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Periods of rigid double octic Calabi–Yau threefolds

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To the memory of Professor Józef Siciak

Abstract. We compute numerical approximations of the period integrals for eleven rigid double octic Calabi–Yau threefolds and compare them with the periods of the corresponding weight four cusp forms, finding commensurabilities as expected. These give information on the correspondences of these varieties with the associated Kuga–Sato modular threefolds.

Introduction. Let X be a rigid Calabi–Yau threefold and $\omega \in H^{3,0}(X)$ a regular 3-form on X. For a 3-cycle $\gamma \in H_3(X,\mathbb{Z})$ on X we can form the *period integral*

$$\int_\gamma \omega \in \mathbb{C}.$$

The set of these period integrals forms a lattice

$$\Lambda := \Big\{ \int_\gamma \omega : \gamma \in H_3(X,\mathbb{Z}) \Big\} \subset \mathbb{C}$$

and hence determines an elliptic curve

$$\mathbb{C}/\Lambda = H^{3,0}(X)^*/H_3(X,\mathbb{Z}) =: J^2(X),$$

which is just an example of an *intermediate Jacobian* of Griffiths [9].

It is now known that any rigid Calabi–Yau threefold defined over $\mathbb{Q}$ is *modular* [8], [6], in the sense that one has an equality of L-functions:

$$L(H^3(X),s) = L(f,s).$$

Here $f \in S_4(\Gamma_0(N))$ is a weight four cusp form for some level N. It is known more generally that a cusp form $f \in S_{k+2}(\Gamma_0(N))$ can be interpreted as a

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k-form on the associated Kuga–Sato variety, which is (a desingularisation of) the k-fold fibre product of the universal elliptic curve over the modular curve $X_0(N)$ [5]. So in our case one expects the equality of L-functions to come from a correspondence between the rigid Calabi–Yau and the Kuga–Sato variety Y, which resolves the fibre product $E \times_C E$ of the universal elliptic curve over the modular curve $C := X_0(N)$. Now the integrals over a positive imaginary half-line,

$$\int_0^{i\infty} f(\tau)\tau^k \, d\tau,$$

determine the periods of the Kuga–Sato variety Y, and the correspondence between X and Y would imply that the period lattice of X is commensurable to the lattice derived from the modular form.

In this note we shall compute numerical approximations of period integrals for certain rigid double octic Calabi–Yau threefolds, i.e. Calabi–Yau threefolds constructed as a resolution of a double cover of projective space $\mathbb{P}^3$, branched along a surface of degree eight.

More specifically, we will look at the eleven arrangements of eight planes defined by linear forms with rational coefficients, described in the PhD thesis of C. Meyer [11]. These arrangements define eleven non-isomorphic rigid Calabi–Yau threefolds. Meyer also determined the weight four cusp forms f for these eleven rigid double octics using the counting of points in $\mathbb{F}_p$ for small primes p.

Each arrangement of real planes defines a partition of the real projective space $\mathbb{P}^3(\mathbb{R})$ into *polyhedral cells* and using these cells one can construct certain *polyhedral* 3*-cycles* on the desingularisation of the double octic. Using the explicit equations for the planes of the arrangement, one can write the period integral as an explicit sum of multiple integrals, which can be integrated numerically.

It has turned out to be difficult to identify a complete basis of $H_3(X,\mathbb{Z})$ in terms of polyhedral cycles. But any two non-proportional periods of a rigid Calabi–Yau threefold define a subgroup of finite index of Λ and hence an elliptic curve *isogenous* to the intermediate Jacobian $J^2(X)$. From such a numerical lattice one can compute the lattice constants g_2 and g_3, and hence the Weierstrass equation and the j-invariant of the curve defined by the lattice spanned by the polyhedral 3-cycles.

We expect that a more refined topological analysis of the above situation will lead to more precise information on the nature of the correspondences between these varieties.

Numerical calculations of periods of algebraic varieties have been found useful in other contexts in arithmetical algebraic geometry; for recent examples see [7, 12].

1. Double octic Calabi–Yau threefolds. By a *double octic* we understand a variety X given as a double cover

$$\pi : X \to \mathbb{P}^3$$

of $\mathbb{P}^3$, ramified over a surface $D \subset \mathbb{P}^3$ of degree eight. Such a double octic X can be given by an equation in weighted projective space $\mathbb{P}(1,1,1,1,4)$ of the form

$$u^2 = F(x,y,z,t),$$

where the polynomial F defines the ramification divisor D. If the surface D is smooth, then X is a smooth Calabi–Yau threefold, but we will be dealing here with the case that D is a union of eight planes, so the polynomial factors into a product of linear forms:

$$F = L_1 L_2 L_3 L_4 L_5 L_6 L_7 L_8.$$

The associated double octic X is then singular along the lines of intersection of the eight planes $D_i := \{L_i = 0\}$. When these planes have the property that

no six intersect in a point, no four intersect along a line,

one can construct a Calabi–Yau desingularisation of X. To do so, one first constructs a sequence of blow-ups with smooth centers

$$T : \widetilde{\mathbb{P}^3} \to \mathbb{P}^3$$

and a divisor $\widetilde{D}$ in $\widetilde{\mathbb{P}^3}$ such that

- $\widetilde{D}$ is *non-singular* (in particular reduced),
- $\widetilde{D}$ is *even* as an element of the Picard group $\mathrm{Pic}(\widetilde{\mathbb{P}^3})$,

by blowing up the singularities of D in the following order:

(1) fivefold points,
(2) triple lines,
(3) fourfold points,
(4) double lines.

In the first two cases we replace the branch divisor by its reduced inverse image: the strict transform plus the exceptional divisor. In the last two cases we replace the branch divisor by its strict transform.

The double cover

$$\widetilde{\pi} : \widetilde{X} \to \widetilde{\mathbb{P}^3}$$

of $\widetilde{\mathbb{P}^3}$ branched along $\widetilde{D}$ is now a smooth Calabi–Yau manifold, which we will call the *double octic Calabi–Yau threefold* of the arrangement (see [11, 4.1]).

We will also need to consider a particular *partial resolution* $\widehat{X}$ of X, obtained as a double cover of a space $\widehat{\mathbb{P}^3}$ by performing only the blow-ups in fivefold points, triple lines and double curves, so leaving out step (3) in the above procedure. Note that there are two types of fourfold points called p_4^0 and p_4^1 in [11]. A fourfold point is of type p_4^1 if it lies on a triple line, it is of type p_4^0 if it is an intersection point of four planes containing it, and it is generic otherwise. A fourfold point of type p_4^1 gets removed in step (2), but fourfold points of type p_4^0 produce *ordinary double points* if we blow up consecutively the intersections of the strict transforms of the relevant planes. After the blow-up of the first double line, the strict transforms of the remaining two planes (not containing this line) intersect along a union of two intersecting lines. So we have to blow up four lines and a *cross*, the latter producing a node on the threefold $\widehat{\mathbb{P}^3}$. The space doubly covering $\widehat{\mathbb{P}^3}$ and ramified over a divisor $\widehat{D}$ is a variety $\widehat{X}$ with twice as many nodes.

We can obtain the same effect by splitting p_4^0 points into four triple points, resolving and then degenerating back. To see this in local coordinates (x, y, z) we can assume that the four planes we are considering have equations

$$x = 0, \quad y = 0, \quad z = 0, \quad x + y + z = 0.$$

We shift the fourth plane to

$$x + y + z = \epsilon,$$

resolve this and then specialise back to $\epsilon = 0$.

Let us first blow up the two disjoint lines

$$x = y = 0, \quad z = x + y + z - \epsilon = 0.$$

In one of the affine charts the blow-up of $\mathbb{P}^3$ is given by the equation

$$x(y + 1) - z(v - 1) - \epsilon.$$

The threefold is smooth unless $\epsilon = 0$ when it acquires a node at $x = 0$, $y = -1, z = 0, v = 1$. Since the surface $x = 0, z = 0$ is a Weil divisor on the threefold which is not Cartier (it is a component of the exceptional locus of the blow-up), the node admits a projective small resolution. Since the node does not lie on the branch divisor, it gives two nodes on the double cover. Consequently, we get the partial resolution $\widehat{X}$.

Since $\widehat{X}$ and $\widetilde{X}$ (resp. $\widehat{\mathbb{P}^3}$ and $\widetilde{\mathbb{P}^3}$) are blow-ups of the same varieties, there exist birational maps

$$\sigma : \widetilde{X} \to \widehat{X} \quad \text{and} \quad S : \widetilde{\mathbb{P}^3} \to \widehat{\mathbb{P}^3}.$$

In fact one can check in local coordinates that S is the blow-up of the intersection of two exceptional loci of R in $\widehat{\mathbb{P}^3}$, so σ and S are projective small resolutions of nodes.

The situation is summarized in the following diagram:

$$\begin{array}{ccccc} \widetilde{X} & \xrightarrow{\sigma} & \widehat{X} & \xrightarrow{\rho} & X \\ \downarrow{\scriptstyle\widetilde{\pi}} & & \downarrow{\scriptstyle\widehat{\pi}} & & \downarrow{\scriptstyle\pi} \\ \widetilde{\mathbb{P}^3} & \xrightarrow{S} & \widehat{\mathbb{P}^3} & \xrightarrow{R} & \mathbb{P}^3 \end{array}$$

The vertical maps are two-fold covers, the map $\sigma : \widetilde{X} \to \widehat{X}$ is a small resolution of the nodes of $\widehat{X}$, and $\rho : \widehat{X} \to X$ is a partial resolution of the double octic variety X. The composition RS is the map $T : \widetilde{\mathbb{P}^3} \to \mathbb{P}^3$ we started with, and $\tau := \rho\sigma : \widetilde{X} \to X$ is a resolution of singularities. In fact the partial resolutions $\widehat{\mathbb{P}^3}$ and $\widehat{X}$ depend on the choice of the order of blow-up of lines, but $\widetilde{\mathbb{P}^3}$ and $\widetilde{X}$ do not.

We will be concerned with eleven special arrangements that were studied by C. Meyer [11]. The resolutions $\widetilde{X}$ of the associated double octics lead to eleven different rigid Calabi–Yau varieties. For the convenience of the reader, we here list the arrangement numbers, the second Betti number and the equations from [11].

No.	$b_2(\widetilde{X})$	Equation	λ
1	70	$xyzt(x+y)(y+z)(z+t)(t+x)$	-1
3	62	$xyzt(x+y)(y+z)(y-t)(x-y-z+t)$	1
19	54	$xyzt(x+y)(y+z)(x-z-t)(x+y+z-t)$	2
32	50	$xyzt(x+y)(y+z)(x-y-z-t)(x+y-z+t)$	-1
69	50	$xyzt(x+y)(x-y+z)(x-y-t)(x+y-z-t)$	-1
93	46	$xyzt(x+y)(x-y+z)(y-z-t)(x+z-t)$	2
238	44	$xyzt(x+y+z-t)(x+y-z+t)(x-y+z+t)(-x+y+z+t)$	1
239	40	$xyzt(x+y+z)(x+y+t)(x+z+t)(y+z+t)$	1
240	40	$xyzt(x+y+z)(x+y-z+t)(x-y+z+t)(x-y-z-t)$	-2
241	40	$xyzt(x+y+z+t)(x+y-z-t)(y-z+t)(x+z-t)$	1
245	38	$xyzt(x+y+z)(y+z+t)(x-y-t)(x-y+z+t)$	-2

The meaning of the λ in the last column will be explained in Section 6.

2. 3-cycles on a double octic. In the above table eleven examples of rigid double octic Calabi–Yau threefolds defined over $\mathbb{Q}$ are given. In all these examples the eight planes are given by equations with integral coefficients.

In general, an arrangement defined by real planes gives a decomposition of $\mathbb{P}^3(\mathbb{R})$ into a finite number of *polyhedral cells*, oriented by a fixed orientation of $\mathbb{P}^3(\mathbb{R})$. By combining these cells one can construct certain *polyhedral cycles* on the smooth model $\widetilde{X}$. To explain this, let us fix one of these cells C and consider its double covering $\mathbf{C}$, that is, its preimage under the 2-fold covering map $\pi : X \to \mathbb{P}^3$. Then $\mathbf{C}$ is a 3-cycle in X and determines an element in $H_3(X, \mathbb{Z})$, up to a sign determined by a choice of orientation.

QUESTION. Do the 3-cycles $\mathbf{C}$ generate $H_3(X, \mathbb{Z})$?

Note that $\mathbf{C}$ in general does not lift to a 3-cycle on the desingularisation $\widetilde{X}$, as the canonical map

$$\tau_* : H_3(\widetilde{X}, \mathbb{Z}) \to H_3(X, \mathbb{Z})$$

is not surjective in general. To see this geometrically, we follow the behaviour of the cycle $\mathbf{C}$ under the blow-up maps and see that it gets transformed into a chain on $\widetilde{X}$ that we will still denote by $\mathbf{C}$. Its boundary $\partial\mathbf{C}$, as a chain on $\widetilde{X}$, is a sum of 2-cycles contained in the exceptional loci,

$$\partial\mathbf{C} = \bigcup_i \mathit{\Gamma}_i.$$

Now observe that $\mathbf{C}$ is *anti-symmetric* with respect to the covering map π, and as a consequence, these cycles $\mathit{\Gamma}_i$ are anti-symmetric as well. On the other hand, a component of the exceptional divisor corresponding to a fivefold point or a triple line is fixed by the involution, while a component of the exceptional locus corresponding to a double line is a blow-up of a conic bundle. So in all the three cases the involution of the double cover acts trivially on the second homology group of that divisor. Hence each 2-cycle $\mathit{\Gamma}_i$ contained in the exceptional divisor corresponding to a double line, a triple line or a fivefold point is a boundary, i.e. there is a 3-chain $\mathbf{C}_i$ such that $\partial\mathbf{C}_i = \mathit{\Gamma}_i$. Hence, if we subtract from $\mathbf{C}$ the chains $\mathbf{C}_i$ we get a chain with boundary contained in the exceptional divisors corresponding to the fourfold points. So we see that $\mathbf{C}$ can be lifted to a cycle $\widehat{\mathbf{C}}$ on the partial resolution $\widehat{X}$ and we have shown:

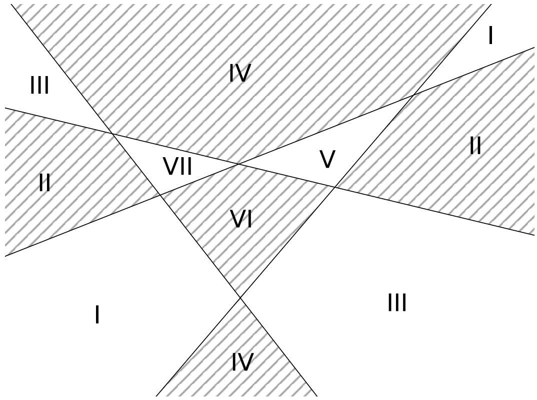
PROPOSITION. *The map* $\rho_* : H_3(\widehat{X}, \mathbb{Z}) \to H_3(X, \mathbb{Z})$ *is surjective.*

Special role of fourfold points. We need to analyse the situation of a fourfold point of type p_4^0 in more detail. Near p the space $\mathbb{R}^3$ is decomposed into $2^4 - 2 = 14$ cells, which are in one-to-one correspondence with the consistent sign patterns

$$(\mathrm{sign}(L_1), \mathrm{sign}(L_2), \mathrm{sign}(L_3), \mathrm{sign}(L_4))$$

where the linear forms L_i define the planes meeting at p. As L_i are dependent real linear forms, exactly one pair of sign patterns is inconsistent and defines an empty set. Each cell has an *opposite cell*, obtained by reversing all signs.

If we blow up the point p, the exceptional divisor is a copy of $\mathbb{P}^2$, on which we find four lines in general position, corresponding to the four planes through p; these four lines decompose the real projective plane into seven regions, which can be coloured into three 'black' and four 'white' regions.



On the double cover we find an exceptional divisor E that is a double cover of this $\mathbb{P}^2$ ramified along these four lines, and each of the seven regions R determines a 2-cycle $\mathbf{R}$ in E.

These regions are in one-to-one correspondence with the pairs of opposite cells. If C is a cell corresponding to a region R, and $\mathbf{C}$ is the chain on the blow-up, then the boundary of this chain is precisely the 2-cycle corresponding to R:

$$\partial\mathbf{C} = \pm\mathbf{R}.$$

What we learn from this is that we can cancel this boundary term of a cell by adding to it the boundary term of the opposite cell!

Hence, we can define a group PC^3 of *polyhedral cycles* consisting of elements

$$\sum_C n_C\mathbf{C}, \qquad n_C \in \mathbb{Z},$$

for which for each cell C and each fourfold point $p \in \bar{C}$ of type p_4^0 one has

$$n_C = n_{C^{\mathrm{opp}}},$$

where C^{opp} is the cell opposite to C.

3. Determination of $H_3(\widehat{X}, \mathbb{Z})$. Denote by X_t a smoothing of $\widehat{X}$. By [2], the deformations of $\widetilde{X}$ correspond to deformations of the arrangements of eight planes that preserve the incidences between the planes in D. The deformations of the arrangement that preserves all the incidences except for the fourfold points correspond to *smoothings* of $\widehat{X}$.

By the work of J. Werner [14, Ch. II], the nodal variety $\widehat{X}$ is homotopy equivalent to its small resolution $\widetilde{X}$ with 3-cells glued along the exceptional lines (which are topological 2-spheres). The nodal variety $\widehat{X}$ is also homotopy equivalent to its smoothing with 4-cells glued along the vanishing 3-cycles. As a consequence, one arrives at the following equations relating topological invariants of X_t, $\widetilde{X}$ and $\widehat{X}$:

$$\begin{aligned}
&b_4(X_t) + 2p_4^0 + b_3(\widehat{X}) = b_4(\widehat{X}) + b_3(X_t),\\
&b_2(X_t) = b_2(\widehat{X}),\\
&b_3(\widetilde{X}) + 2p_4^0 + b_2(\widehat{X}) = b_3(\widehat{X}) + b_2(\widetilde{X}),\\
&b_4(\widetilde{X}) = b_4(\widehat{X}),
\end{aligned}$$

and hence

$$b_3(\widehat{X}) = b_3(\widetilde{X}) + b_2(X_t) - b_2(\widetilde{X}) + 2p_4^0 = b_4(\widetilde{X}) + b_3(X_t) - b_4(X_t) - 2p_4^0$$

where p_4^0 is the number of (smoothed) fourfold points in D that do not lie on a triple line.

For the eleven rigid double octics from [11] we get

No.	$b_3(\widehat{X})$	$b_3(X_t)$	p_4^0
1	3	4	1
3	5	8	3
19	6	10	4
32	7	12	5
69	7	12	5
93	8	14	6
238	11	20	12
239	11	20	10
240	11	20	10
241	11	20	10
245	11	20	9

4. An example. In order to find two independent cycles, we draw projections of intersections of all arrangement planes onto the (x, y)-plane, and consider the equations of planes not perpendicular to it as functions of z. The easiest case is arrangement no. 1, which has a single p_4^0 point. We will go through some details of this example.

The equation of this arrangement is

$$xyzt(x + y)(y + z)(z + t)(x + t) = 0$$

and the only p_4^0 point is $(1 : -1 : 1 : -1)$. The affine change of variables

$$t \mapsto t - x$$

maps this point to the plane at infinity. The arrangement is then given in affine coordinates by the equation

$$xyz(1 - x)(x + y)(y + z)(-x + z + 1) = 0,$$

while the p_4^0 point is the point at infinity $(1 : -1 : 1 : 0)$. The planes defined by the first, second, fourth and fifth factors of the above product are perpendicular to the (x, y)-plane and intersect this plane in the lines $x = 0$, $y = 0$, $x = 1$ and $x + y = 0$. The planes in the arrangement that are not perpendicular to the (x, y)-plane can be seen as graphs over the (x, y)-plane and are given by

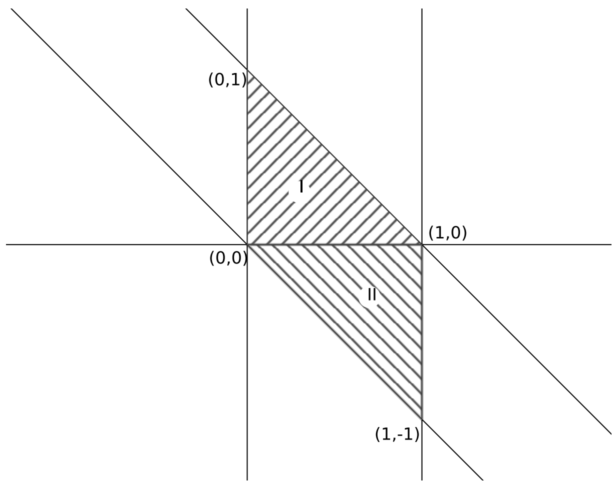
$$
\begin{aligned}
z &= f_3(x, y) := 0, \\
z &= f_6(x, y) := -y, \\
z &= f_7(x, y) := x - 1.
\end{aligned}
$$

The projections of the lines of intersection of these planes are given by

$$
\begin{aligned}
f_3 = f_6 : &\quad y = 0, \\
f_3 = f_7 : &\quad x = 1, \\
f_6 = f_7 : &\quad x + y = 1.
\end{aligned}
$$

In the (x, y)-plane we have two bounded domains

$$
\begin{aligned}
\mathrm{I} : &\quad x > 0,\ y > 0,\ x + y < 1, \\
\mathrm{II} : &\quad x < 1,\ y < 0,\ x + y > 0.
\end{aligned}
$$



For points (x, y) in these regions, the functions f_3, f_6, f_7 satisfy the following inequalities:

$$
\begin{aligned}
\mathrm{I} : &\quad f_7 < f_6 < f_3, \\
\mathrm{II} : &\quad f_7 < f_3 < f_6.
\end{aligned}
$$

Consequently, the domains lying over triangle I are given by

$$
\begin{aligned}
&x > 0,\ y > 0,\ x + y < 1,\ z > x - 1,\ z < -y, \\
&x > 0,\ y > 0,\ x + y < 1,\ z > -y,\ z < 0.
\end{aligned}
$$

As the “right” (horizontal) edge of triangle II is the projection of the intersection of planes no. 3 and 7, the only cycle lying over that triangle is given by

$$x < 1,\ y < 0,\ x + y > 0,\ z > x - 1,\ z < 0.$$

The other domain

$$x < 1,\ y < 0,\ x + y > 0,\ z > 0,\ z < -y$$

is not bounded by arrangement planes; if we want to use it we would have to add the unbounded domain “across the edge”

$$x > 1,\ x + y > 0,\ x + y < 1.$$

Instead we can choose a domain over triangle II:

$$x > 0,\ y < 0,\ x + y > 0,\ z > x - 1,\ z < 0.$$

5. Period integrals. When we are given a degree eight polynomial $F(x, y, z, t)$, the double octic $X \subset \mathbb{P}^4(1, 1, 1, 1, 4)$ defined by the equation

$$u^2 - F(x, y, z, t) = 0$$

comes with a preferred section $\omega \in \Gamma(X, \omega_X)$ of its sheaf of dualising differentials. In the affine chart $t \neq 0$ it can be written as

$$\omega := \frac{dxdydz}{u} = \frac{dxdydz}{\sqrt{F}}.$$

The period integrals of X are thus of the form

$$\int_{\gamma} \omega = \int_{\gamma} \frac{dx\,dy\,dz}{\sqrt{F}}$$

where γ is a 3-cycle in X.

If in particular F defines a real arrangement of eight planes and we have a bounded cell C in $\mathbb{R}^3$ yielding a 3-cycle $\widetilde{\mathbf{C}}$ in the Calabi–Yau threefold $\widetilde{X}$, the period integral

$$\int_{\widetilde{\mathbf{C}}} \omega$$

is just equal to the triple integral

$$2 \iiint_C \frac{dx\,dy\,dz}{\sqrt{F}}.$$

In the case of arrangement no. 1 considered in Section 4, the two period integrals are given by

$$\int_0^1\int_0^{1-x}\int_{x-1}^{-y} \frac{1}{\sqrt{xyz(1-x)(x+y)(y+z)(-x+z+1)}}\,dz\,dy\,dx,$$
$$\int_0^1\int_{-x}^{0}\int_{x-1}^{0} \frac{1}{\sqrt{xyz(1-x)(x+y)(y+z)(-x+z+1)}}\,dz\,dy\,dx.$$

To compute such integrals numerically, we used Maple. However, the function F can have zeros of multiplicity 5 at a vertex of a polyhedron of integration and thus the integrand is unbounded. As a result, a direct numerical integration usually does not yield a satisfactory precision in reasonable time. We used the following *simple trick* which allows us to get 12-digit precision without much effort, which is sufficient for our purposes. Using an affine coordinate change, we reduce computations to the case of integration over a cube $0 \le x, y, z \le 1$, with the function F vanishing only for $xyz = 0$. Then substituting $(x, y, z) \mapsto (x^k, y^k, z^k)$ in the triple integral transforms the integral to integration of a *bounded* function.

Note that depending on the sign of F in a given polyhedral cell C, we get either a real or a purely imaginary number. The computation time in the latter case can be reduced considerably by just using the function $-F$! It should be noted that if we multiply F by a constant factor λ, the corresponding period integral changes by a factor of $\sqrt{\lambda}$. In particular, if we change the sign of F, the real and imaginary periods are interchanged.

No.	Real integrals	Imaginary integrals
1	55.9805041334, 111.961008267	$69.3694986501i$
3	80.3028893419, 160.60577868	$41.4134587444i$, $82.8269174889i$,
		$124.240376233i$, $289.89421121i$
19	72.1085316451, 144.217063291,	$72.1085316451i$, $144.217063291i$,
	216.325594935	$216.325594935i$
32	55.9805041335, 111.961008267	$34.6847493250i$, $69.3694986501i$,
		$138.738997300i$, $208.1084959i$
69	55.9805041335, 111.9610083,	$34.6847493252i$, $138.738997300i$,
	223.922016533	$277.4779945i$
93	55.9805041334	$17.3423746625i$, $69.3694986502i$,
		$138.738997300i$
238	55.9805041334, 111.961008267	$34.6847493250i$
239	48.5252148713, 145.575644614	$35.2275632784i$, $105.682689835i$
240	43.7468074540, 131.240422363,	$28.8234453872i$, $57.6468907743i$,
241	223.922016533	$69.3694986503i$
245	21.8734037270, 87.4936149079,	$28.8234453872i$, $115.293781548i$
	131.240422362	

In the case of arrangement no. 1 everything works nicely, but in the other cases the picture of the decomposition of $\mathbb{P}^3$ becomes much more complicated and more fourfold points of type p_4^0 need to be taken into account. We wrote a simple MAPLE code to produce a linear-cylindric decomposition and form cycles from the polyhedral cells. Then we used several changes of variables moving each of the planes of the arrangement to infinity, which allowed us to compute the integrals for all cycles. In all cases ratios of any two real and any two complex integrals were rational numbers (with numerator and denominator ≤ 6). The table above summarises all different period integrals that appeared in our calculations.

It should be kept in mind that all period integrals get multiplied by a common factor if we change the polynomial F defining the arrangement. For these calculations we used the equations F as listed in [11] scaled by λ from the last column of the table at the end of Section 1.

For each arrangement, the computed period integrals generate a lattice in $\mathbb{C}$, which in turn defines an elliptic curve. This lattice might be a proper sublattice of $H_3(\widetilde{X}, \mathbb{Z})$, but in any case it defines an elliptic curve that is isogenous to the intermediate Jacobian $J^2(\widetilde{X})$ of the corresponding Calabi–Yau threefold. In the following table we list the ratio τ of the lattice generators, the j-invariant $j(\tau)$ and the coefficients of the classical Weierstrass equation

$$y^2 = 4x^3 - g_2 x - g_3$$

of the elliptic curve, which are easily computed numerically via

$$g_2 = 60 \sum_{0 \neq m \in \Lambda} \frac{1}{m^4}, \qquad g_6 = 140 \sum_{0 \neq m \in \Lambda} \frac{1}{m^6}.$$

This is a standard functionality in MAGMA.

No.	τ/i	$j(\tau)$	g_2	g_3
1	1.23917245341	3236.13720434	142.879810750	224.378572683
3	0.515715674539	196267.167917	1838.35630102	−15102.274126
19	1	1728	189.072720130	0
32	0.619586226703	26112.0318779	889.658497527	−4934.98162416
69	0.619586226703	26112.0318779	889.658497527	−4934.98162416
93	0.309793113352	643142260.966	14101.0467615	−322251.215146
238	0.619586226704	26112.0318791	889.658497527	−4934.98162416
239	0.725964086338	6517.46790207	487.190579154	−1774.06947556
240	0.658869688205	14612.0507801	701.139041736	−3355.01890381
241	0.309793113354	643142256.756	14101.0467615	−322251.215146
245	1.31773937641	4737.95402281	137.802991416	248.136467781

6. Comparison with modular periods. Recall that for a Hecke eigenform $f \in S_k(\Gamma_0(N))$ with q-expansion

$$f = \sum_{n=1}^{\infty} a_n q^n$$

the L-function is defined by the series

$$L(f,s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

It converges for $\mathrm{Re}(s) > 1 + k/2$, and the completed L-function

$$\Lambda(f,s) := (\sqrt{N}/2\pi)^s \Gamma(s) L(f,s)$$

satisfies the functional equation

$$\Lambda(f,s) = w i^k \Lambda(f, k-s)$$

where w is the sign of f under the Atkin–Lehner involution. We note that

$$\Lambda(f,s) = (\sqrt{N})^s \int_0^{\infty} f(it) t^s \,\frac{dt}{t}.$$

In all cases we consider, $k = 4$ and $w = 1$ as $L(f,2) \neq 0$, so that the functional equation just reads

$$\Lambda(f,s) = \Lambda(f, 4-s).$$

This means in particular that

$$\Lambda(f,1) = \Lambda(f,3),$$

from which we get the equality

$$L(f,3) = \frac{(2\pi)^2}{N} \frac{\Gamma(1)}{\Gamma(3)} L(f,1) = \frac{2\pi^2}{N} L(f,1).$$

Furthermore, we see from the functional equation that $L(f,k) = 0$ for $k = 0, -1, -2, \ldots$.

By direct point counting (and correcting for the singularities of course), C. Meyer was able to determine cusp forms

$$f = \sum_{n=1}^{\infty} a_n q^n \in S_4(\Gamma_0(N))$$

such that

$$a_p = \mathrm{Tr}(\mathrm{Fr}_p : H^3(X) \to H^3(X)).$$

In other words, one has

$$L(H^3(X), s) = L(f,s).$$

The result is summarised in the following table (we multiplied the equation of the octic arrangement by λ to obtain modular forms of minimal level).

Form	q-expansion	Arrangements
6/1	$q - 2q^2 - 3q^3 + 4q^4 + 6q^5 + 6q^6 - 16q^7 + O(q^8)$	240, 245
8/1	$q - 4q^3 - 2q^5 + 24q^7 - 11q^9 - 44q^{11} + O(q^{12})$	1, 32, 69, 93, 238, 241
12/1	$q + 3q^3 - 18q^5 + 8q^7 + 9q^9 + 36q^{11} + O(q^{12})$	239
32/1	$q + 22q^5 - 27q^9 + O(q^{12})$	19
32/2	$q + 8q^3 - 10q^5 + 16q^7 + 37q^9 - 40q^{11} + O(q^{12})$	3

It is remarkable that only five different modular forms appear.

In cases where two varieties X, X' have the same modular form, one expects there exists a correspondence ϕ between X and X' that explains it. In almost all cases such a correspondence was found; a notable exception is no. 93, for which no correspondence to any other double octic of level eight is known ([1]).

It is gratifying to see that the numerical evaluation of the period integrals leads to the very same grouping of our examples.

Here we summarise the calculations of the critical L-values.

f	$L(f,1)$	$L(f,2)$
6/1	0.22162391559067350824671004425	0.50971042336159397988737819140
8/1	0.35450068373096471876555989149	0.69003116312339752511910542021
12/1	0.61457902590673022954002802969	0.93444013814191444281042898230
32/1	1.82653044425089816105284840591	1.43455365630418076432004680798
32/2	2.03409594950627923591429024672	1.64778916742512594127684239683

If we compare the real and imaginary periods of the double octics with the special L values, we get, at least at the numerical level, nice proportionalities with

$$\pi L(f,2), \qquad \pi^2 L(f,1)$$

for the corresponding modular form.

	Form 6/1	
240	$43.7468074540\ldots = 20\pi^2 L(f,1)$	$28.8234453871\ldots = 18\pi L(f,2)$
245	$21.8734037270\ldots = 10\pi^2 L(f,1)$	$28.8234453871\ldots = 18\pi L(f,2)$
	Form 8/1	
1	$55.9805041334\ldots = 16\pi^2 L(f,1)$	$69.3694986501\ldots = 32\pi L(f,2)$
32	$55.9805041334\ldots = 16\pi^2 L(f,1)$	$34.6847493250\ldots = 16\pi L(f,2)$
69	$55.9805041334\ldots = 16\pi^2 L(f,1)$	$34.6847493250\ldots = 16\pi L(f,2)$
93	$55.9805041334\ldots = 16\pi^2 L(f,1)$	$17.3423746625\ldots = 8\pi L(f,2)$
238	$55.9805041334\ldots = 16\pi^2 L(f,1)$	$34.6847493250\ldots = 16\pi L(f,2)$
	Form 12/1	
239	$48.5252148713\ldots = 8\pi^2 L(f,1)$	$35.2275632785\ldots = 12\pi L(f,2)$

Form 32/1		
19	$72.1085316452\ldots = 4\pi^2 L(f,1)$	$72.1085316452\ldots = 16\pi L(f,2)$
Form 32/2		
3	$80.3028893419\ldots = 4\pi^2 L(f,1)$	$41.4134587443\ldots = 8\pi L(f,2)$

7. Outlook. The above calculations show that it is possible to verify numerically the relation between the periods of a rigid Calabi–Yau and the corresponding L-values of the attached modular form. However, one would like to push these calculations to a higher level. One important problem that was left untouched by our calculations is the complete determination of the group of 3-cycles in terms of polyhedral cycles. We identified some polyhedral cycles, but there is no guarantee that these generate the whole third homology group $H_3(\widetilde{X},\mathbb{Z})$. It follows from Poincaré duality that this group is generated by any two cycles with intersection ± 1. This leads to the question of how to determine the intersection number $\langle \delta, \gamma \rangle$ between two polyhedral cycles δ, γ in purely combinatorial terms of the cells appearing in δ and γ, and their mutual position inside the arrangement.

Apart from the eleven double octics with rational coefficients, there are two others with coefficients in $\mathbb{Q}(\sqrt{-3})$ and one in $\mathbb{Q}(\sqrt{5})$ (cf. [3, 4]). However, in these cases the method presented has not given any reasonable approximation of the period integrals yet.

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