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NULL HYPERSURFACES EVOLVED BY THEIR MEAN CURVATURE IN A LORENTZIAN MANIFOLD

BY

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Abstract. We use null isometric immersions to introduce time-dependent null hypersurfaces, in a Lorentzian manifold, evolving in the direction of their mean curvature vector (a vector transversal to the null hypersurface). We prove an existence result for such hypersurfaces in a short-time interval. Then, we discuss the evolution of some induced geometric objects. Consequently, we prove under certain geometric conditions that some of the above objects will blow-up in finite time. Also, several examples are given to illustrate the main ideas.

1. Introduction. Flow of Riemannian hypersurfaces by functions of their mean curvatures (for example, mean curvature flow, inverse mean curvature flow, and many more) has been an interesting area of research in the past 30 years. Numerous results have been obtained: for instance, Huisken [H98] showed that inverse mean curvature evolution can be used to relate the size of a black hole and its total energy. On the other hand, the same author [H84] showed that the solution to mean curvature evolution with some specified initial data remains smooth, compact and convex until it shrinks to a "round point" in a finite time; that is, the asymptotic shape of the evolving hypersurface just before it disappears is a sphere. Gage and Hamilton [GH86] used curve shortening flow to prove on isoperimetric inequality for convex planar domains, and Andrews [A97] used affine mean curvature flow to prove affine isoperimetric inequalities. More results and details on Riemannian mean curvature evolution can be found in the above mentioned papers and references therein.

Lightlike hypersurfaces are closely linked to black hole theory and the study of trapped surfaces. In fact, it has been shown in [DB96] that black hole horizons can be represented by null hypersurfaces. The classical approach to the above phenomena has been the use of expansion scalars (null mean curvature) from null geodesics, via Raychaudhuri's equation (see [DS10] for

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details). Literature shows that our universe is expanding in time. This can be supported by the fact that very distant stars appear with red-shifts. By definition, a *red-shift* is a change in color when an object that emits light is moving away from the observer [DS10]. It is called a red-shift because the shifts are towards the red part of the spectrum. Heavenly bodies like stars are moving away from us radially, so we say that at one time the universe was smaller. Going back far enough in time, it is evident from the above explanation that the universe may have been something the size of an atom, what we call the Primordial Atom [DS10]. Along with the expanding universe, since the black holes are surrounded by a local mass distribution and expand by the inflow of galactic debris as well as electromagnetic and gravitational radiation, their area increases in a given physical situation. Hence black holes are also expanding in time. Therefore, although the classical isolated black holes have been extensively studied (see [DS10] and references therein), they do not represent a realistic model in the context of an expanding universe. To address this issue of representing expanding black holes, Ashtekar–Krishnan [AK03] introduced a concept of dynamical horizons which are a special type of spacelike hypersurfaces of a spacetime whose asymptotic states are the isolated horizons.

Motivated by the fact that our universe (and its constituents) is changing with time, we introduce a geometric evolution of null hypersurfaces along their null mean curvature. In particular, this kind of evolution has a potential to explain, among other applications, the idea of trapped surfaces (hence the study of black holes since trapped surfaces are linked to black holes). To the best of our understanding, this is the first piece of work in this direction on null hypersurfaces in Lorentzian manifolds, despite the fact that the study of null hypersurfaces was started long time ago by Duggal–Bejancu [DB96] and Kupeli [K96], and up-to-date results may be found in the books [DB96, DS10]. This motivated other researchers to investigate the geometry of null submanifolds (see for example [A09, AET03, DL14, J13, MS16]).

The paper is arranged as follows: In Section 2 we quote some basic geometric tools necessary for the development of other sections. In Section 3 we define the concept of null mean curvature flow (MCF) for null hypersurfaces, using null isometric immersions (see Definition 3.2), and give some supporting physical example. Section 4 is devoted to existence results and geometry of induced objects on the evolving hypersurface.

2. Preliminaries. Consider an $(n+1)$-dimensional null hypersurface (M, g), $n \geq 2$, of an $(n+2)$-dimensional Lorentzian manifold $(\overline{M}, \overline{g})$, where g is a degenerate metric on M, induced by the Lorentzian metric $\overline{g}$ of $\overline{M}$. In the null hypersurface case, basic differences occur mainly due to the fact that the normal vector bundle $TM^{\perp}$ is the same as the null tangent bundle along a non-zero differentiable radical distribution $\operatorname{Rad} TM$ of M, defined by

$$\operatorname{Rad} T_pM = T_pM^{\perp} = \{E_p \in T_pM : g(E_p, X) = 0, \forall X \in T_pM\},$$

where $\dim(\operatorname{Rad} TM) = 1$. There exists a Riemannian screen distribution [DB96], denoted $S(TM)$, on M which is complementary to the radical distribution such that we have the orthogonal direct sum $TM = TM^{\perp} \perp S(TM)$. Throughout this paper, $\Gamma(\Xi)$ will denote the $\mathcal{F}(M)$-module of differentiable sections of a vector bundle Ξ. The manifolds we consider are supposed to be paracompact, smooth and connected. Therefore, the existence of $S(TM)$ is secured. However, in general, $S(TM)$ is not canonical (thus not unique) and the null geometry depends on its choice. But it is known [DB96, DS10] that $S(TM)$ is canonically isomorphic to the bundle $TM/TM^{\perp}$ considered by Kupeli [K96].

From [DB96, p. 79, Theorem 1.1], we know that for a screen distribution $S(TM)$ on M there exists a unique vector bundle $tr(TM)$ such that for any non-zero local normal section $E \in \Gamma(\operatorname{Rad} TM|_{\mathcal{U}})$ on $\mathcal{U} \subset M$ there exists a unique section N of $tr(TM)|_{\mathcal{U}}$ satisfying $\overline{g}(E, N) = 1$ and $\overline{g}(N, Z) = 0$ for all $Z \in \Gamma(S(TM)|_{\mathcal{U}})$. Then we have the decomposition $T\overline{M}|_M = TM \oplus tr(TM)$.

Let ∇ and ∇^* denote the induced connections on M and $S(TM)$ respectively, and P be the projection of TM onto $S(TM)$. Then the local Gauss–Weingarten equations of M and $S(TM)$ are the following [DB96]:

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y) = \nabla_X Y + B(X, Y)N, \tag{2.1}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^t_X N = -A_N X + \tau(X)N, \tag{2.2}$$

$$\nabla_X PY = \nabla^*_X PY + h^*(X, PY) = \nabla^*_X PY + C(X, PY)E, \tag{2.3}$$

$$\nabla_X E = -A^*_E X + \nabla^{*t}_X E = -A^*_E X - \tau(X)E, \quad A^*_E E = 0, \tag{2.4}$$

for all $X, Y \in \Gamma(TM)$, $E \in \Gamma(TM^{\perp})$ and $N \in \Gamma(tr(TM))$, where $\overline{\nabla}$ is the Levi-Civita connection on $\overline{M}$. In the above setting, B is the local second fundamental form of M, and C is the local second fundamental form on $S(TM)$. A_N and A^*_E are the shape operators on TM and $S(TM)$ respectively, while τ is a 1-form on TM. The above shape operators are related to their local fundamental forms by

$$g(A^*_E X, Y) = B(X, Y), \quad g(A_N X, PY) = C(X, PY), \tag{2.5}$$

$$\overline{g}(A^*_E X, N) = 0, \quad \overline{g}(A_N X, N) = 0, \quad \forall X, Y \in \Gamma(TM). \tag{2.6}$$

From (2.6) we notice that A^*_E and A_N are both screen-valued operators.

Let $\vartheta = \overline{g}(N, \cdot)$ be a 1-form metrically equivalent to N defined on $\overline{M}$. Take $\lambda = i^*\vartheta$ to be its restriction on M, where $i : M \to \overline{M}$ is the inclusion map. Then it is easy to show that

$$(\nabla_X g)(Y, Z) = B(X, Y)\lambda(Z) + B(X, Z)\lambda(Y), \quad \forall X, Y, Z \in \Gamma(TM), \tag{2.7}$$

which indicates that ∇ is generally *not* a metric connection with respect to g. However, the induced connection ∇^* on $S(TM)$ is a metric connection. For more details about null hypersurfaces see the books [DB96] and [DS10].

The degeneracy of the induced metric tensor g on M is associated with several challenges in the study of null geometry (for instance see [DB96] and [DS10]). Prior to these books, Katsuno [K81] introduced a non-degenerate metric (called the *associated metric*) on null hypersurfaces in 4-dimensional Lorentzian manifolds. This metric was extended to null hypersurfaces in $(n+2)$-dimensional Lorentzian manifolds in [A09] and used to define some geometric operators on null hypersurfaces. It has also been linked to null rigging techniques on null hypersurfaces (for example, see [GO16] and other references therein).

More precisely, for a null hypersurface (M, g) of a Lorentzian manifold $(\overline{M}, \overline{g})$, the *associated metric* $\widehat{g}$ is given by

$$\widehat{g}(X,Y) = g(X,Y) + \lambda(X)\lambda(Y), \quad \forall X, Y \in \Gamma(TM). \tag{2.8}$$

The metric $\widehat{g}$ is invertible and its inverse, $g^{[\cdot,\cdot]}$, was called the *pseudo-inverse* of g (see [AET03]). Also, observe that $\widehat{g}$ coincides with g if the latter is non-degenerate. The metric $\widehat{g}$ has been used to define (on M) the usual operators such as *gradient*, *divergence*, *d'Alembertian* (see [AET03]), which one cannot afford with the degenerate metric g. In case $\widehat{g}$ coincides with g on M, we define the gradient $\nabla^s f$, Hessian $\mathrm{Hess}^s(f)$, and d'Alembertian $\Delta^s f$ of a smooth function f on $\mathcal{U} \subset M$ with respect to the screen distribution $S(TM)$ as

$$\begin{aligned} \nabla^s f = g^{\alpha\beta} X_\alpha(f) X_\beta, \quad \mathrm{Hess}^s(f) = X_\alpha(X_\beta(f)) - (\nabla^*_{X_\alpha} X_\beta)(f), \\ \Delta^s(f) = \mathrm{tr}^s(\mathrm{Hess}^s(f)) = g^{\alpha\beta}(X_\alpha(X_\beta(f)) - (\nabla^*_{X_\alpha} X_\beta)(f)), \end{aligned} \tag{2.9}$$

where $\{X_1, \ldots, X_n\}$ is a basis of $S(TM)$, and $\mathrm{tr}^s(\cdot)$ denotes the trace with respect to $S(TM)$. *Throughout this paper, we assume that M carries the associated metric $\widehat{g}$*, and $\mathrm{tr}(\cdot)$ will denote the trace over M with respect to $\widehat{g}$.

Denote by R and $\overline{R}$ the curvature tensors of M and $\overline{M}$ respectively. Then, by [DB96, p. 94], we have

$$\begin{aligned} \overline{g}(\overline{R}(X,Y)Z, PW) = g(R(X,Y)Z, PW) + B(X,Z)C(Y,PW) \\ - B(Y,Z)C(X,PW), \quad \forall X, Y, Z, W \in \Gamma(TM). \end{aligned} \tag{2.10}$$

In what follows, we shall make use of the following convention on the range of indices: $1 \leq \alpha, \beta, \gamma, \mu, \sigma \leq n$, $0 \leq a, b, c \leq n$ and $0 \leq i, j, k \leq n+1$.

3. Null mean curvature flow. The concept of null hypersurfaces (and generally, submanifolds) can be presented alternatively by using a special isometric immersion [DB96]. Let (M, g) be a 1-null manifold of dimension

$n+1$ and index $q-1$, with $n>0$ and $q>0$. Let $S(TM)$ be the screen distribution of M as in the previous section.

Suppose there exist a vector bundle $\varLambda$ of rank 1 over M such that $T\overline{M} = TM \oplus \varLambda$ is a semi-Riemannian vector bundle with a semi-Riemannian metric $\overline{g}$ satisfying

(C$_1$) $\overline{g}(X,Y) = g(X,Y)$, $\overline{g}(Z,V) = \overline{g}(V,V') = 0$

for any $X, Y \in \varGamma(TM)$, $Z \in \varGamma(S(TM))$ and $V, V' \in \varGamma(\varLambda)$. Since $\overline{g}$ is non-degenerate on E, it follows that $\overline{g}(U,V) \neq 0$ for any non-zero vector fields $U \in \varGamma(\operatorname{Rad} TM)$ and $V \in \varGamma(\varLambda)$. Suppose there exists a torsion-free linear connection ∇' on M and a linear connection ∇^{tr} on $\varLambda$ satisfying

(C$_2$) $\overline{g}(\nabla'_X U, V) + \overline{g}(U, \nabla^{tr}_X V) = X(\overline{g}(U,V))$

for any $X \in \varGamma(TM)$, $U \in \varGamma(\operatorname{Rad} TM)$ and $V \in \varGamma(\varLambda)$. As

$$TM = S(TM) \perp \operatorname{Rad} TM, \tag{3.1}$$

we have the decompositions

$$\nabla'_X PY = \nabla^{*\prime}_X PY + h^{*\prime}(X, PY), \quad \forall X, Y \in \varGamma(TM), \tag{3.2}$$

$$\nabla'_X U = -A^{*\prime}_U X + \nabla^{*tr}_X U, \quad \forall X \in \varGamma(TM), \quad U \in \varGamma(\operatorname{Rad} TM), \tag{3.3}$$

where P is the projection morphism of TM onto $S(TM)$, $\{\nabla^{*\prime}_X PY, A^{*\prime}_U X\}$ and $\{h^{*\prime}(X,PY), \nabla^{*tr}_X U\}$ belong to $S(TM)$ and $\operatorname{Rad} TM$ respectively. It follows that $\nabla^{*\prime}$ and ∇^{*tr} are linear connections on the vector bundles $S(TM)$ and $\operatorname{Rad} TM$, respectively. On the other hand, $h^{*\prime}$ and $A^{*\prime}$ are $\mathcal{F}(M)$-bilinear forms on $\varGamma(TM) \times \varGamma(S(TM))$ and on $\varGamma(\operatorname{Rad} TM) \times \varGamma(TM)$ respectively. Moreover, we suppose that:

(C$_3$) $A^{*\prime}_U U = 0$, $g(A^{*\prime}_U X, Y) = g(X, A^{*\prime}_U Y)$,
(C$_4$) $(\nabla'_X g)(PY, PZ) = (\nabla'_X g)(U,U) = 0$, $(\nabla'_X g)(PY, U) = g(A^{*\prime}_U X, PY)$,
(C$_5$) $(\nabla'_X A^{*\prime})(U,Y) = (\nabla'_Y A^{*\prime})(U,X)$,

for all $X, Y, Z \in \varGamma(TM)$ and $U \in \varGamma(\operatorname{Rad} TM)$, where $(\nabla'_X A^{*\prime})(U,Y) = \nabla'_X A^{*\prime}_U Y - A^{*\prime}_{\nabla^{*tr}_X U} Y - A^{*\prime}_U \nabla'_X Y$. Denote by R' the curvature tensor of ∇' and further suppose that

(C$_6$) $g(R'(X,Y)Z, PW) = g(A^{*\prime}_{h^{*\prime}(X,PW)} Y, Z) - g(A^{*\prime}_{h^{*\prime}(Y,PW)} X, Z)$,
(C$_7$) $\overline{g}(R'(X,Y)Z, V) = 0$ for all $X, Y, Z, W \in \varGamma(TM)$ and $V \in \varGamma(\varLambda)$.

Let $(M, g, S(TM))$ be a 1-null simply connected $(n+1)$-dimensional manifold of index $q-1$, endowed with a vector bundle $\varLambda$ and geometric objects $\overline{g}$, ∇', ∇^{tr}, $h^{*\prime}$ and $A^{*\prime}$ satisfying conditions (C$_1$)–(C$_7$). Then there exists a null isometric immersion $F: M^{n+1} \to \overline{M}^{n+2}$ [DB96, Theorem 4.1] satisfying

$$g(X,Y) = \overline{g}(F_* X, F_* Y), \quad \forall X, Y \in \varGamma(TM), \tag{3.4}$$

and a vector bundle isomorphism $\overline{F} : tr(TM) \to tr(TF(M))$ such that

$$F_*(\nabla'_X Y) = \nabla_{F_* X} F_* Y, \quad F_*(A^{*\prime}_U X) = A^*_{F_* U} F_* X, \tag{3.5}$$

$$F_*(h^{*\prime}(X, PY)) = h^*(F_* X, F_* PY), \quad \overline{F}(\nabla^{tr}_X V) = \nabla^t_X \overline{F} V, \tag{3.6}$$

for all $X, Y \in \Gamma(TM)$, $U \in \Gamma(\mathrm{Rad}\, TM)$ and $V \in \Gamma(tr(TM))$, where $tr(TF(M))$ is the null transversal vector bundle of $F(M)$ with respect to $F_* S(TM)$, and ∇, ∇^t, h^*, A^* are the geometric objects induced on $F(M)$ with respect to the immersion F. Certainly, $F(M)$ is nothing but a 1-null submanifold of $\overline{M}^{n+2}$ and F preserves both the radical and screen distribution, that is, F_* maps respectively the radical subspace and the screen of the domain to that of the base (see details in [DB96, p. 104]). More precisely,

$$\mathrm{Rad}\, TF(M) = F_* \,\mathrm{Rad}\, TM, \quad S(TF(M)) = F_* S(TM). \tag{3.7}$$

Next, suppose that $F(M)$ is a null hypersurface as described above. By the method of [DB96], the null mean curvature vector $\mathbf{H}$ of $F(M)$ at $p \in M$ is a smooth vector field transversal to $F(M)$ and given by

$$\mathbf{H} = \mathrm{tr}^s(h) = \mathrm{tr}^s(B) N = \mathcal{S} N, \tag{3.8}$$

where $N \in \Gamma(tr(FTM))$ and $\mathcal{S} := \mathrm{tr}^s(B) = \mathrm{tr}^s(A^*_E)$. The function $\mathcal{S}$ is called the *null mean curvature* of $F(M)$. Thus, $F(M)$ will be said to have constant mean curvature if $\mathcal{S}$ is constant.

In the following example, we calculate $\mathcal{S}$ for a null Monge hypersurface.

Example 3.1 (Null Monge hypersurface of $\mathbb{R}^{n+2}_1$). Consider a non-zero smooth function $G : \Sigma \to \mathbb{R}$, where Σ is an open set in $\mathbb{R}^{n+2}_1$. It is well-known [DB96, DS10] that $M = \{(x_0, \ldots, x_{n+1}) \in \mathbb{R}^{n+2}_1 : x_0 = G(x_1, \ldots, x_{n+1})\}$ is a Monge hypersurface. Moreover, it is easy to check that M is a null hypersurface if and only if G is a solution of the partial differential equation $\sum_{i=1}^{n+1} (G'_{x_i})^2 = 1$. Then $\mathrm{Rad}\, TM$ and $tr(TM)$ are respectively spanned by the global vector fields

$$E = \frac{\partial}{\partial x_0} + \sum_{i=1}^{n+1} G'_{x_i} \frac{\partial}{\partial x_i} \quad \text{and} \quad N = \frac{1}{2}\Big(-\frac{\partial}{\partial x_0} + \sum_{i=1}^{n+1} G'_{x_i} \frac{\partial}{\partial x_i} \Big).$$

The corresponding screen distribution is given by $\{Z_1, \ldots, Z_n\}$, where $Z_\alpha = G'_{x_{n+1}} \frac{\partial}{\partial x_\alpha} - G'_{x_\alpha} \frac{\partial}{\partial x_{n+1}}$ for $\alpha \in \{1, \ldots, n\}$. Differentiating $\sum_{i=1}^{n+1} (G'_{x_i})^2 = 1$ we get $\sum_{i=1}^{n+1} G'_{x_i} G''_{x_i x_j} = 0$ for all $i, j \in \{1, \ldots, n+1\}$. Thus, $\overline{\nabla}_E E = \sum_{j=1}^{n+1} \sum_{i=1}^{n+1} G'_{x_i} G''_{x_i x_j} \frac{\partial}{\partial x_j} = 0$. Also, by simple calculations we have

$$\overline{\nabla}_{Z_\alpha} E = \sum_{i=1}^{n+1} (G'_{x_{n+1}} G''_{x_\alpha x_i} - G'_{x_\alpha} G''_{x_{n+1} x_i}) \frac{\partial}{\partial x_i}. \tag{3.9}$$

Since $\frac{\partial}{\partial x_\beta} = (G'_{x_{n+1}})^{-1}(Z_\beta - G'_{x_\beta}\frac{\partial}{\partial x_{n+1}})$, (3.9) simplifies to

$$\overline{\nabla}_{Z_\alpha}E = \sum_{\beta=1}^{n}(G''_{x_\alpha x_\beta} - (G'_{x_{n+1}})^{-1}G'_{x_\alpha}G''_{x_{n+1}x_\beta})Z_\beta + \sum_{\beta=1}^{n}(G''_{x_\alpha x_\beta}G'_{x_\beta} - (G'_{x_{n+1}})^{-1}G'_{x_\beta}G''_{x_{n+1}x_\beta})\frac{\partial}{\partial x_{n+1}} + (G'_{x_{n+1}}G''_{x_\alpha x_{n+1}} - G'_{x_\alpha}G''_{x_{n+1}x_{n+1}})\frac{\partial}{\partial x_{n+1}}. \tag{3.10}$$

Using the fact that $\sum_{\beta=1}^{n}(G''_{x_\alpha x_\beta}) + G'_{x_{n+1}}G''_{x_\alpha x_{n+1}} = 0$ in (3.10) we deduce

$$\overline{\nabla}_{Z_\alpha}E = \sum_{\beta=1}^{n}(G''_{x_\alpha x_\beta} - (G'_{x_{n+1}})^{-1}G'_{x_\alpha}G''_{x_{n+1}x_\beta})Z_\beta. \tag{3.11}$$

From (3.11) and $\sum_{i=1}^{n+1}G'_{x_i}G''_{x_i x_j} = 0$ it follows that the mean curvature function $\mathcal{S}$ is given by

$$\mathcal{S} = \sum_{\alpha=1}^{n}(G''_{x_\alpha x_\alpha} - (G'_{x_{n+1}})^{-1}G'_{x_\alpha}G''_{x_{n+1}x_\alpha}) = \sum_{i=1}^{n+1}G''_{x_i x_i}.$$

3.1. Velocity on M. In order to introduce mean curvature flow for null hypersurfaces we need to recall from [O83, pp. 10–11] some basic facts about *velocity* in manifolds. Let $\sigma : I \to \mathbb{R}$ be a smooth curve in a manifold M, where I is an open interval in $\mathbb{R}$. In fact, I has a coordinate system consisting of the identity map v of I. At each $t \in \mathbb{R}$, one can picture the coordinate vector $(d/dv)(t) \in T_t\mathbb{R}$ as the unit vector at t in the positive v-direction. Then, the velocity vector of σ at $t \in I$ is given by

$$\frac{\partial}{\partial t}\sigma(t) = \sigma_*\left(\left.\frac{d}{dv}\right|_t\right) \in T_{\sigma(t)}M, \tag{3.12}$$

where σ_* denotes the differential of σ. Intuitively speaking, $\frac{\partial}{\partial t}\sigma(t)$ is the vector rate of change of σ at $t \in I$. Let $x_0, x_1, \ldots, x_n$ be a coordinate system in M at a point $\sigma(t)$ of σ. Then we can express $\frac{\partial}{\partial t}\sigma(t)$ in coordinate form as

$$\frac{\partial}{\partial t}\sigma(t) = \sum_{a=0}^{n}\frac{d(x_a \circ \sigma)}{dv}(t)\left.\frac{\partial}{\partial x_a}\right|_{\sigma(t)}. \tag{3.13}$$

Moreover, if $F : M \to \overline{M}$ is a map, then F carries σ to the curve $F \circ \sigma : I \to \overline{M}$ in $\overline{M}$. Furthermore, the differential map of F *preserves velocities*, that is,

$$F_*\left(\frac{\partial}{\partial t}\sigma(t)\right) = \frac{\partial}{\partial t}(F \circ \sigma)(t), \quad \forall t \in I. \tag{3.14}$$

A curve σ is *regular* provided $\frac{\partial}{\partial t}\sigma(t) \neq 0$ for all t. If $[w, y]$ is a closed interval in $\mathbb{R}$, then a curve segment $\sigma : [w, y] \to M$ is a function that has a smooth extension to an open interval containing $[w, y]$. Thus, $\frac{\partial}{\partial t}\sigma(t)$ is well-defined even at the endpoints w and y. For other properties of the velocity vector $\frac{\partial}{\partial t}\sigma(t)$, see [O83].

Next, we adopt the concept of mean curvature flow for Riemannian hypersurfaces given in [H90]. For null hypersurfaces, we introduce the null mean curvature flow (MCF) according to the following definition.

Definition 3.2. Let $(M^{n+1}, g, S(TM))$ be a compact null hypersurface of a semi-Riemannian manifold $(\overline{M}^{n+2}, \overline{g})$. Assume that $F_0 : M^{n+1} \times [0, T) \to \overline{M}^{n+2}$ smoothly immerses M as a null hypersurface in $\overline{M}$ such that conditions (C_1)–(C_7) and (3.4)–(3.7) hold. We say that $M_0 := F_0(M)$ *evolves by its null mean curvature vector* if there is a whole family $F(\cdot, t)$ of smooth immersions with corresponding hypersurfaces $M_t := F(\cdot, t)(M)$ such that

$$\frac{\partial F}{\partial t}(p, t) = \mathbf{H}(p, t), \quad p \in M^{n+1}, \quad F(\cdot, 0) = F_0, \tag{3.15}$$

where $\mathbf{H}(p, t) := \mathcal{S}(p, t)N$ is the mean curvature vector of M_t at $F(p, t)$.

It is important to notice that in the null MCF case, the flow of the surface M_t is towards the transversal direction with a pseudo-speed equal to its *null mean curvature* $\mathcal{S} := \mathrm{tr}^s(B) = \mathrm{tr}^s(A^*_E)$, as opposed to the flow towards the normal direction in the Riemannian (or semi-Riemannian) case. In fact, if $\sigma : [0, T) \to M$ is a smooth curve (also, σ can be seen as a *particle* in M) on an interval $[0, T) \subset \mathbb{R}$, then the action of the null isometric immersion F on M, that is, $F : M \to \overline{M}$, gives rise to a new curve (particle) $F \circ \sigma : [0, T) \to \overline{M}$ in $\overline{M}$. Then, for a given time $t \in [0, T)$ and $p = \sigma(t) \in M$, the velocity vector is given by (3.14) as $\frac{\partial}{\partial t}(F \circ \sigma)(t)$, which is a vector in $\overline{M}$ with tangential component $\left(\frac{\partial}{\partial t}(F \circ \sigma)(t)\right)^{\top}$ and transversal component $\left(\frac{\partial}{\partial t}(F \circ \sigma)(t)\right)^{t}$. If at every $t \in [0, T)$ we set $\left(\frac{\partial}{\partial t}(F \circ \sigma)(t)\right)^{t} = \mathbf{H} = \mathcal{S}N$, then we obtain the null MCF in the definition above. This choice makes sense due to the fact that tangential velocity has no influence on the shape of the evolving null hypersurface but it only controls the parametrization of the hypersurface (see [M12, p. 10] for details).

Example 3.3. In what follows, we show how a double null foliation of Minkowski spacetime evolves under null MCF. Let S_0 be an embedded 2-surface and $\mathit{\Lambda}_0$, $\underline{\mathit{\Lambda}}_0$ be the null hypersurfaces spanned by outgoing and incoming null geodesics normal to S_0. Let $\varrho : S_0 \to \mathbb{R}$ be a smooth function (the null lapse, see [A13]) on S_0, and E' be a null vector field normal to S_0 (and tangential to $\mathit{\Lambda}_0$). Also let N' be the null vector field normal to S_0 and tangent to the null geodesics of $\underline{\Lambda}_0$ such that

$$\overline{g}(E', N') = -\varrho^{-2}. \tag{3.16}$$

Let us extend the fields E', N'on Λ_0, $\underline{\Lambda}_0$ respectively so that $\overline{\nabla}_{E'}E' = 0$ and $\overline{\nabla}_{N'}N' = 0$. Also, extend ϱ to a function on the null hypersurfaces Λ_0 and $\underline{\Lambda}_0$, and consider the vector fields $E'' = \varrho^2 E'$ and $N'' = \varrho^2 N'$. Define a function $\underline{u}$ on Λ_0 by $E''\underline{u} = 1$, with $\underline{u} = 0$ on S_0. Similarly, define a function u on $\underline{\Lambda}_0$ by $N''u = 1$, with $u = 0$ on S_0.

Let $S_{\tilde{\tau}}$ be the embedded 2-surface on Λ_0 such that $\underline{u} = \tilde{\tau}$, and similarly, let $\underline{S}_{\tilde{\tau}}$ be the embedded 2-surface on $\underline{\Lambda}_0$ such that $u = \tilde{\tau}$. We also define N' on Λ_0 so that N' is null and normal to $S_{\tilde{\tau}}$ and $\overline{g}(E', N') = -\varrho^{-2}$.

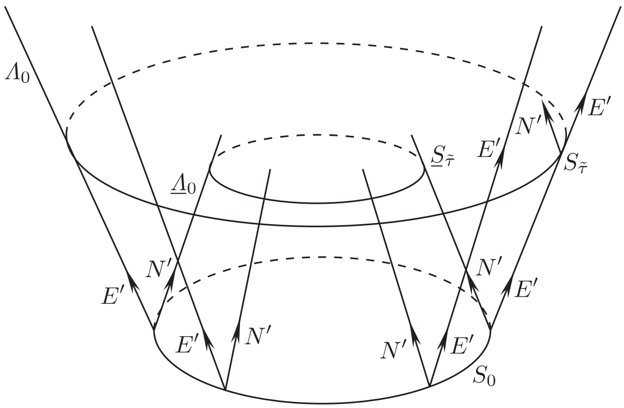

Fig. 1. Double conical null foliation

Consider the affinely parametrized null geodesics which emanate from the points on $S_{\tilde{\tau}}$, with initial tangent vector N'. These geodesics span null hypersurfaces which we denote by $\underline{C}_{\tilde{\tau}}$. Hence, $\Lambda_0 \cap \underline{C}_{\tilde{\tau}} = \underline{S}_{\tilde{\tau}}$. Similarly, we define E' globally (and the hypersurfaces $C_{\tilde{\tau}}$ such that their normal is E'). Next, extend E'', N'' to global vector fields such that $E'' = \varrho^2 E'$ and $N'' = \varrho^2 N'$. Also, extend $u, \underline{u}$ to global functions such that

$$E''u = 0 \quad \text{and} \quad N''\underline{u} = 0. \tag{3.17}$$

Therefore, $C_{\tilde{\tau}} = \{u = \tilde{\tau}\}$ and $\underline{C}_{\tilde{\tau}} = \{\underline{u} = \tilde{\tau}\}$, and hence u, $\underline{u}$ are optical functions (see [A13] for more details). Moreover, u, $\underline{u}$ satisfy

$$\overline{\nabla}\underline{u} = -N', \quad \overline{\nabla}u = -E', \quad E''\underline{u} = 1, \quad N''u = 1, \tag{3.18}$$

where $\overline{\nabla}u$, $\overline{\nabla}\underline{u}$ are the gradients of u, $\underline{u}$ respectively. A proof of the above relations can be found in [A13]. Observe from the above explanations that we have foliated the spacetime with null hypersurfaces, seen as *level surfaces* of u and $\underline{u}$.

Next, we formulate our evolution model from the functions u and $\underline{u}$ as follows. From the previous explanations, we have $\underline{\Lambda}_0 = \{(x, u(x)) : x \in \mathbb{R}^3\} = \text{Graph}(u) \subset \overline{M}_1^4$ and $\Lambda_0 = \{(x, \underline{u}(x)) : x \in \mathbb{R}^3\} = \text{Graph}(\underline{u}) \subset \overline{M}_1^4$. Then we have the null immersions $\underline{F}(p,t) = (x(p,t), u(x(p,t),t))$ and $F(p,t) = (x(p,t), \underline{u}(x(p,t),t))$ for $\underline{\Lambda}_0, \Lambda_0$ respectively. At this point, suppose that we want to follow up the evolution of the future cone Λ_0, starting from S_0, along its transversal direction. Then observe from Figure 1 that this is essentially a flow along the past cone $\underline{\Lambda}_0$, as it is transversal to Λ_0. Also, all points of $\underline{\Lambda}_0$ are moved along the transversal direction during this flow. By direct calculations, we have

$$\frac{\partial \underline{F}}{\partial t} = \left(\frac{\partial x}{\partial t}, \frac{\partial u}{\partial x_a}\frac{\partial x_a}{\partial t} + \frac{\partial u}{\partial t}\right) = \left(\frac{\partial x}{\partial t}, \overline{g}\left(\overline{\nabla} u, \frac{\partial x}{\partial t}\right) + \frac{\partial u}{\partial t}\right). \tag{3.19}$$

From (3.17) and (3.18) we observe that the vector $(N'', N''\underline{u}) = (\varrho^2 N', 0) = (-\varrho^2 \overline{\nabla}\underline{u}, 0)$ is null and always transversal to the future cone Λ_0, and tangent to the past cone $\underline{\Lambda}_0$. Thus, we can write (3.19), using the definition of null MCF, as

$$\left(\frac{\partial x}{\partial t}, \overline{g}\left(\overline{\nabla} u, \frac{\partial x}{\partial t}\right) + \frac{\partial u}{\partial t}\right) = \mathcal{S}(-\varrho^2 \overline{\nabla}\underline{u}, 0). \tag{3.20}$$

Then from (3.20) we get

$$\frac{\partial x}{\partial t} = -\mathcal{S}\varrho^2 \overline{\nabla}\underline{u} \quad \text{and} \quad \overline{g}\left(\overline{\nabla} u, \frac{\partial x}{\partial t}\right) + \frac{\partial u}{\partial t} = 0. \tag{3.21}$$

From (3.21), (3.16) and (3.18) we deduce that

$$\frac{\partial u}{\partial t} = \mathcal{S}\varrho^2 \overline{g}(\overline{\nabla} u, \overline{\nabla}\underline{u}) = -\mathcal{S}\varrho^2 \varrho^{-2} = -\mathcal{S}. \tag{3.22}$$

Finally, using (2.4), (3.21) and (3.22), we have $\frac{\partial x}{\partial t} = -\varrho^2 \operatorname{div}(\overline{\nabla} u)\overline{\nabla}\underline{u}$ and $\frac{\partial u}{\partial t} = -\operatorname{div}(\overline{\nabla} u)$. These PDE's describe the flow of the above cones under null MCF.

4. Short-time existence and evolution of geometric quantities. For a Lorentzian ambient manifold, the null MCF equation (3.15) represents a strictly non-linear PDE. To show this, let $(\mathcal{U}; x_0, \ldots, x_n)$ be local coordinates around $p \in \mathcal{U} \subset M^{n+1}$ at $t \in [0, T)$, such that the local vector fields $X_0 = E = \frac{\partial}{\partial x_0}, X_1 = \frac{\partial}{\partial x_1}, \ldots, X_n = \frac{\partial}{\partial x_n}$ and $N = \frac{\partial}{\partial x_{n+1}}$ span $T\overline{M}$ at time t. Let us write $F := F(p,t)$, $\mathbf{H} := \mathbf{H}(p,t)$ and $E_a := F_* X_a$ for any $a \in \{0, \ldots, n\}$. As F is a null isometric immersion, [DB96, p. 109] shows that F is an *affine immersion*, that is, for any $a, b \in \{0, \ldots, n\}$ we have $\overline{\nabla}_{E_a} E_b = F_*(\nabla_{X_a} X_b) + B(X_a, X_b)N$. Thus, since $B(X_0, X_0) = B(E, E) = 0$, the null mean curvature vector $\mathbf{H}$ can be written, for all $\alpha, \beta, \gamma, \mu \in \{1, \ldots, n\}$, as

$$\begin{aligned}\mathbf{H} &= (g^{\alpha\beta}\overline{\nabla}_{E_\alpha}E_\beta)^t = g^{\alpha\beta}\overline{\nabla}_{E_\alpha}E_\beta - (g^{\alpha\beta}\overline{\nabla}_{E_\alpha}E_\beta)^\top \\ &= g^{\alpha\beta}\overline{\nabla}_{E_\alpha}E_\beta - \overline{g}(g^{\alpha\beta}\overline{\nabla}_{E_\alpha}E_\beta, E_\gamma)g^{\gamma\mu}E_\mu,\end{aligned} \tag{4.1}$$

where $(\cdot)^\top$ and $(\cdot)^t$ are the tangential and transversal components of $(\cdot)$ respectively. Let $\overline{\Gamma}^a_{\alpha\beta}$ denote the ath component of $\overline{\nabla}_{E_\alpha}E_\beta$ in the basis $\{E_0, \ldots, E_n\}$; then (4.1) and (3.15) give $\frac{\partial F}{\partial t} = g^{\alpha\beta}\overline{\Gamma}^a_{\alpha\beta}(E_a - g^{\gamma\mu}g_{a\gamma}E_\mu)$, which is a system of non-linear PDE's because $(g^{\alpha\beta})$, for $\alpha, \beta \in \{1, \ldots, n\}$, is a positive definite matrix which only depends on the first derivatives of F. Moreover, using the local theory of parabolic PDE's, the geometric equation in (3.15) can be shown to be equivalent to a quasilinear scalar equation, which may be solved for some short interval of time using linearization techniques. This short-time solution may then be continued as long as it does not become singular [T96].

Suppose that the null hypersurface (M, g) moves by null mean curvature and let F be the smooth immersion in Definition 3.2. Notice that the null immersion F has no well-defined d'Alembertian operator, but we can compute its d'Alembertian-type operator $\Delta^g F$ via the d'Alembertians of its coordinate functions with respect to $\widehat{g}$ (see [AET03]). Alternatively, we can compute $\Delta^g F$ from the Hessian $\mathrm{Hess}^g(F)$ (regarded as a vector-valued function on $M_0 = F(M)$). Let (M, g) be a null hypersurface of $(\overline{M}, \overline{g})$; then

$$\Delta^g F = \mathrm{tr}(\mathrm{Hess}^g(F)), \tag{4.2}$$

where $\mathrm{Hess}^g(F) = X(F_*Y) - F_*(\nabla'_X Y)$ for all $X, Y \in \Gamma(TM)$, and the trace is taken on M_0 with respect to $\widehat{g}$. In fact, using (3.5) we have

$$\begin{aligned}\mathrm{Hess}^g(F) &= X_a(F_*X_b) - F_*(\nabla'_{X_a}X_b) \\ &= X_a(F_*X_b) - (\nabla_{F_*X_a}F_*X_b), \quad \forall, a, b \in \{0, \ldots, n\}.\end{aligned} \tag{4.3}$$

Contracting (4.3) with respect to $\widehat{g}$, we get

$$\begin{aligned}\mathrm{tr}(\mathrm{Hess}^g(F)) &= g^{[ab]}(X_a(F_*X_b) - (\nabla_{F_*X_a}F_*X_b)) \\ &= g^{[ab]}B(F_*X_a, F_*X_b)N.\end{aligned} \tag{4.4}$$

As F is also an affine immersion [DB96, p. 109], (4.4) vanishes when $a = 0$ or $b = 0$ due to the fact that the local second fundamental form B of a null hypersurface is degenerate. Thus, from (4.4) and (4.1), we deduce that

$$\Delta^g F = \mathrm{tr}(\mathrm{Hess}^g(F)) = \mathbf{H}. \tag{4.5}$$

Now, from (4.5) we state the following.

PROPOSITION 4.1. *Let (M, g) be a null hypersurface in a Lorentzian manifold $(\overline{M}, \overline{g})$ and $F_0 : M^{n+1} \to \overline{M}^{n+2}$ a given immersion. Then the null MCF* *equation in Definition* 3.2 *is given by*

$$\frac{\partial F}{\partial t} = \Delta^g F. \tag{4.6}$$

Using Proposition 4.1 we can state and prove an existence theorem for the null MCF in Definition 3.2 for a short time interval, for a *compact null hypersurface.*

THEOREM 4.2 (Short-time existence). *Let* (M, g) *be a compact null hypersurface of a Lorentzian manifold* $(\overline{M}, \overline{g})$ *and* $F_0 : M^{n+1} \to \overline{M}^{n+2}$ *a given immersion. There exists a constant* $T > 0$ *and a unique smooth family of immersions* $F(\cdot, t) : M^{n+1} \to \overline{M}^{n+1}$ *such that*

$$\frac{\partial F}{\partial t}(p, t) = \boldsymbol{H}(p, t), \qquad F(\cdot, 0) = F_0, \tag{4.7}$$

where $p \in M$ *and* $t \in [0, T)$.

Proof. We start off by observing that $F(\cdot, 0) = F_0$ is an immersion, so $F(p, t)$ would be an immersion for some small t (see for instance [GP74, p. 35]). Hence, we only focus on the existence of a solution of the PDE above. Let us consider a vector field $U = u^a \frac{\partial}{\partial x_a}$ such that $\frac{\partial \tilde{F}}{\partial t} = \Delta^g \tilde{F} + u^a \frac{\partial \tilde{F}}{\partial x_a}$ has a solution for the initial data F_0, $\tilde{F} : M \times [0, T) \to \overline{M}^{n+1}$. We establish that the same happens to the null mean curvature flow with initial data F_0. Consider a family $\varphi_t : M \to M$ of diffeomorphisms of M. Set $F(p) := \tilde{F}(\varphi_t(p), t)$, where $\tilde{F}$ is as mentioned above. Then at a point $p \in M$, we have

$$\begin{aligned} \frac{\partial F}{\partial t}(p) &= \frac{\partial \tilde{F}}{\partial x_a}(\varphi_t(p), t) \frac{\partial \varphi_t^a}{\partial t}(p) + \frac{\partial \tilde{F}}{\partial t}(\varphi_t(p), t) \\ &= \Delta^g \tilde{F}(\varphi_t(p), t) + \frac{\partial \tilde{F}}{\partial x_a}(\varphi_t(p), t)\left(u^a + \frac{\partial \varphi_t^a}{\partial t}(p)\right). \end{aligned} \tag{4.8}$$

Therefore, by (4.8), to get a solution to the null MCF equation it is enough to find a family φ_t such that

$$\frac{\partial \varphi_t}{\partial t} = -U, \qquad \varphi_0 = \mathrm{id}. \tag{4.9}$$

Equation (4.9) is an initial value problem for a system of ODE's, so we can find a solution to it. Moreover, taking $T > 0$ small enough we can assume that φ_t is a diffeomorphism for any $t \in [0, T]$, the reason being that the initial data is a diffeomorphism (the identity) and the fact that the diffeomorphisms from a compact manifold into itself form a stable class (see [GP74, p. 35]). Therefore, $F(p) := \tilde{F}(\varphi_t(p), t)$ represents a solution of the null MCF equation with initial data F_0. In fact $\frac{\partial F}{\partial t}(p) = \Delta^g \tilde{F}(\varphi_t(p), t) = \Delta^g F(p)$ and $F(p, 0) = \tilde{F}(\varphi_0(p), 0) = \tilde{F}(\mathrm{id}(p), 0) = F_0(p)$.

Finally, we show that $\frac{\partial \tilde{F}}{\partial t} = \Delta^g \tilde{F} + u^a \frac{\partial \tilde{F}}{\partial x_a}$ has a solution. Consider the vector U whose coordinates are $u^a = g^{[bc]}(\Gamma^a_{bc} - (\Gamma_0)^a_{bc})$, where Γ^a_{bc} are the coefficients of the induced connection on M, in which $(\Gamma_0)^a_{bc}$ denotes the coefficient at $t = 0$. Then by (4.4) we have

$$\frac{\partial \tilde{F}}{\partial t} = \Delta^g \tilde{F} + u^a \frac{\partial \tilde{F}}{\partial x_a} = g^{[bc]}\left(\frac{\partial^2 \tilde{F}}{\partial x_b \partial x_c} - (\Gamma_0)^a_{bc}\frac{\partial \tilde{F}}{\partial x_a}\right), \tag{4.10}$$

which is a system of quasilinear parabolic PDE's because $(g^{[bc]})$ is a positive-definite matrix which only depends on the first derivatives of $\tilde{F}$. Hence the local theory of parabolic PDE's (see, for instance, [T96]) and the fact that M is compact give us the existence and uniqueness of the solution in a short time interval $[0, T)$. ∎

Now, we discuss the extrinsic geometry of M_t. To that end set $E_a := F_* X_a = \frac{\partial F}{\partial x_a}$, $\tau_a := \tau(E_a)$, $B_{\alpha\beta} = B(E_\alpha, E_\beta)$, $C_{\alpha\beta} = C(E_\alpha, E_\beta)$ and $\tilde{\mathcal{S}} := \sum_{\alpha=1}^{n} C(E_\alpha, E_\alpha)$.

THEOREM 4.3. *Let (M, g) be a screen integrable null hypersurface of a Lorentzian manifold $(\overline{M}, \overline{g})$. Let $F_0 : M \to \overline{M}$ be a given immersion. On any solution M_t of the null MCF* (3.15)*, the induced objects g, B, C, $\mathcal{S}$, and the null normal E evolve as follows:*

$$\frac{\partial}{\partial t} g_{\alpha\beta} = -2\mathcal{S}C_{\alpha\beta}, \tag{4.11}$$

$$\frac{\partial}{\partial t} g^{\alpha\beta} = 2\mathcal{S}C^{\alpha\beta}, \tag{4.12}$$

$$\frac{\partial}{\partial t} E = -\nabla^s \mathcal{S} - \mathcal{S} g^{\alpha\beta}\tau_\alpha E_\beta, \tag{4.13}$$

$$\begin{aligned}\frac{\partial}{\partial t} B_{\alpha\beta} = \operatorname{Hess}^s(\mathcal{S}) - \mathcal{S}B\left(\frac{\partial}{\partial x_\alpha}, A_N \frac{\partial}{\partial x_\beta}\right) + \frac{\partial}{\partial x_\beta}(\mathcal{S})\tau_\alpha \\ + \frac{\partial}{\partial x_\alpha}(\mathcal{S})\tau_\beta + \mathcal{S}\frac{\partial}{\partial x_\alpha}(\tau_\beta) + \mathcal{S}\tau_\alpha\tau_\beta - \mathcal{S}\Gamma^{*\gamma}_{\alpha\beta}\tau_\gamma,\end{aligned} \tag{4.14}$$

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{S} = \Delta\mathcal{S} + \mathcal{S}\mathcal{H} + g^{\alpha\beta}\Big(\frac{\partial}{\partial x_\beta}(\mathcal{S})\tau_\alpha + \frac{\partial}{\partial x_\alpha}(\mathcal{S})\tau_\beta + \mathcal{S}\frac{\partial}{\partial x_\alpha}(\tau_\beta) \\ + \mathcal{S}\tau_\alpha\tau_\beta - \mathcal{S}\Gamma^{*\gamma}_{\alpha\beta}\tau_\gamma)\Big),\end{aligned} \tag{4.15}$$

where $\mathcal{H} := C^{\alpha\beta}B_{\alpha\beta} = \operatorname{tr}^s(A^*_E \circ A_N)$, for any $\alpha, \beta, \gamma \in \{1, \ldots, n\}$.

Proof. By virtue of (3.4), the null MCF equation (3.15) and the local symmetry of the Levi-Civita connection, we derive

$$\frac{\partial}{\partial t}g_{\alpha\beta} = \frac{\partial}{\partial t}\overline{g}(E_\alpha, E_\beta) = \overline{g}\left(\frac{\partial}{\partial t}E_\alpha, E_\beta\right) + \overline{g}\left(E_\alpha, \frac{\partial}{\partial t}E_\beta\right)$$
$$= \overline{g}\left(\frac{\partial}{\partial x_\alpha}\mathbf{H}, E_\beta\right) + \overline{g}\left(E_\alpha, \frac{\partial}{\partial x_\beta}\mathbf{H}\right). \tag{4.16}$$

As F preserves both the radical and screen subspaces (see [DB96, p. 104]), we can see, by (3.7), that E_α belongs to the screen distribution in M_t for all $\alpha \in \{1, \ldots, n\}$. From the above facts and as the mean curvature vector $\mathbf{H}$ is transversal to M_t, we get $\overline{g}(E_\alpha, \mathbf{H}) = 0$. Hence, from the Gauss–Codazzi equations (2.1), (2.3), we get

$$\overline{g}\left(\frac{\partial}{\partial x_\alpha}\mathbf{H}, E_\beta\right) = -\overline{g}\left(\mathbf{H}, \frac{\partial}{\partial x_\alpha}E_\beta\right) = -\mathcal{S}C_{\alpha\beta}. \tag{4.17}$$

Then (4.11) follows from (4.16) and (4.17) and the symmetry of C in α and β.

Relation (4.12) follows easily from (4.11) by the product rule.

Next, we prove (4.13). Decomposing the vector field $\frac{\partial}{\partial t}E$ in the basis $\{E_1, \ldots, E_n\}$ and observing $\overline{g}(E, E_a) = 0$ for all $a \in \{0, \ldots, n\}$, we have

$$\frac{\partial}{\partial t}E = g^{\alpha\beta}\overline{g}\left(\frac{\partial}{\partial t}E, E_\alpha\right)E_\beta$$
$$= g^{\alpha\beta}\left(\frac{\partial}{\partial t}\overline{g}(E, E_\alpha) - \overline{g}\left(E, \frac{\partial}{\partial t}E_\alpha\right)\right)E_\beta$$
$$= -g^{\alpha\beta}\left(\frac{\partial}{\partial x_\alpha}\overline{g}(E, \mathbf{H}) - \overline{g}\left(\frac{\partial}{\partial x_\alpha}E, \mathbf{H}\right)\right)E_\beta, \tag{4.18}$$

in which we have used the symmetry of the Levi-Civita connection. But $\overline{g}(E, \mathbf{H}) = \overline{g}(E, \mathcal{S}N) = \mathcal{S}\overline{g}(E, N) = \mathcal{S}$ and by (2.1) and (2.4) the second term in parenthesis in (4.18) simplifies as

$$\overline{g}\left(\frac{\partial}{\partial x_\alpha}E, \mathbf{H}\right) = \mathcal{S}\overline{g}\left(\frac{\partial}{\partial x_\alpha}E, N\right) = -\mathcal{S}\tau_\alpha. \tag{4.19}$$

Inserting (4.19) back into (4.18), we get

$$\frac{\partial}{\partial t}E = -g^{\alpha\beta}\frac{\partial}{\partial x_\alpha}(\mathcal{S})E_\beta = -\nabla^s\mathcal{S} - \mathcal{S}g^{\alpha\beta}\tau_\alpha E_\beta,$$

which proves (4.13).

From the Gauss formula (2.1) we derive

$$\frac{\partial}{\partial t}B_{\alpha\beta} = \frac{\partial}{\partial t}\overline{g}\left(\frac{\partial}{\partial x_\alpha}E_\beta, E\right)$$
$$= \overline{g}\left(\frac{\partial^2}{\partial t\partial x_\alpha}E_\beta, E\right) + \overline{g}\left(\frac{\partial}{\partial x_\alpha}E_\beta, \frac{\partial}{\partial t}E\right). \tag{4.20}$$

As we are working in local coordinates and as $\overline{R}=0$ (this follows easily from the Gauss–Codazzi equations and (C_7); see details in [DB96, p. 103]), one gets $\frac{\partial^2}{\partial t\partial x_\alpha}E_\beta=\frac{\partial^2}{\partial x_\alpha\partial t}E_\beta=\frac{\partial^2}{\partial x_\alpha\partial x_\beta}\mathbf{H}$ and so (4.20) gives

$$\frac{\partial}{\partial t}B_{\alpha\beta}=\overline{g}\Big(\frac{\partial^2}{\partial x_\alpha\partial x_\beta}\mathbf{H},E\Big)+\overline{g}\Big(\frac{\partial}{\partial x_\alpha}E_\beta,\frac{\partial}{\partial t}E\Big). \tag{4.21}$$

Now, using the Gauss–Codazzi relations (2.1)–(2.2) together with $\frac{\partial}{\partial x_\beta}\mathbf{H}=\frac{\partial}{\partial x_\beta}(\mathcal{S}N)=\big(\frac{\partial}{\partial x_\beta}\mathcal{S}\big)N+\mathcal{S}\big(\frac{\partial}{\partial x_\beta}N\big)$ gives

$$\begin{aligned}\frac{\partial}{\partial t}B_{\alpha\beta}&=\overline{g}\Big(\frac{\partial}{\partial x_\alpha}\Big(\Big(\frac{\partial}{\partial x_\beta}(\mathcal{S})\Big)N\Big),E\Big)-\mathcal{S}B\Big(\frac{\partial}{\partial x_\alpha},A_N\frac{\partial}{\partial x_\beta}\Big)\\&\quad+\frac{\partial}{\partial x_\alpha}(\mathcal{S})\tau_\beta+\mathcal{S}\frac{\partial}{\partial x_\alpha}(\tau_\beta)+\mathcal{S}\tau_\alpha\tau_\beta+\overline{g}\Big(\frac{\partial}{\partial x_\alpha}E_\beta,\frac{\partial}{\partial t}E\Big)\\&=\frac{\partial}{\partial x_\alpha}\Big(\frac{\partial}{\partial x_\beta}(\mathcal{S})\Big)-\mathcal{S}B\Big(\frac{\partial}{\partial x_\alpha},A_N\frac{\partial}{\partial x_\beta}\Big)+\frac{\partial}{\partial x_\beta}(\mathcal{S})\tau_\alpha\\&\quad+\frac{\partial}{\partial x_\alpha}(\mathcal{S})\tau_\beta+\mathcal{S}\frac{\partial}{\partial x_\alpha}(\tau_\beta)+\mathcal{S}\tau_\alpha\tau_\beta+\overline{g}\Big(\frac{\partial}{\partial x_\alpha}E_\beta,\frac{\partial}{\partial t}E\Big).\end{aligned} \tag{4.22}$$

Then applying (4.13) to (4.22) we get

$$\begin{aligned}\frac{\partial}{\partial t}B_{\alpha\beta}&=\frac{\partial}{\partial x_\alpha}\Big(\frac{\partial}{\partial x_\beta}(\mathcal{S})\Big)-\mathcal{S}B\Big(\frac{\partial}{\partial x_\alpha},A_N\frac{\partial}{\partial x_\beta}\Big)+\frac{\partial}{\partial x_\beta}(\mathcal{S})\tau_\alpha\\&\quad+\frac{\partial}{\partial x_\alpha}(\mathcal{S})\tau_\beta+\mathcal{S}\frac{\partial}{\partial x_\alpha}(\tau_\beta)+\mathcal{S}\tau_\alpha\tau_\beta\\&\quad-g^{\gamma\mu}\overline{g}\Big(\frac{\partial}{\partial x_\alpha}E_\beta,\frac{\partial}{\partial x_\gamma}(\mathcal{S})E_\mu\Big)-\mathcal{S}g^{\gamma\mu}\tau_\gamma\overline{g}\Big(\frac{\partial}{\partial x_\alpha}E_\beta,E_\mu\Big)\\&=\frac{\partial}{\partial x_\alpha}\Big(\frac{\partial}{\partial x_\beta}(\mathcal{S})\Big)-\mathcal{S}B\Big(\frac{\partial}{\partial x_\alpha},A_N\frac{\partial}{\partial x_\beta}\Big)+\frac{\partial}{\partial x_\beta}(\mathcal{S})\tau_\alpha\\&\quad+\frac{\partial}{\partial x_\alpha}(\mathcal{S})\tau_\beta+\mathcal{S}\frac{\partial}{\partial x_\alpha}(\tau_\beta)+\mathcal{S}\tau_\alpha\tau_\beta-\Gamma^{*\gamma}_{\alpha\beta}\frac{\partial}{\partial x_\gamma}(\mathcal{S})-\mathcal{S}\Gamma^{*\gamma}_{\alpha\beta}\tau_\gamma,\end{aligned} \tag{4.23}$$

where $\Gamma^{*\gamma}_{\alpha\beta}$ are the coefficients of the metric connection ∇^*. Then from (4.23) we deduce that

$$\begin{aligned}\frac{\partial}{\partial t}B_{\alpha\beta}&=\mathrm{Hess}^s(\mathcal{S})-\mathcal{S}B\Big(\frac{\partial}{\partial x_\alpha},A_N\frac{\partial}{\partial x_\beta}\Big)+\frac{\partial}{\partial x_\beta}(\mathcal{S})\tau_\alpha\\&\quad+\frac{\partial}{\partial x_\alpha}(\mathcal{S})\tau_\beta+\mathcal{S}\frac{\partial}{\partial x_\alpha}(\tau_\beta)+\mathcal{S}\tau_\alpha\tau_\beta-\mathcal{S}\Gamma^{*\gamma}_{\alpha\beta}\tau_\gamma,\end{aligned}$$

where $\mathrm{Hess}^s(\mathcal{S}):=\frac{\partial}{\partial x_\alpha}\big(\frac{\partial}{\partial x_\beta}(\mathcal{S})\big)-\Gamma^{\gamma}_{*\alpha\beta}\frac{\partial}{\partial x_\gamma}(\mathcal{S})$, which proves (4.14).

Finally, we infer from (4.12) and (4.14) that

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{S} &= \left(\frac{\partial}{\partial t}g^{\alpha\beta}\right)B_{\alpha\beta} + g^{\alpha\beta}\left(\frac{\partial}{\partial t}B_{\alpha\beta}\right)\\ &= 2\mathcal{S}C^{\alpha\beta}B_{\alpha\beta} + g^{\alpha\beta}\Big(\mathrm{Hess}^s(\mathcal{S}) - \mathcal{S}g(A_N E_\beta, A^*_E E_\alpha) + \frac{\partial}{\partial x_\beta}(\mathcal{S})\tau_\alpha\\ &\qquad + \frac{\partial}{\partial x_\alpha}(\mathcal{S})\tau_\beta + \mathcal{S}\frac{\partial}{\partial x_\alpha}(\tau_\beta) + \mathcal{S}\tau_\alpha\tau_\beta - \mathcal{S}\Gamma^{*\gamma}_{\alpha\beta}\tau_\gamma\Big)\\ &= \Delta^s\mathcal{S} + 2\mathcal{S}C^{\alpha\beta}B_{\alpha\beta} - \mathcal{S}\,\mathrm{tr}(A^*_E\circ A_N)\\ &\qquad + g^{\alpha\beta}\left(\frac{\partial}{\partial x_\beta}(\mathcal{S})\tau_\alpha + \frac{\partial}{\partial x_\alpha}(\mathcal{S})\tau_\beta + \mathcal{S}\frac{\partial}{\partial x_\alpha}(\tau_\beta) + \mathcal{S}\tau_\alpha\tau_\beta - \mathcal{S}\Gamma^{*\gamma}_{\alpha\beta}\tau_\gamma)\right),\end{aligned}$$

which proves (4.15). ∎

Next, we will derive the evolution equation for the squared norm (with respect to the screen distribution) $|A^*_E|^2_s$ of the screen shape operator A^*_E. To that end, let us consider the orthonormal basis $\{X_1,\dots,X_n\}$, where $X_\alpha = \frac{\partial}{\partial x_\alpha}$, of $S(TM)$ around a point $p\in M$ such that the induced Levi-Civita connection ∇^* of $S(TM)$ satisfies $(\nabla^*_{X_\alpha}X_\beta)(p)=0$ and $C(X_0,X_\alpha)=0$. Observe that the above assumptions implies that locally the curvature R^* of $S(TM)$ vanishes. *Furthermore, we will assume that the* 1*-form* τ *vanishes on* $S(TM)$. As an example we have the following.

Example 4.4 (Null cone of $\mathbb{R}^{n+2}_1$). Let $\mathbb{R}^{n+2}_1$ be the space $\mathbb{R}^{n+2}$ endowed with a semi-Euclidean metric

$$\overline{g}(x,y) = -x_0y_0 + \sum_{a=0}^{n+1} x_a y_a \qquad \Big(x = \sum_{A=0}^{n+1} x^A\partial x_A\Big),$$

where $\partial x_A := \frac{\partial}{\partial x_A}$. Then the null cone Λ^{n+1}_0 is given by the equation $x_0^2 = \sum_{a=1}^{n+1} x_a^2$, $x_0\neq 0$. It is well-known (see for example [DB96, DS10]) that Λ^{n+1}_0 is a null hypersurface of $\mathbb{R}^{n+2}_1$, in which the radical distribution is spanned by a global vector field $E=\sum_{A=0}^{n+1} x_A\partial x_A$ on Λ^{n+1}_0. The transversal bundle is spanned by a global section N given by $N = \frac{1}{2x_0^2}\{-x_0\partial x_0 + \sum_{a=1}^{n+1} x_a\partial x_a\}$. Moreover, E being the position vector field, one gets $\overline{\nabla}_X E = \nabla_X E = X$ for any $X\in\Gamma(TM)$. Consequently, $A^*_E X + \tau(X)E + X = 0$. Noticing that the operator A^*_E is screen-valued, we infer from the last relation that

$$A^*_E X = -PX, \qquad \tau(X) = -\overline{g}(X,N) = -\lambda(X) \tag{4.24}$$

for any $X\in\Gamma(TM)$. Next, any $X\in\Gamma(S(T\Lambda^{n+1}_0))$ can be expressed as $X=\sum_{a=1}^{n+1}\tilde{X}_a\partial x_a$, where $\{\tilde{X}_1,\dots,\tilde{X}_{n+1}\}$ satisfy $\sum_{a=1}^{n+1} x_a\tilde{X}_a = 0$. From the second relation of (4.24) we clearly see that $\tau(X)=0$ for any $X\in\Gamma(S(TM))$.

Next, we turn our attention to the evolution equation of $|A^*_E|^2_s$. To that end, we will need the following result.

PROPOSITION 4.5. *Let (M, g) be a screen integrable null hypersurface of a Lorentzian manifold $(\overline{M}, \overline{g})$. The local second fundamental form B satisfies*

$$\Delta^s B_{\alpha\beta} = \mathrm{Hess}^s(\mathcal{S}) - \mathcal{H}B_{\alpha\beta} + \mathcal{S}C(X_\alpha, A^*_E X_\beta)$$

for all $\alpha, \beta \in \{1, \ldots, n\}$, where $\mathcal{H} := \mathrm{tr}^s(A^*_E \circ A_N)$.

Proof. By the assumption $\overline{R} = 0$, the Gauss–Codazzi equation (3.1) of null hypersurfaces given in [DB96, p. 93] implies

$$(\nabla_X h)(Y, Z) = (\nabla_Y h)(X, Z) \tag{4.25}$$

for all $X, Y, Z \in \Gamma(TM)$, where

$$(\nabla_X h)(Y, Z) = \nabla^t_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{4.26}$$

As $\nabla^*_{X_\alpha} X_\beta = 0$, we have

$$h(\nabla_{X_\alpha} X_\beta, X_\gamma) = h(\nabla^*_{X_\alpha} X_\beta, X_\gamma) + C(X_\alpha, X_\beta)h(E, X_\gamma) = 0,$$

in which we have taken into account (2.3) and the fact $h(E, X) = 0$ for any $X \in \Gamma(TM)$. Thus, (4.25) and (4.26) imply

$$\nabla^t_{X_\gamma} h(X_\alpha, X_\beta) = \nabla^t_{X_\alpha} h(X_\gamma, X_\beta). \tag{4.27}$$

Differentiating (4.27) and applying the definition of curvature, we get

$$\begin{aligned} \nabla^t_{X_\mu}\nabla^t_{X_\gamma} h(X_\alpha, X_\beta) &= \nabla^t_{X_\mu}\nabla^t_{X_\alpha} h(X_\gamma, X_\beta) \\ &= \nabla^t_{X_\alpha}\nabla^t_{X_\mu} h(X_\gamma, X_\beta) - (R(X_\alpha, X_\mu)h)(X_\beta, X_\gamma), \end{aligned} \tag{4.28}$$

where

$$\begin{aligned} (R(X_\alpha, X_\mu)h)(X_\beta, X_\gamma) &= R^t(X_\alpha, X_\mu)h(X_\beta, X_\gamma) \\ &\quad - h(R(X_\alpha, X_\mu)X_\beta, X_\gamma) - h(X_\beta, R(X_\alpha, X_\mu)X_\gamma), \end{aligned} \tag{4.29}$$

and R^t is the curvature tensor of the transversal bundle, given by

$$R^t(X, Y)N = \nabla^t_X\nabla^t_Y N - \nabla^t_Y\nabla^t_X N - \nabla^t_{[X,Y]}N \tag{4.30}$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(tr(TM))$. Then applying (4.28)–(4.30) to $\nabla^t_X N = \tau(X)N$, we get

$$\begin{aligned} \nabla^t_{X_\mu}\nabla^t_{X_\gamma} h(X_\alpha, X_\beta) &= \nabla^t_{X_\alpha}\nabla^t_{X_\mu} h(X_\beta, X_\gamma) + h(R(X_\alpha, X_\mu)X_\beta, X_\gamma) \\ &\quad + h(X_\beta, R(X_\alpha, X_\mu)X_\gamma) \\ &= \nabla^t_{X_\alpha}\nabla^t_{X_\beta} h(X_\mu, X_\gamma) + h(R(X_\alpha, X_\mu)X_\beta, X_\gamma) \\ &\quad + h(X_\beta, R(X_\alpha, X_\mu)X_\gamma), \end{aligned} \tag{4.31}$$

where in the last equality we have used (4.27). Now, as $h(X, Y) = B(X, Y)N$ and $B(X, Y) = g(A^*_E X, Y)$ for any $X, Y \in \Gamma(TM)$, and because of the assumption $\tau = 0$ on the screen distribution, (4.31) reduces to

$$
\begin{aligned}
X_\mu(X_\gamma(B_{\alpha\beta})) &= X_\alpha(X_\beta(B_{\mu\gamma})) + B(R(X_\alpha, X_\mu)X_\beta, X_\gamma) \\
&\quad + B(X_\beta, R(X_\alpha, X_\mu)X_\gamma) \\
&= X_\alpha(X_\beta(B_{\mu\gamma})) + g(A^*_E X_\gamma, R(X_\alpha, X_\mu)X_\beta) \\
&\quad + g(A^*_E X_\beta, R(X_\alpha, X_\mu)X_\gamma).
\end{aligned} \tag{4.32}
$$

Next, as $\overline{R} = 0$, from the Gauss–Codazzi relation (2.10) we get

$$
g(R(X,Y)Z, PW) = B(Y,Z)C(X,PW) - B(X,Z)C(Y,PW) \tag{4.33}
$$

for any $X, Y, Z, W \in \Gamma(TM)$. Applying (4.33) to (4.32) reduces it to

$$
\begin{aligned}
X_\mu(X_\gamma(B_{\alpha\beta})) &= X_\alpha(X_\beta(B_{\mu\gamma})) + B_{\mu\beta}C(X_\alpha, A^*_E X_\gamma) \\
&\quad - B_{\alpha\beta}C(X_\mu, A^*_E X_\gamma) + B_{\mu\gamma}C(X_\alpha, A^*_E X_\beta) \\
&\quad - B_{\alpha\gamma}C(X_\mu, A^*_E X_\beta).
\end{aligned} \tag{4.34}
$$

We are working locally around a point $p \in M$, so $\tau([X_\alpha, X_\beta]) = 0$. Taking this into account, together with the assumption $\tau(X_\alpha) = 0$ for all $\alpha \in \{1, \dots, n\}$, and $\overline{R} = 0$, we deduce from the Gauss–Codazzi equation [DB96, (3.12), p. 95] that

$$
C(X_\alpha, A^*_E X_\gamma) = C(X_\gamma, A^*_E X_\alpha). \tag{4.35}
$$

Finally, inserting (4.35) in (4.34) and taking trace with respect to μ and γ, we obtain the desired result. ∎

THEOREM 4.6. *Let (M, g) be a screen integrable null hypersurface of a Lorentzian manifold $(\overline{M}, \overline{g})$. Under null MCF, the squared norm $|A^*_E|^2_s$ of the screen shape operator A^*_E evolves according to*

$$
\frac{\partial |A^*_E|^2_s}{\partial t} = \Delta^s |A^*_E|^2_s + 2\mathcal{H}|A^*_E|^2_s - 2|\nabla^s B|^2_s,
$$

where $\mathcal{H} := \operatorname{tr}^s(A^*_E \circ A_N)$.

Proof. Consider $g^{\alpha\beta}(p) = \delta_{\alpha\beta}$. Then $|A^*_E|^2_s = g^{\alpha\mu}g^{\gamma\beta}B_{\alpha\mu}B_{\gamma\beta} = B_{\alpha\beta}B_{\alpha\beta}$. Using this relation and Proposition 4.5 we have

$$
\begin{aligned}
\Delta^s |A^*_E|^2_s &= 2B_{\alpha\beta}\Delta^s B_{\alpha\beta} + 2|\nabla^s B|^2_s \\
&= 2B_{\alpha\beta}\mathrm{Hess}^s(\mathcal{S}) - 2\mathcal{H}|A^*_E|^2_s + 2|\nabla^s B|^2_s + 2\mathcal{S}\operatorname{tr}^s(A^{*2}_E \circ A_N).
\end{aligned} \tag{4.36}
$$

On the other hand, we can compute the time derivative of $|A^*_E|^2_s$ from $|A^*_E|^2_s = g^{\alpha\beta}g^{\gamma\mu}B_{\alpha\gamma}B_{\beta\mu}$ and the evolution equations of g and B in Theorem 4.3 with $\tau = 0$ on the screen distribution as follows:

$$
\begin{aligned}
\frac{\partial |A^*_E|^2_s}{\partial t} &= 2\mathcal{S}C^{\alpha\beta}g^{\gamma\mu}B_{\alpha\gamma}B_{\beta\mu} + 2\mathcal{S}g^{\alpha\beta}C^{\gamma\mu}B_{\alpha\gamma}B_{\beta\mu} \\
&\quad + g^{\alpha\beta}g^{\gamma\mu}\{\mathrm{Hess}^s(\mathcal{S}) - \mathcal{S}B(X_\alpha, A_N X_\gamma)\}B_{\beta\mu} \\
&\quad + g^{\alpha\beta}g^{\gamma\mu}B_{\alpha\gamma}\{\mathrm{Hess}^s(\mathcal{S}) - \mathcal{S}B(X_\beta, A_N X_\mu)\}.
\end{aligned} \tag{4.37}
$$

Rearranging some indices in (4.37), we get

(4.38)
$$\begin{aligned}\frac{\partial|A_E^*|_s^2}{\partial t} &= 4\mathcal{S}C^{\alpha\beta}g^{\gamma\mu}B_{\alpha\gamma}B_{\beta\mu} + 2g^{\alpha\beta}g^{\gamma\mu}\{\mathrm{Hess}^s(\mathcal{S}) - \mathcal{S}B(X_\alpha, A_N X_\gamma)\}B_{\beta\mu}\\ &= 4\mathcal{S}\,\mathrm{tr}^s(A_E^{*2}\circ A_N) + 2B_{\alpha\gamma}\mathrm{Hess}^s(\mathcal{S}) - 2\mathcal{S}\,\mathrm{tr}^s(A_E^{*2}\circ A_N)\\ &= 2\mathcal{S}\,\mathrm{tr}^s(A_E^{*2}\circ A_N) + 2B_{\alpha\gamma}\mathrm{Hess}^s(\mathcal{S}).\end{aligned}$$

Putting together (4.36) and (4.38) we get the desired result. ∎

In order to discuss some geometric implications of the evolution equations in Theorems 4.3 and 4.6, we need the following well-known concept in null geometry.

A null hypersurface (M, g) of a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is called *screen conformal* [DB96, p. 51] if there exist a non-vanishing smooth function ψ on a neighborhood $\mathcal{U}$ in M such that $A_N = \psi A_E^*$, or equivalently

$$C(X, PY) = \psi B(X, Y), \quad \forall X, Y \in \Gamma(TM). \tag{4.39}$$

We say that M is *screen homothetic* if ψ is a constant function on M.

Example 4.7. Consider the null cone in Example 4.4. By straightforward calculation, one gets $\overline{g}(\nabla_E X, E) = -\sum_{a=1}^{n+1} x_a X_a = 0$, which implies that $\nabla_E X \in \Gamma(S(T\Lambda_0^{n+1}))$. Hence, $A_N E = 0$. Using Gauss–Codazzi equations, we calculate $C(X, Y) = \overline{g}(\nabla_X Y, N) = \overline{g}(\overline{\nabla}_X Y, N) = -\frac{1}{2x_0^2}g(X, Y)$ for any $X, Y \in \Gamma(S(T\Lambda_0^{n+1}))$. Consequently,

$$A_N X = -\frac{1}{2x_0^2}PX, \quad \forall X, Y \in \Gamma(S(T\Lambda_0^{n+1})). \tag{4.40}$$

Combining (4.24) and (4.40) we deduce that

$$A_N X = \frac{1}{2x_0^2}A_E^* X, \quad \forall X \in \Gamma(S(T\Lambda_0^{n+1})). \tag{4.41}$$

Hence, Λ_0^{n+1} is a screen globally conformal null hypersurface of $\mathbb{R}_1^{n+2}$, with a positive conformal factor $\psi = \frac{1}{2x_0^2}$ globally defined on Λ_0^{n+1}.

Using the previous concept and Theorem 4.6 we state the following.

Corollary 4.8. *Under the hypothesis of Theorem* 4.6*, if the initial null hypersurface* M_0 *is screen conformal and the* 1*-form* τ *vanishes on* $S(TM_0)$*, then*

$$\frac{\partial|A_E^*|_s^2}{\partial t} = \Delta^s|A_E^*|_s^2 + 2\psi|A_E^*|_s^4 - 2|\nabla^s B|_s^2.$$

From Corollary 4.8 we notice that the *quadratic* term in $|A_E^*|_s^2$ will cause *finite time blow-up* in the evolution equation of $|A_E^*|_s^2$. To understand the long-term behavior of solutions near such blow-ups (or singularities) requires one to obtain a priori estimates. Such estimates can be integral or pointwise. In the latter case, the following *maximum principle* is always used.

THEOREM 4.9 ([M12]). *Assume that $g(t)$, for $t \in [0,T)$, is a family of Riemannian metrics on a manifold M, with a possible boundary ∂M, such that the dependence on t is smooth. Let $u : M \times [0,T) \to \mathbb{R}$ be a smooth function satisfying*

$$\frac{\partial u}{\partial t} \leq \Delta_{g(t)} u + \overline{g}(X(p,u,\nabla u,t),\nabla u)_{g(t)} + f(u), \tag{4.42}$$

where X and f are respectively a continuous vector field and a locally Lipschitz function in their arguments. Suppose that for every $t \in [0,T)$ there exists a value $\delta > 0$ and a compact subset $\mathcal{K} \subset M \setminus \partial M$ such that at every time $t' \in (t-\delta, t+\delta) \cap [0,T')$ the maximum of $u(\cdot,t')$ is attained at least at one point of $\mathcal{K}$ (this is clearly true if M is compact without boundary). Set $u_{\max}(t) := \max_{p\in M} u(p,t)$. Then the function $u_{\max}$ is locally Lipschitz, hence differentiable at almost every $t \in [0,T)$, and at every differentiability time we have

$$\frac{du_{\max}(t)}{dt} \leq f(u_{\max}(t)). \tag{4.43}$$

Consequently, if $v : [0,T') \to \mathbb{R}$ is a solution of the ODE

$$\frac{dv(t)}{dt} = f(v(t)), \qquad v(0) = u_{\max}(0), \tag{4.44}$$

for $T' \leq T$, then $u \leq v$ in $M \times [0,T')$. Moreover, if M is connected and at some time $\tilde{t} \in (0,T')$ we have $u_{\max}(\tilde{t}) = v(\tilde{t})$, then $u = v$ in $M \times [0,\tilde{t}]$, that is, $u(\cdot,t)$ is constant in space.

Notice that analogous results hold for the minimum of the solution of the opposite partial differential inequality. Moreover, the maximum principle for elliptic equations easily follows as the special case where all the quantities around do not depend on the time variable t. We will apply the above maximum principle to some of evolution equations derived earlier, when the initial null hypersurface M_0 is *compact* and *mean convex*. The hypersurface M_0 is mean convex if $\mathcal{S} \geq 0$ everywhere. It is well-known that mean convexity is preserved by the mean curvature flow [M12].

THEOREM 4.10. *Let M_0 be a mean convex, screen conformal null hypersurface. If $|A^*_E|^2_s$ is not bounded as $t \to T < \infty$ during the null MCF of a compact null hypersurface, then it must satisfy the following lower bound for its blow-up rate:*

$$\max_{p\in M} |A^*_E|^2_s(p,t) \geq \frac{1}{2\psi(T-t)}$$

for every $t \in [0, T)$. *Hence,*

$$\lim_{t \to T} \max_{p \in M} |A_E^*|_s^2(p, t) = \infty.$$

Proof. From Corollary 4.8 and the maximum principle (Theorem 4.9), we deduce that

$$\frac{\partial |A_E^*|^2_{s\,\max}}{\partial t} \leq 2\psi |A_E^*|^4_{s\,\max}.$$

Notice that when the ambient space $\overline{M}$ is Lorentzian, then the screen distribution of M_0 is Riemannian. Consequently, $|A_E^*|^2_{s\,\max}$ is always positive, otherwise we would get the case $A_E^* = 0$ (that is, M_0 is totally geodesic in $\overline{M}$), rendering M_0 a hyperplane in $\overline{M}$, thereby contradicting the compactness assumption on M_0.

More precisely, there are no compact hypersurfaces with zero mean curvatures [M12]. Therefore, we can divide both sides by $|A_E^*|^4_{s\,\max}$ obtaining the following differential inequality for the locally Lipschitz function $\frac{1}{|A_E^*|^2_{s\,\max}}$, holding at almost every $t \in [0, T)$:

$$-\frac{d}{dt} \frac{1}{|A_E^*|^2_{s\,\max}} \leq 2\psi.$$

Integration of the above inequality in $[t, t'] \subset [0, T)$ gives

$$\frac{1}{|A_E^*(\cdot, t)|^2_{s\,\max}} - \frac{1}{|A_E^*(\cdot, t')|^2_{s\,\max}} \leq 2\psi(t' - t).$$

Suppose that A_E^* is not bounded in $[0, T)$, that is, there exists a sequence of times $t_i' \to T$ such that $|A_E^*(\cdot, t_i')|^2_{s\,\max} \to \infty$. Considering these times t_i' in the above inequality and letting $i \to \infty$, we get

$$\frac{1}{|A_E^*(\cdot, t)|^2_{s\,\max}} \leq 2\psi(T - t). \quad \blacksquare$$

The above result shows that there will also be *singularities* in null MCF if the initial null hypersurface M_0 is screen conformal since the quantity $|A_E^*|_s^2$ blows up in finite time. When this happens we say that T is a *singular* time for the null MCF. Moreover, we have the following.

THEOREM 4.11. *Given a compact, immersed screen conformal null hypersurface* M_0 *in* $\overline{M}$, *there exists a unique null MCF defined on a maximal interval* $[0, T_{\max})$. *Moreover,* $T_{\max}$ *is given by*

$$\max_{p \in M} |A_E^*|_s^2 \geq \frac{1}{2\psi(T_{\max} - t)}.$$

Let us turn to the evolution equation of $\mathcal{S}$. From Theorem 4.3 with $\tau = 0$ on the screen distribution, we have

$$\frac{\partial \mathcal{S}}{\partial t} = \Delta^s \mathcal{S} + \mathcal{S}\mathcal{H}, \quad \text{where} \quad \mathcal{H} := \mathrm{tr}^s(A_E^* \circ A_N). \tag{4.45}$$

THEOREM 4.12. *Let M_0 be a mean convex and compact screen conformal null hypersurface of $\overline{M}$. Under null MCF, the minimum $\mathcal{S}_{\min}$ of $\mathcal{S}$ is increasing, hence $\mathcal{S}$ is positive for every positive time.*

Proof. Consider the interval $(t_0, t_1) \subset \mathbb{R}^+$ and suppose for contradiction that $\mathcal{S}_{\min}(t) < 0$ and $\mathcal{S}_{\min}(t_0) = 0$ for any t in this interval. Assume that $|A_E^*|_s^2$ is bounded on (t_0, t_1), that is, $|A_E^*|_s^2 \leq A$. Then relation (4.45) and the maximum principle imply that

$$\frac{\partial \mathcal{S}_{\min}}{\partial t} \geq \psi A \mathcal{S}_{\min}, \qquad \forall t \in (t_0, t_1). \tag{4.46}$$

Integrating (4.46) over $[s, t] \subset (t_0, t_1)$ we get

$$\mathcal{S}_{\min}(t) \geq \exp\Big(\int_s^t \psi A\Big) \mathcal{S}_{\min}(s). \tag{4.47}$$

Then, letting $s \to t_0^+$ in (4.47) gives $\mathcal{S}_{\min}(t) \geq 0$ for all $t \in (t_0, t_1)$, which is a contradiction. Next, let $k_1, \ldots, k_n$ be the eigenvalues of A_E^* with respect to $\{X_1, \ldots, X_n\}$. It is easy to show that

$$|A_E^*|_s^2 - \frac{1}{n}\mathcal{S}^2 = \frac{1}{n} \sum_{1 \leq \alpha \leq \beta \leq n} (k_\beta - k_\alpha)^2, \tag{4.48}$$

from which we deduce $|A_E^*|_s^2 - \frac{1}{n}\mathcal{S}^2 \geq 0$. Using this inequality and (4.45) we get

$$\frac{\partial \mathcal{S}}{\partial t} = \Delta \mathcal{S} + \mathcal{S}\mathcal{H} \geq \Delta \mathcal{S} + \frac{\psi}{n}\mathcal{S}^3. \tag{4.49}$$

Let $u = -\mathcal{S}$, $X = 0$ and $f(x) = \frac{\psi}{n}x^3$. Then if $\mathcal{S}_{\min}(0) = 0$, the ODE has a solution $v(t) = 0$. Hence, if for some time $\tilde{t}$ we have $\mathcal{S}_{\min}(\tilde{t}) = 0$, then $\mathcal{S}_{\min}(\cdot, \tilde{t})$ is constant on M_t and equal to zero. But there are no compact hypersurfaces with zero mean curvature (see [M12]). Hence, $\mathcal{S}_{\min}$ is always increasing during the flow, and $\mathcal{S}$ is positive at every positive time. ∎

Next, suppose that M_0 admits a symmetric induced Ricci tensor. Denote by $\tilde{R}$ the corresponding scalar curvature. As $\overline{R} = 0$, we deduce from of [DS10, p. 69, (2.4.12)] that

$$\tilde{R} = \operatorname{tr}^s(A_E^*)\operatorname{tr}^s(A_N) - \operatorname{tr}^s(A_E^* \circ A_N). \tag{4.50}$$

THEOREM 4.13. *If the scalar curvature $\tilde{R}$ is positive and bounded on a screen homothetic initial null hypersurface M_0 with $\psi > 0$, then it remains bounded for all positive times t.*

Proof. As M_0 is screen homothetic, (4.50) gives

$$\tilde{R} = \psi \mathcal{S}^2 - \psi |A_E^*|_s^2, \tag{4.51}$$

where ψ is a constant function on M_0. Differentiating (4.51) and using the evolution equations of $\mathcal{S}$ and $|A_E^*|_s^2$ (see Theorems 4.3, 4.6 and Corollary 4.8), we get

$$\begin{aligned}\frac{\partial \tilde{R}}{\partial t} &= 2\psi\mathcal{S}(\Delta^s\mathcal{S} + \psi\mathcal{S}|A_E^*|_s^2) - \psi(\Delta^s|A_E^*|_s^2 + 2\psi|A_E^*|_s^4 - 2|\nabla^s B|_s^2) \\ &= \Delta^s(\psi\mathcal{S}^2) - \Delta^s(\psi|A_E^*|_s^2) + 2\psi^2\mathcal{S}^2|A_E^*|_s^2 - 2\psi^2|A_E^*|_s^4 \\ &\quad - 2\psi|\nabla^s\mathcal{S}|_s^2 + 2\psi|\nabla^s B|_s^2 \\ &= \Delta^s\tilde{R} + 2\psi|A_E^*|_s^2\tilde{R} + 2\psi(|\nabla^s B|_s^2 - |\nabla^s\mathcal{S}|_s^2).\end{aligned} \tag{4.52}$$

Observe that the term $|\nabla^s B|_s^2 - |\nabla^s\mathcal{S}|_s^2$ is non-positive and hence (4.52) gives

$$\frac{\partial \tilde{R}}{\partial t} \leq \Delta^s\tilde{R} + 2\psi|A_E^*|_s^2\tilde{R}. \tag{4.53}$$

Thus, taking $T' < T$, if ω is the maximum of $|A_E^*|_s^2$ on $M \times [0, T']$, we deduce from (4.53) that $\frac{\partial \tilde{R}}{\partial t} \leq \Delta^s\tilde{R} + 2\psi\omega\tilde{R}$ on $M \times [0, T']$. By the maximum principle (Theorem 4.9), we have $\frac{d\tilde{R}_{\max}}{dt} \leq 2\psi\omega|\tilde{R}|_{\max}$, from which the result follows by integration and the arbitrariness of T'. ∎

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