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The extremal function for the complex ball for generalized notions of degree and multivariate polynomial approximation

T. Bloom (Toronto), L. Bos (Verona),
N. Levenberg (Bloomington, IN), S. Ma'u (Auckland)
and F. Piazzon (Padova)

Dedicated to the memory of Professor Józef Siciak

Abstract. We discuss the Siciak–Zakharyuta extremal function of pluripotential theory for the unit ball in $\mathbb{C}^d$ for spaces of polynomials with the notion of degree determined by a convex body P. We then use it to analyze the approximation properties of such polynomial spaces, and how these may differ depending on the function f to be approximated.

1. Introduction. The classical Bernstein–Walsh theorem relates the order of approximation of an analytic function in terms of its analyticity inside of level sets of the Siciak–Zakharyuta extremal function.

Theorem 1.1. *Let $K \subset \mathbb{C}^d$ be compact with V_K continuous. Let $R > 1$, and let $\Omega_R := \{z : V_K(z) < \log R\}$. Let f be continuous on K. Then*

$$\limsup_{n\to\infty} D_n(f,K)^{1/n} \leq 1/R$$

if and only if f is the restriction to K of a function holomorphic in Ω_R.

Here for $K \subset \mathbb{C}^d$ compact,

$$V_K(z) = \max\left[0, \sup\left\{\frac{1}{\deg(p)}\log|p(z)| : \|p\|_K := \max_{\zeta\in K}|p(\zeta)| \leq 1\right\}\right], \tag{1.1}$$

where p is a nonconstant holomorphic polynomial, is the Siciak–Zakharyuta extremal function for K, and for a continuous complex-valued function f on K,

$$D_n(f,K) := \inf\{\|f-p_n\|_K : p_n \in \mathcal{P}_n\}$$

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is the error in best uniform approximation to f on K by polynomials of degree at most n. We write $\mathcal{P}_n$ for the space of holomorphic polynomials of degree at most n.

Recently Trefethen [Tre17] has argued that polynomial approximation on the hypercube $K = [-1,1]^d \subset \mathbb{R}^d$ by the space of polynomials of what he refers to as *euclidean degree* at most n can be quite advantageous. By this is meant the space of polynomials

$$\Big\{p \in \mathbb{R}[x] : p(x) = \sum_{|\alpha|_2 \leq n} a_\alpha x^\alpha,\ x \in \mathbb{R}^d,\ a_\alpha \in \mathbb{R}\Big\}$$

where for the multi-index $\alpha \in \mathbb{Z}_+^d$, $|\alpha|_2 := \sqrt{\sum_{i=1}^d \alpha_i^2}$ is the usual euclidean norm of α.

Generalizations of the notion of the degree of a polynomial and the associated extremal functions have been given by Bayraktar [Bay17]. Indeed, given a convex body $P \subset (\mathbb{R}^+)^d = [0,\infty)^d$ we may define a *P-extremal function $V_{P,K}$* associated to K. Specifically, we suppose that $P \subset (\mathbb{R}^+)^d$ is a compact convex set in $(\mathbb{R}^+)^d$ with nonempty interior P^o. We also require that $P \subset (\mathbb{R}^+)^d$ has the property that

$$\Sigma \subset kP \quad \text{for some } k \in \mathbb{Z}^+ \tag{1.2}$$

where

$$\Sigma := \{(x_1,\dots,x_d) \in \mathbb{R}^d : x_1,\dots,x_d \geq 0,\ x_1 + \cdots + x_d \leq 1\}$$

is the standard (unit) simplex.

Associated with P, following [Bay17], we consider the finite-dimensional polynomial spaces

$$\mathrm{Poly}(nP) := \Big\{p(z) = \sum_{J \in nP \cap (\mathbb{Z}^+)^d} c_J z^J : c_J \in \mathbb{C}\Big\}$$

for $n = 1,2,\dots$. Here $J = (j_1,\dots,j_d)$. For $P = \Sigma$ we have $\mathrm{Poly}(n\Sigma) = \mathcal{P}_n$, the usual space of holomorphic polynomials of degree at most n in $\mathbb{C}^d$.

A more general class of examples is given by $P_q := \{(x_1,\dots,x_d) \in (\mathbb{R}^+)^d : (x_1^q + \cdots + x_d^q)^{1/q} \leq 1\}$, the (nonnegative) portion of the ℓ^q ball in $(\mathbb{R}^+)^d$, $1 \leq q \leq \infty$.

Note that $P_1 = \Sigma$ and hence $\mathrm{Poly}(nP_1) = \mathcal{P}_n$ while P_2 is that part of the euclidean ball in the positive "octant" and so $\mathrm{Poly}(nP_2)$ corresponds to the space of polynomials of "euclidean degree" at most n considered by Trefethen.

Clearly there exists a minimal positive integer $A = A(P) \geq 1$ such that $P \subset A\Sigma$. Thus

$$\mathrm{Poly}(nP) \subset \mathrm{Poly}(An\Sigma) = \mathcal{P}_{An} \quad \text{for all } n. \tag{1.3}$$

We let $d_n := \dim(\text{Poly}(nP))$ and note that by (1.3), $d_n = O(n^d)$. It follows from convexity of P that

$$p_n \in \text{Poly}(nP),\ p_m \in \text{Poly}(mP) \implies p_n \cdot p_m \in \text{Poly}((n+m)P).$$

Now, recall that the *indicator function* of a convex body P is

$$\phi_P(x_1, \ldots, x_d) := \sup_{(y_1,\ldots,y_d)\in P} (x_1 y_1 + \cdots + x_d y_d).$$

For the P we consider, $\phi_P \geq 0$ on $(\mathbb{R}^+)^d$ with $\phi_P(0) = 0$. Define the *logarithmic indicator function*

$$H_P(z) := \sup_{J\in P} \log|z^J| := \phi_P(\log|z_1|, \ldots, \log|z_d|).$$

Here $|z^J| := |z_1|^{j_1} \cdots |z_d|^{j_d}$ for $J = (j_1, \ldots, j_d) \in P$ (the components j_k need not be integers). From (1.2), we have

$$H_P(z) \geq \frac{1}{k} \max_{j=1,\ldots,d} \log^+|z_j| = \frac{1}{k} \max_{j=1,\ldots,d} [\max(0, \log|z_j|)].$$

We use H_P to define generalizations of the Lelong classes $L(\mathbb{C}^d)$, the set of all plurisubharmonic (psh) functions u on $\mathbb{C}^d$ with the property that $u(z) - \log|z| = O(1)$ as $|z| \to \infty$, and

$$L^+(\mathbb{C}^d) = \{u \in L(\mathbb{C}^d) : u(z) \geq \log^+|z| + C_u\}$$

where C_u is a constant depending on u. Here

$$|z| := \|z\|_2 = (|z_1|^2 + \cdots + |z_d|^2)^{1/2}.$$

We remark that, a priori, for a set $E \subset \mathbb{C}^d$, one defines the *global extremal function*

$$V_E(z) := \sup\{u(z) : u \in L(\mathbb{C}^d),\ u \leq 0 \text{ on } E\}.$$

It is a theorem, due to Siciak and to Zakharyuta (see [Kli93, Theorem 5.1.7]), that for $K \subset \mathbb{C}^d$ compact, V_K coincides with the function in (1.1). Moreover,

$$V_K^* \in L^+(\mathbb{C}^d), \quad \text{where} \quad V_K^*(z) := \limsup_{\zeta\to z} V_K(\zeta),$$

precisely when K is nonpluripolar. Define

$$L_P = L_P(\mathbb{C}^d) := \{u \in \text{PSH}(\mathbb{C}^d) : u(z) - H_P(z) = O(1),\ |z| \to \infty\},$$
$$L_{P,+} = L_{P,+}(\mathbb{C}^d) := \{u \in L_P(\mathbb{C}^d) : u(z) \geq H_P(z) + C_u\}.$$

Then $L_\Sigma = L(\mathbb{C}^d)$ and $L_{\Sigma,+} = L^+(\mathbb{C}^d)$. Given $E \subset \mathbb{C}^d$, the *P-extremal function of E* is given by $V_{P,E}^*(z) := \limsup_{\zeta\to z} V_{P,E}(\zeta)$ where

$$V_{P,E}(z) := \sup\{u(z) : u \in L_P(\mathbb{C}^d),\ u \leq 0 \text{ on } E\}.$$

For $P = \Sigma$, we recover $V_E = V_{\Sigma,E}$. We will restrict to the case where $E = K \subset \mathbb{C}^d$ is compact. In this case, Bayraktar [Bay17] proved a Siciak–

Zakharyuta type theorem showing that $V_{P,K}$ can be obtained using polynomials. Note that $\frac{1}{n}\log|p_n| \in L_P$ for $p_n \in \mathrm{Poly}(nP)$.

PROPOSITION 1.2 ([Bay17]). *Let $K \subset \mathbb{C}^d$ be compact and nonpluripolar. Then*

$$V_{P,K} = \lim_{n\to\infty} \frac{1}{n} \log \Phi_n$$

pointwise on $\mathbb{C}^d$ where

$$\Phi_n(z) := \sup\{|p_n(z)| : p_n \in \mathrm{Poly}(nP),\ \|p_n\|_K \le 1\}.$$

If $V_{P,K}$ is continuous, the convergence is locally uniform on $\mathbb{C}^d$.

Note that $V_{P,K} = 0$ on the polynomial hull $\hat{K}$ of K (and only there). Also, if V_K is continuous, so is $V_{P,K}$ (see [BL17, discussion after Proposition 2.3]).

The degree of approximation of analytic functions by polynomials in $\mathrm{Poly}(nP)$ is given by a generalization of the Bernstein–Walsh theorem, proved in [BL17]. Set

$$D_n = D_n(f, K, P) := \inf\{\|f - p_n\|_K : p_n \in \mathrm{Poly}(nP)\}.$$

THEOREM 1.3 ([BL17]). *Let K be compact and assume $V_{P,K}$ is continuous. Let $R > 1$, and let $\Omega_R = \Omega_{R(P,K)} := \{z : V_{P,K}(z) < \log R\}$. Let f be continuous on K. Then f is the restriction to K of a function holomorphic in $\Omega_{R(P,K)}$ if and only if*

$$\limsup_{n\to\infty} D_n(f, P, K)^{1/n} \le 1/R.$$

[BL17] also gives a formula for the P-extremal function of a product set. We make the following definition: we call a convex body $P \subset (\mathbb{R}^+)^d$ a *lower set* if for each $n = 1, 2, \ldots$, whenever $(j_1, \ldots, j_d) \in nP \cap (\mathbb{Z}^+)^d$ we have $(k_1, \ldots, k_d) \in nP \cap (\mathbb{Z}^+)^d$ for all $k_l \le j_l$, $l = 1, \ldots, d$.

PROPOSITION 1.4 ([BL17]). *Let $P \subset (\mathbb{R}^+)^d$ be a lower set and $E_1, \ldots, E_d \subset \mathbb{C}$ be compact and nonpolar. Then*

$$V^*_{P,E_1\times\cdots\times E_d}(z_1, \ldots, z_d) = \phi_P(V^*_{E_1}(z_1), \ldots, V^*_{E_d}(z_d)). \tag{1.4}$$

Bos and Levenberg use this formula to explain the (sometimes) advantageous approximation properties of polynomial spaces of euclidean degree at most n discovered by Trefethen.

In this work we discuss the case of $K = B_2 := \{z \in \mathbb{C}^d : \|z\|_2 \le 1\}$, the complex unit ball in $\mathbb{C}^d$, as an example of a nonproduct set. Relying on the approach discussed in the next section, we get an explicit formula for V_{P_∞,B_2} in Proposition 3.9. We analyze the approximation properties on $K = B_2$ of polynomial spaces $\mathrm{Poly}(nP_q)$ as in Theorem 1.3 and see how these may differ depending on the function f in Section 3. In Section 4 we compute the Monge–Ampère measure $(dd^c V^*_{P_\infty,B_2})^2 = \mu_{P_\infty,B_2}$ (Proposition 4.2) and give a probabilistic application following [Bay17].

The genesis of this work took place at the Dolomites Research Week in Approximation, September 4–8, 2017.

2. Computing extremal functions. To compute extremal functions, in particular V_{P,B_2} for various P, we will generalize the approach of Bloom [Blo97], for which we will require a generalized version of a theorem of Zeriahi [Zer85] (see also [Blo97, Theorem 3.2]) that allows one to compute the extremal function by means of orthogonal polynomials. Hence consider a compact set $K \subset \mathbb{C}^d$ and let μ be a finite Borel measure supported on K satisfying a Bernstein–Markov inequality, i.e., for every $\epsilon > 0$ there exists a constant $C(\epsilon) > 0$ such for all $p \in \mathbb{C}[z]$,

$$\|p\|_K \le C(\epsilon)(1+\epsilon)^{\deg(p)}\|p\|_{L^2(\mu)}. \tag{2.1}$$

Here $\deg(p)$ denotes the usual degree of p. Such measures always exist [BL13]. However, for the convex body P we may introduce

DEFINITION 2.1. For $p \in \mathbb{C}[z]$ ($z \in \mathbb{C}^d$), we set

$$\deg_P(p) := \inf\{n \in \mathbb{Z}^+ : p \in \mathrm{Poly}(nP)\}$$

and for $\alpha \in (\mathbb{Z}^+)^d$,

$$|\alpha|_P := \inf\{t \in \mathbb{R}^+ : z^\alpha \in \mathrm{Poly}(tP)\}.$$

For general $\theta \in (\mathbb{R}^+)^d$ we use the same notation to denote

$$|\theta|_P := \inf\{t > 0 : \theta \in tP\},$$

which need not be an integer.

We note that

$$\deg_P(z^\alpha) - 1 \le |\alpha|_P \le \deg_P(z^\alpha). \tag{2.2}$$

We remark that by our assumption (1.2) on P we may equivalently replace the classical degree ($\deg(p) = \deg_\Sigma(p)$) in (2.1) by $\deg_P(p)$.

To define the orthogonal polynomials we impose an ordering on the multinomial indices $\alpha \in (\mathbb{Z}^+)^d$, which is consistent with the degree, i.e.,

$$\alpha \le \beta \implies \deg_P(z^\alpha) \le \deg_P(z^\beta).$$

We then let

$$\{p_\alpha(z) = p_\alpha(z,\mu) : \alpha \in (\mathbb{Z}^+)^d\}$$

be the family of orthonormal polynomials obtained by the Gram–Schmidt process with inner product given by μ applied to the monomials $\{z^\alpha : \alpha \in (\mathbb{Z}^+)^d\}$ so ordered.

THEOREM 2.2 (Generalized Zeriahi [Zer85]). *Under the above assumptions,*

$$V_{P,K}(z) = \limsup_{\alpha} \frac{1}{|\alpha|_P} \log|p_\alpha(z)| \quad \text{for } z \in \mathbb{C}^d \setminus \hat{K}$$

where $\hat{K}$ denotes the polynomial hull of K.

Proof. The argument is a straightforward generalization of that of Zeriahi. We give the details for the sake of completeness.

First note that by our assumption that μ satisfies a Bernstein–Markov inequality (2.1),

$$\limsup_{\alpha} \frac{1}{\deg_P(p_\alpha)} \log|p_\alpha(z)| \le V_{P,K}(z), \qquad z \in \mathbb{C}^d \setminus \widehat{K}.$$

Then, as $\deg_P(p_\alpha) = \deg_P(z^\alpha)$, from (2.2) we also have

$$\limsup_{\alpha} \frac{1}{|\alpha|_P} \log|p_\alpha(z)| \le V_{P,K}(z), \qquad z \in \mathbb{C}^d \setminus \widehat{K}.$$

To show the reverse inequality, first recall that by Proposition 1.2,

$$V_{P,K}(z) = \lim_{n\to\infty}\left(\sup\left\{\frac{1}{n}\log|p(z)| : p \in \mathrm{Poly}(nP) \text{ and } \|p\|_K \le 1\right\}\right). \tag{2.3}$$

Now, let $q \in \mathrm{Poly}(nP)$ be such that $\|q\|_K \le 1$. We expand q in its orthogonal series with respect to the basis $\{p_\alpha : \deg_P(p_\alpha) \le n\}$, i.e.,

$$q(z) = \sum_{\alpha\in nP} c_\alpha p_\alpha(z) \quad \text{where} \quad c_\alpha = \int_K q(z)\overline{p_\alpha(z)}\,d\mu(z).$$

Since $\|q\|_K \le 1$ we have

$$|c_\alpha| \le \int_K |p_\alpha(z)|\,d\mu(z) \le \sqrt{\mu(K)}$$

by the Cauchy–Schwarz inequality. Thus

$$\begin{aligned} |q(z)| &\le \dim(\mathrm{Poly}(nP))\sqrt{\mu(K)}\max_{\alpha\in nP}|p_\alpha(z)| \\ &= d_n\sqrt{\mu(K)}\max_{\alpha\in nP}|p_\alpha(z)|. \end{aligned} \tag{2.4}$$

Now fix a $z_0 \in \mathbb{C}^d \setminus \widehat{K}$ so that $V_{P,K}(z_0) > 0$ and let α_n be the largest multi-index in our ordering such that

$$|p_{\alpha_n}(z_0)| = \max_{\alpha\in nP}|p_\alpha(z_0)|.$$

Since the chosen ordering respects $\deg_P$, we see that $n \le m$ implies that $\deg_P(z^{\alpha_n}) \le \deg_{\mathrm{P}}(z^{\alpha_m})$, i.e., the sequence $\{\deg_P(z^{\alpha_n})\}$ is increasing. Further, $\lim_{n\to\infty}\deg_P(z^{\alpha_n}) = \infty$, for if not, say $\deg_P(z^{\alpha_n}) \le M$ for all n, then by (2.4) for any polynomial $q(z)$ satisfying $\|q\|_K \le 1$ we have

$$|q(z_0)| \le d_M\sqrt{\mu(K)}\max_{\alpha\in MP}|p_\alpha(z_0)|,$$

so that by (2.3), $V_{P,K}(z_0) = 0$, a contradiction.

Thus we also have $\lim_{n\to\infty}|\alpha_n|_P = \infty$ and, in view of (2.3) and (2.4),

$$V_{P,K}(z_0) \le \limsup_{n\to\infty}\frac{1}{n}\log|p_{\alpha_n}(z_0)|.$$

But, as noted previously, $|\alpha_n|_P \le \deg_P(z^{\alpha_n}) \le n$ and so also

$$\limsup_{\alpha} \frac{1}{|\alpha|_P} \log |p_\alpha(z)| \ge V_{P,K}(z_0),$$

proving the result. ∎

For $K = B_2$ normalized monomials will be used in Theorem 2.2 in the next section. It is worth noting that, for slightly more general compact sets K, the monomials are also Chebyshev polynomials. Specifically, consider $K \subset \mathbb{C}^d$ compact. Let $<_l$ be the lexicographic ordering on the multi-indices $\alpha \in (\mathbb{Z}^+)^d$ given by $\alpha >_l \beta$ if $|\alpha| > |\beta|$ or if $|\alpha| = |\beta|$ and $\alpha_i = \beta_i$ for $i = 1, \dots, r$ and $\alpha_{r+1} > \beta_{r+1}$ for some r. Here, and in the rest of this section, for $\alpha = (\alpha_1, \dots, \alpha_d) \in (\mathbb{R}^+)^d$ we write $|\alpha| := \alpha_1 + \dots + \alpha_d$.

For each multi-index α we define a collection $\mathcal{Q}(\alpha)$ of polynomials as follows:

$$\mathcal{Q}(\alpha) := \Big\{ q(z) = z^\alpha + \sum_{\beta <_l \alpha} c_\beta z^\beta : c_\alpha \in \mathbb{C} \Big\}.$$

Let $b_\alpha = \inf_{q \in Q(\alpha)} \|q\|_K$, let

$$B_\Sigma = \{\theta \in (\mathbb{R}^+)^d : |\theta| = 1\},$$

and write $B_\Sigma^\circ = \{\theta \in B_\Sigma : \theta_i > 0,\ i = 1, \dots, d\}$ for its (relative) interior.

The following result is due to Zakharyuta [Zah75].

Theorem 2.3. *For $\theta \in B_\Sigma^\circ$ the limit*

$$b(\theta, K) := \lim_{\alpha/|\alpha| \to \theta} b_\alpha^{1/|\alpha|}$$

exists and $\log b(\theta, K)$ *is convex on* B_Σ°.

The number $b(\theta, K)$ is called the *directional Chebyshev constant* (with direction θ) for K.

For a Bernstein–Markov measure μ on K (see (2.1)) we set

$$h_\alpha := \inf_{q \in \mathcal{Q}(\alpha)} \|q\|_{L^2(\mu)}.$$

Then we have

Proposition 2.4. *For $\theta \in B_\Sigma$,*

$$\lim_{\alpha/|\alpha| \to \theta} b_\alpha^{1/|\alpha|} = \lim_{\alpha/|\alpha| \to \theta} h_\alpha^{1/|\alpha|}$$

in the sense that one of the limits exists if and only if the other does and in that case they are equal.

We say that a polynomial q_0 *realizes* b_α, i.e., q_0 is a Chebyshev polynomial for K of index α, if $q_0 \in \mathcal{Q}(\alpha)$ and $\|q_0\|_K = b_\alpha$ (and similarly for h_α). Now assume that K is invariant under the torus action

$$z = (z_1, \dots, z_d) \mapsto (e^{it_1} z_1, \dots, e^{it_d} z_d), \quad t_1, \dots, t_d \in \mathbb{R},$$

and that μ is also invariant under the torus action. This will be the case in the next section. Then the monomials are mutually orthogonal and any polynomial which realizes h_α is a monomial. Moreover, we have

PROPOSITION 2.5. *Let K be invariant under the torus action. For each multi-index α the monomial z^α realizes b_α, i.e., z^α is a Chebyshev polynomial for K.*

Proof. Let q_0 be a polynomial which realizes b_α. Suppose that $\alpha = (\alpha_1, \ldots, \alpha_d)$ and $\alpha_1 > 0$. Let

$$q_1(z) := \frac{1}{\alpha_1} \sum_{j=1}^{\alpha_1} q_0(e^{2\pi i j/\alpha_1} z_1, z_2, \ldots, z_d).$$

Then q_1 is homogeneous in z_1 of degree α_1, $q_1 \in \mathcal{Q}(\alpha)$, and

$$\|q_1\|_K \le \|q_0\|_K,$$

so $\|q_1\|_K = b_\alpha$.

Then repeat successively the averaging procedure for each of the remaining variables z_j for which $\alpha_j > 0$. We obtain the monomial z^α and we see that $\|z^\alpha\|_K = b_\alpha$. ■

Note that for K invariant under the torus action and $P \subset (\mathbb{R}^+)^d$ a convex body, the polynomial spaces $\mathrm{Poly}(nP)$ as well as the extremal function $V_{P,K}(z)$ are invariant under the torus action.

3. The case of K the unit ball in $\mathbb{C}^d$. Here we take

$$K = B_d := \{z \in \mathbb{C}^d : |z| \le 1\}$$

where $|z| := \|z\|_2 = (|z_1|^2 + \cdots + |z_d|^2)^{1/2}$ denotes the euclidean norm of $z \in \mathbb{C}^d$ and μ will be Lebesgue measure on K. It is well-known that (2.1) holds in this setting.

It is known (see e.g. [Rud08]) that the monomials z^α are mutually orthogonal and indeed

$$p_\alpha(z) = c_\alpha z^\alpha, \quad \alpha \in (\mathbb{Z}^+)^d, \quad \text{with} \quad c_\alpha^2 := \frac{(|\alpha| + d)!}{\alpha! \pi^d},$$

are orthonormal polynomials. Here $|\alpha| := \sum_{j=1}^d \alpha_j$ and $\alpha! := \prod_{j=1}^d (\alpha_j!)$.

Now, for $0 \ne z \in \mathbb{C}^d$ let

$$I(z) := \{i : z_i \ne 0\}.$$

Fixing a convex body $P \subset (\mathbb{R}^+)^d$, by Theorem 2.2, in this case the P-extremal function is determined by the limsup of the sequence of normalized monomials. However, by compactness, any sequence of normalized multi-indices $\alpha(j)/|\alpha(j)|_P$ has a limit point, and hence we first consider such convergent sequences.

LEMMA 3.1. *Suppose that $\{\alpha(j) \in (\mathbb{Z}^+)^d\}$ is an infinite sequence of distinct multi-indices, ordered as above, such that $i \notin I(z) \Rightarrow \alpha_i(j) = 0$, and that*

$$\lim_{j\to\infty} \frac{\alpha(j)}{|\alpha(j)|_P} = \theta \in (\mathbb{R}^+)^d.$$

Necessarily then $|\theta|_P = 1$ and $i \notin I(z) \Rightarrow \theta_i = 0$.

We have

$$\lim_{j\to\infty} \frac{1}{|\alpha(j)|_P} \log|c_{\alpha(j)} z^{\alpha(j)}| = \frac{1}{2}\Big\{\sum_{i\in I(z)} \theta_i \log(|z_i|^2) - \sum_{i\in I(z)} \theta_i \log(\theta_i) + \Big(\sum_{i\in I(z)} \theta_i\Big)\log\Big(\sum_{i\in I(z)} \theta_i\Big)\Big\}.$$

Proof. This is a straightforward calculation based on Stirling's formula, $\log(m!) = m\log(m) - m + O(\log(m))$ and the fact that, by construction, $\lim_{j\to\infty} |\alpha(j)|_P = \infty$. ∎

We define

$$F_d(\theta; z) := \frac{1}{2}\Big\{\sum_{i\in I(z)} \theta_i \log(|z_i|^2) - \sum_{i\in I(z)} \theta_i \log(\theta_i) + \Big(\sum_{i\in I(z)} \theta_i\Big)\log\Big(\sum_{i\in I(z)} \theta_i\Big)\Big\}.$$

PROPOSITION 3.2. *For $z \in \mathbb{C}^d \setminus K$,*

$$V_{P,K}(z) = \max_{\theta\in(\mathbb{R}^+)^d,\, |\theta|_P=1} F_d(\theta; z).$$

Proof. First note that restricted to any hyperplane of the form $\{w \in \mathbb{C}^d : w_i = 0\}$, the unit ball, the extremal function and, at least for points z such that $i \notin I(z)$, the functional $F_d(\theta; z)$ all reduce to the same corresponding lower-dimensional problem. Hence we may, without loss of generality, assume that $z_i \neq 0$, $1 \le i \le d$, i.e., $I(z) = \{1, \dots, d\}$, and

$$F_d(\theta; z) = \frac{1}{2}\Big\{\sum_{i=1}^d \theta_i \log(|z_i|^2) - \sum_{i=1}^d \theta_i \log(\theta_i) + \Big(\sum_{i=1}^d \theta_i\Big)\log\Big(\sum_{i=1}^d \theta_i\Big)\Big\}.$$

The proof is now straightforward as $K = B_d$ is polynomially convex, and by Theorem 2.2 we have

$$V_{P,K}(z) = \limsup_{\alpha} \frac{1}{|\alpha|_P} \log|c_\alpha z^\alpha| \quad \text{for } z \in \mathbb{C}^d \setminus K.$$

Further, every convergent subsequence of $\{\alpha/|\alpha_P|\}$ has its limit in $\{\theta \in (\mathbb{R}^+)^d : |\theta|_P = 1\}$ and every such θ is the limit of such a subsequence. If we combine this with Lemma 3.1, the result follows. ∎

For the sake of completeness we will now verify the known formula for the extremal function in the case of the classical degree, i.e., when $P = \Sigma$, the standard unit simplex.

PROPOSITION 3.3. *For $P = \Sigma$,*

(i) $$\max_{\theta\in(\mathbb{R}^+)^d,\, \sum_{i=1}^d \theta_i=1} F_d(\theta; z) = \log|z|, \qquad \forall z \in \mathbb{C}^d,\ z \neq 0,$$

(ii) $$V_K(z) := V_{\Sigma,K}(z) = \log|z|, \qquad z \notin K.$$

Proof. Formula (ii) follows immediately from (i). To show (i) we proceed by induction on the dimension. If $d = 1$, then $\theta_1 = 1$ and trivially

$$F_1(\theta; z) = \tfrac{1}{2}\{1 \times \log(|z|^2) - 0\} = \log|z|.$$

We suppose then that the result holds for up to dimension $d-1$ and must prove that it also holds for dimension d. Again, we may assume without loss that $z_i \neq 0$, $1 \le i \le d$. We maximize

$$\begin{aligned} F_d(\theta; z) &= \frac{1}{2}\Big\{\sum_{i=1}^d \theta_i \log(|z_i|^2) - \sum_{i=1}^d \theta_i \log(\theta_i) + \Big(\sum_{i=1}^d \theta_i\Big) \log\Big(\sum_{i=1}^d \theta_i\Big)\Big\} \\ &= \frac{1}{2}\Big\{\sum_{i=1}^d \theta_i \log(|z_i|^2) - \sum_{i=1}^d \theta_i \log(\theta_i)\Big\} \end{aligned}$$

over the set $\{\theta \in (\mathbb{R}^+)^d : \sum_{i=1}^d \theta_i = 1\}$.

Consider first the interior ($\theta_i > 0$ for all i) critical point(s) given by Lagrange multipliers as the solution of

$$\log(|z_i|^2) - (1 + \log(\theta_i)) = \lambda, \qquad 1 \le i \le d,$$

or equivalently

$$\begin{aligned} \log(|z_i|^2) - \log(\theta_i) = {} & \log(|z_d|^2) - \log(\theta_d),\ 1 \le i \le d \\ & \iff \log(|z_i|^2/\theta_i) = \log(|z_d|^2/\theta_d),\ 1 \le i \le d \\ & \iff |z_i|^2/\theta_i = |z_d|^2/\theta_d,\ 1 \le i \le d \\ & \iff \theta_i = \theta_d \frac{|z_i|^2}{|z_d|^2},\ 1 \le i \le d. \end{aligned}$$

Taking the sum of both sides we see that

$$1 = \sum_{i=1}^d \theta_i = \theta_d \frac{1}{|z_d|^2} \sum_{i=1}^d |z_i|^2 = \theta_d \frac{1}{|z_d|^2} |z|^2$$

so that

$$\theta_d = \frac{|z_d|^2}{|z|^2}$$

and

$$\theta_i = \theta_d \frac{|z_i|^2}{|z_d|^2} = \frac{|z_i|^2}{|z|^2}, \qquad 1 \le i \le d.$$

Substituting these values of the θ_i into the expression for F_d we obtain the critical value of

$$F_d(\theta;z) = \frac{1}{2}\left\{\sum_{i=1}^{d}\frac{|z_i|^2}{|z|^2}\log(|z_i|^2) - \sum_{i=1}^{d}\frac{|z_i|^2}{|z|^2}\log\left(\frac{|z_i|^2}{|z|^2}\right)\right\} = \log|z|, \tag{3.1}$$

after simplification.

The other competitors for the maximum are on the boundary of our constraint set $\{\theta \in (\mathbb{R}^+)^d : \sum_{i=1}^d \theta_i = 1\}$, i.e., when one or more of the θ_i are zero. But in this case we reduce to a lower-dimensional version of the same problem, and by our induction assumption the maximum of $F_d(\theta;z)$ is then

$$\frac{1}{2}\log\Big(\sum_{i,\,\theta_i\neq 0}|z_i|^2\Big)$$

which is less than the value at the interior critical point. Hence the maximum is indeed $\log|z|$ and we are done. ∎

We next collect some basic facts about the function F_d for a general P.

Proposition 3.4. *Assume again that $z_i \neq 0$, $1 \le i \le d$. Then*

(i) *$F_d(\theta;z)$ is homogeneous of order one in θ so that*

$$F_d(\theta;z) = \sum_{i=1}^{d}\theta_i\frac{\partial}{\partial\theta_i}F_d(\theta;z);$$

(ii) $\nabla F_d(\theta;z) := \big(\frac{\partial F_d}{\partial\theta_1},\ldots,\frac{\partial F_d}{\partial\theta_d}\big) \neq 0 \in \mathbb{R}^d$ *for $z \notin K$;*

(iii) *at any interior point, $\theta_i > 0$, $1 \le i \le d$, the Hessian of $F_d(\theta;z)$ is non-positive definite;*

(iv) *if $|z| < 1$ then $F_d(\theta;z) < 0$;*

(v) *if $|z| = 1$ then $\max_{\theta\in(\mathbb{R}^+)^d,\,|\theta|_P=1} F_d(\theta;z) = 0$;*

(vi) *if $|z| > 1$ then $\max_{\theta\in(\mathbb{R}^+)^d,\,|\theta|_P=1} F_d(\theta;z) > 0$.*

Proof. Item (i) is completely elementary and so we leave out the details. For (ii) we calculate

$$2\frac{\partial}{\partial\theta_i}F_d(\theta;z) = \log(|z_i|^2) - \log(\theta_i) + \log\Big(\sum_{i=1}^{d}\theta_i\Big)$$

so that $\nabla F_d(\theta;z) = 0$ iff

$$\log(|z_i|^2) = \log\Big(\theta_i/\Big(\sum_{j=1}^{d}\theta_j\Big)\Big),\ 1\le i\le d \iff |z_i|^2 = \theta_i/\Big(\sum_{j=1}^{d}\theta_j\Big),\ 1\le i\le d,$$

and hence, taking the sum, we must have

$$\sum_{i=1}^{d}|z_i|^2 = \Big(\sum_{i=1}^{d}\theta_i\Big)/\Big(\sum_{j=1}^{d}\theta_j\Big) = 1.$$

To see (iii), we easily calculate

$$2\frac{\partial^2}{\partial\theta_i\partial\theta_j}F_d(\theta;z) = \begin{cases} 1/S - 1/\theta_i & \text{if } j=i, \\ 1/S & \text{if } j \neq i, \end{cases}$$

where $S := \sum_{i=1}^d \theta_i$. Hence (twice the) Hessian, H_F, say, is

$$2H_F = \frac{1}{S}uu^t - D$$

where

$$u := \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^d \quad \text{and} \quad D := \begin{bmatrix} 1/\theta_1 & 0 & \cdot & \cdot & 0 \\ 0 & 1/\theta_2 & 0 & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & 0 & 1/\theta_d \end{bmatrix} \in \mathbb{R}^{d\times d},$$

which we recognize as a rank one perturbation of the negative definite diagonal matrix $-D$. More specifically, it is easy to verify that

$$H_F v = 0 \in \mathbb{R}^d \quad \text{where} \quad v := \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_d \end{bmatrix}$$

and that for $0 \neq w \in \mathbb{R}^d$ with $w^t u = 0$,

$$w^t(2H_F)w = -w^t D w < 0,$$

so that H_F is singular and negative definite on a $(d-1)$-dimensional subspace of $\mathbb{R}^d$.

Properties (iv), (v) and (vi) can be easily verified using the homogeneity. Indeed, for any $\theta \in (\mathbb{R}^+)^d$ with $|\theta|_P = 1$, there is a $\theta' \in (\mathbb{R}^+)^d$ with $\sum_{i=1}^d \theta'_i = 1$ and $t_\theta > 0$ such that $\theta = t_\theta \theta'$ and hence

$$F_d(\theta;z) = F_d(t_\theta\theta';z) = t_\theta F_d(\theta';z),$$

and the result follows from the classical case, Proposition 3.3. ■

For brevity's sake let

$$B_P := \{\theta \in (\mathbb{R}^+)^d : |\theta|_P = 1\}$$

denote the constraint set.

LEMMA 3.5. *Suppose that B_P is smooth near its boundary. Then, if $z_i \neq 0$, $1 \leq i \leq d$, the maximum of $F_d(\theta;z)$ over B_P is never attained at a boundary point, i.e., where one or more of the θ_i are zero.*

Proof. We just note that as

$$2\frac{\partial}{\partial\theta_i}F_d(\theta;z) = \log(|z_i|^2) - \log(\theta_i) + \log\Big(\sum_{i=1}^{d}\theta_i\Big)$$

it follows that

$$\lim_{\theta_i\to 0^+}\frac{\partial}{\partial\theta_i}F_d(\theta;z) = \infty,$$

while the other partials with respect to $\theta_j \neq 0$ are finite. Hence a sufficiently small positive perturbation of $\theta_i = 0$ will result in an increase in the value of $F_d(\theta;z)$. ∎

REMARK 3.6. Note this means, e.g., that for such P we can never have $V_{P,B_2}(z_1,z_2) = V_{B_1}(z_1)$ for a point $(z_1,z_2)\in\mathbb{C}^2$ with $z_2\neq 0$.

LEMMA 3.7. *Suppose that B_P is strictly convex (i.e. $|(x+y)/2|_P < (|x|_P+|y|_P)/2$, $x\neq y$). Then if $z_i\neq 0$, $1\le i\le d$, and $z\notin K$, the maximum of $F_d(\theta;z)$ over B_P is uniquely attained.*

Proof. Suppose for the sake of a contradiction that the maximum is attained at two distinct points $\theta',\theta''\in B_P$. By Lemma 3.5 both θ',θ'' are in the interior of B_P. Now, by Proposition 3.4(iii), $F_d(\theta;z)$ is a concave function so that

$$F_d((\theta'+\theta'')/2;z) \ge (F_d(\theta';z)+F_d(\theta'';z))/2 = \max_{\theta\in B_P}F_d(\theta;z).$$

Note that, as $|(\theta'+\theta'')/2|_P < (|\theta'|_P+|\theta''|_P)/2 = 1$,

$$t := 2/|\theta'+\theta''|_P > 1.$$

Then $\theta := t(\theta'+\theta'')/2\in B_P$ and

$$\begin{aligned} F_d(\theta;z) &= F_d(t(\theta'+\theta'')/2;z) = tF_d((\theta'+\theta'')/2;z)\\ &\ge t\max_{\theta\in B_P}F_d(\theta;z) > \max_{\theta\in B_P}F_d(\theta;z), \end{aligned}$$

a contradiction. ∎

In case the P-norm is a smooth function, the maximum can be characterized by Lagrange multipliers. For simplicity let $g(\theta) := |\theta|_P$ so that $B_P = \{\theta\in(\mathbb{R}^+)^d : g(\theta)=1\}$. Then the Lagrange multiplier equations are

$$\frac{\partial}{\partial\theta_i}F_d(\theta;z) = \lambda\frac{\partial}{\partial\theta_i}g(\theta),\qquad 1\le i\le d.$$

Taking the sum of both sides we obtain by homogeneity

$$F_d(\theta;z) = \sum_{i=1}^{d}\theta_i\frac{\partial}{\partial\theta_i}F_d(\theta;z) = \lambda\sum_{i=1}^{d}\theta_i\frac{\partial}{\partial\theta_i}g(\theta).$$

But as $g(\theta)$ is a norm, it is also homogeneous of order one, and so by the Euler identity, the sum on the right-hand side reduces to $g(\theta)=1$ for $\theta\in B_P$.

In other words, the Lagrange multiplier is

$$\lambda = F_d(\theta; z). \tag{3.2}$$

3.1. The case of P the unit ℓ^q ball, $1 \le q \le \infty$. For $1 \le q < \infty$,

$$g(\theta) = |\theta|_P = \left\{\sum_{i=1}^d \theta_i^q\right\}^{1/q} =: |(\theta_1, \dots, \theta_d)|_q$$

is smooth and hence the associated extremal function may be found by solving the Lagrange multipliers equations

$$\begin{aligned}\frac{\partial}{\partial\theta_i} F_d(\theta; z) = \lambda \frac{\partial}{\partial\theta_i} g(\theta) &= F_d(\theta; z)\frac{\theta_i^{q-1}}{g(\theta)^{q-1}} \\ &= F_d(\theta; z)\theta_i^{q-1}, \qquad 1 \le i \le d,\end{aligned}$$

using (3.2). These d equations are actually dependent as the sum of both sides multiplied by θ_i reduces to the tautology $F_d(\theta; z) = F_d(\theta; z)$. Hence we solve the system

$$\begin{aligned}\frac{\partial}{\partial\theta_i} F_d(\theta; z) &= F_d(\theta; z)\theta_i^{q-1}, \qquad 1 \le i \le d-1, \\ g(\theta) &= 1.\end{aligned}$$

For $z_i \ne 0$, $1 \le i \le d$, a unique solution is guaranteed by Lemma 3.7. In the case of $d = 2$ this is particularly easy to find numerically and in Figure 1 we show several contours for $q = 1, 2, 4$. One notices immediately that on

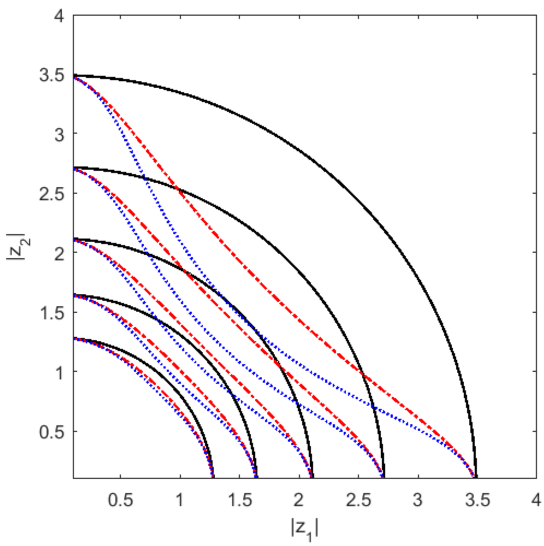

Fig. 1. Extremal function contour plots for levels $.25, .5, .75, 1, 1.25$. The solid curves correspond to $q = 1$, the dot-dashed ones to $q = 2$ and the dotted ones to $q = 4$.

the diagonal $|z_1| = |z_2|$ the extremal functions are notably different, whereas they have the same values on the complex lines $z_1 = 0$ and $z_2 = 0$ (as here they all reduce to the same univariate extremal function; cf. the proof of Proposition 3.2).

This has some interesting consequences for the approximation of functions from $\mathrm{Poly}(nP)$. Let P_q denote the intersection of the unit ℓ^q ball with $(\mathbb{R}^+)^d$.

In the following three examples K denotes the euclidean unit ball B_2 in $\mathbb{C}^2$.

Example 1. Consider

$$f_1(z_1, z_2) := \frac{1}{1 - z_1/2} + \frac{1}{1 - z_2/2}.$$

As the monomials form an orthogonal basis, best L^2 approximations are equivalent to Taylor expansions. In this case we have

$$f_1(z_1, z_2) = \sum_{k=0}^{n} \left(\frac{z_1^k}{2^k} + \frac{z_2^k}{2^k} \right) + \left\{ \frac{(z_1/2)^{n+1}}{1 - z_1/2} + \frac{(z_2/2)^{n+1}}{1 - z_2/2} \right\}$$

for $|z_1|, |z_2| < 2$, in particular on K. But note that for *any* $q \geq 1$, the degrees of z_1^k and z_2^k are both k. In particular, the best L^2 approximation for f_1 on K of degree n, for *any* $q \geq 1$, is

$$p_n(z_1, z_2) := \sum_{k=0}^{n} \left(\frac{z_1^k}{2^k} + \frac{z_2^k}{2^k} \right).$$

In other words, for L^2 approximation, there is no advantage in a higher value of q despite the fact that the spaces $\mathrm{Poly}(nP_q)$ are of increasing dimension in q.

Less obvious examples may be analyzed by means of the extremal function. Indeed, by Theorem 1.3 the order of uniform approximation to a function $f(z)$ holomorphic on a neighborhood of K by $\mathrm{Poly}(nP)$ is given (essentially) by

$$D_n(f, K, P) = O(R^{-n})$$

where

$$\log(R) := \inf_{z \in S(f)} V_{P,K}(z)$$

and $S(f) := \{z \in \mathbb{C}^d : f \text{ is not holomorphic at } z\}$ is the singular set of f.

Example 2. Consider the bivariate Runge type function

$$f_2(z_1, z_2) := \frac{1}{a^2 + z_1^2 + z_2^2}, \qquad a > 1.$$

Its singular set is given by

$$S(f_2) := \{z \in \mathbb{C}^2 : z_1^2 + z_2^2 = -a^2\}.$$

LEMMA 3.8. *We have*

$$\min_{z\in S(f_2)} V_{P_q,K}(z) = \log(a), \qquad q \geq 1,$$

attained at (*among other points*) $z_1 = ia$, $z_2 = 0$.

Proof. Consider first the classical case $q = 1$ when $P_q = \Sigma$. By Proposition 3.3 the extremal function is $\log^+ |z|$. Hence it suffices to show that

$$\min_{z_1^2+z_2^2=-a^2} (|z_1|^2 + |z_2|^2) = a^2.$$

To see this we calculate

$$\begin{aligned}
\min_{z_1^2+z_2^2=-a^2} (|z_1|^2 + |z_2|^2) &= \min_{z_1\in\mathbb{C}} (|z_1|^2 + |a^2 + z_1^2|^2) \\
&= \min_{r\geq 0,\, \theta\in[0,2\pi]} (r^2 + \{a^4 + 2a^2r^2\cos(2\theta) + r^4\}^{1/2}) \\
&\qquad\qquad\qquad (\text{writing } z_1 = r\exp(i\theta)) \\
&= \min_{r\geq 0} [r^2 + \{a^4 - 2a^2r^2 + r^4\}^{1/2}] \quad (\text{for } \theta = \pi/2) \\
&= \min_{r\geq 0} [r^2 + \{(a^2 - r^2)^2\}^{1/2}] \\
&= \min_{r\geq 0} [r^2 + |a^2 - r^2|] \\
&= \min_{r\geq 0} \begin{cases} a^2 & \text{if } r \leq a \\ 2r^2 - a^2 & \text{if } r \geq a \end{cases} \\
&= a^2.
\end{aligned}$$

A particular minimum point is given by $r = a$, $\theta = \pi/2$, i.e., $z_1 = ia$, for which $z_2^2 = -a^2 - z_1^2 = 0$.

For any other value of $\infty \geq q > 1$, we note that $\mathrm{Poly}(nP_q) \supset \mathrm{Poly}(nP_1)$ and hence the approximation error satisfies

$$D_n(f_2, K, P_q) \leq D_n(f_2, K, P_1),$$

and so comparing the orders of error decay we must have

$$\min_{z\in S(f_2)} V_{P_q,K}(z) \geq \min_{z\in S(f_2)} V_{P_1,K}(z) = \log(a).$$

On the other hand, $V_{P_q,K}(ia, 0) = V_{P_1,K}(ia, 0) = \log(a)$ and so also

$$\min_{z\in S(f_2)} V_{P_q,K}(z) \leq \log(a). \ \blacksquare$$

In other words the rates of decay of the uniform approximation errors to f_2 are also the same for all choices of $q \geq 1$; there is no approximation value added despite the fact that the dimensions of the spaces $\mathrm{Poly}(nP_q)$ are increasing in q. This behavior is illustrated numerically in Figure 2 where we show the L^2 best approximation error for f_2 with $a = 2$ as a function of n for $q = 1$ and $q = 4$.

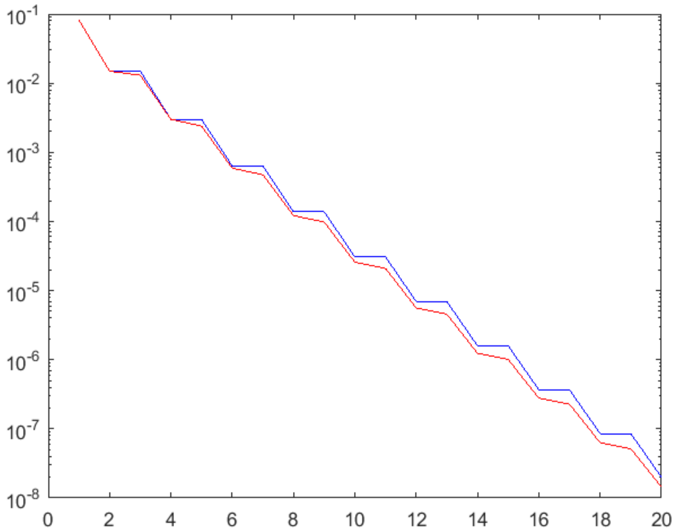

Fig. 2. Errors in best L^2 approximation to f_2; blue curve for $q = 1$ and red (lower) curve for $q = 4$

Example 3. There is no gain in approximating f_1 or f_2 by the spaces $\text{Poly}(nP_q)$, $q > 1$, precisely because there is a singular point on the coordinate hyperplane $z_2 = 0$ where the extremal functions all reduce to the same univariate extremal function for all $q \geq 1$. We now give an example of a function whose singular set does not approach the coordinate hyperplanes and for which the approximation order of $\text{Poly}(nP_q)$ is strictly increasing in q. Specifically, let

$$f_3(z_1, z_2) = \frac{1}{1 - z_1 z_2}.$$

The best L^2 approximation is again easy to calculate by means of a Taylor series, which in this case is just a geometric series:

$$f_3(z_1, z_2) = \frac{1}{1 - z_1 z_2} = \sum_{k=0}^{m} z_1^k z_2^k + \frac{(z_1 z_2)^{m+1}}{1 - z_1 z_2}$$

for $|z_1 z_2| < 1$, in particular on K. The *uniform* norm of the error on K is easily bounded by

$$\max_{|z| \leq 1} \left| \frac{(z_1 z_2)^{m+1}}{1 - z_1 z_2} \right| \leq \frac{2^{-(m+1)}}{1 - 1/2} = 2^{-m}. \tag{3.3}$$

If we take $m = n/2$ (ignoring round-offs) then we approximate f_3 by a polynomial

$$p_n(z_1, z_2) := \sum_{0 \le k \le n/2} z_1^k z_2^k$$

of classical degree n. Its uniform error is then $O(2^{-n/2})$, implying that

$$\min_{z \in S(f_3)} V_{P_1,K}(z) \le \log(\sqrt{2}).$$

On the other hand, for $z_0 = (1,1) \in S(f_3)$, from Proposition 3.2,

$$\begin{aligned} &V_{P_1,K}(z_0) \\ &\ge F_2(\theta; z_0) \quad \text{with } \theta = (1/2, 1/2) \\ &= \frac{1}{2}\left\{\frac{1}{2}\log(1) + \frac{1}{2}\log(1) - \left(\frac{1}{2}\log\left(\frac{1}{2}\right) + \frac{1}{2}\log\left(\frac{1}{2}\right)\right) + \left(\frac{1}{2} + \frac{1}{2}\right)\log(1)\right\} \\ &= \log(\sqrt{2}) \end{aligned}$$

so that also

$$\min_{z \in S(f_3)} V_{P_1,K}(z) \ge \log(\sqrt{2})$$

and we may conclude that

$$\min_{z \in S(f_3)} V_{P_1,K}(z) = \log(\sqrt{2})$$

and that the rate of decay of the uniform error, $O(2^{-n/2})$, is optimal for $q = 1$.

For values of $q > 1$, note that $|(k,k)|_q \le n$ iff $k \le n/2^{1/q}$. Hence

$$p_n(z) := \sum_{0 \le k \le n/2^{1/q}} z_1^k z_2^k \in \mathrm{Poly}(nP_q)$$

with uniform error on K of $O(2^{-n/2^{1/q}} = O((2^{2^{-1/q}})^{-n})$ by (3.3). Again, this implies that

$$\min_{z \in S(f_3)} V_{P_q,K}(z) \ge \log(2^{2^{-1/q}}).$$

On the other hand, again for $z_0 = (1,1) \in S(f_3)$,

$$\begin{aligned} V_{P_q,K}(z_0) &\ge F_2(\theta; z_0) \quad \text{with } \theta = (2^{-1/q}, 2^{-1/q}) \\ &= \log(2^{2^{-1/q}}) \end{aligned}$$

and we may conclude that in general

$$\min_{z \in S(f_3)} V_{P_q,K}(z) = \log(2^{2^{-1/q}}),$$

and the optimal rate of decay of the uniform error is $O((2^{2^{-1/q}})^{-n})$. For example, as $q \to \infty$ this rate approaches $O(2^{-n})$, considerably better than the $O(2^{-n/2})$ for the classical case. Note also that this advantage persists even when the difference in the dimensions of the various polynomial spaces is taken into account. Indeed, for the classical total degree the dimension of the bivariate polynomials of degree at most n is $N := (n+2)(n+1)/2$ so that the decay of the error in terms of the dimension is $O(2^{-n/2}) = O(2^{-\sqrt{N/2}})$. On the other hand, for the tensor-product case, $q = \infty$, the dimension is $N = (n+1)^2$ so that the error decays like $O(2^{-n}) = O(2^{-\sqrt{N}})$.

We do not believe that there is a closed formula for the extremal function for $1 < q < \infty$. However for $q = \infty$ we may show

PROPOSITION 3.9. *Suppose that $d = 2$. Then for $|z| \geq 1$,*

$$V_{P_\infty,B_2}(z) = \begin{cases} \frac{1}{2}\{\log(|z_2|^2) - \log(1-|z_1|^2)\} & \text{if } |z_1|^2 \leq 1/2 \text{ and } |z_2|^2 \geq 1/2, \\ \frac{1}{2}\{\log(|z_1|^2) - \log(1-|z_2|^2)\} & \text{if } |z_1|^2 \geq 1/2 \text{ and } |z_2|^2 \leq 1/2, \\ \log(|z_1|) + \log(|z_2|) + \log(2) & \text{if } |z_1|^2 \geq 1/2 \text{ and } |z_2|^2 \geq 1/2. \end{cases}$$

Proof. If $z_1 = 0$ the first case of the formula reduces to $\log(|z_2|)$, i.e., the univariate extremal function in z_2, as is correct. Similarly, if $z_2 = 0$ the second case reduces to $\log(|z_1|)$, i.e., the univariate extremal function in z_1, as is correct. Hence we suppose that $z_1, z_2 \neq 0$ and we maximize $F_d(\theta; z)$ over the constraint set

$$B_{P_\infty} = \{(\theta_1, \theta_2) : 0 \leq \theta_1, \theta_2 \leq 1\}.$$

Lemma 3.5 informs us that the maximum cannot be attained at a boundary point of B_{P_∞}, i.e., when either $\theta_1, \theta_2 = 0$.

Consider first the upper edge of the constraint set, $\theta_2 = 1$, $0 \leq \theta_1 \leq 1$. The boundary value at $\theta_1 = \theta_2 = 1$ is

$$\begin{aligned} F_d((1,1);z) &= \frac{1}{2}\{\log(|z_1|^2) + \log(|z_2|^2) - 0 + 2\log(2)\} \\ &= \log(|z_1|) + \log(|z_2|) + \log(2) \end{aligned} \tag{3.4}$$

and is a candidate for the maximum (while, as mentioned above, $\theta_1 = 0$, $\theta_2 = 1$ is not). Competitors are given by critical points along this edge. Hence we calculate

$$\begin{aligned} \frac{\partial}{\partial\theta_1} F_d((\theta_1,1);z) &= \frac{1}{2}\{\log(|z_1|^2) - \log(\theta_1) + \log(\theta_1+1)\} = 0 \\ &\iff \log(|z_1|^2) = \log(\theta_1) - \log(\theta_1+1) = \log(\theta_1/(\theta_1+1)) \\ &\iff |z_1|^2 = \theta_1/(\theta_1+1) \\ &\iff \theta_1 = |z_1|^2/(1-|z_1|^2) \quad (|z_1| \neq 1). \end{aligned}$$

Now it is easy to check that $\theta_1 = |z_1|^2/(1-|z_1|^2) \in [0,1]$ iff $|z_1|^2 \leq 1/2$, i.e., we have a competitor critical point in this case and otherwise we do not. If indeed $|z_1|^2 \le 1/2$, then we calculate

$$\begin{aligned}
&F_d((|z_1|^2/(1-|z_1|^2),1);z)\\
&\quad = \frac{1}{2}\bigg\{\frac{|z_1|^2}{1-|z_1|^2}\log(|z_1|^2)+\log(|z_2|^2)-\frac{|z_1|^2}{1-|z_1|^2}\log\bigg(\frac{|z_1|^2}{1-|z_1|^2}\bigg)\\
&\qquad + \bigg(\frac{|z_1|^2}{1-|z_1|^2}+1\bigg)\log\bigg(\frac{|z_1|^2}{1-|z_1|^2}+1\bigg)\bigg\}\\
&\quad = \frac{1}{2}\{\log(|z_2|^2)-\log(1-|z_1|^2)\}
\end{aligned}$$

after some simplification.

Now, we claim that this critical value, for $|z_1|^2 \le 1/2$, is greater than the corner value (3.4). Indeed,

$$\begin{aligned}
&\frac{1}{2}\{\log(|z_2|^2)-\log(1-|z_1|^2)\} \ge \log(|z_1|)+\log(|z_2|)+\log(2)\\
&\iff \frac{1}{2}\{\log(|z_2|^2)-\log(1-|z_1|^2)\} \ge \frac{1}{2}\{\log(|z_1|^2)+\log(|z_2|^2)+\log(4)\}\\
&\iff -\log(1-|z_1|^2) \ge \log(|z_1|^2)+\log(4)\\
&\iff \log(4|z_1|^2(1-|z_1|^2)) \le 0\\
&\iff 4|z_1|^2(1-|z_1|^2) \le 1,
\end{aligned}$$

which clearly holds. In summary, we have shown that

$$\max_{0\le\theta_1\le1,\,\theta_2=1} F_d(\theta;z) = \begin{cases} \frac{1}{2}\{\log(|z_2|^2)-\log(1-|z_1|^2)\} & \text{if } |z_1|^2 \le 1/2,\\ \log(|z_1|)+\log(|z_2|)+\log(2) & \text{if } |z_1|^2 \ge 1/2.\end{cases}$$

We immediately obtain the maximum value on the right edge by symmetry:

$$\max_{0\le\theta_2\le1,\,\theta_1=1} F_d(\theta;z) = \begin{cases} \frac{1}{2}\{\log(|z_1|^2)-\log(1-|z_2|^2)\} & \text{if } |z_2|^2 \le 1/2,\\ \log(|z_1|)+\log(|z_2|)+\log(2) & \text{if } |z_2|^2 \ge 1/2,\end{cases}$$

and the result follows. ∎

4. Computing the extremal measure. Returning to our general setting of a compact, nonpluripolar set $K \subset \mathbb{C}^d$ and a convex body $P \subset (\mathbb{R}^+)^d$, recall that d_n is the dimension of $\mathrm{Poly}(nP)$. We write

$$\mathrm{Poly}(nP) = \mathrm{span}\{e_1,\dots,e_{d_n}\}$$

where $\{e_j(z) := z^{\alpha(j)}\}_{j=1}^{d_n}$ are the standard basis monomials. For $\zeta_1,\dots,\zeta_{d_n}$ in $\mathbb{C}^d$, let

$$\mathrm{VDM}(\zeta_1,\dots,\zeta_{d_n}) := \det\,[e_i(\zeta_j)]_{i,j=1}^{d_n} = \det\begin{bmatrix} e_1(\zeta_1) & \dots & e_1(\zeta_{d_n}) \\ \vdots & \ddots & \vdots \\ e_{d_n}(\zeta_1) & \dots & e_{d_n}(\zeta_{d_n}) \end{bmatrix},$$

and for a compact subset $K \subset \mathbb{C}^d$ let

$$V_n = V_n(K) := \max_{\zeta_1,\dots,\zeta_{d_n}\in K} |\mathrm{VDM}(\zeta_1,\dots,\zeta_{d_n})|.$$

Points $z_1^{(n)},\dots,z_{d_n}^{(n)} \in K$ achieving the maximum are called *Fekete points* of order n for K, P. It was shown in [BBL17] that the limit

$$\delta(K) := \delta(K,P) := \lim_{n\to\infty} V_n^{1/l_n}$$

exists where

$$l_n := \sum_{j=1}^{d_n} \deg(e_j) = \sum_{j=1}^{d_n} |\alpha(j)|_{P_1}$$

is the sum of the P_1-degrees of a set of these basis monomials for $\mathrm{Poly}(nP)$. The quantity $\delta(K)$ is called the *P-transfinite diameter* of K. One of the key results in [BBL17] was the following:

THEOREM 4.1. *Let $K \subset \mathbb{C}^d$ be compact and nonpluripolar. For each n, take points $z_1^{(n)},\dots,z_{d_n}^{(n)}$ in K for which*

$$\lim_{n\to\infty} |\mathrm{VDM}(z_1^{(n)},\dots,z_{d_n}^{(n)})|^{1/l_n} = \delta(K)$$

(*asymptotically P-Fekete arrays*) *and let $\mu_n := \frac{1}{d_n}\sum_{j=1}^{d_n} \delta_{z_j^{(n)}}$. Then*

$$\mu_n \to \frac{1}{d!\,\mathrm{Vol}(P)}(dd^cV_{P,K}^*)^d \qquad \textit{weak-}*.$$

Here $\mathrm{Vol}(P)$ denotes the $\mathbb{R}^d$-Lebesgue measure of P and $(dd^cV_{P,K}^*)^d$ is the *complex Monge–Ampère measure* of $V_{P,K}^*$ (see [Kli93]).

This shows the significance of being able to find the "target" measure $\mu_{P,K} := (dd^cV_{P,K}^*)^d$. It is important to observe that $\mu_{P,K}$ has support in K. In this section, we begin with calculations of $\mu_{P,K}$ for certain P and the unit d-torus in $\mathbb{C}^d$ and then we use the calculations of the previous section to compute $\mu_{P,K}$ for certain P and the unit ball in $\mathbb{C}^2$.

We first recall two results (see [Bay17] or [BBL17]).

1. For $P \subset (\mathbb{R}^+)^d$ a convex body and $K = T^d$, the unit d-torus in $\mathbb{C}^d$, we have

   $$V_{P,T^d}(z) = H_P(z) = \max_{J\in P} \log(|z^J|) \in L_P^+.$$

   If $P = \Sigma = P_1$, then $V_{P,T^d}(z) = \max_{j=1,\dots,d} \log^+(|z_j|)$.

2. Let $\omega := dd^c \max_{j=1,\dots,d} \log^+(|z_j|)$. We normalize so that $\int_{\mathbb{C}^d} \omega^d = 1$. Then for any $u \in L_P^+$ we have

$$\int_{\mathbb{C}^d} (dd^c u)^d = \int_{\mathbb{C}^d} (dd^c H_P)^d = d!\,\mathrm{Vol}(P), \tag{4.1}$$

where $\mathrm{Vol}(P)$ denotes the euclidean volume of $P \subset (\mathbb{R}^+)^d$. In particular, $\mu_{P,K}(K) = d!\,\mathrm{Vol}(P)$.

For simplicity, we take $d = 2$, i.e., we work in $\mathbb{C}^2$ and start with $T = \{(z_1, z_2) : |z_1| = |z_2| = 1\}$. We know that

$$V_{P,T}(z_1, z_2) = H_P(\log^+|z_1|, \log^+|z_2|).$$

Then $V_{P_1,T}(z_1, z_2) = \max[\log^+|z_1|, \log^+|z_2|]$ and $\mu_{P_1,T}$ is normalized Haar measure on T. Note that $\mu_{P,T}(T) = 2\,\mathrm{Vol}(P) = 1$. At the other extreme, for $P_\infty = [0,1] \times [0,1]$,

$$\begin{aligned} V_{P_\infty,T}(z_1, z_2) &= \log^+|z_1| + \log^+|z_2| \\ &= \max[0, \log|z_1|, \log|z_2|, \log|z_1| + \log|z_2|]. \end{aligned}$$

We see that near the face $|z_1| = 1$, $|z_2| < 1$, $V_{P_\infty,T}(z_1, z_2) = \log^+|z_1|$, which is maximal there ($(dd^c \log^+|z_1|)^2 = 0$); ditto for the face $|z_2| = 1$, $|z_1| < 1$. Thus, as we knew, $\mu_{P_\infty,T}$ is supported in T but the total mass is $2\,\mathrm{Vol}([0,1] \times [0,1]) = 2$. Indeed, for any $1 \le q \le \infty$, we have

$$V_{P_q,T}(z_1, z_2) = [(\log^+|z_1|)^{q'} + (\log^+|z_2|)^{q'}]^{1/q'}$$

where $1/q + 1/q' = 1$. By invariance under $(z_1, z_2) \mapsto (e^{i\theta_1} z_1, e^{i\theta_2} z_2)$, $\mu_{P_q,T}$ is a multiple of normalized Haar measure on T; precisely, $\mu_{P_q,T}(T) = 2\,\mathrm{Vol}(P_q)$.

We now turn to the case of the closed euclidean ball B_2 and P_∞. We have shown that for $|z| \ge 1$,

$$V_{P_\infty,B_2}(z) = \begin{cases} \frac{1}{2}\{\log(|z_2|^2) - \log(1 - |z_1|^2)\} & \text{if } |z_1|^2 \le 1/2 \text{ and } |z_2|^2 \ge 1/2, \\ \frac{1}{2}\{\log(|z_1|^2) - \log(1 - |z_2|^2)\} & \text{if } |z_1|^2 \ge 1/2 \text{ and } |z_2|^2 \le 1/2, \\ \log|z_1| + \log|z_2| + \log(2) & \text{if } |z_1|^2 \ge 1/2 \text{ and } |z_2|^2 \ge 1/2. \end{cases} \tag{4.2}$$

PROPOSITION 4.2. *For $K = B_2$ and $P = P_\infty$, the measure $(dd^c V_{P,K}^*)^2 = \mu_{P_\infty,B_2}$ is Haar measure on the torus $\{|z_1| = 1/\sqrt{2},\ |z_2| = 1/\sqrt{2}\}$ with total mass 2.*

Proof. The function $\frac{1}{2}[\log(|z_2|^2) - \log(1 - |z_1|^2)]$ is pluriharmonic in a neighborhood U_a of any point $a \in \partial B_2 \cap \{|z_2| > 1/\sqrt{2} > |z_1|\}$, and negative on $U_a \cap B_2$. Utilizing [Kli93, Proposition 3.8.1], we conclude that on U_a, $V_{P_\infty,B_2} = \max\{\frac{1}{2}[\log(|z_2|^2) - \log(1 - |z_1|^2)], 0\}$ is maximal. Similarly, V_{P_∞,B_2} is maximal in a neighborhood of any point of $\partial B_2 \cap \{|z_1| > 1/\sqrt{2} > |z_2|\}$. Thus there is no Monge–Ampère mass on these portions of ∂B_2; we only have mass on the torus in ∂B_2 where $|z_1|, |z_2| = 1/\sqrt{2}$. Invariance of (4.2) under $(z_1, z_2) \mapsto (e^{i\theta_1} z_1, e^{i\theta_2} z_2)$ implies that $(dd^c V_{P,K}^*)^2$ is a multiple of Haar measure on this torus. The total mass is 2 by (4.1) since the volume of P is 1. ∎

Thus for $P = P_1$, μ_{P_1,B_2} is supported on the entire topological boundary of B_2, while μ_{P_∞,B_2} is supported on a torus.

As an interesting application, let $\{p_\alpha(z) = c_\alpha z^\alpha\}$ be the orthonormal polynomials for Lebesgue measure on $K = B_2$ as in Section 2. Following [Bay17], we can consider, given P, random Poly(nP) polynomials of the form $P_n(z) = \sum_{\alpha \in nP} a_\alpha p_\alpha(z)$ where the coefficients a_α are independent, identically distributed complex-valued random variables. For simplicity, we assume that they are complex Gaussian random variables with distribution

$$\phi(t) dm(t) = \frac{1}{\pi} e^{-|t|^2} dm(t)$$

where dm denotes Lebesgue measure on $\mathbb{C}$. We really want to consider random polynomial mappings $F_n(z) = (P_n(z), Q_n(z))$. Thus we get a probability measure Prob_n on $\mathcal{F}_n$, the random polynomial mappings with $P_n, Q_n \in$ Poly(nP). We can identify $\mathcal{F}_n$ with $\mathbb{C}^{d_n} \times \mathbb{C}^{d_n}$. Given $F_n \in \mathcal{F}_n$, let $\Delta_{F_n} := \{P_n = Q_n = 0\}$ and

$$\tilde{Z}_{F_n} := \frac{1}{n^2} \sum_{\mathbf{p} \in \Delta_n} \delta_{\mathbf{p}}.$$

Thus $\tilde{Z}_{F_n}$ is, up to a constant, the normalized zero measure on the (finite) zero set $\{P_n = Q_n = 0\}$. For generic F_n, $\left(dd^c \frac{1}{n} \log |F_n|\right)^2$ is well-defined and

$$\tilde{Z}_{F_n} = \left(dd^c \frac{1}{n} \log |F_n|\right)^2 = \left(dd^c \left[\frac{1}{2n} \log(|P_n|^2 + |Q_n|^2)\right]\right)^2.$$

The expectation $\mathbf{E}(\tilde{Z}_{F_n})$ is a measure on $\mathbb{C}^2$ defined, for $\psi \in C_c(\mathbb{C}^2)$, as

$$\begin{aligned}\left(\mathbf{E}(\tilde{Z}_{F_n}), \psi\right)_{\mathbb{C}^2} &:= \int_{\mathbb{C}^{d_n} \times \mathbb{C}^{d_n}} (\tilde{Z}_{F_n}, \psi)_{\mathbb{C}^2} \, d\,\mathrm{Prob}_n \\ &= \frac{1}{\pi^{2d_n}} \int_{\mathbb{C}^{d_n} \times \mathbb{C}^{d_n}} (\tilde{Z}_{F_n}, \psi)_{\mathbb{C}^2} e^{-\sum_{\alpha \in nP} |a_\alpha|^2} \prod_{\alpha \in nP} dm(a_\alpha)\end{aligned}$$

where $(\tilde{Z}_{F_n}, \psi)_{\mathbb{C}^2}$ denotes the action of the measure $\tilde{Z}_{F_n}$ on ψ. In this setting, Bayraktar proved that

$$\lim_{n \to \infty} \mathbf{E}(\tilde{Z}_{F_n}) = (dd^c V_{P,K})^2$$

as measures. Forming the product probability space of sequences of random polynomial mappings

$$\mathcal{P} := \bigotimes_{n=1}^{\infty}(\mathcal{F}_n, \mathrm{Prob}_n) = \bigotimes_{n=1}^{\infty}(\mathbb{C}^{d_n} \times \mathbb{C}^{d_n}, \mathrm{Prob}_n),$$

almost surely (a.s.) in $\mathcal{P}$ we have

$$\frac{1}{n}\log|F_n| = \frac{1}{2n}\log(|P_n|^2 + |Q_n|^2) \to V_{K,P}(z)$$

pointwise in $\mathbb{C}^2$ and in $L^1_{\mathrm{loc}}(\mathbb{C}^2)$. Moreover, a.s. in $\mathcal{P}$ we have

$$\left(dd^c\frac{1}{n}\log|F_n|\right)^2 = \left(dd^c\left[\frac{1}{2n}\log(|P_n|^2 + |Q_n|^2)\right]\right)^2 \to (dd^c V_{P,K})^2$$

as measures.

COROLLARY 4.3. *With $K = B_2$,*

(1) *for $P = P_1 = \Sigma$, $\mathbf{E}(\tilde{Z}_{F_n}) \to \mu_{P_1,B_2}$, normalized surface area measure on ∂B_2;*
(2) *for $P = P_\infty$, $\mathbf{E}(\tilde{Z}_{F_n}) \to \mu_{P_\infty,B_2}$, a multiple of Haar measure on the torus $\{|z_1| = |z_2| = 1/\sqrt{2}\}$,*

with analogous statements for the a.s. results.

QUESTION 4.4. For $P = P_q$ with $1 \leq q \leq \infty$, we have $\mathbf{E}(\tilde{Z}_{F_n}) \to \mu_{P_q,B_2}$. However, we do not have any knowledge of the support of μ_{P_q,B_2} for $1 < q < \infty$. How does the support of μ_{P_q,B_2} vary with q?

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T. Bloom
Department of Mathematics
University of Toronto
Toronto, Canada
E-mail: bloom@math.utoronto.edu

L. Bos
Department of Computer Science
University of Verona
Verona, Italy
E-mail: leonardpeter.bos@univr.it

N. Levenberg
Department of Mathematics
Indiana University
Bloomington, IN, U.S.A.
E-mail: nlevenbe@indiana.edu

S. Ma'u
Department of Mathematics
University of Auckland
Auckland, New Zealand
E-mail: smau@auckland.ac.nz

F. Piazzon
Department of Mathematics
University of Padova
Padova, Italy
E-mail: fpiazzon@math.unipd.it