(k_1,0,\ldots,0)\\ k_1\in\mathbb{Z}^1_*}} \widehat{\chi}_{\mathcal{D}}(tAk)(\widehat{\rho}(\epsilon k) - 1).$$

For the first term, we have

$$\sum_{\substack{k=(k_1,0,\ldots,0)\\ k_1\in\mathbb{Z}^1_*}} \widehat{\chi}_{\mathcal{D}}(tAk) = (a_1 t)^{-1-\nu} g_1(t) + O((a_1 t)^{-1-\nu-1/\eta}), \tag{3.9}$$

where

$$g_1(t) = C_5 \sum_{k_1\in\mathbb{Z}^1_*} \sin(-2\pi t a_1 |k_1| + \pi\nu/2)|k_1|^{-1-\nu}.$$

Here the periodic real function $g_1(t)$ satisfies $\limsup_{t\to\infty} |g_1(t)| > 0$. And for the second term, we get

$$\sum_{\substack{k=(k_1,0,\ldots,0)\\ k_1\in\mathbb{Z}^1_*}} \widehat{\chi}_{\mathcal{D}}(tAk)(\widehat{\rho}(\epsilon k) - 1) = O((a_1 t)^{-1-\nu}(\epsilon^\nu + \epsilon)). \tag{3.10}$$

By using (3.9) and (3.10), we have

$$\limsup_{t\to\infty} \left| (t \pm a_*\epsilon)^d \sum_{\substack{k=(k_1,0,\ldots,0)\\ k_1\in\mathbb{Z}^1_*}} \widehat{\chi}_{\mathcal{D}}((t \pm a_*\epsilon)Ak)\widehat{\rho}(\epsilon k) \right| / (t^d (a_*^{-1} t)^{-1-\nu})$$
$$= \limsup_{t\to\infty} (a_1/a_*^{-1})^{-1-\nu} |g_1(t \pm a_*\epsilon)|,$$

where $\limsup_{t\to\infty} |g_1(t \pm a_*\epsilon)|$ is positive and independent of the matrix $A$. This finishes the proof of the lower bound, and hence the entire proof of the theorem. ∎

Let $A_j = \operatorname{diag}(a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_d)$ be a $(d-1)\times(d-1)$ matrix, let $A_{j,l} = \operatorname{diag}(a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_{l-1}, a_{l+1}, \ldots, a_d)$ for $1 \le j < l \le d$ be a $(d-2)\times(d-2)$ matrix, and set $\mathcal{D}_{j,l} = \mathcal{D}_j \cap \mathcal{D}_l \subset \mathbb{R}^d$. For notational simplicity we may abuse the notation $\mathcal{D}_j$ and $\mathcal{D}_{j,l}$ for subsets of $\mathbb{R}^{d-1}$ and $\mathbb{R}^{d-2}$ respectively, although they are technically both subsets of $\mathbb{R}^d$.

Corollary 3.2. *If $a_*^{-1}t \geq 1$ then*

$$\#\Big(\mathbb{Z}^d \cap tA\Big(\bigcup_{j=1}^d \mathcal{D}_j\Big)\Big) - \sum_{j=1}^d a_j^{-1}|\mathcal{D}_j|t^{d-1} = O\big(a_*^{1+\frac{d-1}{\omega}} t^{(d-1)(1-\frac{1}{\omega})} + a_*^{2-\frac{2}{d+1}} t^{d-2+\frac{2}{d+1}}\big), \tag{3.11}$$

*where the implicit constant depends only on the domain $\mathcal{D}$.*

*Proof.* Since

$$\Big|\#\Big(\mathbb{Z}^d \cap tA\Big(\bigcup_{j=1}^d \mathcal{D}_j\Big)\Big) - \sum_{j=1}^d \#(\mathbb{Z}^{d-1} \cap tA_j\mathcal{D}_j)\Big| \leq \sum_{1\leq j<l\leq d} \#(\mathbb{Z}^{d-2} \cap tA_{j,l}\mathcal{D}_{j,l}),$$

we have

$$\Big|\#\Big(\mathbb{Z}^d \cap tA\Big(\bigcup_{j=1}^d \mathcal{D}_j\Big)\Big) - \sum_{j=1}^d a_j^{-1}|\mathcal{D}_j|t^{d-1}\Big| \leq \sum_{1\leq j<l\leq d} \#(\mathbb{Z}^{d-2} \cap tA_{j,l}\mathcal{D}_{j,l}) + \sum_{j=1}^d \Big|\#(\mathbb{Z}^{d-1} \cap tA_j\mathcal{D}_j) - a_j^{-1}|\mathcal{D}_j|t^{d-1}\Big|. \tag{3.12}$$

We rearrange $\#(\mathbb{Z}^{d-1} \cap tA_j\mathcal{D}_j)$ as

$$\#\big(\mathbb{Z}^{d-1} \cap (ta_j^{-\frac{1}{d-1}})(a_j^{\frac{1}{d-1}}A_j)\mathcal{D}_j\big)$$

and apply (3.2) to get

$$\begin{aligned}\big|\#(\mathbb{Z}^{d-1} \cap tA_j\mathcal{D}_j) - a_j^{-1}|\mathcal{D}_j|t^{d-1}\big| &\lesssim a_j^{-1}\big(a_*^{1+\frac{d-2}{\omega}} t^{(d-2)(1-\frac{1}{\omega})} + a_*^{2-\frac{2}{d}} t^{d-3+\frac{2}{d}}\big) \\ &\lesssim a_*^{1+\frac{d-1}{\omega}} t^{(d-1)(1-\frac{1}{\omega})} + a_*^{2-\frac{2}{d+1}} t^{d-2+\frac{2}{d+1}},\end{aligned} \tag{3.13}$$

where in the last inequality we use the definition of $a_*$ and $a_*^{-1}t \geq 1$. By a similar argument we have

$$\#(\mathbb{Z}^{d-2} \cap tA_{j,l}\mathcal{D}_{j,l}) \lesssim a_*^{1+\frac{d-1}{\omega}} t^{(d-1)(1-\frac{1}{\omega})} + a_*^{2-\frac{2}{d+1}} t^{d-2+\frac{2}{d+1}}, \tag{3.14}$$

where $1 \leq j < l \leq d$. Finally, we apply (3.12)–(3.14) to obtain the asymptotic formula (3.11). ∎

In the following theorem, we get the numbers of positive and nonnegative lattice points in $tA\mathcal{D}$.

THEOREM 3.3. *If* $a_*^{-1}t \ge 1$ *then*

$$\#(\mathbb{N}^d \cap tA\mathcal{D}) = 2^{-d}|\mathcal{D}|t^d - 2^{-d}\sum_{j=1}^{d} a_j^{-1}|\mathcal{D}_j|t^{d-1} + O\big(a_*^{1+\frac{d-1}{\omega}} t^{(d-1)(1-\frac{1}{\omega})} + a_*^{2-\frac{2}{d+1}} t^{d-2+\frac{2}{d+1}}\big) \tag{3.15}$$

*and*

$$\#(\mathbb{Z}_+^d \cap tA\mathcal{D}) = 2^{-d}|\mathcal{D}|t^d + 2^{-d}\sum_{j=1}^{d} a_j^{-1}|\mathcal{D}_j|t^{d-1} + O\big(a_*^{1+\frac{d-1}{\omega}} t^{(d-1)(1-\frac{1}{\omega})} + a_*^{2-\frac{2}{d+1}} t^{d-2+\frac{2}{d+1}}\big), \tag{3.16}$$

*where the implicit constants depend only on the domain* $\mathcal{D}$.

*Proof.* Since the domain $\mathcal{D}$ is symmetric, we have

$$\#(\mathbb{N}^d \cap tA\mathcal{D}) = 2^{-d}\Big(\#(\mathbb{Z}^d \cap tA\mathcal{D}) - \#\Big(\mathbb{Z}^d \cap tA\Big(\bigcup_{j=1}^{d} \mathcal{D}_j\Big)\Big)\Big).$$

So (3.15) follows from (3.2) and (3.11).

To prove (3.16), let

$$K(S) = \{(k_1, \dots, k_d) \in \mathbb{Z}_+^d : k_i = 0 \text{ if } i \in S, \text{ otherwise } k_i \in \mathbb{N}\},$$

where $S \in P_j(\mathbb{N}_d)$ for $1 \le j \le d$. Here $\mathbb{N}_d = \{1, \dots, d\}$ and $P_j(\mathbb{N}_d)$ is the collection of all subsets of $\mathbb{N}_d$ having $j$ elements. Thus

$$\#(\mathbb{Z}_+^d \cap tA\mathcal{D}) = \#(\mathbb{N}^d \cap tA\mathcal{D}) + \sum_{j=1}^{d} \sum_{S\in P_j(\mathbb{N}_d)} \#(K(S) \cap tA\mathcal{D}). \tag{3.17}$$

Note that

$$\sum_{S\in P_1(\mathbb{N}_d)} \#(K(S) \cap tA\mathcal{D}) = \sum_{j=1}^{d} \#(\mathbb{N}^{d-1} \cap tA_j\mathcal{D}_j) \tag{3.18}$$

$$= \sum_{j=1}^{d} 2^{-(d-1)}\Big(\#(\mathbb{Z}^{d-1} \cap tA_j\mathcal{D}_j) - \#\Big(\mathbb{Z}^{d-1} \cap tA_j\Big(\bigcup_{l=1,\, l\neq j}^{d} (\mathcal{D}_l \cap \mathcal{D}_j)\Big)\Big)\Big),$$

where

$$\#\Big(\mathbb{Z}^{d-1} \cap tA_j\Big(\bigcup_{l=1,\, l\neq j}^{d} (\mathcal{D}_l \cap \mathcal{D}_j)\Big)\Big) \lesssim \sum_{1\le j<l\le d} \#(\mathbb{Z}^{d-2} \cap tA_{j,l}\mathcal{D}_{j,l}). \tag{3.19}$$

When $j \geq 2$,

$$\sum_{S \in P_j(\mathbb{N}_d)} \#(K(S) \cap tA\mathcal{D}) \lesssim \sum_{1 \leq l < m \leq d} \#(\mathbb{Z}^{d-2} \cap tA_{l,m}\mathcal{D}_{l,m}). \tag{3.20}$$

By (3.17)–(3.20) and (3.13)–(3.15) we get the asymptotic formula (3.16). ∎

**4. Proofs of Theorems 1.6 and 1.7.** The proofs of Theorem 1.6 and 1.7 are very similar to each other and also to those in [5, Section 4]. For completeness we briefly prove Theorem 1.6 in this section. See [5] for details.

*Proof of Theorem 1.6.* By applying (3.15) for the following matrix

$$B = \operatorname{diag}\left(\frac{|\mathcal{D}_1|}{(|\mathcal{D}_1| \cdots |\mathcal{D}_d|)^{1/d}}, \dots, \frac{|\mathcal{D}_d|}{(|\mathcal{D}_1| \cdots |\mathcal{D}_d|)^{1/d}}\right)$$

we get

$$\begin{aligned} \#(\mathbb{N}^d \cap tB\mathcal{D}) &= 2^{-d}|\mathcal{D}|t^d - 2^{-d}d(|\mathcal{D}_1| \cdots |\mathcal{D}_d|)^{1/d}t^{d-1} \\ &\quad + O(t^{(d-1)(1-\frac{1}{\omega})} + t^{d-2+\frac{2}{d+1}}) && (4.1) \\ &\geq 2^{-d}|\mathcal{D}|t^d - Ct^{d-1} && (4.2) \end{aligned}$$

for sufficiently large $t$, where $C = 2^{1-d}d(\prod_{j=1}^d |\mathcal{D}_j|)^{1/d}$.

For every sufficiently large $t$ let $A(t)$ be a fixed optimal stretching matrix (see (1.10)). Thus

$$\#(\mathbb{N}^d \cap tB\mathcal{D}) \leq \#(\mathbb{N}^d \cap tA(t)\mathcal{D}). \tag{4.3}$$

Denote $a_*(t) = \|A(t)^{-1}\|_\infty$. Then $t/a_*(t) \geq 1$, as otherwise $tA(t)\mathcal{D}$ contains no positive lattice point. Therefore [5, Proposition 2.1] gives

$$\#(\mathbb{N}^d \cap tA(t)\mathcal{D}) \leq 2^{-d}|\mathcal{D}|t^d - ca_*(t)t^{d-1} \tag{4.4}$$

for some constant $c > 0$ depending only on $\mathcal{D}$. Combining (4.2)–(4.4) yields

$$a_*(t) \leq C/c.$$

Then (3.15) gives

$$\begin{aligned} \#(\mathbb{N}^d \cap tA(t)\mathcal{D}) = 2^{-d}|\mathcal{D}|t^d - 2^{-d}\sum_{j=1}^d a_j(t)^{-1}|\mathcal{D}_j|t^{d-1} \\ + O\big(t^{(d-1)(1-\frac{1}{\omega})} + t^{d-2+\frac{2}{d+1}}\big). \end{aligned}$$

Plugging this asymptotics and (4.1) into (4.3) yields

$$\sum_{j=1}^d a_j(t)^{-1}\frac{|\mathcal{D}_j|}{(|\mathcal{D}_1| \cdots |\mathcal{D}_d|)^{1/d}} \leq d + O(t^{-\frac{d-1}{\omega}} + t^{-\frac{d-1}{d+1}}).$$

Then (1.11) follows easily from [5, Appendix, Lemma B.1]. ∎

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