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A NEW SOLUTION TO HOM-YANG–BAXTER EQUATIONS

BY

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Abstract. The paper is concerned with solutions to Hom-Yang–Baxter equations. Relying on the work of Li et al. and D. Yau, we first introduce the concept of weak Hom-Hopf algebras whose structure maps satisfy an α-twisted version of (co)associativity and antipode. Then some non-trivial examples and properties are investigated. We also construct a braided almost Hom-bialgebra, that is, a quantum double of weak Hom-Hopf algebras.

1. Introduction. In [MS1], the authors provided a new way for constructing a subclass of quasi-Lie algebras, the Hom-Lie algebras, by extending the fundamental construction of Lie algebras from associative algebras via commutator bracket multiplication. For this, they defined the notions of Hom-associative algebras generalizing associative algebras to a situation where the associativity law is twisted by a linear map, and showed that the commutator product defined using the multiplication in a Hom-associative algebra leads naturally to Hom-Lie algebras. Several authors have investigated deeper properties of quasi-Lie algebras and Hom-Lie algebras. In all that research, the fact that it requires a broad insight into various Hom-algebraic structures seems to be essential. As a consequence, the notions of Hom-bialgebras and Hom-Hopf algebras have been introduced and some of their properties extending those of bialgebras and Hopf algebras have been explored.

In [Y1, Y2] D. Yau provides a novel method of constructing a solution to the Hom-Yang–Baxter equation, which is a non-associative version of the Yang–Baxter equation, by introducing the notions of (co)braided Hom-bialgebras. A survey on weak Hopf algebras and Hom-Hopf algebras can be found in [CG, Y2]. It has turned out that many results of Hom-Hopf algebra theory have weaker versions.

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The basic motivation of the present paper is to investigate the above construction in the case of weak Hom-Hopf algebras. To this end, we proceed as follows. In Section 1, we recall some basic definitions from [CG, L1]. We also introduce the notion of weak Hom-Hopf algebra and give some non-trivial examples and properties. In Section 2, drawing inspiration from the work of D. Yau [Y1, Y2], we construct a braided almost Hom-bialgebra, that is, a quantum double of weak Hom-Hopf algebras which guarantees the existence of solutions of Hom-Yang–Baxter equations.

Convention. We work over a fixed field k. Modules, tensor products and linear maps are all taken over k. For a k-space A, we denote its dual space by $\widehat{A}$. Unless otherwise stated, id_A will denote the identity map from A to itself and τ_A will denote the usual twist map. Given a bilinear map $\mu : A^{\otimes 2} \to A$, we write $\mu(x, y)$ as xy. We will use the version of Sweedler's notation, $\Delta : A \to A^{\otimes 2}$, $\Delta(x) = \sum x_1 \otimes x_2$ for any $x \in A$.

In what follows, we first recall some concepts and results used in this paper. For unexplained concepts and notations about weak Hopf algebras and Hom-Hopf algebras we refer to [CG, Y2].

Let $\mathcal{C}$ be a category. A new category $\mathcal{H}(\mathcal{C})$ is introduced as follows: Objects are couples (M, μ), with $M \in \mathcal{C}$ and $\mu \in \mathrm{Aut}_{\mathcal{C}}(M)$. A morphism $f : (M, \mu) \to (N, \nu)$ is a morphism $f : M \to N$ in $\mathcal{C}$ such that $\nu \circ f = f \circ \mu$.

It is shown in [C] that the category $\widetilde{\mathcal{H}}(\mathcal{C}) = (\mathcal{H}(\mathcal{C}), \otimes, (I, I), \widetilde{a}, \widetilde{l}, \widetilde{r})$ is monoidal if $\mathcal{C} = (\mathcal{C}, \otimes, I, a, l, r)$ is a monoidal category.

Definition 1.1 ([CG]). A *monoidal Hom-algebra* is an object $(A, \alpha) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$ together with a k-linear map $m_A : A \otimes A \to A$ and an element $1_A \in A$ such that for any $a, b, c \in A$,

$$\alpha(ab) = \alpha(a)\alpha(b), \tag{1}$$

$$1_A a = a 1_A = \alpha(a), \tag{2}$$

$$\alpha(a)(bc) = (ab)\alpha(c), \tag{3}$$

$$\alpha(1_A) = 1_A. \tag{4}$$

A *morphism of Hom-associative algebras* is a linear map of the underlying k-modules that commutes with the twisting maps and the multiplications.

Definition 1.2 ([CG]). A *monoidal Hom-coalgebra* is an object $(C, \gamma) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$ together with k-linear maps $\Delta : C \to C \otimes C$ and $\varepsilon : C \to k$ such that for any $x \in C$,

$$\sum \gamma(x)_1 \otimes \gamma(x)_2 = \sum \gamma(x_1) \otimes \gamma(x_2), \tag{5}$$

$$\varepsilon(\gamma(x)) = \varepsilon(x), \tag{6}$$

$$\sum \varepsilon(x_1)x_2 = \sum \varepsilon(x_2)x_1 = \gamma^{-1}(x), \tag{7}$$

$$\sum \gamma^{-1}(x_1) \otimes x_{21} \otimes x_{22} = \sum x_{11} \otimes x_{12} \otimes \gamma^{-1}(x_2). \tag{8}$$

A *morphism of Hom-coassociative coalgebras* is a linear map of the underlying k-modules that commutes with the twisting maps and the comultiplications.

DEFINITION 1.3 ([CG]). A *monoidal Hom-bialgebra* $H=(H,\alpha,m,\eta,\Delta,\varepsilon)$ is a bialgebra in the symmetric monoidal category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$. This means that (H,α,m,η) is a monoidal Hom-algebra, $(H,\alpha,\Delta,\varepsilon)$ is a monoidal Hom-coalgebra, and Δ and ε are morphisms of monoidal Hom-algebras, that is,

$$\sum (xy)_1 \otimes (xy)_2 = \sum x_1y_1 \otimes x_2y_2, \tag{9}$$

$$\Delta(1) = 1 \otimes 1, \tag{10}$$

$$\varepsilon(xy) = \varepsilon(x)\varepsilon(y), \tag{11}$$

$$\varepsilon(1) = 1. \tag{12}$$

A *morphism of Hom-bialgebras* is a Hom-algebra morphism and also a Hom-coalgebra morphism.

DEFINITION 1.4 ([L1]). A *weak Hopf algebra* is a bialgebra H endowed with $S \in \mathrm{Hom}_k(H,H)$ such that $\sum x_1S(x_2)x_3 = x$ and $\sum S(x_1)x_2S(x_3) = S(x)$ for any $x \in H$, where S is called a *weak antipode*.

DEFINITION 1.5 ([L1]). Let (H,μ,η) be a k-algebra and (H,Δ,ε) be a k-coalgebra. If $\Delta(xy) = \Delta(x)\Delta(y)$ for all $x, y \in H$, then one calls $(H,\mu,\eta,\Delta,\varepsilon)$ an *almost bialgebra*.

Motivated by the work of Li et al. [L1, LZ] and D. Yau [Y1, Y2], we introduce the following.

DEFINITION 1.6. Let (H,μ,η,α) be a monoidal Hom-algebra and let $(H,\Delta,\varepsilon,\alpha)$ be a monoidal Hom-coalgebra. If Δ is a Hom-algebra map, then $(H,\mu,\eta,\Delta,\varepsilon,\alpha)$ is called an *almost Hom-bialgebra*.

DEFINITION 1.7. Let $(H,\mu,\eta,\Delta,\varepsilon,\alpha)$ be a Hom-bialgebra and $S: H \to H$ a Hom-map satisfying $\sum S(h)_1 \otimes S(h)_2 = \sum S(h_2) \otimes S(h_1)$ and $S(hg) = S(g)S(h)$ for all $h, g \in H$.

If the following conditions are satisfied, then $(H,\mu,1,\Delta,\varepsilon,\alpha,S)$ is called a *weak Hom-Hopf algebra* and S is called a *weak Hom-antipode*:

$$\sum(x_{11}S(x_{12}))\alpha(x_2) = \alpha^3(x), \qquad x \in H, \tag{13}$$

$$\sum(S(x_{11})x_{12})\alpha(S(x_2)) = \alpha^3(S(x)), \qquad x \in H. \tag{14}$$

A *morphism of weak Hom-Hopf algebras* is a Hom-bialgebra morphism commuting with the weak Hom-antipode.

REMARK 1.8. For $\alpha = \mathrm{id}_H$, a weak Hom-Hopf algebra H is the usual weak Hopf algebra defined in [L1].

THEOREM 1.9. *Let $(H,\mu,\eta,\Delta,\varepsilon,S)$ be a weak Hopf algebra. If $\alpha : H \to H$ is a morphism of weak Hopf algebras, then $H_\alpha = (\mu_\alpha,\eta,\Delta_\alpha,\varepsilon,S,\alpha)$ is a weak Hom-Hopf algebra, where $\mu_\alpha = \alpha\circ\mu$ and $\Delta_\alpha = \Delta\circ\alpha$.*

Proof. It was proved in [Y2] that $H_\alpha = (\mu_\alpha,\eta,\Delta_\alpha,\varepsilon,\alpha)$ is a Hom-bialgebra. We have only to show that S is a weak Hom-antipode.

The equality (13) is a consequence of the following computation, for any $h \in H$:

$$\begin{aligned}&(\mu_\alpha\otimes\alpha)(\mathrm{id}_H\otimes S\otimes\mathrm{id}_H)(\Delta_\alpha\otimes\mathrm{id}_H)\Delta_\alpha(h)\\ &\quad=\sum\alpha\{\alpha[\alpha(h_1)S(\alpha(h_2))]\alpha^2(h_3)\}=\sum\alpha(\alpha^2(h_1)S(\alpha^2(h_2))\alpha^2(h_3))\\ &\quad=\alpha(\alpha^2(h))=\alpha^3(h).\end{aligned}$$

We conclude the proof by proving (14). In fact, for any $h\in H$,

$$\begin{aligned}&\mu_\alpha(\mu_\alpha\otimes\alpha)(S\otimes\mathrm{id}_H\otimes S)(\Delta_\alpha\otimes\mathrm{id}_H)\Delta_\alpha(h)\\ &\quad=\sum\alpha\{\alpha[S(\alpha(h_1))\alpha(h_2)]\alpha^2(S(h_3))\}\\ &\quad=\sum\alpha(S(\alpha^2(h_1))\alpha^2(h_2)S(\alpha^2(h_3)))=\alpha(\alpha^2(S(h)))=\alpha^3(S(h)).\ \blacksquare\end{aligned}$$

The following are some examples of weak Hom-Hopf algebras.

EXAMPLE 1.1 (Hom-Clifford monoid algebra). It is stated in [L1] that kG is a weak Hopf algebra if G is a Clifford semigroup. Assume $\alpha : G \to G$ is a monoid morphism; it extends naturally to a weak Hopf algebra morphism on kG where $\alpha(\sum c_i g) = \sum c_i\alpha(g)$ for any $g \in G$. By Theorem 1.9, $(kG)_\alpha$ is a weak Hom-Hopf algebra.

EXAMPLE 1.2 (weak Hom-Hopf algebra associated to a Taft alebra). Cheng [C] has given three kinds of weak Hopf algbras $H^1_{n^2}(q)$, $H^2_{n^2}(q)$ and $H^3_{n^2}(q)$, where $n > 2$ is an integer and q is a primitive nth root of unity. We recall them here for completeness.

The weak Hopf algebra $H^1_{n^2}(q)$ is generated by $\{g,x\}$ satisfying

$$g^{n+1}=g,\qquad x^n=0,\qquad gx=qxg, \tag{15}$$

with comultiplication and weak antipode given by

$$\Delta(g)=g\otimes g,\qquad \Delta(x)=g\otimes x+x\otimes 1, \tag{16}$$

$$S(1)=1,\qquad S(g)=g^{n-1},\qquad S(x)=qxg^{n-1}. \tag{17}$$

The weak Hopf algebra $H^2_{n^2}(q)$ is generated by $\{g,x\}$ satisfying

$$g^{n+1}=g,\qquad x^n=0,\qquad gxg^{n-1}=qx, \tag{18}$$

with comultiplication given by

$$\Delta(g) = g \otimes g, \qquad \Delta(x) = g \otimes x + x \otimes g^n, \tag{19}$$

and weak antipode defined as in (17).

The weak Hopf algebra $H^3_{n^2}(q)$ is generated by $\{g, x\}$ satisfying (15), with comultiplication and weak antipode given in (19) and (17) respectively.

Pick an invertible scalar λ and consider the map

$$\alpha^j_\lambda : H^j_{n^2}(q) \to H^j_{n^2}(q), \qquad \alpha^j_\lambda(g) = g, \qquad \alpha^j_\lambda(x) = \lambda x, \qquad j = 1, 2, 3.$$

It is easy to prove that α^j_λ is a weak Hopf algebra morphism. By Theorem 1.9, $(H^j_{n^2}(q))_{\alpha^j_\lambda}$ is a weak Hom-Hopf algebra.

Example 1.3 (Hom-$wsl_q(2)$). It is well-known that there exists a weak quantum group $wsl_q(2)$ generated by E_w, F_w and K_w, $\overline{K}_w$ subject to the relations

$$K_w\overline{K}_w = \overline{K}_wK_w, \qquad K_w\overline{K}_wK_w = K_w, \qquad \overline{K}_wK_w\overline{K}_w = \overline{K}_w, \tag{20}$$

$$K_wE_w = q^2E_wK_w, \qquad \overline{K}_wE_w = q^{-2}E_w\overline{K}_w, \tag{21}$$

$$K_wF_w = q^{-2}F_wK_w, \qquad \overline{K}_wF_w = q^2F_w\overline{K}_w, \tag{22}$$

$$E_wF_w - F_wE_w = \frac{K_w\overline{K}_w}{q - q^{-1}}. \tag{23}$$

The coalgebra structure of $wsl_q(2)$ is given by

$$\Delta_w(E_w) = E_w \otimes K_w + 1 \otimes E_w, \qquad \Delta_w(F_w) = F_w \otimes 1 + \overline{K}_w \otimes F_w, \tag{24}$$

$$\Delta_w(K_w) = K_w \otimes K_w, \qquad \Delta_w(\overline{K}_w) = \overline{K}_w \otimes \overline{K}_w, \tag{25}$$

$$\varepsilon_w(K_w) = \varepsilon_w(\overline{K}_w) = 1, \qquad \varepsilon_w(E_w) = \varepsilon_w(F_w) = 0. \tag{26}$$

The weak antipode of $wsl_q(2)$ is given by

$$\begin{aligned} &S_w(E_w) = -E_w\overline{K}_w, \qquad S_w(F_w) = -K_wF_w, \\ &S_w(K_w) = \overline{K}_w, \qquad S_w(\overline{K}_w) = K_w, \qquad S_w(1) = 1. \end{aligned} \tag{27}$$

Pick an invertible scalar λ and define a map $\alpha : wsl_q(2) \to wsl_q(2)$ by $\alpha(E_w) = \lambda E_w$, $\alpha(F_w) = \lambda^{-1}F_w$, $\alpha(K_w) = K_w$, $\alpha(\overline{K}_w) = \overline{K}_w$. It is straightforward to check that α preserves (20)–(27), and as a consequence α is a weak Hopf algebra morphism. By Theorem 1.9 there exists a weak Hom-Hopf algebra $(wsl_q(2))_\alpha$.

We finish this section with some useful properties.

Theorem 1.10. *Let $(H, \mu, \eta, \Delta, \varepsilon, S)$ and $(H', \mu', \eta', \Delta', \varepsilon', S')$ be weak Hopf algebras, and let $\alpha : H \to H$ and $\alpha' : H' \to H'$ be weak Hopf algebra morphisms. If $f : H \to H'$ is a weak Hopf algebra morphism such that $f \circ \alpha = \alpha' \circ f$, then $f : (H, \mu, \eta, \Delta, \varepsilon, S, \alpha) \to (H', \mu', \eta', \Delta', \varepsilon', S', \alpha')$ is a weak Hom-Hopf algebra morphism.*

THEOREM 1.11. *Let $H_\alpha = (H, \mu, \eta, \Delta, \varepsilon, S, \alpha)$ be a finite-dimensional weak Hom-Hopf algebra. Then its dual $H^*_\alpha = (H^*, \Delta^*, \varepsilon^*, \mu^*, \eta^*, S^*, \alpha^*)$ is also a weak Hom-Hopf algebra.*

Proof. It is proved in [Y2] that $H^*_\alpha = (H^*, \Delta^*, \varepsilon^*, \mu^*, \eta^*, \alpha^*)$ is a Hom-bialgebra. Then, by Definition 1.7, we only need to prove that S^* satisfies (13) and (14). In fact, for any $f \in H^*$ and $h \in H$,

$$\begin{aligned}\sum((f_{11}S^*(f_{12}))\alpha^*(f_2))(h) &= \sum f((h_{11}S(h_{12}))\alpha(h_2))\\ &= f(\alpha^3(h)) = (\alpha^*)^3(f)(h),\\ \sum((S^*(f_{11})f_{12})\alpha^*(S^*(f_2)))(h) &= \sum f((S(h_{11})h_{12})\alpha(S(h_2)))\\ &= f(\alpha^3(S(h))) = (\alpha^*)^3(S^*(f))(h). \blacksquare\end{aligned}$$

THEOREM 1.12. *Let $H_\alpha = (H, \mu, \eta, \Delta, \varepsilon, S, \alpha)$ be a weak Hom-Hopf algebra. Then $H^m_\alpha = (H, \mu^m, \eta, \Delta^m, \varepsilon, S, \alpha^{2^m})$ is a weak Hom-Hopf algebra for any integer $m \geq 0$, where $\mu^m = \alpha^{2^m-1} \circ \mu$ and $\Delta^m = \Delta \circ \alpha^{2^m-1}$.*

Proof. For simplicity we write $\mu^m(x \otimes y) = x \cdot_m y$ and $\Delta^m(x) = \sum x^m_1 \otimes x^m_2$. It is shown in [Y2] that $H^m_\alpha = (H, \mu^m, \eta, \Delta^m, \varepsilon, \alpha^{2^m})$ is a Hom-bialgebra. To finish the proof we only need to show that (13) and (14) are true. In fact, for any $h \in H$,

$$\begin{aligned}&\sum(h^m_{11} \cdot_m S(h^m_{12})) \cdot_m h^m_2\\ &\quad= \sum \alpha^{2m-1}(\alpha^{2m-1}(\alpha^{2m-1}(h_{11})S(\alpha^{2m-1}(h_{12})))\alpha^{2^m}(\alpha^{2m-1}(h_2)))\\ &\quad= \sum \alpha^{2m-1}(\alpha^{2m-1}((\alpha^{2m-1}(h_{11})S(\alpha^{2m-1}(h_{12})))\alpha^{2m-1}(h_2)))\\ &\quad= \sum \alpha^{2(2m-1)}(\alpha^3(\alpha^{2m-1}(h))) = (\alpha^{2^m})^3(h),\\ &\sum(S(h^m_{11}) \cdot_m h^m_{12}) \cdot_m S(h^m_2)\\ &\quad= \sum \alpha^{2m-1}(\alpha^{2m-1}(\alpha^{2m-1}(S(h_{11}))\alpha^{2m-1}(h_{12}))\alpha^{2^m}(\alpha^{2m-1}(S(h_2))))\\ &\quad= \sum \alpha^{2m-1}(\alpha^{2m-1}((\alpha^{2m-1}(S(h_{11}))\alpha^{2m-1}(h_{12}))\alpha^{2m-1}(S(h_2))))\\ &\quad= \sum \alpha^{2(2m-1)}(\alpha^3(\alpha^{2m-1}(S(h)))) = (\alpha^{2^m})^3(S(h)). \blacksquare\end{aligned}$$

2. The main results. In this section, we construct a braided almost Hom-bialgebra, that is, a quantum double of weak Hom-Hopf algebras.

DEFINITION 2.1. A *braided weak Hom-Hopf algebra* is a weak Hom-Hopf algebra $H_\alpha = (H, \mu, \eta, \Delta, \varepsilon, S, \alpha)$ with an element $R = \sum R^1 \otimes R^2 \in H \otimes H$ satisfying the following equalities:

$$(\Delta \otimes \alpha)(R) = R_{13}R_{23}, \quad (\alpha \otimes \Delta)(R) = R_{13}R_{12}, \tag{28}$$

$$[(\tau \circ \Delta)(h)]R = R\Delta(h), \quad h \in H, \tag{29}$$

where $R_{12} = R \otimes 1,\ R_{23} = 1 \otimes R,\ R_{13} = (\tau \circ \mathrm{id}_H)(R_{23})$.

In particular, if $H_\alpha = (H, \mu, \eta, \Delta, \varepsilon, \alpha)$ is an almost Hom-bialgebra, then $H_\alpha = (H, \mu, \eta, \Delta, \varepsilon, \alpha, R)$ is called a *braided almost Hom-bialgebra*.

As a consequence of Theorems 1.9 and 1.12, we have the following results.

PROPOSITION 2.2. *Let* $H_\alpha = (H, \mu, \eta, \Delta, \varepsilon, S, \alpha, R)$ *be a braided weak Hom-Hopf algebra. Then* R *satisfies the quantum Hom-Yang–Baxter equations* $(R_2 R_{13})R_{23} = R_{23}(R_{13}R_{12})$ *and* $R_{12}(R_{13}R_{23}) = (R_{23}R_{13})R_{12}$.

PROPOSITION 2.3. *Let* $(H, \mu, \eta, \Delta, \varepsilon, S, R)$ *be a braided weak Hopf algebra and* $\alpha : H \to H$ *a morphism of weak Hopf algebras. Then*

$$H_\alpha = (H, \mu_\alpha, \eta, \Delta_\alpha, \varepsilon, S, \alpha, R)$$

is a braided weak Hom-Hopf algebra.

PROPOSITION 2.4. *Let* $H_\alpha = (H, \mu, \eta, \Delta, \varepsilon, S, R, \alpha)$ *be a braided weak Hom-Hopf algebra and* $\alpha : H \to H$ *a surjective map. Then*

$$H_\alpha^{m,j} = (H, \mu^m, \eta, \Delta^m, \varepsilon, S, R^{\alpha^j}, \alpha^{2^m})$$

is a braided weak Hom-Hopf algebra for any integers $m, j \geq 0$*, where* $\mu^m = \alpha^{2^m-1} \circ \mu$*,* $\Delta^m = \Delta \circ \alpha^{2^m-1}$ *and* $R^{\alpha^j} = (\alpha^j \otimes \alpha^j)R$.

We now give an example of a braided weak Hom-Hopf algebra.

EXAMPLE 2.1. Let G be a finite commutative semigroup so that kG is a commutative and cocommutative weak Hopf algebra. We now consider a function $r : G \times G \to k$ satisfying $\sum_{xy=v} r(u,x)r(w,y) = \delta_{u,w} r(u,v)$ and $\sum_{xy=u} r(x,v)r(y,w) = \delta_{v,w} r(u,v)$, where $\delta_{u,w}$ is the Kronecker delta. As is well known, r gives rise to a braided structure on kG. Define $R = r(u,v)u \otimes v$. Then (28) is an immediate consequence of the above conditions, and (29) follows from the fact that kG is cocommutative and commutative. By Proposition 2.3 and Example 1.1, there exists a braided weak Hom-Hopf algebra.

We are now ready for the dual version of Definition 2.1.

DEFINITION 2.5. A *cobraided weak Hom-Hopf algebra* is a weak Hom-Hopf algebra $H_\alpha = (H, \mu, \eta, \Delta, \varepsilon, S, \alpha)$ with a bilinear form $r : H \otimes H \to k$ satisfying the following conditions:

$$r(xy \otimes \alpha(z)) = \sum r(\alpha(x) \otimes z_1) r(\alpha(y) \otimes z_2), \tag{30}$$

$$r(\alpha(x) \otimes yz) = \sum r(x_1 \otimes \alpha(z)) r(x_2 \otimes \alpha(y)), \tag{31}$$

$$\sum y_1 x_1 r(x_2 \otimes y_2) = \sum r(x_1 \otimes y_1) x_2 y_2, \qquad x, y, z \in H. \tag{32}$$

Similar to Propositions 2.3 and 2.4, we have the following.

PROPOSITION 2.6. *Let* $(H, \mu, \eta, \Delta, \varepsilon, S, r)$ *be a cobraided weak Hopf algebra and* $\alpha : H \to H$ *a morphism of weak Hopf algebras. Then* $H_\alpha = (H, \mu_\alpha, \eta, \Delta_\alpha, \varepsilon, S, \alpha, r)$ *is a cobraided weak Hom-Hopf algebra.*

Proposition 2.7. *Let* $H_\alpha = (H, \mu, \eta, \Delta, \varepsilon, \alpha, S, r)$ *be a cobraided weak Hom-Hopf algebra and* $\alpha : H \to H$ *a surjective map. Then*

$$H_\alpha^{m,j} = (H, \mu^m, \eta, \Delta^m, \varepsilon, S, r^{\alpha^j}, \alpha^{2^m})$$

is a cobraided weak Hom-Hopf algebra for any integer $m, j \geq 0$*, where* $\mu^m = \alpha^{2^m-1} \circ \mu$*,* $\Delta^m = \Delta \circ \alpha^{2^m-1}$ *and* $r^{\alpha^m} = r \circ (\alpha^m \otimes \alpha^m)$*.*

We now give an example of a cobraided weak Hom-Hopf algebra.

Example 2.2. Let G be a finite commutative semigroup so that kG is a commutative and cocommutative weak Hopf algebra. A bilinear form r is a cobraiding of kG if and only if $xyr(y \otimes x) = r(y \otimes x)yx$, $r(xy \otimes z) = r(x \otimes z)r(y \otimes z)$ and $r(x \otimes yz) = r(x \otimes z)r(x \otimes y)$. By Proposition 2.6 and Example 1.1, there exists a cobraided weak Hom-Hopf algebra.

The remainder of this section will be devoted to the construction of a braided almost Hom-bialgebra.

From now on, we assume that $H_\alpha = (H, \mu, \eta, \Delta, \varepsilon, \alpha, S)$ is a finite-dimensional cocommutative weak Hom-Hopf algebra with bijective S such that $(S * \mathrm{id})(H) \subseteq C(H)$ and $\alpha^2 = \mathrm{id}$.

Define $D(H) = \widehat{H}^{\mathrm{cop}} \bowtie H$ as a k-module with multiplication and comultiplication given respectively by

$$(f \bowtie a)(g \bowtie b) = \sum fg([S^{-1}(a_{22})?]a_1) \bowtie a_{21}b, \qquad f, g \in \widehat{H}, \qquad a, b \in H,$$
$$\Delta(f \bowtie a) = \sum f_2 \bowtie a_1 \otimes f_1 \bowtie a_2, \qquad \varepsilon_{D(H)}(f \bowtie a) = f(1)\varepsilon(a).$$

For convenience, the multiplication is also given by

$$(f \bowtie a)(g \bowtie b) = \sum \langle g_1, S^{-1}(a_{22})\rangle \langle g_{22}, a_1\rangle f g_{21} \bowtie a_{21}b.$$

We now state the main result of this paper.

Theorem 2.8. $(D(H), \alpha^* \otimes \alpha)$ *is a braided almost Hom-bialgebra with the above multiplication and comultiplication and with* $R = (\varepsilon \bowtie \alpha(u_i)) \otimes (u^i \bowtie 1)$ *as braiding, where* $\{u_i\}$ *and* $\{u^i\}$ *are dual bases in* H *and* $\widehat{H}$*.*

Proof. First we show that this is a Hom-associative algebra. In fact, for any $f, g, l \in \widehat{H}$ and $a, b, c \in H$,

$$\begin{aligned}
&(\alpha^* \otimes \alpha)(f \bowtie a)[(g \bowtie b)(l \bowtie c)] \\
&\quad = \sum \langle g_1, S^{-1}(\alpha(a_{222}))\rangle \langle g_{22}, \alpha(a_{11})\rangle \langle l_1, S^{-1}(b_{22})\rangle \langle l_{22}, b_1\rangle \\
&\qquad \cdot \langle l_{21\underline{1}}, S^{-1}(\alpha(a_{221}))\rangle \langle l_{21\underline{22}}, \alpha(a_{12})\rangle \alpha^*(f)(g_{21} l_{21\underline{21}}) \bowtie \alpha(a_{21})(b_{21}c) \\
&\quad \overset{(5)}{=} \sum \langle g_1, S^{-1}(\alpha(a_{222}))\rangle \langle g_{22}, \alpha(a_{11})\rangle \langle l_1, S^{-1}(b_{22})\rangle \langle l_{\underline{22}}, b_1\rangle \\
&\qquad \cdot \langle l_{2\underline{11}1}, S^{-1}(a_{221})\rangle \langle l_{2\underline{12}}, a_{12}\rangle \alpha^*(f)(g_{21} l_{2\underline{11}2}) \bowtie \alpha(a_{21})(b_{21}c) \\
&\quad \overset{(5)}{=} \sum \langle g_1, S^{-1}(\alpha(a_{222}))\rangle \langle g_{22}, \alpha(a_{11})\rangle \langle l_1, S^{-1}(b_{22})\rangle \langle l_{2\underline{2}2}, \alpha(b_1)\rangle \\
&\qquad \cdot \langle l_{2\underline{11}}, S^{-1}(\alpha(a_{221}))\rangle \langle l_{2\underline{2}1}, a_{12}\rangle \alpha^*(f)(g_{21}\alpha^*(l_{2\underline{12}})) \bowtie \alpha(a_{21})(b_{21}c)
\end{aligned}$$

$$
\begin{aligned}
&\overset{(5)}{=} \sum\langle g_1, S^{-1}(\alpha(a_{222}))\rangle\langle g_{22},\alpha(a_{11})\rangle\langle l_{\underline{1}}, S^{-1}(b_{22})\rangle\langle l_{\underline{22}22}, b_1\rangle\\
&\quad\cdot\langle l_{\underline{21}}, S^{-1}(a_{221})\rangle\langle l_{\underline{22}21},\alpha(a_{12})\rangle\alpha^*(f)(g_{21}\alpha^*(l_{\underline{22}1}))\bowtie\alpha(a_{21})(b_{21}c)\\
&\overset{(5)}{=} \sum\langle g_1,S^{-1}(\alpha(a_{2\underline{22}}))\rangle\langle g_{22},\alpha(a_{\underline{11}})\rangle\langle l_{11},S^{-1}(\alpha(b_{22}))\rangle\\
&\quad\cdot\langle l_{222},\alpha(b_1)\rangle\langle l_{12},S^{-1}(a_{\underline{2}21})\rangle\langle l_{221},a_{\underline{12}}\rangle\alpha^*(f)(g_{21}l_{21})\bowtie\alpha(a_{\underline{2}1})(b_{21}c)\\
&\overset{(5)}{=} \sum\langle g_1,S^{-1}(a_{2\underline{22}2})\rangle\langle g_{22},a_1\rangle\langle l_{11},S^{-1}(\alpha(b_{22}))\rangle\langle l_{222},\alpha(b_1)\rangle\\
&\quad\cdot\langle l_{12},S^{-1}(\alpha(a_{2\underline{22}1}))\rangle\langle l_{221},a_{2\underline{1}}\rangle\alpha^*(f)(g_{21}l_{21})\bowtie a_{2\underline{21}}(b_{21}c)\\
&\overset{(5)}{=} \sum\langle g_1,S^{-1}(\alpha(a_{2\underline{22}}))\rangle\langle g_{22},a_1\rangle\langle l_{11},S^{-1}(\alpha(b_{22}))\rangle\langle l_{222},\alpha(b_1)\rangle\\
&\quad\cdot\langle l_{12},S^{-1}(a_{2\underline{21}})\rangle\langle l_{221},\alpha(a_{2\underline{1}1})\rangle\alpha^*(f)(g_{21}l_{21})\bowtie a_{2\underline{1}2}(b_{21}c)\\
&\overset{(5)}{=} \sum\langle g_1,S^{-1}(a_{22})\rangle\langle g_{22},a_1\rangle\langle l_{11},S^{-1}(\alpha(b_{22}))\rangle\langle l_{222},\alpha(b_1)\rangle\\
&\quad\cdot\langle l_{12},S^{-1}(a_{21\underline{2}})\rangle\langle l_{221},a_{21\underline{11}}\rangle\alpha^*(f)(g_{21}l_{21})\bowtie\alpha(a_{21\underline{12}})(b_{21}c)\\
&\overset{(5)}{=} \sum\langle g_1,S^{-1}(a_{22})\rangle\langle g_{22},a_1\rangle\langle l_{11},S^{-1}(\alpha(b_{22}))\rangle\langle l_{222},\alpha(b_1)\rangle\\
&\quad\cdot\langle l_{12},S^{-1}(\alpha(a_{2122}))\rangle\langle l_{221},\alpha(a_{211})\rangle\alpha^*(f)(g_{21}l_{21})\bowtie\alpha(a_{2121})(b_{21}c)\\
&= [(f\bowtie a)(g\bowtie b)](\alpha^*\otimes\alpha)(l\bowtie c),
\end{aligned}
$$

$$
\begin{aligned}
(f\bowtie a)(\varepsilon\bowtie 1) &= \sum\langle\varepsilon_1,S^{-1}(a_{22})\rangle\langle\varepsilon_{22},a_1\rangle f\varepsilon_{21}\bowtie a_{21}1\\
&= \alpha^*(f)\bowtie\alpha(a) = (\alpha^*\otimes\alpha)(f\bowtie a),\\
(\varepsilon\bowtie 1)(f\bowtie a) &= \sum\langle f_1,S^{-1}(1)\rangle\langle f_{22},1\rangle f_{21}\varepsilon\bowtie 1a = (\alpha^*\otimes\alpha)(f\bowtie a).
\end{aligned}
$$

On the other hand, it is easy to show that $(D(H),\alpha^*\otimes\alpha)$ is also a Hom-coassociative coalgebra.

We need to show that the comultiplication Δ is a Hom-algebra morphism. For any $f,g,u,v\in\widehat{H}$ and $a,b,x,y\in H$, and $\theta := x\otimes u\otimes y\otimes v$, we compute

$$
\begin{aligned}
&\langle\Delta(f\bowtie a)\Delta(g\bowtie b),\theta\rangle\\
&\quad\overset{(5)}{=} \sum f(y_1x_1)g_1(S^{-1}(a_{222}))g_{21}(y_2([a_{21}S^{-1}(\alpha(a_{122}))]\alpha(x_2)))\\
&\qquad\cdot g_{22}(\alpha(a_{11}))u(a_{121}b_1)v(a_{221}b_2)\\
&\quad= \sum f(y_1x_1)g_1(S^{-1}(a_{222}))g_{21}(y_2(\alpha(x_2)[a_{21}S^{-1}(\alpha(a_{122}))]))\\
&\qquad\cdot g_{22}(\alpha(a_{11}))u(a_{121}b_1)v(a_{221}b_2)\\
&\quad\overset{(5)}{=} \sum f(y_1x_1)g_1(S^{-1}(a_{222}))g_{21}(\alpha(y_2x_2)[\alpha(a_{21})S^{-1}(a_{122})])\\
&\qquad\cdot g_{22}(\alpha(a_{11}))u(a_{121}b_1)v(a_{221}b_2)\\
&\quad\overset{(5)}{=} \sum f(y_1x_1)g_1(S^{-1}(a_{222}))g_2((y_2x_2)([\alpha(a_{21})S^{-1}(a_{122})]\alpha(a_{11})))\\
&\qquad\cdot u(a_{121}b_1)v(a_{221}b_2)\\
&\quad= \sum f(y_1x_1)g(S^{-1}(a_{22})[(y_2x_2)(S^{-1}(S(a_{111})[\alpha(a_{112})S(\alpha(a_{12}))])])\\
&\qquad\cdot u(a_{211}b_1)v(a_{212}b_2)
\end{aligned}
$$

$$\overset{(14)}{=} \sum f(y_1x_1)g(S^{-1}(a_{22})[(y_2x_2)a_1])u(a_{211}b_1)v(a_{212}b_2)$$
$$= \langle \Delta((f \bowtie a)(g \bowtie b)), \theta \rangle,$$

which means that

$$\Delta((f \bowtie a)(g \bowtie b)) = \Delta(f \bowtie a)\Delta(g \bowtie b).$$

To finish the proof we only need to prove the braiding. We begin by proving (28). For any $a, b, c \in H, f, g, l \in \widehat{H}$, and $\vartheta := a \otimes f \otimes b \otimes g \otimes c \otimes l$,

$$\begin{aligned}
\langle \Delta(R^1) \otimes \alpha(R^2), \vartheta \rangle &= \sum \varepsilon(a)\varepsilon(b)f(\alpha(u_{i1}))g(\alpha(u_{i2}))u^i(\alpha(c))l(1) \\
&= \sum \varepsilon(a)\varepsilon(b)f(\alpha^2(c_1))g(\alpha^2(c_2))l(1) = \sum \varepsilon(a)\varepsilon(b)f(c_1)g(c_2)l(1) \\
&= \sum \varepsilon(a)f(u_i)u^i{}_1(b)g(1)u^i{}_2(c)l(1) \\
&= \langle R_{13}R_{23}, \vartheta \rangle,
\end{aligned}$$

$$\begin{aligned}
\langle \alpha(R^1) \otimes \Delta(R^2), \vartheta \rangle &= \sum \varepsilon(a)f(\alpha(u_i)\alpha(u_j))g(1)u^i(\alpha(c)u^j(\alpha(b))l(1) \\
&\overset{(11)}{=} \sum \varepsilon(S^{-1}(\alpha(u_{i22}))(a\alpha(u_{i1})))f(\alpha(u_{i21})\alpha(u_j))g(1)u^i(\alpha(c)u^j(\alpha(b))l(1) \\
&= \langle R_{13}R_{12}, \vartheta \rangle,
\end{aligned}$$

and we are done.

Finally, (29) is a consequence of the following computation: for any $a, b, x \in H,\ f, g, l \in \widehat{H}$, and $\delta := x \otimes g \otimes b \otimes l$,

$$\begin{aligned}
&\langle \Delta^{\text{op}}(f \bowtie a)R, \delta \rangle \\
&\overset{(11)}{=} \sum f(\alpha(x)b_1)g(a_2[S^{-1}(\alpha(a_{122}))(\alpha(b_2)\alpha(a_{11}))])l(\alpha(a_{121})) \\
&\overset{(5)}{=} \sum f(\alpha(x)b_1)g((\alpha(a_2)S^{-1}(\alpha(a_{122})))(b_2a_{11}))l(\alpha(a_{121})) \\
&\overset{(5)}{=} \sum f(\alpha(x)b_1)g([(a_2S^{-1}(a_{122}))b_2]\alpha(a_{11}))l(\alpha(a_{121})) \\
&\overset{(5)}{=} \sum f(\alpha(x)b_1)g(\alpha(b_2)[(a_2S^{-1}(a_{122}))a_{11}])l(\alpha(a_{121})) \\
&\overset{(14)}{=} \sum f(\alpha(x)b_1)g(\alpha(b_2)a_1)l(\alpha(a_2)) \\
&\overset{(5)}{=} \sum f(\alpha(x)[(b_2S^{-1}(b_{122}))b_{11}])g(b_{121}a_1)l(\alpha(a_2)) \\
&= \sum f([x(b_2S^{-1}(b_{122}))]\alpha(b_{11}))g(b_{121}a_1)l(\alpha(a_2)) \\
&\overset{(5)}{=} \sum f([(b_2S^{-1}(b_{122}))x]\alpha(b_{11}))g(b_{121}a_1)l(\alpha(a_2)) \\
&\overset{(5)}{=} \sum f(\alpha(b_2S^{-1}(\alpha(b_{122})))(xb_{11}))g(b_{121}a_1)l(\alpha(a_2)) \\
&= \sum f(b_2[S^{-1}(\alpha(u_m))(\alpha(xu_i))])g(\alpha(u_j)a_1)u^i(b_{11})u^j(b_{121})u^m(b_{122})l(\alpha(a_2)) \\
&= \sum f(b_2[S^{-1}(\alpha(u_{i22}))(\alpha(xu_{i1}))])g(\alpha(u_{i21})a_1)u^i(b_1)l(\alpha(a_2)) \\
&= \langle R\Delta(f \bowtie a), \delta \rangle,
\end{aligned}$$

hence $\Delta^{\text{op}}(f \bowtie a)R = R\Delta(f \bowtie a)$, which completes the proof. ∎

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