STUDIA MATHEMATICA

Online First version

Injectivity of spaces of bounded vector sequences and spaces of operators

by

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Abstract. Various Banach or Fréchet spaces which are either vector-valued sequence spaces or components of some closed operator ideals are considered. Complete solutions are given to the problem of their injectivity or embeddability as complemented subspaces in dual Fréchet spaces. The problem of complementability of components of a given closed operator ideal in the corresponding components of another closed operator ideal is considered as well and some new results are obtained. All these results are obtained by application of a "universal" method introduced in this paper.

Foreword. This paper had been written over a period of several years in the mid-1990's, reaching the form of a fairly complete manuscript. However, it did not quite satisfy us, and we decided to shelve it "for a while", as we optimistically thought then. For Paweł's own words on this, see [21, p. 38]. But, alas, no quick return happened and, as time passed by, the usual challenges of life as well as still new research projects that kept on attracting us, prevailed. Suddenly came the worst: Paweł died in August of 2016. Now, at the beginning of 2019, it is high time to decide about the fate of the paper. Although, to some people, it may look a bit outdated, I strongly believe that its ideas and methods, mostly due to Paweł, are still fresh, worthy of presentation, and certainly will soon find followers. At this point I would also like to stress that without a tremendous technical and editorial help from Michał Goliński and Tomasz Ciaś who may rightly be considered Paweł's mathematical descendants, this exposition, very close to what Paweł imagined it to be, would simply be impossible.

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2010 *Mathematics Subject Classification*: Primary 46M10, 46A32, 46B25, 46B45, 47L05, 47L20; Secondary 46M15.

Key words and phrases: Fréchet space, Banach space, vector-valued sequence space, complemented subspace, dual space, injective space, closed operator ideal, space of compact operators.

Received 18 February 2019; revised 26 March 2019.

Published online 28 June 2019.

DOI: 10.4064/sm190218-2-4

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1. Introduction. At the time of writing this paper a long-standing conjecture of J. Lindenstrauss that, for all Banach spaces E and F, the space $\mathcal{K}(E, F)$ of compact operators is either equal to or else uncomplemented in the space $\mathcal{L}(E, F)$ of all operators, was still open. In spite of efforts of numerous mathematicians (for instance, see [2], [31], [32], [34], [35], [44], [45], [47], [50], [75], [77]), which certainly motivated the conjecture, the answer was known to be affirmative only for several particular cases, for instance, if E contains a complemented copy of l_1 [47, Lemma 3], or F contains a copy of c_0 [35, Cor. 1]. Finally, in 2011, the conjecture was answered in the negative by Spiros Argyros and Richard Haydon [1]. For a comprehensive story about this and similar earlier problems, the reader may like to have a look at Timothy Gowers's post on his blog [37]; see also Remark 10.14(e) below. Analogous problems have been considered for other pairs of classical closed operator ideals instead of $\mathcal{K}$ and $\mathcal{L}$. Thus, for instance, the answer is negative for the pairs consisting of the ideals $\mathcal{W}$ (of weakly compact operators) and $\mathcal{L}$, $\mathcal{S}$ (of strictly singular operators) and $\mathcal{L}$, as well as $\mathcal{SP}$ (of separable operators) and $\mathcal{L}$ (see [52, Ex. 1.d.2], [38], and [73]). However, the question remained open for other pairs of classical closed operator ideals, and even for the pairs mentioned above one may ask if there are any positive particular cases (for example, if any analogues of Kalton's or Feder's results are true). Something was known in this direction (see [2], [29], [33] and [50]) but the problem has never been systematically studied and the known results were often based on some "ad hoc" assumptions.

The main purpose of the present paper is to give a unified and more systematic approach to the problems of the above type for "concrete" spaces of operators—an approach which turns out to be quite elementary, improves some known results, yields many new ones, and allows an easy generalization to the case of Fréchet and DF-spaces. Roughly speaking, it is based on two reductions. First, the problems for spaces of operators are reduced to analogous problems for spaces of bounded vector sequences and these, in the second step, are reduced, essentially, to the old Phillips [64] theorem stating that c_0 is uncomplemented in l_∞. More precisely, we will use a slightly strengthened version of Phillips' theorem (see the Key Lemma 5.1).

For instance, in Section 10 we obtain analogues of Kalton's and Feder's results for many pairs of classical closed operator ideals, like the ideals of weakly compact, limited, Rosenthal, completely continuous, and unconditionally converging operators (Corollaries 10.10–10.13). In fact, we exhibit some "nice" classes of operator ideals which can replace the ideals of compact and of all operators in Kalton's and Feder's results. "Nice" means here that the class is wide and admits an easy criterion deciding if a given ideal belongs to the class or not. It should be noted that the original proof of Feder gives no indication that his result may remain true for ideals other than $\mathcal{K}$.

The problem described above is related to a somewhat more specific problem of whether a space in question is complemented in its bidual (or some analogue of the bidual in the setting of locally convex spaces). The latter problem, in turn, is quite close to the problem of injectivity, i.e., the question if the given space is complemented in every space containing it isomorphically. Our method also applies to these problems and gives even better (in fact, final) results. Namely, we show that, among the previously mentioned spaces of operators, the injective spaces occur only in "trivial" cases, that is, when the space in question coincides with a space whose injectivity is already well-known (see Corollaries 10.6 and 10.8).

As already mentioned, our method depends heavily on a study of spaces of bounded vector-valued sequences satisfying some natural conditions. (These include, for instance, the spaces of (weakly) relatively compact or weakly conditionally compact sequences.) Accordingly, the first part of the paper is devoted to quite an extensive study of such spaces. Apart from the basic results which are later used for spaces of operators, we also obtain several results of independent interest. In particular, we show that various classical spaces $k(E)$ of E-valued sequences are injective only in trivial cases, i.e., when $k(E) = l_\infty(E)$ and E is injective (Corollary 6.5). We also prove that, for many pairs of such spaces, complementability of the smaller space in the bigger one implies that they are actually equal (Corollary 6.2). It is worth mentioning that an application of our method to spaces of continuous vector-valued functions has already been given in [19]. We work in quite a general setting, covering the case of Fréchet and DF-spaces. We should emphasize, however, that most of our results are new also for Banach spaces, the case probably most interesting and important. For the sake of clarity, let us point out that the assumptions like quasi- or sequential completeness, $\aleph_0$-barrelledness and weak angelicity are satisfied for all Banach spaces. The authors do not consider their approach entirely original: at least some of the ideas we follow can be traced back to many papers on the subject published so far. In this context we should mention the papers of Arterburn and Whitley [2], Kalton [47], and Emmanuele [29], [33], but the true origin is our earlier joint work [17].

Now, we briefly summarize the content of the paper. It splits naturally into two parts: a part dealing with spaces of sequences, and a part devoted to spaces of operators, the latter depending heavily on the former. Section 2 fixes some terminology and notation, and recalls some basic facts. Section 3 presents a number of concrete spaces of bounded vector sequences. In Section 4 we introduce and discuss some important general properties of such spaces. The most useful, and of utmost importance for us, is the property of l_∞-subsequential determinedness (l_∞-SD), defined in Subsection 4A. The next section contains the essentials of our method: Theorem 5.3, which is the main tool of our paper, and its consequences—the crucial complementability results for general spaces of bounded vector sequences. In Section 6 we apply these results to the concrete spaces described in Section 3 (see for instance Corollaries 6.2 and 6.5).

In the next two sections we investigate functorial "constructions" of spaces of bounded vector sequences, and try to draw up borderlines for our method in the sequence space case. In Section 7 we show that for $E = l_p$ $(1 \le p < \infty)$ or $E = c_0$ there are only few sufficiently "nice" spaces of bounded E-valued sequences (see Proposition 7.2), where "nice" means that we can apply the results of Section 5 to the space in question. In Section 8 we study the so-called bs-functors which assign to every space E (from a given class of locally convex spaces) a space $k(E)$ of bounded E-valued sequences. A particular attention is devoted to those bs-functors which, in a certain natural sense, are generated by "nice" spaces of bounded l_∞-valued sequences. Their importance for us lies in the fact that, under fairly general and natural assumptions, for such bs-functors k the spaces $k(E)$ are "automatically" l_∞-SD (see Corollary 8.10). We also show that there exist maximal bs-functors k yielding spaces $k(E)$ to which our method applies (Theorem 8.14).

The last two sections form the "operator core" of the paper. In Section 9 we develop our method for general spaces of operators, relying on previous sequential results. Then, in Section 10, we show that the method applies to many classical operator ideals. In particular, we show when a component of such an ideal is injective as a Banach or Fréchet space (Corollary 10.6), complemented in a dual space (Corollary 10.8), or complemented in the corresponding component of another, bigger ideal (Corollaries 10.10–10.12).

The reader who is interested only in the operator results may omit Sections 7 and 8, and from Section 6 will need only Theorem 6.1.

2. Preliminaries. We refer the reader to [43], to [72], [12], and to [65] for the standard terminology, notation, and basic facts concerning locally convex spaces, vector-valued sequences and operator ideals, respectively. Here we only explain some special conventions and notation, and recall a few facts that will be used throughout the paper.

We will constantly use the following abbreviations:

- TVS for *topological vector space*;
- LCS for *locally convex space*;
- QC for *quasi-complete*;
- SC for *sequentially complete.*

Other abbreviations, like SH, SD, l_∞-SD, etc. will be introduced in Section 4.

Given a LCS E, we denote by E' its dual space, and we often write

- σ for $\sigma(E,E')$, the weak topology of E;
- σ' for $\sigma(E',E)$, the weak* topology of E';
- μ' for $\mu(E',E)$, the Mackey topology of E'.

An important convention is that whenever the dual E' is treated as a LCS, we always mean the *strong dual* of E, thus $E' = (E', \beta(E',E))$.

We denote by $E'^{\times}$ the strong dual of the bornological space associated with the equicontinuous bornology on E'. We recall that E is canonically embedded in $E'^{\times}$, and every operator $T\colon E \to F$ has a second adjoint $T'^{\times}\colon E'^{\times} \to F'^{\times}$ which is an extension of T; for more details see [16, p. 9]. Another fact worth recalling is that E is (isomorphic to) a complemented subspace of the dual of a bornological LCS iff E is complemented in $E'^{\times}$ [16, Prop. 1.3]. We say E is a *dual Fréchet space* if it is isomorphic to the (strong) dual of a bornological DF-space. If E is a Banach space, then $E'^{\times}$ is simply the strong bidual E'' of E. A Banach space is complemented in the dual of a bornological LCS iff it is complemented in a dual Fréchet space iff it is complemented in a dual Banach space.

A LCS E is called *injective* if it is complemented in every LCS containing it, or equivalently, if every operator into E can be extended to any LCS containing its domain. It is known that if E is a Banach space, then all the spaces occurring in this definition can be taken Banach as well, and likewise if E is a Fréchet space. For more information see [51], [62] and [20].

By an *operator* we always mean a continuous linear operator (from one LCS to another). We denote by $\mathcal{L}$ the class of all operators; other classes of operators are defined in Sections 9 and 10. Of these we will encounter in the earlier sections only the class $\mathcal{W}$ of all operators mapping bounded sequences to relatively weakly compact sequences. Whenever a space $A(E,F)$ of operators is considered, it is understood that $A(E,F)$ is a subspace of $\mathcal{L}(E,F)$, the space of all operators from E to F. Moreover, we then let

$$A'(E,F) := \{T'\colon T \in \mathcal{A}(E,F)\}$$

stand for the corresponding space of dual (adjoint) operators from F' to E'.

By a *category* of LCSs we mean a subcategory $\mathcal{E}$ of the category $\mathcal{LCS}$ of all LCSs which we always assume to be full (all operators between the spaces in $\mathcal{E}$ are morphisms in $\mathcal{E}$) and such that if $E \in \mathcal{E}$, $F \in \mathcal{LCS}$ and $E \simeq F$, then $F \in \mathcal{E}$. The assumptions in our results are chosen general enough to cover at least the cases where the locally convex spaces under consideration are Fréchet spaces or complete DF-spaces. Let us therefore recall at this point that both Fréchet spaces and quasi-complete DF-spaces are complete, satisfy the Eberlein–Šmulian Theorem [59] (i.e., they are weakly angelic) and are $\aleph_0$-barrelled.

We now explain some notation used when dealing with sequences. If $M \subset \mathbb{N}$, then

$$\mathfrak{f}(M) := \text{the family of all finite subsets of } M,$$
$$[M] := \text{the family of all infinite subsets of } M,$$
$$e_M := \text{the characteristic function of } M, \text{ denoted } e_m \text{ when } M = \{m\},$$
$$\pi_M := \text{the increasing bijection from } \mathbb{N} \text{ onto } M \text{ (for } M \text{ infinite).}$$

If a sequence is denoted by a single letter, say x, then its terms are denoted by the same letter with an index (usually n) which is tacitly assumed to range over $\mathbb{N}$; thus $x = (x_n)$ or, more precisely, $(x_n)_{n\in\mathbb{N}}$. Sequences with terms in a vector space E are called *E-sequences.* If x is a sequence in a vector space and a is a sequence of scalars, then $ax := (a_n x_n)$; in particular $e_M x = (e_M(n)x_n)_{n\in\mathbb{N}}$ for $M \subset \mathbb{N}$. If $M \in [\mathbb{N}]$, then $x\pi_M \equiv x \circ \pi_M = (x_{\pi_M(n)})_{n\in\mathbb{N}} = (x_{m_n})_{n\in\mathbb{N}}$, sometimes written also as $(x_n)_{n\in M}$, where $M = \{m_1, m_2, \ldots\}$ and $m_1 < m_2 < \cdots$.

We adopt the convention that if a topological property (P) is defined for sets, then a sequence (x_n) is said to have property (P) if its range $\{x_n : n \in \mathbb{N}\}$ has property (P).

Finally, for every $M \in [N]$,

$$l_\infty(M) := \{a = (a_n) \in l_\infty \colon a_n = 0 \text{ for all } n \notin M\};$$

of course, this subspace of l_∞ is (isometrically) isomorphic to l_∞.

3. Some concrete spaces of bounded vector sequences. Let E be a LCS. Then $l_\infty(E)$ denotes the (locally convex) space of bounded E-sequences equipped with the uniform convergence topology. If also F is a LCS and $S \in \mathcal{L}(E, F)$, then the induced operator $\hat{S} \colon l_\infty(E) \to l_\infty(F)$ is defined by $\hat{S}((x_n)_{n\in\mathbb{N}}) = (Sx_n)_{n\in\mathbb{N}}$. Note that if S is an isomorphism or isomorphic embedding, so is $\hat{S}$. Also, $\hat{S}(e_M x) = e_M \hat{S}x$ and $\hat{S}(x\pi_M) = (\hat{S}x)\pi_M$.

3A. Basic examples. Given a linear Hausdorff topology τ on E, weaker than the original one, we define the following (topological) subspaces of $l_\infty(E)$:

$$c_0(E, \tau) = \{(x_n) \in l_\infty(E) \colon x_n \to 0 \ (\tau)\},$$
$$\kappa(E, \tau) = \{(x_n) \in l_\infty(E) \colon (x_n) \text{ is relatively } \tau\text{-compact}\},$$
$$p\kappa(E, \tau) = \{(x_n) \in l_\infty(E) \colon (x_n) \text{ is } \tau\text{-precompact}\},$$
$$c\kappa(E, \tau) = \{(x_n) \in l_\infty(E) \colon (x_n) \text{ is conditionally } \tau\text{-compact}\}.$$

We stress that these spaces are considered with the topology induced from $l_\infty(E)$, where E carries its *original* topology (and not τ). We will be particularly interested in the cases where τ is the original topology of E (in which case we suppress τ), or τ is the weak topology $\sigma := \sigma(E, E')$ of E.

Likewise, when dealing with analogously defined spaces of sequences in the dual space E', we will usually take for τ the weak* topology $\sigma' := \sigma(E', E)$ or the Mackey topology $\mu' := \mu(E', E)$ of E'.

We are now going to introduce some additional, less common, spaces of sequences. But first, for notational convenience, let us set

$$e(E') = \{(x'_n) \in l_\infty(E') \colon (x'_n) \text{ is equicontinuous}\},$$
$$e_0(E') = e(E') \cap c_0(E', \sigma'),$$

and recall the elementary facts that

- $\mathcal{L}(E, l_\infty) =$ the maps $S\colon E \to l_\infty\,;\ x \mapsto (\langle x'_n, x\rangle)$, where $(x'_n) \in e(E')$;
- $\mathcal{L}(E, c_0) =$ the maps $S\colon E \to c_0\,;\ x \mapsto (\langle x'_n, x\rangle)$, where $(x'_n) \in e_0(E')$.

3B. Limited sets and the Gelfand–Phillips property. A set $A \subset E$ is said to be *limited* [12] if $\sup_{x\in A} |\langle x, x'_n\rangle| \to 0$ for every sequence $(x'_n) \in e_0(E')$; equivalently, if every $T \in \mathcal{L}(E, c_0)$ maps A to a relatively compact set in c_0. Clearly, 'precompact' implies 'limited' implies 'bounded'. If every limited set in E is precompact, then E is said to have the *Gelfand–Phillips property*. We denote by $l(E)$ the space of limited sequences in E. Thus, by definition and easy verification,

$$\begin{aligned} l(E) &= \{x \in l_\infty(E) \colon \hat{T}(x) \in \kappa(c_0),\ \forall T \in \mathcal{L}(E, c_0)\} \\ &= \{(x_n) \in l_\infty(E) \colon \langle x_n, x'_n\rangle \to 0,\ \forall (x'_n) \in e_0(E')\}. \end{aligned}$$

3C. wuC series, uc operators; properties (V), (V*) of Pełczyński. We denote by c_{00} the subspace $\mathrm{lin}\,(e_n)$ of c_0. If the spaces E and F occurring in the discussion below are SC, then c_{00} can be replaced by c_0, and 'unconditionally Cauchy' by 'unconditionally convergent'. For every sequence (x_n) in a LCS E, the following conditions are equivalent (see e.g. [12]):

- the series $\sum_n x_n$ is *weakly unconditionally Cauchy* (wuC), that is, $\sum_n |\langle x_n, x'\rangle| < \infty$ for all $x' \in E'$;
- the sequence (x_n) is *perfectly bounded* (after W. Orlicz), that is, the set $\{\sum_{n\in M} x_n \colon M \in \mathfrak{f}(\mathbb{N})\}$ is bounded in E;
- there exists an operator $R\colon c_{00} \to E$ with $Re_n = x_n$ for all n.

An operator S from E to another LCS F is said to be *unconditionally converging* (uc) if, for every wuC series $\sum_n x_n$ in E, the series $\sum_n S(x_n)$ is unconditionally Cauchy (uC) in F. The requirement that $\sum_n (x_n)$ is uC can be replaced here by '$S(x_n) \to 0$', as well as by '$\{\sum_{n\in M} S(x_n) \colon M \in (\mathbb{N})\}$ is precompact'. Remember that $S(x_n) \to 0$ holds iff $\sup\{|\langle S(x_n), y'\rangle| \colon y' \in B\} = \sup\{|\langle x_n, x'\rangle| \colon x' \in S'(B)\} \to 0$ for every equicontinuous set (or sequence) B in F'. With this in mind, a subset A of E' is said to have *Pełczyński's property* (V) if, for every wuC series $\sum_n x_n$ in E, we have $\sup\{|\langle x_n, x'\rangle| \colon x' \in A\} \to 0$. Equivalently, for every operator $R\colon c_{00} \to E$

one has $\sup\{|\langle e_{M_k}, R'x'\rangle| : x' \in A\} \to 0$ for all disjoint sequences (M_k) of finite subsets of $\mathbb{N}$. Thus, by a well known compactness criterion in Banach spaces with Schauder bases [24, IV.5.5], A has property (V) iff $R'(A)$ is relatively compact in l_1 for every $R \in \mathcal{L}(c_{00}, E)$. Note that any set $A \subset E'$ with property (V) is (strongly) bounded, for otherwise we could find $(z_n) \in l_\infty(E)$ such that $\sup\{|\langle 2^{-n} z_n, x'\rangle| : x' \in A\} > 1$ for every n, in spite of the fact that the series $\sum_n 2^{-n} z_n$ is wuC. On the other hand, all relatively weakly compact sets in E' have property (V), as is immediately seen from the above l_1-characterization of (V)-sets and the Schur property of l_1. If the converse holds, the space E is said to have property (V).

Now, back to operators, it is clear that an operator $S: E \to F$ is uc iff its adjoint $S': F' \to E'$ maps equicontinuous sets in F' to (equicontinuous) sets in E' having property (V). (Of course, 'sets' can be replaced by 'sequences'.) Applying this to the identity operator on E, we see that every wuC series in E is uC iff every equicontinuous subset of E' has property (V). Let us recall a well known result of Bessaga and Pełczyński [12, Ch. V, Th. 8 and Ex. 8] which, in its general form, states that an operator $S: E \to F$ is uc iff S fixes no copy of c_{00}, i.e., E has no subspace E_0 isomorphic to c_{00} and such that $S|E_0$ is an isomorphic embedding. In particular, E has no subspace isomorphic to c_{00} iff every wuC series in E is uC.

We let $v(E')$ stand for the space of all sequences in E' with property (V). By what was said above,

$$v(E') = \{(x'_n) \in l_\infty(E') : (R'x'_n) \in \kappa(l_1),\ \forall R \in \mathcal{L}(c_{00}, E)\}$$
$$= \Big\{(x'_n) \in l_\infty(E') : \langle x_n, x'_n\rangle \to 0 \text{ for every wuC series } \sum_n x_n \text{ in } E\Big\},$$

where the second equality follows by an easy reformulation of the definition of property (V) for sequences. We also know that $\kappa(E', \sigma) \subset v(E')$, and it is clear that $v(E') \cap e(E')$ consists precisely of those E'-sequences which represent uc operators from E to l_∞.

3D. Characterizations of wuC series in dual spaces. For a sequence (x'_n) in E' consider the following conditions:

(1′) The series $\sum_n x'_n$ is wuC in $E' = (E', \beta(E', E))$ (thus weakly $= \sigma(E', E'')$);
(2′) the sequence (x'_n) is perfectly bounded in $E' = (E', \beta(E', E))$;
(3′) there exists an operator $R: E \to l_1$ with $R'e'_n = x'_n$ for all n, where e'_n denotes the nth unit vector in $l_\infty = l'_1$;
(4′) the sequence (x'_n) is *perfectly equicontinuous*, meaning that the set $\{\sum_{n\in M} x'_n : M \in \mathfrak{f}(\mathbb{N})\}$ is equicontinuous.

Then, as is easily seen,

$$(3')\Leftrightarrow(4')\Rightarrow(2')\Leftrightarrow(1'),$$

and $(2')\Rightarrow(4')$ holds when E is quasi-$\aleph_0$-barrelled. Also note that $(4')$ implies that the series $\sum_n x'_n$ is weak* subseries convergent and the set $\{\sum_{n\in M} x'_n : M\subset\mathbb{N}\}$ is equicontinuous (and weak* compact). We say that a (bounded) subset A of E has *Pełczyński's property* (V*) if

$$\sup\{|\langle x, x'_n\rangle|: x\in A\}\to 0$$

for every perfectly equicontinuous sequence (x'_n) in E'.

Now, for every operator $T\colon F\to E$, the following are equivalent:

- T maps bounded sets (sequences) in F to sets (sequences) in E having property (V*);
- T' maps perfectly equicontinuous sequences in E' to uC series in F'.

Using the equivalence of $(3')$ and $(4')$ above, one readily verifies that A has property (V*) iff $R(A)$ is relatively compact in l_1 for every $R\in\mathcal{L}(E,l_1)$. In consequence, all relatively (as well as conditionally) weakly compact sets in E have property (V*). We denote by $v^*(E)$ the space of all sequences in E with property (V*); thus

$$v^*(E)=\{x\in l_\infty(E)\colon \hat{R}(x)\in\kappa(l_1),\ \forall R\in\mathcal{L}(E,l_1)\},$$

and $\kappa(E,\sigma)\subset v^*(E)$, $c\kappa(E,\sigma)\subset v^*(E)$.

The following proposition collects some elementary but useful characterizations of the spaces of sequences defined above.

PROPOSITION 3.1. *Let E be a* LCS.

(a) $p\kappa(E)=\{x\in l_\infty(E)\colon \hat{T}(x)\in\kappa(l_\infty),\ \forall T\in\mathcal{L}(E,l_\infty)\}$.
(b) $\kappa(E)=\{x\in l_\infty(E)\colon \hat{T}(x)\in\kappa(l_\infty),\ \forall T\in\mathcal{L}(E,l_\infty)\}$ *if E is* QC.
(c) $\kappa(E,\sigma)=\{x\in l_\infty(E)\colon \hat{T}(x)\in\kappa(l_\infty,\sigma),\ \forall T\in\mathcal{L}(E,l_\infty)\}$ *if E is* QC.
(d) $c\kappa(E,\sigma)=\{x\in l_\infty(E)\colon \hat{T}(x)\in c\kappa(l_\infty,\sigma),\ \forall T\in\mathcal{L}(E,l_\infty)\}$ *if E is metrizable.*
(e) $l(E)=\{x\in l_\infty(E)\colon \hat{T}(x)\in\kappa(c_0),\ \forall T\in\mathcal{L}(E,c_0)\}$.
(f) $c_0(E',\mu')=\{x'\in l_\infty(E')\colon \hat{S}(x')\in c_0(l_\infty),\ \forall S\in\mathcal{W}'(l_1,E)\}$.
(g) $p\kappa(E',\mu')=\{x'\in l_\infty(E')\colon \hat{S}(x')\in\kappa(l_\infty),\ \forall S\in\mathcal{W}'(l_1,E)\}$.
(h) $v(E')=\{x'\in l_\infty(E')\colon \hat{S}(x')\in\kappa(l_1),\ \forall S\in\mathcal{L}'(c_{00},E)\}$.
(i) $v^*(E)=\{x\in l_\infty(E)\colon \hat{T}(x)\in\kappa(l_1),\ \forall T\in\mathcal{L}(E,l_1)\}$.

Proof. We skip the easy proofs of (a), (b) and (d); they are similar to that of (c) given below. Parts (e), (h) and (i) are already known from Subsections 3B, 3C and 3D. As for the other parts, we only show that the sets on the right are contained in those on the left.

(c): Let (x_n) be an E-sequence such that $(Tx_n)\in\kappa(l_\infty,\sigma)$ for all $T\in\mathcal{L}(E,l_\infty)$. We may consider E a subspace of a product $F=\prod_{j\in J}F_j$ of

Banach spaces. Let $P_j\colon F \to F_j$ be the natural projections. For each j, by the injectivity of l_∞ there exists an operator $R_j\colon F_j \to l_\infty$ whose restriction to $\overline{\operatorname{lin}}\{P_j(x_n)\colon n \in \mathbb{N}\}$ is an isomorphic embedding. Now, by assumption, R_jP_j maps (x_n) to a relatively weakly compact sequence in l_∞, whence $(P_jx_n)_{n\in\mathbb{N}}$ is relatively weakly compact in F_j. Let C_j be a weakly compact set in F_j containing $(P_jx_n)_{n\in\mathbb{N}}$. Then $C := \prod_{j\in J} C_j$ is weakly compact in F. Let A denote the closed convex hull of (x_n). Since A is complete in E, it is closed, hence weakly closed, in F. Therefore, $A \cap C$ is a weakly compact subset of E, and it contains (x_n).

(f): Let $(x'_n) \in l_\infty(E')$. Suppose $(x'_n) \notin c_0(E', \mu')$ so that

$$\sup_{x\in A} |\langle x, x'_n\rangle| \nrightarrow 0$$

for some absolutely convex weakly compact set $A \subset E$. Then there is a sequence (x_n) in A such that if B is its closed absolutely convex hull, then we still have $r_n := \sup_{x\in B} |\langle x, x'_n\rangle| \nrightarrow 0$. Let $S\colon l_1 \to E$ be the operator with $Se_n = x_n$ for all n. Then S is weakly compact and $\|S'x'_n\| = r_n \nrightarrow 0$ in l_∞.

(g): Let $(x'_n) \in l_\infty(E')$ and suppose $(x'_n) \notin p\kappa(E', \mu')$. By passing to a subsequence, we may assume that there is an absolutely convex weakly compact set $A \subset E$ such that $\sup_{x\in A} |\langle x, x'_n - x'_m\rangle| > 1$ whenever $m \neq n$. Then we can find a double sequence $(x_{mn})_{m,n\in\mathbb{N}}$ in A with $|\langle x_{mn}, x'_m - x'_n\rangle| > 1$ whenever $m \neq n$. We finish as in the proof of (f) above. ∎

Remark 3.2. Similar characterizations hold obviously for the spaces $l_\infty(E)$, $c_0(E)$, $c_0(E, \sigma)$, and $c_0(E', \sigma')$. In fact, except for $c_0(E)$, one-dimensional operators from E or into E are sufficient in these cases. More familiar variants of parts (a) through (d) (resp., (f) and (g)) of Proposition 3.1 are obtained by using operators from E to *arbitrary* Banach spaces (resp., weakly compact operators from *arbitrary* Banach spaces to E). Thus modified, analogues of (a)–(e) and (g) are valid also for sets.

For Banach spaces, part (a) of the proposition below is due to Bourgain and Diestel [10]; its extension to Fréchet spaces has been noted in [55].

Proposition 3.3. *Let E be a* lcs. *Then:*

(a) $l(E) \subset r(E)$, *where*

$$r(E) := \{x \in l_\infty(E)\colon \hat{T}(x) \in c\kappa(l_\infty, \sigma),\ \forall T \in \mathcal{L}(E, l_\infty)\}.$$

In particular, $l(E) \subset c\kappa(E, \sigma) = r(E)$ *if E is metrizable.*

(b) $p\kappa(E', \mu') \subset v(E')$ *if E is* qc.

Proof. (a): Suppose $(x_n) \in l(E) \setminus r(E)$ so that there exists $T \in \mathcal{L}(E, l_\infty)$ for which $(y_n) := (Tx_n) \notin c\kappa(l_\infty, \sigma)$. By Rosenthal's l_1-theorem we may assume (y_n) is equivalent to the unit basis (e_n) of l_1. Let us identify (y_n) with (e_n) and $\overline{\operatorname{lin}}\,(y_n)$ with l_1. Then (see [10, proof of Proposition, item 4] or

[18, proof of Lemma 1]) the inclusion map $i\colon l_1 \to c_0$ can be extended to an operator $U\colon l_\infty \to c_0$. Since $UT \in \mathcal{L}(E, c_0)$ and $(UTx_n) \notin \kappa(c_0)$, we obtain $(x_n) \notin l(E)$; a contradiction.

(b): Let (x_n) be a weakly null sequence in E (in particular, it is so when the series $\sum_n x_n$ is wuC), and let $A \subset E'$ be μ'-precompact. Since E is QC, the closed absolutely convex hull of (x_n) is weakly compact, hence there exists an operator $T\colon l_1 \to E$ with $Te_n = x_n$ for all n, and $T \in \mathcal{W}(l_1, E)$. By Proposition 3.1(g), $T'(A)$ is relatively compact in l_∞. However, since (x_n) is weakly null, we have $T'(E') \subset c_0$ so that $T'(A)$ is relatively compact in c_0. In consequence (see [24, IV.5.5]), $\sup\{|\langle x_n, x'\rangle|\colon x' \in A\} \to 0$ as $n \to \infty$. Thus A has property (V). ■

We finish the present section with some characterizations of sequences in $v^*(E)$ and $r(E)$ for SC spaces E. We will need the following auxiliary result; it is hard to locate its first appearance in the literature, the references we have are [58, Th. 1.4] and [25, Th. 13]. We are grateful to Professor A. Pełczyński for suggesting the change of basis trick used in the simple proof given below.

PROPOSITION 3.4. *Every non-relatively-compact bounded sequence (x_n) in l_1 contains a basic subsequence which is equivalent to the standard basis (e_n) of l_1 and spans a complemented subspace of l_1.*

Proof. By passing to a subsequence, we may assume that (x_n) is $\sigma(l_1, c_0)$-convergent to some $z \in l_1$. Consider the case when $z \neq 0$; without loss of generality we may assume that the first coordinate of z is 1. Then $z \notin \overline{\mathrm{lin}}\,\{e_n\colon n \geq 2\}$, hence the sequence $z, e_2, e_3, \ldots$ is a basis of l_1 equivalent to the standard basis. Therefore, by applying the automorphism sending the new basis to the standard one, we may finally assume that the sequence (x_n) converges coordinatewise—relative to the standard basis—to e_1.

The rest of the proof is rather standard; see [52, Sec. 1.a] for the facts that are tacitly used below. For every $n \in \mathbb{N}$, denote by a_n the first coordinate of x_n and define $y_n = x_n - a_n e_1$. Then the sequence (y_n) converges to 0 coordinatewise, but not in norm. Hence, by passing to a suitable subsequence, we may assume that there is a block sequence (u_n) such that $\sum_n \|y_n - u_n\| < \infty$. Since (u_n) is equivalent to the basis (e_n) and its closed span is complemented in l_1, the same holds true for the sequence $(y_n)_{n\geq n_0}$ for some n_0; we may of course assume $n_0 = 1$. Thus (y_n) is equivalent to (e_n) and its closed span Y is complemented in l_1. It is easy to see that the sequence $(x_n) \equiv (a_n e_1 + y_n)$ admits the same lower l_1-estimate as (y_n), hence it is equivalent to the basis (e_n), too. Finally, the subspace $\overline{\mathrm{lin}}\,(x_n)$, as the graph of the operator $\sum_n \eta_n y_n \to (\sum_n a_n \eta_n) e_1$, is complemented in $\mathrm{lin}\,\{e_1\} \oplus Y$, and the latter is obviously complemented in l_1.

The argument employed in the preceding paragraph, but without passing to the auxiliary sequence (y_n), works also in the case $z = 0$. ■

In the case of Banach spaces, the characterization of sets (or sequences) with property (V*) in (a) below is due to Bombal [3, Prop. 1.1] and Emmanuele [28].

PROPOSITION 3.5. *Let E be a* SC LCS *and let $(x_n) \in l_\infty(E)$. Then*

(a) $(x_n) \in v^*(E)$ *iff (x_n) has no subsequence equivalent to the standard basis of l_1 and spanning a complemented subspace of E.*
(b) $(x_n) \in r(E)$ *iff (x_n) has no subsequence equivalent to the standard basis of l_1.*

Proof. We only prove the sufficiency parts.

(a): Suppose $(x_n) \notin v^*(E)$ so that there is $T \in \mathcal{L}(E, l_1)$ for which $(Tx_n) \notin \kappa(l_1)$. By the proposition above, (x_n) has a subsequence (y_n) such that (Ty_n) is a basic sequence equivalent to the standard basis in l_1 and there is a projection P from l_1 onto the closed span of (Ty_n). By the lifting property of l_1, $T|\overline{\mathrm{lin}}\,(y_n)$ is an isomorphism, and $T^{-1} \circ P \circ T$ is a projection from E onto $\overline{\mathrm{lin}}\,(y_n)$.

(b): Suppose $(x_n) \notin r(E)$ so that there is $T \in \mathcal{L}(E, l_\infty)$ such that $(Tx_n) \notin c\kappa(l_\infty, \sigma)$. By Rosenthal's l_1-theorem, (x_n) has a subsequence (y_n) such that (Ty_n) is a basic sequence equivalent to the standard basis in l_1. By the lifting property of l_1, (y_n) has the same property. ∎

4. General spaces of bounded vector sequences. We establish here some additional notation and terminology that will be used throughout the paper in connection with spaces of bounded vector sequences. The most important property of such spaces introduced below is l_∞-subsequential determinedness (l_∞-SD).

Let E be a LCS.

4A. Basic definitions. By a *space of bounded E-sequences* we shall understand any topological (not necessarily linear) subspace K of $l_\infty(E)$. Such a space K is called

- *closed* if K is a closed subspace of $l_\infty(E)$;
- *invariant* if $\hat{T}(K) \subset K$ for every $T \in \mathcal{L}(E)$;
- *l_∞-stable* if $ax \in K$ whenever $x \in K$ and $a \in l_\infty$;
- *subsequence hereditary* (SH) if $x\pi_M \in K$ whenever $x \in K$ and $M \in [\mathbb{N}]$;
- *subsequentially determined* (SD) if K is SH and contains all $x \in l_\infty(E)$ which are *sporadically in* K, i.e. for every $N \in [\mathbb{N}]$ there exists $M \in [N]$ with $x\pi_M \in K$.

We will also need a stronger variant of the last property. Let $\mathcal{A}$ be a set of operators from E to a LCS G. Then a space K as above is said to be $\mathcal{A}$-*subsequentially determined* ($\mathcal{A}$-SD) if K is SH and contains all $x \in l_\infty(E)$ such that $\hat{S}(x)$ is sporadically in $\hat{S}(K)$ for every $S \in \mathcal{A}$. Note that, in particular, K

must then contain all $x \in l_\infty(E)$ such that $\hat{S}x \in \hat{S}(K)$ for every $S \in \mathcal{A}$. The spaces K which are $\mathcal{L}(E,G)$-SD are briefly called G-SD. The particular cases where G is l_∞ or ω are the most important for us. Clearly, K is ω-SD [l_∞-SD] iff K is SH and contains all sequences $(x_n) \in l_\infty(E)$ such that for every [equicontinuous] sequence (x'_n) in E' and every $N \in [\mathbb{N}]$ there exist $M \in [N]$ and $(y_m)_{m\in M} \in K$ with $\langle x_m, x'_n\rangle = \langle y_m, x'_n\rangle$ for all $m \in M$ and $n \in \mathbb{N}$.

4B. Additional definitions. For a space K of (strongly) bounded E'-sequences, and any set $\mathcal{A} \subset \mathcal{L}(G,E)$ as above, we shall say that K is *dually* $\mathcal{A}$-SD if K is $\mathcal{A}'$-SD, where $\mathcal{A}' := \{T' : T \in \mathcal{A}\}$. A particular case of this, the most important for us, is when $\mathcal{A} = \mathcal{L}(\lambda_1, E)$, where λ_1 denotes the subspace $\operatorname{lin}(e_n)$ of l_1; then we say that K is *dually* l_∞-SD. Note that $\mathcal{L}'(\lambda_1, E) =$ the space of operators $S\colon E' \to l_\infty$ of the form $S(x') = (\langle x_n, x'\rangle)_{n\in\mathbb{N}}$, where $(x_n) \in l_\infty(E)$. Also note that if E is SC, then l_1 can be used instead of λ_1 because $\mathcal{L}'(\lambda_1, E) = \mathcal{L}'(l_1, E)$. Thus K is dually l_∞-SD iff it contains all sequences $(x'_n) \in l_\infty(E')$ such that for every $(x_n) \in l_\infty(E)$ and every $N \in [\mathbb{N}]$ there exist $M \in [N]$ and $(y'_m)_{m\in M} \in K$ with $\langle x_n, x'_m\rangle = \langle x_n, y'_m\rangle$ for all $n \in \mathbb{N}$ and $m \in M$. (In other words, the sequences $(x'_m)_{m\in M}$ and $(y'_m)_{m\in M}$ coincide on the subspace $\overline{\operatorname{lin}}\,(x_n)$ of E.) Clearly, if K is, for instance, dually $\mathcal{L}(c_{00}, E)$-SD or dually $\mathcal{W}(l_1, E)$-SD, then it is dually l_∞-SD, and this in turn implies that it is l_∞-SD.

REMARKS 4.1. Let E and G be LCSs and let $K \subset l_\infty(E)$.

(a) Since $\overline{\operatorname{lin}}\,\{e_M : M \subset \mathbb{N}\} = l_\infty$ and, for every $x \in l_\infty(E)$, the map $T_x : a \mapsto ax$ from l_∞ to $l_\infty(E)$ is continuous, it is clear that: *If K is a closed linear subspace of $l_\infty(E)$, then it is l_∞-stable iff $e_M x \in K$ whenever $x \in K$ and $M \subset \mathbb{N}$.* Also note that this last condition is satisfied whenever K is SD.

(b) If K is closed, contains the zero sequence and is SD, then $K \supset c_0(E)$.

(c) Let G be a LCS. If $\mathcal{A} \subset \mathcal{B} \subset \mathcal{L}(E,G)$ and K is $\mathcal{A}$-SD, then K is also $\mathcal{B}$-SD. It follows that if G embeds continuously into another LCS H, and K is G-SD, then K is H-SD.

Before stating some important consequences of this, let us recall the following facts: Let G be a Banach (resp., Fréchet) space. Then G' is weak* separable (i.e., G' contains a sequence total on G) iff G embeds continuously into l_∞ (resp., ω); in particular, it is so when G is separable. If G is of infinite dimension, then l_∞ embeds continuously into G. If G is Fréchet and admits no continuous norm, then it contains an isomorphic copy of ω (cf. [66, 4.2.7]). Now, assuming G is of infinite dimension, we have:

1) if G is a Banach space with G' weak* separable, or G is a separable Fréchet space admitting a continuous norm, then K is G-SD iff it is l_∞-SD;
2) if G is a Fréchet space admitting no continuous norm and with G' weak* separable, then K is G-SD iff it is ω-SD.

We also note that if K is l_∞-SD, then it is ω-SD; the converse holds when E is a normed space.

(d) If K is $\mathcal{A}$-SD, then it is SD. Conversely, if K is SD and $\mathcal{A}$ contains a one-one operator, then K is $\mathcal{A}$-SD. In consequence, if E' is weak* separable [and E is normable], then K is SD iff it is ω-SD [l_∞-SD].

(e) Assume E is a normed space. Let D be a subset of E' whose linear span is norm dense. Then it is easy to see that K is l_∞-SD iff K contains all those $x \in l_\infty(E)$ for which the condition formulated at the end of Subsection 4A holds for all sequences (x'_n) in D. An analogous observation can be made about the definition of dually l_∞-SD spaces $K \subset l_\infty(E')$.

(f) Let G be a LCS and let $K \subset l_\infty(G)$ be SD. Then by a straightforward verification it is readily seen that for every LCS E the class $k(E) := \{(x_n) \in l_\infty(E) \colon (Sx_n) \in K,\ \forall S \in \mathcal{L}(E,G)\}$ is G-SD. If additionally G embeds continuously into l_∞, then by remark (c) the classes $k(E)$ are even l_∞-SD (comp. Theorem 8.3(a) below).

REMARKS 4.2. Denote $I = [0,1]$.

(a) Let K be the (nonlinear) space of all bounded sequences (x_n) in the Hilbert space $E := l_2(I)$ such that $\lim_{n\to\infty} \|x_n\chi_A\| = 1$, where A denotes the set of irrational numbers in I. Then K is closed and SD. However, it is not l_∞-SD. In fact, there is even an $x \in l_\infty(E) \setminus K$ such that $\hat{S}(x) \in \hat{S}(K)$ for all $S \in \mathcal{L}(E, l_\infty)$. For every $t \in I$ denote by e_t the tth unit vector of E (i.e., the characteristic function of $\{t\}$), and by e'_t the evaluation functional at t. Consider the constant sequence $x := (e_0, e_0, \dots) \in l_\infty(E)$; of course, $x \notin K$. If C is a countable subset of I, then choose any $s \in A \setminus C$ and note that $z_s := (e_0 + e_s, e_0 + e_s, \dots) \in K$. Moreover, $e'_t(e_0) = e'_t(e_0 + e_s)$ for every $t \in C$. From this (cf. also Remark 4.1(e)) it follows that x is as required.

(b) Define K to be the vector space of all bounded sequences (x_n) in $E := l_1(I) = c_0(I)'$ such that $\lim_{n\to\infty} \sum_{t\in I} x_n(t) = 0$. Then K is closed and SD. In fact, since E embeds isomorphically into l_∞ [61, Prop. 3.3], see also [14, Th. 1] for an even stronger result), K is l_∞-SD, by Remark 4.1(d). However, K is not dually l_∞-SD. This can be easily seen by a similar argument to (a), with e'_t's viewed as elements of the predual space $c_0(I)$ (viz., its unit vectors), and with z_s defined now as the sequence with all terms equal to $e_0 - e_s$.

5. General spaces of bounded vector sequences: Complementability and injectivity. Here we prove our basic tool, Theorem 5.3, and derive some of its immediate consequences. We start with a lemma which is a basis for the whole paper. It is related to the classical result of Phillips on the uncomplementability of c_0 in l_∞ [64, Cor. 7.5]; see also [74, p. 544], [9, Ch. IV, §5, Ex. 5a] and [49, §31.2.2]. An elegant proof based on some ideas of Whitley [79] can be found in [47, Prop. 4].

KEY LEMMA 5.1. *If $T\colon l_\infty \to \omega$ is an operator such that $T|c_0 = 0$, then there exists $M \in [\mathbb{N}]$ for which $T|l_\infty(M) = 0$.*

REMARK 5.2. The above lemma is still valid when one replaces ω by any metrizable TVS not containing a copy of l_∞. This follows quite easily from the extension of Rosenthal's l_∞ result given in [23] and [22].

THEOREM 5.3. *Let E be a LCS and let K be an ω-SD space of bounded E-sequences. If $x \in l_\infty(E)$ and there is an operator $T\colon l_\infty \to l_\infty(E)$ such that $T(l_\infty) \subset K$ and $Ta = ax$ for all $a \in c_0$, then $x \in K$.*

Proof. By assumption, the operators T and $T_x\colon l_\infty \to l_\infty(E)\,;\, a \mapsto ax$ coincide on c_0. Take any $S \in \mathcal{L}(E,\omega)$. Then the operator $\hat{S}\circ T - \hat{S}\circ T_x\colon l_\infty \to l_\infty(\omega)$ vanishes on c_0, and its range space embeds continuously into ω. Therefore, by the Key Lemma, for every $N \in [\mathbb{N}]$ there is $M \in [N]$ such that $\hat{S}\circ T(e_M) = \hat{S}\circ T_x(e_M) = (e_M(n)Sx_n)_{n\in\mathbb{N}}$. Since K is SH, we see that $(Sx_n)_{n\in M} \in \hat{S}(K)$. Finally, since K is ω-SD, we conclude that $x \in K$. ■

COROLLARY 5.4. *Let E be a LCS and let K_1 and K_2 be linear spaces of bounded E-sequences, both containing $c_0(E)$, such that K_1 is ω-SD and K_2 is l_∞-stable. If there exists an operator $T\colon K_2 \to K_1$ such that $T|c_0(E) = \mathrm{id}_{c_0(E)}$, then $K_2 \subset K_1$. In particular, if K_1 is a complemented subspace of K_2, then $K_1 = K_2$.*

Proof. Take an arbitrary $x \in K_2$ and apply Theorem 5.3 to the operator $T\circ T_x\colon l_\infty \to K_1$, where $T_x\colon l_\infty \to K_2\,;\, a \mapsto ax$. ■

COROLLARY 5.5. *Let E be a LCS and let K be an ω-SD linear space of bounded E-sequences. Then K is injective iff $K = l_\infty(E)$ and E is injective.*

Proof. Clearly, if E is injective, then $l_\infty(E)$ is injective. On the other hand, if K is injective, then it is complemented in $l_\infty(E)$, hence closed, so that, in view of Remark 4.1(b), it contains $c_0(E)$. Therefore, $K = l_\infty(E)$ by Corollary 5.4. Finally, E is injective as a complemented subspace of $l_\infty(E)$. ■

COROLLARY 5.6. *Let E be a Fréchet space and let K be an ω-SD Fréchet space of bounded E-sequences. Then K is complemented in a dual Fréchet space iff $K = l_\infty(E)$ and E is complemented in a dual Fréchet space.*

Proof. Suppose that E is complemented in F', where F is a bornological DF-space. Then $l_\infty(E)$ is complemented in $l_\infty(F')$ and $l_\infty(F') \simeq (l_1 \hat{\otimes}_\pi F)'$, where $l_1 \hat{\otimes}_\pi F$ is a bornological DF-space (see [18, Lemma 2]).

Conversely, assume that there is a projection $P\colon K'^{\times} \to K$. Since $K \supset c_0(E)$ (see Remark 4.1(b)), K contains complemented isomorphic copies of E. Hence E is complemented in a dual Fréchet space. To see that $l_\infty(E) = K$, take any $x \in l_\infty(E)$ and consider the map $S\colon c_0 \to K\,;\, a \to ax$. By applying Theorem 5.3 to the operator $P\circ S'^{\times}\colon l_\infty = {c_0'}^{\times} \to K$, we easily see that $x \in K$. ■

6. Applications to concrete spaces of bounded vector sequences. Now, we apply the results of the preceding section to many familiar spaces of bounded vector sequences:

THEOREM 6.1. *Let E be a* LCS.

(A) *The following spaces of bounded E-sequences are closed and l_∞-*SD*:*

(A1) $l_\infty(E)$, $c_0(E)$, $p\kappa(E)$, $c_0(E,\sigma)$, $l(E)$, $v^*(E)$, $r(E)$;
(A2) $c\kappa(E,\sigma)$ *if E is metrizable;*
(A3) $\kappa(E)$ *and* $\kappa(E,\sigma)$ *if E is* QC.

(B) *The following spaces of bounded E'-sequences are closed and dually l_∞-*SD*:*

(B1) $c_0(E',\sigma')$, $c_0(E',\mu')$, $p\kappa(E',\mu')$, *and* $v(E')$;
(B2) $\kappa(E',\mu')$ *if E is quasi-$\aleph_0$-barrelled.*

Proof. The assertions in (A) and (B1) are either straightforward or easily derived with the aid of Proposition 3.1 and Remark 4.1(f) from the fact that the spaces $\kappa(l_\infty)$, $\kappa(l_\infty,\sigma)$, etc. occurring in Proposition 3.1 are closed and SD. (To see that $\kappa(l_\infty,\sigma)$ is SD one needs Eberlein's theorem.)

As for (B2), if E is quasi-$\aleph_0$-barrelled, then $\kappa(E',\mu') = p\kappa(E',\mu')$. Indeed, for such E (strongly) bounded countable subsets of E' are equicontinuous. Therefore, the $\sigma(E',E)$-closed absolutely convex hull of every sequence (x_n) in $l_\infty(E')$ is $\sigma(E',E)$-compact, hence $\mu(F',F)$-complete by [43, 3.2.4]. ∎

Since all the spaces in Theorem 6.1 are obviously l_∞-stable, directly from Corollary 5.4 we now obtain the following.

COROLLARY 6.2. *If K_1 and K_2 are two spaces listed in* (A) *or* (B) *of Theorem* 6.1 *and $K_1 \subset K_2$, then K_1 is complemented in K_2 iff $K_1 = K_2$.*

REMARKS 6.3. (a) For the sake of completeness and future references, we characterize the equalities $K_1 = K_2$, for various pairs of spaces occurring in Theorem 6.1, in terms of some well-known properties of E, F or F'. In some cases these characterizations are just the definitions of the right hand side notions. Let us point out that the inclusions $\subset$ between the pairs of spaces below and in (b), with the exception of (3) and (5), always hold. Most of them are fairly obvious; for those less obvious see Subsection 3C and Proposition 3.3.

(1) $c_0(E) = c_0(E,\sigma)$ *iff E has the sequential Schur property.* (Definition.)
(2) $\kappa(E) = \kappa(E,\sigma)$ *iff E has the Schur property.* (Definition.)
(3) *For E* QC*: $\kappa(E) = c\kappa(E,\sigma)$ iff E has the sequential Schur property and compact sets in E are sequentially compact.*
(4) *For E* QC*: $\kappa(E) = l_\infty(E)$ iff E is a semi-Montel space.* (Definition.)

(5) *For E* QC*: $\kappa(E,\sigma)=c\kappa(E,\sigma)$ iff E is weakly* SC *and weakly compact sets in E are weakly sequentially compact.*
(6) *For E* QC*: $\kappa(E,\sigma)=l_\infty(E)$ iff E is semi-reflexive.* (See [49, 23.3(1)]).
(7) *For E* SC*: $r(E)=l_\infty(E)$ iff E has no subspace isomorphic to l_1.* (This follows easily from Rosenthal's l_1-theorem [12, p. 201] applied to sequences in l_∞.)
(8) *$c_0(E')=c_0(E',\sigma)$ iff E' has the sequential Schur property.* (Definition.)
(9) *For E Banach: $c_0(E')=c_0(E',\mu')$ iff E contains no copy of l_1.* (Apply [18, Lemma 1 and Remark (c)]).
(10) *For E Fréchet or quasi-normable: $c_0(E')=c_0(E',\sigma')$ iff E is a semi-Montel space.* (Follows from [8] and [54].)
(11) *$c_0(E',\sigma)=c_0(E',\sigma')$ iff E has the Grothendieck property.* (Definition.)
(12) *$\kappa(E')=\kappa(E',\sigma)$ iff E' has the Schur property.* (Definition.)
(13) *For E Banach: $\kappa(E')=\kappa(E',\mu')$ iff E contains no copy of l_1.* (Apply [26]; see also [17, Prop. 3].)
(14) *For E* QC *and quasi-$\aleph_0$-barrelled: $\kappa(E',\mu')=l_\infty(E')$ iff E has the Schur property.* (The proof is given below.)
(15) *For E quasi-$\aleph_0$-barrelled: $v(E')=l_\infty(E')$ iff E contains no copy of c_{00}.* (Follows easily from Subsection 3C.)
(16) *$\kappa(E',\sigma)=v(E')$ iff E has property* (V). (Definition; see Subsection 3C.)
(17) *$p\kappa(E)=l(E)$ iff E has the Gelfand–Phillips property.* (Definition; see Subsection 3B.)
(18) *For E Fréchet or quasi-normable: $l(E)=l_\infty(E)$ iff E is a semi-Montel space.* (Follows from [8] and [54].)

Proof of (14). If E has the Schur property then, since E is QC, $\mu(E',E)$ is the same as the topology of uniform convergence on precompact sets in E. Therefore, to see that $\kappa(E',\mu')=l_\infty(E')$, it is enough to apply the Alaoglu–Bourbaki theorem [43, 8.5.2] and the quasi-$\aleph_0$-barrelledness of E. For the converse, suppose E fails the Schur property. Since E is QC, this means that there exists a weakly compact absolutely convex subset A of E which is not precompact. In view of Proposition 3.1(a), we can find a sequence (x_n) in A and an operator $T\colon E\to l_\infty$ such that $(Tx_n)\notin\kappa(l_\infty)$. Since (Tx_n) is contained in the weakly compact subset $T(A)$ of l_∞, we may assume that (Tx_n) is weakly convergent to some $y_0\in T(A)$. As is easily seen, there is no loss of generality in assuming that $y_0=0$. Now, by replacing (x_n) with a subsequence if necessary, we may assume that (Tx_n) is a basic sequence in l_∞. Extending, by the Hahn–Banach theorem, the associated coefficient functionals to all of l_∞, we obtain a bounded sequence $(y_n')\subset(l_\infty)'$ which is biorthogonal to (Tx_n). Then the sequence $(x_n'):=(T'y_n')\subset E'$ is equicontinuous and biorthogonal to (x_n). It follows that $\sup_{x\in A}|\langle x,x_m'-x_n'\rangle|\ge 1$

whenever $m \neq n$ so that (x'_n) is not μ'-precompact. Clearly, this is impossible if $\kappa(E',\mu') = l_\infty(E')$. ■

Note: An alternative proof of (14) can be given in which, assuming that a sequence $(x_n) \in l_\infty(E)$ is not precompact, a result of Grothendieck [39, p. 134] is used to show that (x_n) cannot be relatively weakly compact.

As a supplement to (8) and (12), let us mention that if E is a Banach space, then E' has the Schur property iff E has the Dunford–Pettis property and E contains no copy of l_1 (see [63] or apply [12, XI.Ex. 4].

(b) The list above is obviously incomplete. For example, the authors do not know of any characterizations of the following equalities:

- $c_0(E',\mu') = c_0(E',\sigma')$;
- $\kappa(E') = v(E')$;
- $\kappa(E',\mu') = v(E')$;
- $l(E') = \kappa(E',\mu')$;
- $l(E) = c\kappa(E,\sigma)$;
- $l(E') = v(E')$.

(c) There is an extensive literature on sequential characterizations (in terms of sequences in the space or in its dual) of some properties of Banach spaces, Fréchet spaces or DF-spaces. What was said above summarizes much of the story (see also [4], [8], [11], [12], [18], [56], [60], [63], and a nice review in [6] of the results connected with the Josefson–Nissenzweig theorem for Fréchet spaces).

There are also pairs of vector sequence spaces among those appearing in Theorem 6.1 which are never equal. For the sake of completeness they are listed below.

COROLLARY 6.4. *If $E \neq \{0\}$, then:*

(1) $c_0(E)$ *is not complemented in* $\kappa(E)$.
(2) $c_0(E,\sigma)$ *is not complemented in* $\kappa(E,\sigma)$.
(3) $c_0(E',\mu')$ *is not complemented in* $\kappa(E',\mu')$.
(4) $c_0(E',\sigma')$ *is not complemented in* $l_\infty(E')$.

Proof. All these assertions are fairly direct consequences of the uncomplementability of c_0 in l_∞; we need not appeal to Corollary 6.2. The argument is exactly the same in each case. For instance, in order to show (1), fix $0 \neq x \in E$ and define $K = \{(a_n x) : (a_n) \in l_\infty\}$ and $K_0 = \{(a_n x) : (a_n) \in c_0\}$. Then $l_\infty \simeq K \subset \kappa(E)$, $c_0 \simeq K_0 \subset c_0(E)$, and $K \cap \kappa(E) = K_0$. Hence $c_0(E)$ cannot be complemented in $\kappa(E)$. ■

Our next result identifies the injective spaces among the spaces of sequences we are dealing with. It follows by applying Corollary 5.5 and Theorem 6.1, along with some additional facts, indicated in the proof below. Note that, in view of the preceding corollary, the c_0-type spaces of bounded E- or E'-sequences are never injective (unless $E = \{0\}$).

COROLLARY 6.5.

(1) *Each of the spaces* $\kappa(E)$, $\kappa(E,\sigma)$, $l(E)$, $r(E)$ *is injective iff* $E \simeq \mathbb{K}^{\mathfrak{m}}$ *for some cardinal* $\mathfrak{m}$.
(2) $v^*(E)$ *is injective iff* E *is injective.*
(3) *If* E *is* QC *and quasi-*$\aleph_0$*-barrelled, then* $\kappa(E',\mu')$ *is injective iff* E' *is injective and* E *has the Schur property.*
(4) *If* E *is quasi-*$\aleph_0$*-barrelled, then* $v(E')$ *is injective iff* E' *is injective.*

We note that (1), slightly rephrased, says that the sequence spaces in question are injective iff they coincide with $l_\infty(\mathbb{K}^{\mathfrak{m}}) \simeq l_\infty^{\mathfrak{m}}$.

Proof of Corollary 6.5. In proving (1), we make use of the following result from [20] and [57]: If an injective space is not isomorphic to a product space $\mathbb{K}^{\mathfrak{m}}$, then it must contain a copy of l_∞. One should also observe that, obviously, the spaces $\kappa(l_\infty)$, $\kappa(l_\infty,\sigma)$, $l(l_\infty)$ and $r(l_\infty)$ are proper subspaces of $l_\infty(l_\infty)$. To derive (3), we also appeal to Remark 6.3(a)(14). For (2) and (4), we supply more details.

(2): We need to verify that if E is injective, then $v^*(E) = l_\infty(E)$. This is an easy consequence of Proposition 3.5. An alternative, and more direct, argument runs as follows: We have to show that every operator $T\colon E \to l_1$ is compact (maps bounded sequences to relatively compact sequences). It is known that this holds for $E = l_\infty$ (see Remark 8.15 below). Fix $(x_n) \in l_\infty(E)$ and $T \in \mathcal{L}(E, l_1)$. We may consider E a subspace of a product $\prod_{j\in J} l_\infty(\Gamma_j) =: G$, where each of the sets Γ_j is infinite. Let P be a projection from G onto E; set $S = TP$. There exists a finite subset K of J such that $S = SQ$, where Q denotes the natural projection from G onto its subspace $G_K := \prod_{j\in K} l_\infty(\Gamma_j)$. Since $G_K \simeq l_\infty(\Gamma)$, where $\Gamma := \bigcup_{j\in K}\Gamma_j$, there exists in G_K a copy of l_∞ containing the sequence (Qx_n) (see Remark 8.2 below). In consequence, $(Tx_n) = (SQx_n) \in \kappa(l_1)$.

(4): In view of Remark 6.3(a)(15), $v(E')$ is injective iff E' is injective and E contains no copy of c_{00}. We show that the latter property is already implied by the injectivity of E'. For, suppose c_{00} is a subspace of E. By the Hahn–Banach theorem, we can extend the coordinate functionals on c_{00} to an equicontinuous sequence $(z_n') \subset E'$. Then the operator $S\colon z' \mapsto z'|c_{00}$ maps E' onto $c_{00}' = l_1$, and the operator $T\colon (a_n) \mapsto \sum_n a_n z_n'$ is an isomorphic embedding of l_1 into E'. Finally, the operator TS is a projection from E' onto its subspace $T(l_1) \simeq l_1$. This is impossible: an injective space cannot contain a complemented copy of l_1, a noninjective space. ∎

REMARK 6.6. We conjecture that if E is a Banach space, then $\kappa(E',\mu')$ is injective iff $E \simeq l_1(\Gamma)$ for some Γ.

The abbreviation *dFs-complemented* used below means 'complemented in a dual Fréchet space' or, more precisely, 'isomorphic to a complemented subspace in a dual Fréchet space' (for the latter notion see Section 2). Applying Corollary 5.6 and Theorem 6.1, along with appropriate parts of Remark 6.3(a), we now obtain the following.

COROLLARY 6.7. *Let E be a Fréchet space and F a DF-space.*

(1) *$\kappa(E)$ is dFs-complemented iff E is a Montel space.*
(2) *$\kappa(E,\sigma)$ is dFs-complemented iff E is reflexive.*
(3) *$c\kappa(E,\sigma)$ is dFs-complemented iff E is dFs-complemented and has no subspace isomorphic to l_1.*
(4) *$l(E)$ is dFs-complemented iff E is a Montel space.*
(5) *If F is* QC, *then $\kappa(F',\mu')$ is dFs-complemented iff F has the Schur property.*
(6) *$v(F')$ is dFs-complemented iff F has no subspace isomorphic to c_{00}.*
(7) *$c_0(E)$, $c_0(E,\sigma)$, $c_0(F',\sigma')$ and $c_0(F',\mu')$ are not dFs-complemented if $E\neq\{0\}$.*

7. Invariant spaces of bounded vector sequences. Let E be a LCS. We shall say that a linear space $K\subset l_\infty(E)$ is *nice* if it is closed, invariant, and SD. (Note that the results of Section 5 are applicable for such spaces K.) Recall the basic scheme:

$c_0(E)$ is a subspace of both $\kappa(E)$ and $c_0(E,\sigma)$, and each of the latter spaces is contained in $\kappa(E,\sigma)\subset c\kappa(E,\sigma)\subset r(E)\subset v^*(E)$; moreover, $\kappa(E)\subset l(E)\subset r(E)$; finally, all these spaces are subspaces of $l_\infty(E)$, and all are nice.

We are interested in the position occupied in this scheme by an arbitrary nice space K, and in identifying, if possible, all the nice spaces $K\subset l_\infty(E)$. We will show that for some classical Banach spaces E there are only few such spaces K and that, for an arbitrary SC space E, there always exists a maximal nice space $K\neq l_\infty(E)$.

We first make a few general observations. Let a space $K\subset l_\infty(E)$ be nice. Then K is l_∞-stable (Remark 4.1(a)) and contains $c_0(E)$ (Remark 4.1(b)); thus $c_0(E)$ is the smallest nice space K. Moreover, denoting by C the space of constant E-sequences, it is clear that, by invariance, either $C\subset K$ or $C\cap K=\{0\}$. Also note that the intersection of any family of nice spaces $K\subset l_\infty(E)$ is again a nice space.

PROPOSITION 7.1. *Let $K\subset l_\infty(E)$ be a nice space.*

(a) *If $C\subset K$, then $\kappa(E)\subset K$.*
(b) *If $C\cap K=\{0\}$, then $K\subset c_0(E,\sigma)$.*

(c) *If weakly compact sets in E are sequentially weakly compact and $c_0(E,\sigma) \subset K$, then either $\kappa(E,\sigma) \subset K$ or $K = c_0(E,\sigma)$, depending on whether or not $C \subset K$.*
(d) *If E is injective and $K \not\subset r(E)$, then $K = l_\infty(E)$.*
(e) *If E is* SC *and $K \not\subset v^*(E)$, then $K = l_\infty(E)$.*

In consequence, there is no strictly intermediate nice space between $c_0(E)$ and $\kappa(E)$, and if E is SC *(resp., injective) and of infinite dimension, then $v^*(E)$ (resp., $r(E)$) is the largest proper nice subspace of $l_\infty(E)$.*

Proof. (a) is obvious; (c) follows easily from (b).

(b): Suppose this is false. Then, as K is SH, K must contain a sequence (x_n) such that $0 \neq r_n := \langle x_n, x'\rangle \to r \neq 0$ for some $x' \in E'$. Since K is l_∞-stable, $(y_n) := (r_n^{-1}x_n) \in K$ and $\langle y_n, x'\rangle = 1$ for all n. Now take any $x_0 \in E$ with $\langle x_0, x'\rangle = 1$ and define an operator $T\colon E \to E$ by $Tx = \langle x, x'\rangle x_0$. Then $Tx_n = x_0$ for all n, hence $(x_0, x_0, \dots) \in K$, contrary to the assumption.

(d): As K is SH, from the assumption and Proposition 3.5(b) it follows that there exists a sequence $(x_n) \in K$ which is basic and equivalent to the standard basis of l_1. Using this and the injectivity of E, for every $(y_n) \in l_\infty(E)$ we can find $T \in \mathcal{L}(E)$ such that $Tx_n = y_n$ for all n. By invariance, $(y_n) \in K$. Therefore, $K = l_\infty(E)$.

(e): Use Proposition 3.5(a) and slightly modify the preceding argument (cf. the proof of Theorem 8.14(a) below). ∎

Proposition 7.2.

(a) *There are only three distinct nice spaces $K \subset l_\infty(l_1)$:*

$$c_0(l_1), \quad \kappa(l_1) \quad \textit{and} \quad l_\infty(l_1).$$

(b) *If $1 < p < \infty$, then there are precisely four distinct nice spaces $K \subset l_\infty(l_p)$:*

$$c_0(l_p), \quad \kappa(l_p), \quad c_0(l_p,\sigma) \quad \textit{and} \quad l_\infty(l_p).$$

(c) *There are precisely five distinct nice spaces $K \subset l_\infty(c_0)$:*

$$c_0(c_0), \quad \kappa(c_0), \quad c_0(c_0,\sigma), \quad \kappa(c_0,\sigma) \quad \textit{and} \quad l_\infty(c_0).$$

Remark 7.3. The problem of characterizing nice spaces $K \subset l_\infty(l_\infty)$ remains open. It is especially interesting in view of Corollary 8.9.

Proof of Proposition 7.2. (a): By the Schur property of l_1, we have $c_0(l_1) = c_0(l_1,\sigma)$ and $\kappa(l_1) = \kappa(l_1,\sigma)$. Suppose a nice space $K \subset l_\infty(l_1)$ is different from these two spaces. Then, by Proposition 7.1(c), $\kappa(l_1)$ is a proper subspace of K. Let (x_n) be a non-relatively-compact sequence in K. By Proposition 3.4, we may assume that (x_n) is equivalent to the standard basis of l_1 and that there is a projection P from l_1 onto $\overline{\mathrm{lin}}\,(x_n)$. Now, if $(y_n) \in l_\infty(l_1)$, then there is an operator $S\colon l_1 \to l_1$ with $S(x_n) = y_n$ for all n. In consequence, for the operator $T := SP \in \mathcal{L}(l_1)$ we have $T(x_n) = y_n$ for all n. Therefore, by the invariance of K, it follows that $(y_n) \in K$. Thus $K = l_\infty(l_1)$.

(b): Assume $C \cap K = \{0\}$. Then, by Proposition 7.1(b), $c_0(l_p) \subset K \subset c_0(l_p, \sigma)$. Suppose $K \neq c_0(l_p)$. Then K contains a nonconvergent weakly null sequence (x_n). By passing to a subsequence, we may assume that (x_n) is equivalent to the standard basis of l_p and that there is a projection P from l_p onto $E_0 := \overline{\operatorname{lin}}\,(x_n)$. Now, take any $(y_n) \in c_0(l_p, \sigma)$. Then either $(y_n) \in c_0(l_p) \subset K$ or else (y_n) has a subsequence (z_n) which is equivalent to the standard basis of l_p. Hence there exists an isomorphic embedding $S\colon E_0 \to l_p$ such that $S(x_n) = z_n$ for all n. In consequence, for the operator $T := SP \in \mathcal{L}(l_p)$ we have $T(x_n) = z_n$ for all n. By the invariance of K, $(z_n) \in K$. Since K is SD, we conclude that $c_0(l_p, \sigma) \subset K$. Thus $K = c_0(l_p, \sigma)$.

Now assume that $C \subset K$. Then, by Proposition 7.1(a), $\kappa(l_p) \subset K$. Suppose $K \neq \kappa(l_p)$. Then, since l_p is reflexive and K is SH, K must contain a sequence (x_n) which is weakly, but not norm, convergent. Since $C \subset K$, we may assume that (x_n) is weakly null. Now, by the same reasoning as above, we obtain $c_0(l_p, \sigma) \subset K$. Hence, by Proposition 7.1(c), $l_\infty(l_p) = \kappa(l_p, \sigma) \subset K$.

(c): Let $K \subset l_\infty(c_0)$ be nice. Much as in the first part of the proof of (b) above, we can show that if $C \cap K = \{0\}$, or equivalently $c_0(c_0) \subset K \subset c_0(c_0, \sigma)$, then either $K = c_0(c_0)$ or $K = c_0(c_0, \sigma)$. Assume therefore that $C \subset K$. Then $\kappa(c_0) \subset K$. Suppose the inclusion is proper.

If $K \subset \kappa(c_0, \sigma)$ then, as $C \subset K$, it is clear that K must contain a nonconvergent weakly null sequence. Therefore, $K \cap c_0(c_0, \sigma) \neq c_0(c_0)$. Since also $K \cap c_0(c_0, \sigma)$ is a nice space, from the preceding part of this proof we know that it must coincide with $c_0(c_0, \sigma)$. Hence $c_0(c_0, \sigma) \subset K$; in consequence, $\kappa(c_0, \sigma) \subset K$. Thus $K = \kappa(c_0, \sigma)$.

If $K \not\subset \kappa(c_0, \sigma)$, then we can find a sequence (x_n) in K which is weakly Cauchy but not weakly convergent. Thus (x_n) converges coordinatewise to some $z \in l_\infty \setminus c_0$. Let $x_n = (\xi_{ni})$ and $z = (\zeta_i)$. Since $z \notin c_0$, there is $\varepsilon > 0$ for which the set $N := \{i\colon |\zeta_i| \geq \varepsilon\} = \{k_1 < k_2 < \cdots\}$ is infinite. Define an operator $T\colon c_0 \to c_0$ by the formula $T((\xi_i)_{i\in\mathbb{N}}) = (\zeta_{k_i}^{-1}\xi_{k_i})_{i\in\mathbb{N}}$. Since K is invariant, the sequence (Tx_n) is in K and converges coordinatewise to the constant sequence $(1, 1, \ldots)$. We may therefore assume that our original sequence (x_n) has this property, i.e., $z = (1, 1, \ldots)$.

We now define by induction two strictly increasing sequences (m_n) and (p_n) in $\mathbb{N}$ so that $p_1 = 1$ and

$$\max_{1\leq i\leq p_n} |\xi_{m_n i} - 1| < n^{-1} \quad \text{and} \quad \max_{i\geq p_{n+1}} |\xi_{m_n i}| < n^{-1} \quad \text{for every } n.$$

Consider the operator $S\colon c_0 \to c_0$ given by $S((\xi_i)_{i\in\mathbb{N}}) = ((\xi_{p_i})_{i\in\mathbb{N}})$. Then, denoting $y_n = Sx_{m_n}$ and $w_n = e_1 + \cdots + e_n$, we have $(y_n) \in K$, by the invariance of K. In addition, $\|y_n - w_n\| < n^{-1}$ for every n so that $(y_n - w_n) \in c_0(c_0) \subset K$. It follows that $(w_n) \in K$.

Now take any sequence $(z_n) \in l_\infty(c_0)$. By passing to a subsequence we may assume that it is weakly Cauchy. Then, since c_0 has Pełczyński's property (u) (see [53, Prop. 1.c.2]), there exists a weakly unconditionally convergent series $\sum_{n=1}^{\infty} u_n$ in c_0 such that the sequence $(z_n - v_n)$, where $v_n := \sum_{j=1}^{n} u_j$, is weakly null. Let $T \in \mathcal{L}(c_0)$ be such that $Te_n = u_n$ for every n. Then $(Tw_n) = (v_n)$, hence $(v_n) \in K$, by the invariance of K. Since $(z_n - v_n) \in c_0(c_0, \sigma) \subset K$, also $(z_n) \in K$. Since K is SD, we conclude that $K = l_\infty(c_0)$. ∎

Remark 7.4. Proposition 7.2 is also valid for the spaces $l_p(\Gamma)$ ($1 \le p < \infty$) and $c_0(\Gamma)$, where Γ is any infinite set. This is obvious if one remembers that since the elements of these spaces have countable supports, every sequence in $l_p(\Gamma)$ or $c_0(\Gamma)$ is contained in $l_p(A) \simeq l_p$ or $c_0(A) \simeq c_0$ for some countable set $A \subset \Gamma$.

8. bs-functors. In this section we distinguish a class of functors satisfying some very natural and simple conditions, and "producing" l_∞-SD spaces of bounded vector sequences. The crucial result obtained here is Corollary 8.10; rougly speaking, it means that practically all spaces of bounded sequences one usually considers are of the type needed in Theorem 5.3.

Throughout this section, $\mathcal{E}$ is a fixed category of LCSs (see Section 2 for the additional requirements which we always tacitly impose on $\mathcal{E}$). By a *bs-functor* on $\mathcal{E}$ we understand a map k which assigns to every space $E \in \mathcal{E}$ a vector space $k(E)$ of bounded E-sequences in such a way that

(k1) $\qquad \hat{T}(k(E)) \subset k(F) \quad$ for all E, $F \in \mathcal{E}$ and $T \in \mathcal{L}(E, F)$.

Clearly, if $E, F \in \mathcal{E}$ and $T \in \mathcal{L}(E, F)$, then by the operator associated by k with T one should understand the operator $\hat{T}: k(E) \to k(F)$. Note that, in particular, (k1) implies that each $k(E)$ is invariant.

A bs-functor k on $\mathcal{E}$ is said to be closed, SH, SD or G-SD if $k(E)$ has the respective property for each $E \in \mathcal{E}$.

Proposition 8.1. *Let k be a bs-functor on $\mathcal{E}$.*

(a) *Let $E, F \in \mathcal{E}$. If $T: E \to F$ is an isomorphic embedding, then $\hat{T}$ is an isomorphism from $k(E)$ onto $k(\hat{T}(E)) \subset k(F)$. In particular, if E is a subspace of F, then $k(E) \subset k(F)$. Moreover, if E is a complemented subspace of F, then $k(E) = l_\infty(E) \cap k(F)$.*
(b) *If $E_j \in \mathcal{E}$ ($j \in J$), $E := \prod_{j\in J} E_j \in \mathcal{E}$, and J is finite or $k(E)$ is closed, then*

$$k(E) \simeq \prod_{j\in J} k(E_j)$$

under the natural isomorphism T between $l_\infty(E)$ and $\prod_{j\in J} l_\infty(E_j)$ given by $T(x) = (\hat{P}_j(x))_{j\in J}$, where $P_j\colon E \to E_j$ is the natural projection. In other words, $k(E)$ is precisely the set of all $x \in l_\infty(E)$ such that $\hat{P}_j(x) \in k(E_j)$ for every $j \in J$.

Proof of (b). Let $I_j\colon E_j \to E$ be the natural embeddings. Clearly,

$$\hat{I}_j(k(E_j)) = k(I_j(E_j)) \subset k(E) \quad \text{and} \quad \hat{P}_j(k(E)) \subset k(E_j),$$

for every $j \in J$. Hence for every finite subset A of J,

$$K_A := \prod_{j\in A} k(E_j) \times \{0\}^{J\setminus A} = T\Big(\sum_{j\in A} \hat{I}_j(k(E_j))\Big) \subset T(k(E)) \subset \prod_{j\in J} k(E_j) =: K_J.$$

If J is finite, then $\sum_{j\in J} \hat{I}_j(k(E_j)) = k(E)$ and all the inclusions above (with $A = J$) become equalities. In turn, assume $k(E)$ is closed in $l_\infty(E)$. Since T is an isomorphism, $T(k(E))$ is closed in $\prod_{j\in J} l_\infty(E_j)$ and a fortiori in K_J. Moreover, the union of all the finite products K_A is dense in K_J. It follows that $T(k(E)) = K_J$. ∎

Remark 8.2. We point out here a sufficient condition for a space $k(E)$ to be l_∞-SD, different from those already known from Remark 4.1(c, d).

Let k be a bs-functor on a category $\mathcal{E}$. Assume that a space $E \in \mathcal{E}$ satisfies the following condition:

(†) For every $(x_n) \in l_\infty(E)$ there exists a complemented subspace F of E containing (x_n) and such that $F \in \mathcal{E}$ and $k(F)$ is l_∞-SD.

Then also $k(E)$ is l_∞-SD.

Indeed, let $x = (x_n) \in l_\infty(E)$ be such that $\hat{T}(x)$ is sporadically in $\hat{T}(k(E))$ for every $T \in \mathcal{L}(E, l_\infty)$. Choose F according to (†), and let P be a projection from E onto F. Take any $S \in \mathcal{L}(F, l_\infty)$ and define $T = SP$. Then $\hat{S}(x) = \hat{T}(x)$ so that, by assumption and (k1), $\hat{S}(x)$ is sporadically in $\hat{T}(k(E)) = \hat{S}(\hat{P}(k(E)) \subset \hat{S}(k(F))$. Since $k(F)$ is l_∞-SD, we conclude that $x \in k(F) \subset k(E)$.

For example, let $\mathcal{E}$ contain every closed subspace of each of its members. Then condition (†) holds for every Fréchet space $E \in \mathcal{E}$ with the separable complementation property. (Recall that this means that in E every separable subspace is contained in a complemented separable subspace.) Thus, in particular, it is so when E is a weakly compactly generated Banach space (as is any space $L_p(\mu)$ for $1 < p < \infty$), or when $E = L_1(\mu)$. Condition (†) also holds for the spaces $l_\infty(\Gamma)$ whenever they belong to $\mathcal{E}$. This is because every separable subspace X of $l_\infty(\Gamma)$ is contained in a subspace Y which is isometric to l_∞. To see this, using the separability of X choose a countable subset A of Γ so that the map $J\colon x \mapsto 1_A x$ is an isometric embedding of X into $l_\infty(A)$. Let e'_γ ($\gamma \in \Gamma$) be the evaluation (or coordinate) functionals on $l_\infty(\Gamma)$. For every $\gamma \in \Gamma \setminus A$ consider the functional $e'_\gamma \circ J^{-1}$ on $J(X)$, and let f_γ be its norm preserving extension to $l_\infty(A)$. In addition, let $f_\gamma := e'_\gamma | l_\infty(A)$ for $\gamma \in A$. Finally, define an operator $S\colon l_\infty(A) \to l_\infty(\Gamma)$ by $S(y) = (f_\gamma(y))_{\gamma\in\Gamma}$. Then the subspace $Y := S(l_\infty(A))$ is as required.

A bs-functor k on $\mathcal{E}$ is called *injective* if

(k2) $\quad k(E) = l_\infty(E) \cap k(F)$

whenever $E, F \in \mathcal{E}$ and E is a subspace of F.

It is said to be *productive* if whenever $(E_j)_{j\in J}$ is a family of spaces in $\mathcal{E}$ whose product is also in $\mathcal{E}$, we have $k(\prod_{j\in J} E_j) \simeq \prod_{j\in J} k(E_j)$. (Here, and in other similar situations below, the isomorphism is understood as in Proposition 8.1(b).)

A LCS G is called *injective for the category* $\mathcal{E}$ (briefly $\mathcal{E}$-*injective*) provided that whenever $E, F \in \mathcal{E}$ and E is a subspace of F, every $S \in \mathcal{L}(E,G)$ has an extension $T \in \mathcal{L}(F,G)$.

THEOREM 8.3. *Let G be a* LCS, *and let K be an invariant vector space of bounded G-sequences.*

(a) *The formula*

$$k(E) = \bigcap_{S\in\mathcal{L}(E,G)} \hat{S}^{-1}(K) = \{(x_n) \in l_\infty(E)\colon (Sx_n) \in K,\ \forall S \in \mathcal{L}(E,G)\}$$

defines a bs-functor k on $\mathcal{LCS}$ such that $k(G) = K$. If K is closed, or SH, *or* SD, *then k is closed, or* SH, *or G-*SD, *respectively; and if $c_0(G) \subset K$, then $c_0(E) \subset k(E)$ for every $E \in \mathcal{E}$. Moreover, if G is complete and E is a dense subspace of a* LCS *F, then $k(E) = l_\infty(E) \cap k(F)$.*

(b) *If G is a normed space, then the bs-functor k is productive.*

(c) *If the space G is injective for the category $\mathcal{E}$, then the restriction to $\mathcal{E}$ of the bs-functor k is injective.*

The bs-functor k defined above, as well as its restriction to any subcategory of $\mathcal{LCS}$, will be called *generated* by the space K.

REMARK 8.4. Part (a) of the theorem, via Remark 4.1(f), has already been used in the proof of Theorem 6.1.

Proof of Theorem 8.3. (a): Straightforward verifications.

(b): Assume that $E_j \in \mathcal{E}$ ($j \in J$) and that $E := \prod_{j\in J} E_j \in \mathcal{E}$. Let $P_j\colon E \to E_j$ be the natural projections. We have to show that if $x \in l_\infty(E)$ is such that $\hat{P}_j(x) \in k(E_j)$ for every $j \in J$, then $x \in k(E)$. Let $S \in \mathcal{L}(E,G)$. Then there exist a finite subset A of J and operators $T_j\colon E_j \to G$ ($j \in A$)

such that $S=\sum_{j\in A} T_j\circ P_j$. Now, by assumption, $\hat{P}_j(x)\in k(E_j)$, hence $\hat{T}_j\circ\hat{P}_j(x)\in K$ for every $j\in A$. By linearity, $\hat{S}(x)\in K$.

(c): Let $E,F\in\mathcal{E}$, E a subspace of F, and let $x\in l_\infty(E)\cap k(F)$. Take any $S\in\mathcal{L}(E,G)$. By the $\mathcal{E}$-injectivity of G, S has an extension $T\in\mathcal{L}(F,G)$. Then $\hat{S}x=\hat{T}x\in K$ by the definition of $k(F)$. Therefore, $x\in k(E)$ by the definition of $k(E)$. ∎

Remarks 8.5. (a) An injective bs-functor k on $\mathcal{LCS}$ (and on many other categories as well) is separably determined in the following sense: For every LCS E, we have $k(E)=\bigcup_F k(F)$, where the union is taken over all separable (or closed and separable) subspaces F of E.

(b) The assumption in Theorem 8.3 that K is invariant is needed only if one insists on $k(G)=K$. In general, one can always replace the possibly noninvariant space K by the invariant space $K_1:=\bigcap\{\hat{S}^{-1}(K): S\in\mathcal{L}(G)\}$ because the bs-functors generated by K and K_1 are the same.

Theorem 8.6. *Assume that the category $\mathcal{E}$ contains a space G such that*

(∗) *whenever $E\in\mathcal{E}$ and $(x_n)\in l_\infty(E)$, there exists a subspace F of E which contains the sequence (x_n) and is a member of $\mathcal{E}$, and an operator T: $E\to G$ for which $T|F$ is an isomorphic embedding.*

Let k be an injective bs-functor on $\mathcal{E}$. Then k is generated by the space $k(G)$, that is,

$$k(E)=\{x\in l_\infty(E): \hat{S}(x)\in k(G),\ \forall S\in\mathcal{L}(E,G)\}\qquad \forall E\in\mathcal{E}.$$

Moreover, if

$$G=\prod_{j\in J} H_j,\quad \text{where}\quad H_j\in\mathcal{E}\ (j\in J),\quad \text{and}\quad k(G)\simeq\prod_{j\in J} k(H_j),$$

then

$$k(E)=\bigcap_{j\in J} k_j(E)\qquad \text{for every } E\in\mathcal{E},$$

where k_j denotes the bs-functor generated by the space $k(H_j)$ $(j\in J)$.

In particular, if $G=H^{\mathfrak{m}}$ for some $H\in\mathcal{E}$ and a cardinal number $\mathfrak{m}$, and $k(G)\simeq k(H)^{\mathfrak{m}}$, then k is generated by the space $k(H)$.

Remark 8.7. The conditions $k(G)\simeq\prod_{j\in J} k(H_j)$ or $k(G)\simeq k(H)^{\mathfrak{m}}$ are automatically satisfied when k is productive or $k(G)$ is closed (see Proposition 8.1(b)).

Proof of Theorem 8.6. Fix $E\in\mathcal{E}$ and $x=(x_n)\in l_\infty(E)$, and assume that $\hat{S}x\in k(G)$ for every $S\in\mathcal{L}(E,G)$. Choose F and T according to (∗). Then $\hat{T}(x)\in k(G)$ and $\hat{T}(x)\in l_\infty(T(F))$, hence $\hat{T}(x)\in k(T(F)$ by the injectivity of k. Since $\hat{T}$ is an isomorphism from $k(F)$ onto $k(T(F))$, we conclude that $x\in k(F)\subset k(E)$.

The verification of the other assertions is straightforward. ∎

Let us say that a category $\mathcal{E}$ is $\mathfrak{m}$-*good*, where $\mathfrak{m}$ is a cardinal number, if

- both l_∞ and $l_\infty^{\mathfrak{m}}$ are in $\mathcal{E}$, and whenever $E \in \mathcal{E}$ and $(x_n) \in l_\infty(E)$, there is a subspace F of E containing (x_n) and such that $F \in \mathcal{E}$ and F embeds isomorpically into $l_\infty^{\mathfrak{m}}$.

If $\mathcal{E}$ is $\mathfrak{m}$-good for some $\mathfrak{m}$, we shall call it *good*.

For example, the category of all normed (or Banach) spaces, the category of all metrizable LCSs (or Fréchet spaces), and the category $\mathcal{LCS}$ are 1-, $\aleph_0$- and $2^{\aleph_0}$-good, respectively. In each case, the subspace F in the condition above can be taken to be $\overline{\text{lin}}\,(x_n)$.

The next result is just a special case of Theorem 8.6 for $G := l_\infty^{\mathfrak{m}}$ and $H := l_\infty$.

THEOREM 8.8. *Let k be an injective bs-functor on an $\mathfrak{m}$-good category $\mathcal{E}$ such that $k(l_\infty^{\mathfrak{m}}) \simeq k(l_\infty)^{\mathfrak{m}}$. Then k is generated by the space $k(l_\infty)$.*

In consequence, it is so for every injective and productive (*in particular, closed*) *bs-functor k on any good category $\mathcal{E}$.*

As direct consequences of Theorems 8.3 and 8.8 we have the following corollaries.

COROLLARY 8.9. *Let K be an invariant space of bounded l_∞-sequences. Then, on every good category $\mathcal{E}$, the bs-functor generated by K is a unique injective and productive bs-functor k with $k(l_\infty) = K$.*

COROLLARY 8.10. *Let $\mathcal{E}$ be a good category* (*for instance, the category of all Banach or Fréchet spaces*), *and let k be an injective and productive* (*in particular, closed*) *bs-functor on $\mathcal{E}$. If k is* SD, *then it is l_∞-*SD.

REMARK 8.11. It is an open problem if the same holds for all closed bs-functors (at least on sufficiently nice categories, such as that of all Banach spaces).

By a *dual bs-functor* on the category $\mathcal{E}$ we shall understand a map k which assigns to every space $E \in \mathcal{E}$ a space $k(E')$ of bounded E'-sequences so that

(k1′) $\qquad \hat{S}(k(F')) \subset k(E') \qquad$ for all $E, F \in \mathcal{E}$ and $S \in \mathcal{L}'(E, F)$.

PROPOSITION 8.12. *Let the category $\mathcal{E}$ have the following property: For every $E \in \mathcal{E}$, there is a class $\mathcal{S}_E$ of subspaces of E such that* 1) *each $F \in \mathcal{S}_E$ is spanned by a bounded E-sequence, and* 2) *if $(x_n) \in l_\infty(E)$, then* $\text{lin}\,(x_n)$ *is dense in some $F \in \mathcal{S}_E$. Let k be a* SD *dual bs-functor on $\mathcal{E}$ satisfying the following condition:*

(+) *Each* $k(E')$ *contains all* $x \in l_\infty(E')$ *such that* $(x_n|F) \in k(F')$ *for every* $F \in \mathcal{S}_E$.

Then each $k(E')$ *is dually* l_∞-SD.

REMARK 8.13. This result can usually be applied with $\mathcal{S}_E$ defined to be the class of closed subspaces in E that are spanned by bounded sequences.

Proof of Proposition 8.12. Let $(x'_n) \in l_\infty(E')$ be as required by the dual l_∞-subsequential determinedness (see Subsection 4B). These requirements, in view of the assumptions made on $\mathcal{S}_E$, can be stated as follows: Let $F \in \mathcal{S}_E$; then for every $N \in [\mathbb{N}]$ there exists $M \in [N]$ such that $(x'_m)_{m\in M}$ coincides on F with a sequence $(y'_m)_{m\in M} \in K$. Since, by (k1′), $(y'_m|F) \in k(F')$, we see that $(x'_m|F)_{m\in M} \in k(F')$. Therefore, as $k(F')$ is SD, we must have $(x'_n|F) \in k(F')$, and by appealing to (+) we conclude that $(x'_n) \in K$. ∎

We conclude this section with a result on maximal bs-functors. Let us say that a bs-functor k on $\mathcal{E}$ is *nontrivial* if $k(E) \neq l_\infty(E)$ for some $E \in \mathcal{E}$. As we will see in a moment, under some mild requirements on $\mathcal{E}$ there is a largest nontrivial SH bs-functor on $\mathcal{E}$, and a largest nontrivial SH injective bs-functor on $\mathcal{E}$. We define

$$\kappa_p := \text{the bs-functor on } \mathcal{LCS} \text{ generated by } \kappa(l_p)\ (1 \le p \le \infty),$$
$$r := \text{the bs-functor on } \mathcal{LCS} \text{ generated by } c\kappa(l_\infty,\sigma).$$

By applying Theorem 8.3 it is easily seen that each of the spaces $\kappa_p(E)$ and $r(E)$ is closed and l_∞-SD (cf. Remark 4.1(c)); in addition, $\kappa(E) \subset \kappa_p(E)$ and $\kappa(E,\sigma) \subset r(E)$. Moreover, the bs-functors $\kappa_\infty \equiv p\kappa$ (see Proposition 3.1(a)) and r are injective. Note that r has already appeared in Proposition 3.3 and that, by Proposition 3.1(i), $\kappa_1 = v^*$. For some additional information about κ_p, see Remark 8.15 below.

THEOREM 8.14. *Let the category* $\mathcal{E}$ *consist of* SC *spaces.*

(a) v^* *is the largest nontrivial* SH *bs-functor on* $\mathcal{E}$.
(b) r *is the largest nontrivial and* SH *injective bs-functor on* $\mathcal{E}$ *provided* $l_1 \in \mathcal{E}$.

Proof. (a): Suppose k is a SH bs-functor on $\mathcal{E}$ such that $k(E)\setminus v^*(E) \neq \emptyset$ for some $E \in \mathcal{E}$. We have to show that $k(F) = l_\infty(F)$ for each $F \in \mathcal{E}$. Let $(x_n) \in k(E) \setminus v^*(E)$. Since $k(E)$ is SH, by using Proposition 3.5.(a) and passing to a subsequence we may assume that the sequence (x_n) is equivalent to the standard basis in l_1 and that there is a projection P from E onto its closed span X. Now take an arbitrary $F \in \mathcal{E}$ and let $(z_n) \in l_\infty(F)$. There exists $T \in \mathcal{L}(X,F)$ such that $Tx_n = z_n$ for all n. Then $S := T \circ P \in \mathcal{L}(E,F)$ and $Sx_n = z_n$ for all n. By condition (k1), $(z_n) \in k(F)$. Thus $l_\infty(F) = k(F)$.

(b): Suppose k is a SH injective bs-functor on $\mathcal{E}$ such that $k(E)\setminus r(E) \neq \emptyset$ for some $E \in \mathcal{E}$. We have to show that $k(F) = l_\infty(F)$ for each $F \in \mathcal{E}$. In view of Proposition 3.5(b), from the assumptions it follows easily that the standard basis (e_n) of l_1 is in $k(l_1)$. Since every bounded sequence in any SC LCS is a continuous linear image of (e_n), we conclude as above. ∎

REMARK 8.15. If $1 \le p < \infty$, then

$$\kappa_p(l_r) = \begin{cases} \kappa(l_r) & \text{if } 1 \le r \le p, \\ l_\infty(l_r) & \text{if } p < r < \infty, \end{cases}$$
$$\kappa_p(l_\infty) = \begin{cases} l_\infty(l_\infty) & \text{if } 1 \le p < 2, \\ r(l_\infty) & \text{if } 2 \le p < \infty, \end{cases}$$
$$\kappa_p(c_0) = l_\infty(c_0).$$

Let $1 \le r \le p$ and suppose there exists $(x_n) \in \kappa_p(l_r) \setminus \kappa(l_r)$. We may assume that (x_n) is weak* convergent to some x_0 in l_r. Moreover, since $\kappa_p(l_r)$ contains all the constant l_r-sequences, we may also assume that $x_0 = 0$. Now, (x_n) has a subsequence (y_n) which is equivalent to the standard basis (e_n) of l_r and spans a complemented subspace Y of l_r. Consider the operator $T := SRP\colon l_r \to l_p$, where P is a projection from l_r onto Y, R is the natural isomorphism between Y and l_r, and $S\colon l_r \to l_p$ is the inclusion mapping. Then T maps (y_n) to the standard basis of l_p, a non-relatively-compact sequence, which contradicts the fact that $(y_n) \in \kappa_p(l_r)$.

Now, let us recall that 1) for $r < \infty$, every operator from l_r (or c_0) to l_p is compact iff $p < r$; 2) every operator from l_∞ to l_p is compact iff $1 \le p < 2$ (see [52, Prop. 2.c.3] and [67, Rem. 2, p. 211]). From this all the remaining equalities follow, with one exception: for $2 \le p < \infty$ we only get $\kappa_p(l_\infty) \ne l_\infty(l_\infty)$. In order to complete the proof it is enough to observe that, since l_∞ has the Dunford–Pettis property, every operator $T \in \mathcal{L}(l_\infty, l_p)$ maps sequences in $r(l_\infty) = c\kappa(l_\infty, \sigma)$ to sequences in $\kappa(l_p)$. Hence $r(l_\infty) \subset \kappa_p(l_\infty)$, and we get equality here when $2 \le p < \infty$ by appealing to Proposition 7.1.

In consequence, there are uncountably many distinct bs-functors (even on Banach spaces) which are closed and l_∞-SD.

9. Operator spaces. We now apply our sequential tools from Section 5 to general spaces of operators. In what follows, E, F and G will denote LCSs, and all spaces of operators between LCSs will be equipped with the topology of uniform convergence on bounded sets in the domain. We recall the convention from Section 2 that whenever we consider a space of operators, $A(E,F)$ say, it is understood that $A(E,F)$ is a subspace of $\mathcal{L}(E,F)$. We will often use notation of the following type:

$$T \circ \mathcal{A}(E,F) \circ S := \{T \circ R \circ S\colon R \in \mathcal{A}(E,F)\},$$
$$\mathcal{B}(F,G) \circ \mathcal{A}(E,F) := \{T \circ S\colon T \in \mathcal{B}(F,G),\ S \in \mathcal{A}(E,F)\},$$

where $\mathcal{A}(E,F)$, $\mathcal{B}(F,G)$ are classes of operators, and S, T are operators with appropriate domains and ranges.

We denote by $\mathcal{F}$ and $\mathcal{G}$ the classes of finite rank and approximable operators, respectively. Other classes of operators will be defined in Section 10.

The present section is devoted to the problem of complementability of one space of operators, $A(E,F)$, in another one, $B(E,F)$. We would like to cover all the cases when E and F are Fréchet spaces or spaces dual to Fréchet spaces (or, even more generally, complete DF-spaces). With this in mind, we formulate our results with the weakest assumptions possible, at least weak enough to cover the cases just mentioned. It should be stressed, however, that most of our results are interesting and new also for Banach spaces. In order to make this section easier accessible to those interested solely in the Banach case, let us recall the following: Fréchet spaces and complete DF-spaces are *weakly angelic* [59] (i.e., satisfy the Eberlein–Šmulian Theorem), $\aleph_0$-*barrelled* (i.e., bounded unions of countably many equicontinuous sets in their duals are equicontinuous) and, of course, SC and QC. Moreover, Banach spaces and DF-spaces are quasi-normable (see [43]).

We recall that an operator ideal $\mathcal{A}$ (on a category $\mathcal{E}$) is said to be *injective* (in the sense of the theory of operator ideals) if it has the following property: Whenever $E, F, F_0 \in \mathcal{E}$, $J\colon F_0 \to F$ is a topological embedding and $T \in \mathcal{L}(E,F_0)$, then $T \in \mathcal{A}(E,F_0)$ iff $J_0T \in \mathcal{A}(E,F)$. (Of course, this is to be distinguished from the injectivity of a component $\mathcal{A}(E,F)$ as a LCS.) We also recall that a space $A(E,F)$ of operators is said to be a *right* (resp., *left*) *ideal of operators* if whenever $T \in A(E,F)$, then also $TR \in A(E,F)$ for all $R \in \mathcal{L}(E)$ (resp., $ST \in A(E,F)$ for all $S \in \mathcal{L}(F)$). We are mostly interested in those spaces $A(E,F)$ which are defined (or can be characterized) by a pair (K_E, K_F) of sets of bounded sequences $K_E \subset l_\infty(E)$ and $K_F \subset l_\infty(F)$ in the sense that

$$A(E,F) = \{T \in \mathcal{L}(E,F) \colon \hat{T}(K_E) \subset K_F\},$$

or whose space of dual operators, $A'(E,F)$, can be described in this manner, i.e.,

$$A'(E,F) = \{T \in \mathcal{L}(E,F) \colon \hat{T}'(K_{F'}) \subset K_{E'}\}$$

for some $K_{E'} \subset e(E')$ and $K_{F'} \subset e(F')$.

If K is a SH subset of $l_\infty(F)$, then we define its l_∞-SD *hull* $h(K)$ as the smallest l_∞-SD subset of $l_\infty(F)$ containing K. It is easy to give an explicit formula for $h(K)$:

$$h(K) = \{y \in l_\infty(F) \colon \forall R \in \mathcal{L}(F, l_\infty)\ \forall N \in [\mathbb{N}]\ \exists M \in [N] \colon \hat{R}(y\pi_M) \in \hat{R}(K)\}.$$

Given a SH set $K_E \subset l_\infty(E)$ and a space $A(E,F)$, let

$$\hat{A}(K_E) := \{(Tx_n)\colon T \in A(E,F),\ (x_n) \in K_E\}.$$

We shall say that the space $A(E,F)$ is K_E-l_∞-*determined* if there exists an l_∞-SD set $K_F \subset l_\infty(F)$ such that $A(E,F)$ is defined by the pair (K_E, K_F). If this is so, then the smallest such K_F is of course $h(\hat{A}(K_E))$.

9A. Additional notation and concepts. We denote

$$l_{\infty,1}(E) := \{(Qz_n)\colon Q \in \mathcal{L}(l_1,E),\ (z_n) \in l_\infty(l_1)\} = \{(Qe_n)\colon Q \in \mathcal{L}(l_1,E)\};$$

obviously this is a SH linear subspace of $l_\infty(E)$ consisting precisely of the sequences contained in Banach discs. Note that if E is SC, then $l_{\infty,1}(E) = l_\infty(E)$.

Given a space $A(E,F)$ of operators, we define

$$K_A(F) = \hat{A}(l_{\infty,1}(E)).$$

Clearly, this is a SH subset of $l_\infty(F)$; we denote by $s_{A(E,F)}(F)$, or briefly $s_A(F)$, its l_∞-SD hull. Thus, explicitly,

$$s_A(F) = \{(y_n) \in l_\infty(F)\colon \forall R \in \mathcal{L}(F,l_\infty)\ \forall N \in [\mathbb{N}]\ \exists M \in [N]\ \exists T \in A(E,F)\ \exists Q \in \mathcal{L}(l_1,E)\colon Ry_n = RTQe_n \text{ for all } n \in M\}.$$

We shall say that the space $A(E,F)$ is *surjectively-l_∞-determined* if it contains every operator $T \in \mathcal{L}(E,F)$ such that $\hat{T}(l_{\infty,1}(E)) \subset s_A(F)$. Since $\hat{T}(l_{\infty,1}(E))$ is SH, the property required from T can be expressed as follows:

For every $(x_n) \in l_{\infty,1}(E)$ and $R \in \mathcal{L}(F,l_\infty)$ there exist $M = \{m_1, m_2, \dots\} \in [\mathbb{N}]$, $(u_n) \in l_{\infty,1}(E)$, and $S \in A(E,F)$ such that $(RTx_{m_n}) = (RSu_n)$.

Equivalently: For every pair of operators $Q\colon l_1 \to E$ and $R\colon F \to l_\infty$ there exists $M \in [\mathbb{N}]$ such that

$$RTQ|_{l_1(M)} \in R \circ A(E,F) \circ \mathcal{L}(l_1,E).$$

The corresponding notion for operator ideals is defined as follows: We say that an operator ideal $\mathcal{A}$ is *surjectively-l_∞-determined* if it contains l_1 and, for every component $\mathcal{A}(E,F)$ of $\mathcal{A}$, an operator $T\colon E \to F$ belongs to $\mathcal{A}(E,F)$ provided that for every pair of operators $Q\colon l_1 \to E$ and $R\colon F \to l_\infty$ there exists $M \in [\mathbb{N}]$ such that

$$RTQ|_{l_1(M)} \in R \circ \mathcal{A}(l_1,F).$$

9B. Dually l_∞-SD hulls. If K is a SH subset of E', then we define its *dually l_∞-SD hull* as

$$h'(K) = \{x' \in l_\infty(E')\colon \forall Q \in \mathcal{L}(l_1,E)\ \forall N \in [\mathbb{N}]\ \exists M \in [N]\colon \hat{Q}'(x'\pi_M) \in \hat{Q}'(K)\}.$$

As is easily seen, $h'(K)$ is the smallest dually l_∞-SD subset of $l_\infty(E')$ containing K.

Given a space $B(E,F)$ of operators, we define

$$K_B(E') = \{(x'_n) \in l_\infty(E') \colon \exists T \in B(E,F)\ \exists (y'_n) \in e(F') \colon (x'_n) = (T'y'_n)\},$$

a SH subset of $l_\infty(E')$, and denote by $i_{B(E,F)}(E')$, or briefly $i_B(E')$, its dually l_∞-SD hull. Remembering the correspondence between $e(F')$ and $\mathcal{L}(F, l_\infty)$ (see Subsection 3A), we can express $i_B(E')$ explicitly as follows:

$$i_B(E') = \{(x'_n) \in l_\infty(E') \colon \forall Q \in \mathcal{L}(l_1, E)\ \forall N \in [\mathbb{N}]\ \exists M \in [N]\ \exists T \in B(E,F)$$
$$\exists R \in \mathcal{L}(F, l_\infty) \colon x'_n \circ Q = e'_n \circ R \circ T \circ Q \text{ for all } n \in M\},$$

where the e'_n are the coordinate functionals on l_∞.

We shall say that the space $B(E,F)$ is *injectively-l_∞-determined* if it contains every operator $T \in \mathcal{L}(E,F)$ such that $\hat{T}'(e(F')) \subset i_B(E')$. This requirement can be reformulated as follows:

For every $(y'_n) \in e(F')$ and $Q \in \mathcal{L}(l_1, E)$ there exist $M = \{m_1, m_2, \ldots\} \in [\mathbb{N}]$, $(v'_n) \in e(F')$, and $S \in B(E,F)$ such that $(Q'T'y'_{m_n}) = (Q'S'v'_n)$.

Equivalently: For every pair of operators $R \colon F \to l_\infty$ and $Q \colon l_1 \to E$ there exists $M \in [\mathbb{N}]$ such that

$$P_M RTQ \in \mathcal{L}(F, l_\infty) \circ B(E,F) \circ Q,$$

where P_M is the standard projection from l_∞ onto $l_\infty(M)$.

The corresponding notion for operator ideals is defined as follows: We say that an operator ideal $\mathcal{B}$ is *injectively-l_∞-determined* if, for every component $\mathcal{B}(E,F)$ of $\mathcal{B}$, an operator $T \colon E \to F$ belongs to $\mathcal{B}(E,F)$ provided that for every pair of operators $R \colon F \to l_\infty$ and $Q \colon l_1 \to E$ there exists $M \in [\mathbb{N}]$ such that

$$P_M RTQ \in \mathcal{B}(E, l_\infty) \circ Q.$$

As is easily seen, if an operator ideal is surjectively- or injectively-l_∞-determined so is each of its components. The converse does not seem to be true.

THEOREM 9.1. *Let $A(E,F)$ and $B(E,F)$ be two spaces of operators such that*

$$\mathcal{F}(E,F) \subset A(E,F) \subset B(E,F)$$

and $A(E,F)$ is complemented in $B(E,F)$.

(a) *Assume that $A(E,F)$ is surjectively-l_∞-determined and $B(E,F)$ is a right ideal of operators. If E contains a complemented copy of l_1, then*

$$B(E,F) \cap (\mathcal{L}(l_1, F) \circ P) \subset A(E,F)$$

for every projection $P \colon E \to l_1$.

(b) *Assume that* $A(E,F)$ *is injectively-*l_∞*-determined and* $B(E,F)$ *is a left ideal of operators. If* F *contains a copy of* l_∞*, then*

$$B(E,F)\cap(J\circ\mathcal{L}(E,l_\infty))\subset A(E,F)$$

for every isomorphic embedding $J\colon l_\infty\to F$*.*

REMARK 9.2. We say $P\colon E\to l_1$ is a *projection* if there is a subspace E_1 of E such that $P_1:=P|E_1$ is an isomorphism onto l_1. If we write J_1 for P_1^{-1}, then of course $J_1\circ P$ is a projection (in the usual sense) from E onto $E_1\simeq l_1$.

Proof of Theorem 9.1. (a): Let E_1, P_1 and J_1 be as in the remark above. Let $T\in B(E,F)\cap(\mathcal{L}(l_1,F)\circ P)$. Then

$$T=\sum_{n=1}^{\infty}(e_n'\circ P)\otimes y_n,$$

where (e_n') is the sequence of coefficient functionals on l_1 and $(y_n)\in l_\infty(F)$. (The series converges pointwise on E.)

For every $a=(a_n)\in l_\infty$, let M_a be the multiplication operator $(t_n)\mapsto(a_nt_n)$ on l_1. Then, since $B(E,F)$ is a right ideal of operators, the operator $T_a:=T\circ(P_1^{-1}M_aP)$ is in $B(E,F)$. Therefore, we can define a map

$$U\colon l_\infty\to B(E,F)\quad\text{by}\quad U(a):=T_a=\sum_{n=1}^{\infty}(e_n'\circ P)\otimes a_ny_n,$$

and it is obvious that it is linear and continuous. Next, define an operator

$$W\colon A(E,F)\to l_\infty(F)\quad\text{by}\quad W(Q):=(Qu_n),$$

where $(u_n):=(J_1e_n)$.

Now, denote by V a projection from $B(E,F)$ onto $A(E,F)$ and consider the operator $WVU\colon l_\infty\to l_\infty(F)$. Observe that for $a\in c_0$ the series defining $U(a)=T_a$ converges in $\mathcal{L}(E,F)$. Since $(e_n'\circ P)\otimes y_n\in\mathcal{F}(E,F)\subset A(E,F)$ for all n, it follows that $VU(a)=U(a)\in A(E,F)$ and $WVU(a)=(a_ny_n)$ for every $a=(a_n)\in c_0$. In addition the range of W, hence of WVU as well, is contained in $s_A(F)$, an l_∞-SD subset of $l_\infty(F)$. In consequence, by Theorem 5.3, $(Tu_n)=(y_n)\in s_A(F)$.

Finally, take any $Q\in\mathcal{L}(l_1,E)$. Since $B(E,F)$ is a right ideal of operators, the operator $T_1:=TQP$ is in $B(E,F)$ and, of course, in $\mathcal{L}(l_1,F)\circ P$. Therefore, we can apply to T_1 what was proved above for T, obtaining $(TQe_n)=(T_1u_n)\in s_A(F)$. We have thus shown that $\hat{T}(l_{\infty,1}(E))\subset s_A(F)$. Since $A(E,F)$ is surjectively-l_∞-determined, we conclude that $T\in A(E,F)$.

(b): Let $T\in B(E,F)\cap(J\circ\mathcal{L}(E,l_\infty))$. Then

$$T=JS,\quad\text{where}\quad S\colon E\to l_\infty\,;\,x\mapsto(\langle x,x_n'\rangle)$$

for some equicontinuous sequence (x_n') in E'.

Let P be a projection from F onto $J(l_\infty)$. For every $a = (a_n) \in l_\infty$, let M_a be the multiplication operator $(t_n) \mapsto (a_n t_n)$ on l_∞. Then, since $B(E,F)$ is a left ideal of operators, the operator $T_a := JM_aS = (JM_aJ^{-1}P) \circ T$ is in $B(E,F)$. Therefore, we can define a map

$$U : l_\infty \to B(E,F) \quad \text{by} \quad U(a) := T_a,$$

and it is obvious that it is linear and continuous.

Next, define an operator

$$W : A(E,F) \to l_\infty(E') \quad \text{by} \quad W(Q) := (Q'v_n'),$$

where $v_n' := (J^{-1}P)'e_n'$ ($n \in \mathbb{N}$), and e_n' are the coordinate functionals on l_∞.

Now, denote by V a projection from $B(E,F)$ onto $A(E,F)$ and consider the operator $WVU : l_\infty \to l_\infty(E')$. Observe that if $a = (a_n) \in c_0$, then $U(a) = \sum_{n=1}^\infty a_n x_n' \otimes J(e_n)$, where the series converges in $\mathcal{L}(E,F)$. Since $x_n' \otimes J(e_n) \in \mathcal{F}(E,F) \subset A(E,F)$ for all n, it follows that $VU(a) = U(a) \in A(E,F)$ and $WVU(a) = (a_n x_n')$ for every $a = (a_n) \in c_0$. In addition the range of W, hence of WVU as well, is contained in $i_A(E')$, a dually l_∞-SD subset of $l_\infty(E')$. In consequence, by Theorem 5.3, $(T'v_n') = (x_n') \in i_A(E')$.

Finally, take any $R \in \mathcal{L}(F, l_\infty)$. Since $B(E,F)$ is a left ideal of operators, the operator $T_1 := JRT$ is in $B(E,F)$ and, of course, in $J \circ \mathcal{L}(E, l_\infty)$. Therefore, we can apply to T_1 what was proved above for T, obtaining $((RT)'e_n') = (T_1'v_n') \in i_A(E')$. We have thus shown that $\hat{T}'(e(F')) \subset i_A(E')$. Since $A(E,F)$ is injectively-l_∞-determined, we conclude that $T \in A(E,F)$. ■

Corollary 9.3. *Let $\mathcal{E}$ be a category of* LCSs*, and let $\mathcal{A}$ and $\mathcal{B}$ be operator ideals on $\mathcal{E}$ satisfying $\mathcal{F} \subset \mathcal{A} \subset \mathcal{B}$. Let $E, F \in \mathcal{E}$ and assume that $\mathcal{A}(E,F)$ is complemented in $\mathcal{B}(E,F)$.*

(a) *If E contains a complemented copy of l_1 and $\mathcal{A}$ is surjectively-l_∞-determined, then*

$$\mathcal{A}(G,F) = \mathcal{B}(G,F) \quad \textit{for every } G \in \mathcal{E}.$$

(b) *If F contains a copy of l_∞ and $\mathcal{A}$ is injectively-l_∞-determined, then*

$$\mathcal{A}(E,G) = \mathcal{B}(E,G) \quad \textit{for every } G \in \mathcal{E}.$$

Proof. In view of Theorem 9.1 it is clear that $\mathcal{A}(l_1,F) = \mathcal{B}(l_1,F)$ in case (a), and $\mathcal{A}(E,l_\infty) = \mathcal{B}(E,l_\infty)$ in case (b). From this the desired equalities follow easily by the very definitions of surjective- and injective-l_∞-determinedness of $\mathcal{A}$. ■

Sometimes we can weaken the assumptions concerning the range space. Namely, we can assume that it contains a copy of c_0 instead of the whole copy of l_∞.

We say an operator ideal $\mathcal{A}$ on a category $\mathcal{E}$ containing both c_0 and l_∞ is c_0-*projectively determined* if an operator $T \in \mathcal{L}(E, c_0)$ (where $E \in \mathcal{E}$) is in $\mathcal{A}(E,c_0)$ provided that for every $Q\in\mathcal{L}(l_1,E)$ and every $N\in[\mathbb{N}]$ there is $M\in[N]$ such that

$$JP_MTQ\in\mathcal{A}(E,l_\infty)\circ Q,$$

where $J\colon c_0\to l_\infty$ is the standard embedding.

THEOREM 9.4. *Let $\mathcal{E}$ be a category of* LCS*s consisting of* SC *spaces and containing both c_0 and l_∞. Let $\mathcal{A}$ and $\mathcal{B}$ be closed operator ideals on $\mathcal{E}$ satisfying $\mathcal{G}\subset\mathcal{A}\subset\mathcal{B}$ and such that $\mathcal{A}$ is c_0-projectively determined.*

If $E,F\in\mathcal{E}$, F contains an isomorphic copy of c_0, and $\mathcal{A}(E,F)$ is complemented in $\mathcal{B}(E,F)$, then

$$\mathcal{A}(E,c_0)=\mathcal{B}(E,c_0).$$

In consequence, if $\mathcal{B}$ is injective, then $\mathcal{B}(E,F)\cap(J_0\circ\mathcal{L}(E,c_0))\subset\mathcal{A}(E,F)$ for every isomorphic embedding $J_0\colon c_0\to F$.

Proof. Let $T\in\mathcal{B}(E,c_0)\setminus\mathcal{A}(E,c_0)$. Then there exist $Q\in\mathcal{L}(l_1,E)$ and $N\in[\mathbb{N}]$ such that

$$(\dagger\dagger)\qquad JP_MTQ\notin\mathcal{A}(E,l_\infty)\circ Q\quad\text{for every } M\in[N],$$

where $J\colon c_0\to l_\infty$ is the standard embedding. Of course, T is of the form

$$Tx=\sum_{n=1}^{\infty}x_n'(x)e_n,$$

where (e_n) is the unit vector basis in c_0 and (x_n') is an equicontinuous weak* null sequence in E'.

We can define an operator

$$U\colon l_\infty\to\mathcal{B}(E,F)\ \text{ by }\ U(a)=J_0M_aT=\sum_{n=1}^{\infty}a_nx_n'\otimes J_0e_n,$$

where M_a is the multiplication operator $(t_n)\mapsto(a_nt_n)$ on c_0. Next, define an operator

$$W\colon\mathcal{B}(E,F)\to l_\infty(l_\infty)\quad\text{by}\quad W(S)=(RSQ(e_n^1))_{n\in\mathbb{N}},$$

where (e_n^1) is the unit vector basis in l_1 and $R\in\mathcal{L}(F,l_\infty)$ is an extension of the operator $JJ_0^{-1}\colon J_0(c_0)\to l_\infty$. Now, denote by V a projection from $\mathcal{B}(E,F)$ onto $\mathcal{A}(E,F)$ and consider the operator

$$Z:=WU-WVU\colon l_\infty\to l_\infty(l_\infty)\simeq l_\infty.$$

Clearly, $U(c_0)\subset\mathcal{G}(E,F)\subset\mathcal{A}(E,F)$, hence $Z|c_0=0$, so that, by Key Lemma 5.1, we can find $M\in[N]$ for which $Z|l_\infty(M)=0$. In particular, $RU(e_M)Q(e_n^1)=RV(U(e_M))Q(e_n^1)$ for every $n\in\mathbb{N}$. From this, since $U(e_M)=J_0P_MT$ and $R=JJ_0^{-1}$ on $J_0(c_0)$, it follows that

$$JP_MTQ=R\circ V(U(e_M))\circ Q.$$

But $R\circ V(U(e_M))\in\mathcal{A}(E,l_\infty)$, and we arrive at a contradiction with $(\dagger\dagger)$.

Now, to prove the final assertion, take any $S \in \mathcal{L}(E, c_0)$ such that $T := J_0 S \in \mathcal{B}(E, F)$. Since $\mathcal{B}$ is injective, $S \in \mathcal{B}(E, c_0)$. Hence $S \in \mathcal{A}(E, c_0)$, and finally $T \in \mathcal{A}(E, F)$. ∎

Now we give a simple but useful criterion for operator ideals to be c_0-projectively determined.

Proposition 9.5. *Let $\mathcal{A}$ be an operator ideal on a category $\mathcal{E}$ of* LCSs *consisting of* SC *spaces, with $c_0, l_\infty \in \mathcal{E}$. Let k be an injective bs-functor on $\mathcal{E}$ such that $(x_n) \in k(l_\infty)$ iff for each $N \in [\mathbb{N}]$ there is $M \in [N]$ such that $(P_M x_n)_{n\in\mathbb{N}} \in k(l_\infty)$. Suppose that for each pair $E, F \in \mathcal{E}$ there is $k_1(E) \subset l_\infty(E)$ such that*

$$\mathcal{A}(E, F) = \{T \in \mathcal{L}(E, F) \colon \hat{T}(k_1(E)) \subset k(F)\}.$$

Then $\mathcal{A}(E, F)$ is c_0-projectively determined.

Proof. Let $T \in \mathcal{L}(E, c_0) \setminus \mathcal{A}(E, c_0)$. Then $(Tx_n) \notin k(c_0)$ for some $(x_n) \in k_1(E)$, and by the assumption on k there is $N \in [\mathbb{N}]$ such that $(P_M T x_n) \notin k(c_0) = k(l_\infty) \cap l_\infty(c_0)$ for all $M \in [N]$. Hence for the operator $Q \colon l_1 \to E$ mapping (e_n) to (x_n) and for every $M \in [N]$ we have $(P_M T Q e_n) \notin k(l_\infty)$, and so $J P_M T Q \notin \mathcal{A}(E, l_\infty) \circ Q$. ∎

Now, we consider the problem of injectivity (in the sense of the category of LCSs).

Theorem 9.6.

(a) *Assume that E contains a complemented copy of l_1, F is* SC, *and let $A(E, F)$ be a surjectively-l_∞-determined space of operators such that $\mathcal{G}(E, F) \subset A(E, F)$. If $A(E, F)$ is complemented in the dual to a bornological space, then $A(E, F) = \mathcal{L}(E, F)$ and F is complemented in the dual to a bornological space.*
If $\mathcal{A}$ is a surjectively-l_∞-determined operator ideal on a category $\mathcal{E}$ containing l_1, E and F, then $\mathcal{A}(E, F) = \mathcal{L}(E, F)$ iff $\mathrm{id}_F \in \mathcal{A}(F)$.

(b) *Assume that F contains a copy of l_∞, and let $A(E, F)$ be an injectively-l_∞-determined space of operators such that $\mathcal{G}(E, F) \subset A(E, F)$. If $A(E, F)$ is complemented in the dual to a bornological space, then $A(E, F) = \mathcal{L}(E, F)$ and F is complemented in the dual to a bornological space.*
If $\mathcal{A}$ is an injectively-l_∞-determined operator ideal on a category $\mathcal{E}$ containing l_∞, E and F, then $\mathcal{A}(E, F) = \mathcal{L}(E, F)$ iff $\mathrm{id}_E \in \mathcal{A}(E)$.

Proof. (a): We will show that $s_A(F) = l_\infty(F)$. From this the equality $A(E, F) = \mathcal{L}(E, F)$ will follow immediately because $A(E, F)$ is surjectively-l_∞-determined.

Let $J \colon l_1 \to E$ be an isomorphic embedding, and P a projection from E onto $J(l_1)$.

Fix $(y_n) \in l_\infty(F)$, and define an operator

$$U\colon c_0 \to \mathcal{G}(E,F) \subset A(E,F) \quad \text{by} \quad U((a_n)) = \sum_{n=1}^{\infty} a_n (J^{-1}P)'(e_n') \otimes y_n,$$

where the e_n' are the coefficient functionals on l_1. This definition makes sense because F is SC. Also note that the series converges in $\mathcal{L}(E,F)$.

Next, define an operator

$$W\colon A(E,F) \to l_\infty(F) \quad \text{by} \quad W(T) = (T \circ J(e_n)).$$

Now, consider the diagram

$$l_\infty = {c_0}'^{\times} \xrightarrow{U'^{\times}} A(E,F)'^{\times} \xrightarrow{V} A(E,F) \xrightarrow{W} l_\infty(F),$$

where V is a projection. Let $Z\colon l_\infty \to l_\infty(F)$ be the composition of the maps in this diagram. Note that W, hence also Z, has its range contained in $s_A(F)$, an l_∞-SD subset of $l_\infty(F)$ (see Subsection 9A). Moreover, as is easily seen, $Z((a_n)) = (a_n y_n)$ for all $(a_n) \in c_0$. Thus we may apply Theorem 5.3 and get $(y_n) \in s_A(F)$.

As for the second part of (a), from $\mathcal{A}(E,F) = \mathcal{L}(E,F)$ it follows easily that $\mathcal{A}(l_1,F) = \mathcal{L}(l_1,F)$ and this in turn implies $\mathrm{id}_F \in \mathcal{A}(F)$ (see the proof of Corollary 9.3). The other direction is trivial.

(b): The proof is dual to the previous one (cf. the proof of Theorem 9.1(b)). ■

REMARK 9.7. The assumption that F contains a copy of l_∞ can be replaced in the first part of (b) by a formally weaker condition that F contains a copy of c_0. However, if $A(E,F)$ is complemented in a dual to a bornological space, so is F. Hence if additionally F contains a copy of c_0, then it contains a copy of l_∞ as well (cf. [15, Cor. 2.6]).

COROLLARY 9.8. *Assume E is* QC *and $\aleph_0$-barrrelled.*

- (a) *Let $A(E,F)$ be a surjectively- or injectively-l_∞-determined space of operators such that $\mathcal{G}(E,F) \subset A(E,F)$. Then $A(E,F)$ is injective iff $A(E,F) = \mathcal{L}(E,F)$ and both E' and F are injective.*
- (b) *Let $\mathcal{A}$ be a surjectively- or injectively-l_∞-determined operator ideal on a category $\mathcal{E}$ containing l_1 and l_∞ as well as E and F. Assume that $\mathcal{G} \subset \mathcal{A}$. Then $\mathcal{A}(E,F)$ is injective iff both E' and F are injective and either $\mathrm{id}_F \in \mathcal{A}$ or $\mathrm{id}_E \in \mathcal{A}$.*

Proof. We concentrate on proving (a).

Necessity: If $A(E,F)$ is injective, then E' and F are injective as complemented subspaces of $A(E,F)$; in particular, F is complete. By [20] or [59], either $E' \simeq \mathbb{K}^I$ or E' contains a copy of l_∞. According to [5, Th. 8] (see also [7, Lemma 10], if G is a complete and $\aleph_0$-barrelled LCS and G' contains a copy of c_0, then G contains a complemented copy of l_1. In fact, the proof in [5] requires only quasi-completeness of G instead of completeness. Thus in our situation either $E' \simeq \mathbb{K}^I$ or E contains a complemented copy of l_1. Analogously, either $F \simeq \mathbb{K}^I$ or F contains a copy of l_∞.

It is clear that

(1) if $E' \simeq \mathbb{K}^I$, then $E \simeq \mathbb{K}^{(I)}$, and consequently $\mathcal{G}(E,F) = \mathcal{L}(E,F) \simeq F^I$.

Similarly,

(2) if $F \simeq \mathbb{K}^I$, then $\mathcal{G}(E,F) = \mathcal{L}(E,F) = A(E,F) \simeq (E')^I$.

So, finally, we may assume that E contains a complemented copy of l_1 and F contains a copy of l_∞, and we conclude by applying Theorem 9.6.

Sufficiency: Follows easily from the fact [57] that if E' and F are injective, so is $\mathcal{L}(E,F)$.

Part (b) follows from (a) and Theorem 9.6. Let us only point out that, in the present setting, the necessity part above yields $\mathrm{id}_F \in \mathcal{A}$ in case (1), and $\mathrm{id}_E \in \mathcal{A}$ in case 9B. ∎

10. Applications to concrete operator ideals. We first introduce the notation for the classes of operators between LCSs that will be used below to illustrate the theory developed in the previous section.

As before, let E and F be LCSs.

10A. More spaces of operators

10A(i). *Operators mapping bounded sequences to some compact-like ones.* $\mathcal{K}$, $\mathcal{W}$, and $\mathcal{L}_{\mathrm{ltd}}$ will denote the classes of operators mapping bounded sequences to sequences that are relatively compact, or relatively weakly compact, or limited, respectively. Thus

$$\mathcal{K}(E,F) = \{T \in \mathcal{L}(E,F) : \hat{T}(l_\infty(E)) \subset \kappa(F)\},$$
$$\mathcal{W}(E,F) = \{T \in \mathcal{L}(E,F) : \hat{T}(l_\infty(E)) \subset \kappa(F,\sigma)\},$$
$$\mathcal{L}_{\mathrm{ltd}}(E,F) = \{T \in \mathcal{L}(E,F) : \hat{T}(l_\infty(E)) \subset l(F)\}.$$

If F is QC, then $\mathcal{K}(E,F)$ and $\mathcal{W}(E,F)$ consist of the operators mapping bounded sets to relatively compact or relatively weakly compact sets, respectively. Moreover, by Grothendieck's results [39, Sec. 1.1, Lemmas 1 and 2, and Remark on p. 133], in this case we also have the following dual characterizations:

$$\mathcal{K}(E,F) = \{T \in \mathcal{L}(E,F) : \hat{T}'(e(F')) \subset \kappa(E')\},$$
$$\mathcal{W}(E,F) = \{T \in \mathcal{L}(E,F) : \hat{T}'(e(F')) \subset \kappa(E',\sigma)\}.$$

10A(ii). *Operators fixing no copy of l_1 or c_0.* $\mathcal{R}$ and $\mathcal{BP}$ will denote the classes of operators fixing no copy of l_1 or c_0, respectively. Evidently, an operator $T: E \to F$ is in $\mathcal{R}$ iff $TQ \in \mathcal{R}(l_1, F)$ (or equivalently TQ is strictly singular) for all $Q \in \mathcal{L}(l_1, E)$; and likewise for $\mathcal{BP}$. Since the strictly singular operators from any Banach space G to F form a linear subspace of $\mathcal{L}(G, F)$ [78, Th. 3.4], it follows immediately that $\mathcal{R}(E, F)$ and $\mathcal{BP}(E, F)$ are indeed linear subspaces of $\mathcal{L}(E, F)$. Let E be SC. Then $\mathcal{BP}(E, F)$ coincides with the space of unconditionally converging operators from E to F (cf. Subsection 3C). Moreover (cf. Proposition 3.5(b) and Subsection 3C),

$$\begin{aligned}\mathcal{R}(E, F) &= \{T \in \mathcal{L}(E, F) \colon \hat{T}(l_\infty(E)) \subset r(F)\},\\ \mathcal{BP}(E, F) &= \{T \in \mathcal{L}(E, F) \colon \hat{T}(bp_0(E)) \subset c_0(F)\}\\ &= \{T \in \mathcal{L}(E, F) \colon \hat{T}(bp_0(E)) \subset \kappa(F)\}\\ &= \{T \in \mathcal{L}(E, F) \colon \hat{T}'(e(F')) \subset v(E')\},\end{aligned}$$

where $bp_0(E) := \{(Qe_n) \colon Q \in \mathcal{L}(c_0, E)\}$, the space of perfectly bounded sequences (x_n) in E, i.e. those for which the series $\sum_n x_n$ is wuC. Let us also note that if E is SC and F is metrizable, then $\mathcal{R}(E, F)$ is the space of operators mapping bounded sets to conditionally weakly compact sets (because $r(F) = c\kappa(F, \sigma)$ in this case).

10A(iii). *Operators fixing no complemented copy of c_0 or l_∞.* $\mathcal{SR}$ and $\mathcal{SBP}$ denote the classes of operators fixing no complemented copy of l_1 or a copy l_∞ (automatically complemented, of course), respectively. Let us clarify at this point that we say that an operator $T \colon E \to F$ *fixes a complemented copy* of a space G if there is a subspace X of E such that $X \simeq G$, $T|X$ is an isomorphic embedding, and the subspace $T(X)$ is complemented in F. (Then also X is complemented in E.) Using the fact that l_1 is saturated with complemented isomorphs of l_1, and applying a similar argument to that indicated in 10A(ii), one easily verifies that $\mathcal{SR}(E, F)$ is always a linear subspace of $\mathcal{L}(E, F)$. The same holds for $\mathcal{SBP}(E, F)$, by Rosenthal's l_∞-theorem (see [68]; cf. also [23], [22]). If E is SC, then (cf. Proposition 3.5(a)) an operator $T : E \to F$ belongs to $\mathcal{SR}$ iff $QT \in \mathcal{K}$ for every $Q \in \mathcal{L}(F, l_1)$. In other words,

$$\mathcal{SR}(E, F) = \{T \in \mathcal{L}(E, F) \colon \hat{T}(l_\infty(E)) \subset v^*(F)\}.$$

Now, let F be QC. Then an operator $T : E \to F$ belongs to $\mathcal{SBP}$ iff $TQ \in \mathcal{W}$ for every $Q \in \mathcal{L}(l_\infty, E)$. Indeed, if $TQ \notin \mathcal{W}$, then there is $S \in \mathcal{L}(F, l_\infty)$ such that $STQ \notin \mathcal{W}$ (see Proposition 3.1(b)). Therefore, by [13, Cor. VI.1.3], STQ fixes a copy of l_∞, hence so does T. By a similar argument, using also [13, Th. VI.1.1], one can show that

$$\mathcal{SBP}(E, F) = \{T \in \mathcal{L}(E, F) \colon \hat{T}(bp_\infty(E)) \subset c_0(F)\},$$

where $bp_\infty(E) := \{(Qe_n) \colon Q \in \mathcal{L}(l_\infty, E)\}$.

10A(iv). *Completely continuous or Dunford–Pettis operators.* $\mathcal{V}$ denotes the class of completely continuous (or Dunford–Pettis) operators, i.e., those mapping relatively weakly compact sequences to relatively compact sequences. Thus

$$\mathcal{V}(E,F) = \{T \in \mathcal{L}(E,F)\colon \hat{T}(\kappa(E,\sigma)) \subset \kappa(F)\}.$$

If F is QC, then $\mathcal{V}(E,F)$ is precisely the space of operators mapping weakly compact sets to compact sets.

On the other hand, if E is QC, then $\kappa(E,\sigma) = \{(Qe_n)\colon Q \in \mathcal{W}(l_1,E)\}$. In consequence, if both E and F are QC, then an operator $T\colon E \to F$ is in $\mathcal{V}(E,F)$ iff TQ is in $\mathcal{K}(l_1,F)$ for all $Q \in \mathcal{W}(l_1,E)$, and also iff $STQ \in \mathcal{K}(l_1,l_\infty)$ for all $Q \in \mathcal{W}(l_1,E)$ and $S \in \mathcal{L}(F,l_\infty)$. Thus, using Schauder's theorem, if $T \in \mathcal{V}(E,F)$ then, given any sequence $(y_n') \in e(F')$ (representing an operator $S\colon F \to l_\infty$), we see that $(Q'T'y_n') \in \kappa(l_1)$ for all $Q \in \mathcal{W}(l_1,E)$. That is, in view of Proposition 3.1(g), $(T'y_n') \in p\kappa(E',\mu')$. (Actually, since $(T'y_n')$ is equicontinuous, it is in $\kappa(E',\mu')$.) We have thus shown that $\hat{T}'(e(F')) \subset p\kappa(E',\mu')$. Conversely, let an operator $T\colon E \to F$ satisfy this condition, and suppose $T \notin \mathcal{V}(E,F)$. Hence there is $(x_n) \in \kappa(E,\sigma)$ for which $(Tx_n) \notin \kappa(F)$. Then, by the same construction as in the proof of Remark 6.3(a)(14), we can find a sequence $(y_n') \in e(F')$ which is biorthogonal to (Tx_n). Clearly, the sequence $(T'y_n')$ is then biorthogonal to (x_n), whence it cannot be μ'-precompact. If both E and F are QC then, again by a result of Grothendieck [39, Sec. 1.1, Lemma 2 and Remark on p. 133],

$$\mathcal{V}(E,F) = \{T \in \mathcal{L}(E,F)\colon \hat{T}'(e(F')) \subset p\kappa(E',\mu')\};$$

instead of $p\kappa(E',\mu')$ one can use $\kappa(E',\mu')$ as well.

10A(v). *Weakly completely continuous operators.* $\mathcal{WV}$ denotes the class of weakly completely continuous operators, i.e., those mapping weakly Cauchy sequences to weakly convergent sequences. If F is QC, then an operator $T \in \mathcal{L}(E,F)$ is in $\mathcal{WV}$ iff it maps conditionally weakly compact sets to relatively weakly compact sets.

10A(vi). *Strictly singular or cosingular operators.* $\mathcal{S}$ and $\mathcal{C}$ denote, respectively, the classes of strictly singular and strictly cosingular operators on Banach spaces, i.e., those operators $T : E \to F$, where E and F are Banach spaces, such that there is no infinite-dimensional Banach space G and no isomorphic embedding $j\colon G \to E$ (resp., no quotient map $q\colon F \to G$) for which $T \circ j$ is an isomorphic embedding (resp., $q \circ T$ is a surjective map).

PROPOSITION 10.1. *Let E be a complete LB-space and F a Fréchet space.*

(a) $\mathcal{K}(E,F)$ *and* $\mathcal{W}(E,F)$ *are precisely the classes of compact and weakly compact operators, respectively.*
(b) $\mathcal{L}_{\rm ltd}(E,F)$ *is the class of operators mapping a* 0*-neighborhood to a limited set.*

(c) $\mathcal{R}(E,F)$ *is the class of operators mapping a* 0*-neighborhood to a conditionally weakly compact set.*

Proof. Let us observe that if $\mathcal{A}=\mathcal{K}$, $\mathcal{W}$, $\mathcal{L}_{\mathrm{ltd}}$ or $\mathcal{R}$, then an operator $T\colon E\to F$ is in $\mathcal{A}(E,F)$ iff for every absolutely convex 0-neighborhood U in F the operator $i_U\circ T\colon E\to F_U$ is in $\mathcal{A}$.

Since all these ideals $\mathcal{A}$ are surjective in the category of Banach spaces (see [46, 6.2.6]), $T\in\mathcal{A}(E,F)$ iff $i_U\circ T$ maps a 0-neighborhood to a relatively compact, relatively weakly compact, limited, or conditionally weakly compact set, respectively. This easily implies that T itself maps a 0-neighborhood to a set in the respective family. (Use the fact that, since E is an LB-space, for every sequence (U_n) of 0-neighborhoods in E there is a 0-neighborhood V in E which is absorbed by all U_n; see [46, 2.7.9]. ∎

In the proof of Proposition 10.3 below we will use the following.

LEMMA 10.2. *If A is an absolutely convex noncompact* (*resp., non-weakly-compact*) *closed bounded subset of l_∞, then there is $N\in[\mathbb{N}]$ such $P_M(A)$ is not relatively compact* (*resp., not relatively weakly compact*) *for every $M\in[N]$.*

Proof. We will give a proof only for the weakly compact version.

Let X be the linear span of A; equipped with the gauge functional of A as a norm, X becomes a Banach space. Let $J\colon X\to l_\infty$ denote the identity embedding; clearly, J is a continuous non-weakly-compact operator. Suppose the assertion of the lemma is false so that for every $N\in[\mathbb{N}]$ there is $M\in[N]$ for which $P_M(A)$ is relatively weakly compact. The latter means that the operator $P_M\circ J$ is weakly compact, hence so is its adjoint, by the Gantmacher theorem. In particular, denoting by (e_n') the unit vectors in $l_1\subset l_\infty'$, we have

$$(J'(e_n'))_{n\in M}\in\kappa(X',\sigma(X',X'')).$$

Thus every subsequence of $(J'(e_n'))$ has a relatively weakly compact subsequence. Therefore, by the Eberlein–Šmulian theorem,

$$(J'(e_n'))_{n\in\mathbb{N}}\in\kappa(X',\sigma(X',X'')).$$

In consequence, $J'|_{l_1}\colon l_1\to X'$ is weakly compact. Since J' is weak*-weak* continuous and the unit ball of l_1 is weak* dense in the unit ball of l_∞', it follows that J' is weakly compact. By the Gantmacher theorem again, J is weakly compact; a contradiction. ∎

PROPOSITION 10.3.

(a) *Each of the operator ideals $\mathcal{K}$, $\mathcal{W}$, $\mathcal{L}_{\mathrm{ltd}}$, $\mathcal{R}$, $\mathcal{SR}$ is surjectively-l_∞-determined in any category of* SC *spaces.*
(b) *Each of the operator ideals $\mathcal{K}$, $\mathcal{W}$, $\mathcal{V}$, $\mathcal{BP}$, $\mathcal{WV}$ is injectively-l_∞-determined and projectively-c_0-determined in any category of* QC *spaces.*

(c) *Each of the operator ideals* $\mathcal{K}$, $\mathcal{W}$, $\mathcal{BP}$, $\mathcal{V}$, $\mathcal{WV}$, $\mathcal{SBP}$ *is* c_0*-projectively determined in any category of* SC *spaces.*

(d) *The injective operator ideals* $\mathcal{R}$ *and* $\mathcal{S}$ *in the category of Banach spaces are not injectively-*l_∞*-determined, while the surjective operator ideal* $\mathcal{C}$ *in the category of Banach spaces is not surjectively-*l_∞*-determined.*

Proof. (a): This is immediate from 10A(i)–10A(iii) and Theorem 6.1(A).

(b): To see that these ideals are injectively-l_∞-determined, apply the dual characterizations given in Subsections 10A(i)–10A(iii) along with Theorem 6.1(B).

Only the case of $\mathcal{WV}$ needs some work. Suppose an operator $T\colon E \to F$ is not in $\mathcal{WV}$; thus there is a weakly Cauchy sequence (x_n) in E such that (Tx_n) is not relatively weakly compact. Choose an operator $R\colon F \to l_\infty$ so that also (RTx_n) is not relatively weakly compact (see Proposition 3.1(c)), and define an operator $Q\colon l_1 \to E$ by $Qe_n = x_n$. Applying Lemma 10.2 to the closed absolutely convex hull of (RTx_n) we find $N \in [\mathbb{N}]$ such that for every $M \in [N]$ the operator $P_M RTQ$ is not weakly compact. On the other hand, $\mathcal{WV}(E, l_\infty) \circ Q$ consists only of weakly compact operators. This means that $\mathcal{WV}$ is injectively-l_∞-determined.

(c): In view of Lemma 10.2, the injective bs-functors $k(\cdot) = \kappa(\cdot)$ and $k(\cdot) = \kappa(\cdot, \sigma)$ satisfy the hypothesis on k in Proposition 9.5. Therefore, it is enough to apply this proposition with $k(E) = \kappa(E)$ for $\mathcal{K}$, $\mathcal{V}$ and $\mathcal{BP}$, and with $k(E) = \kappa(E, \sigma)$ for $\mathcal{W}$, $\mathcal{WV}$ and $\mathcal{SBP}$, noting also that

$$\mathcal{SBP}(E, F) = \{T \in \mathcal{L}(E, F)\colon \hat{T}(w(E)) \subset \kappa(F, \sigma)\},$$

where $w(E) := \{\hat{Q}(x)\colon x \in l_\infty(l_\infty),\ Q \in \mathcal{L}(l_\infty, E)\}$.

(d): By [24, Ex. IV.13.43], a bounded sequence (x_n) in l_∞ is weakly Cauchy iff for every $N \in [\mathbb{N}]$ there is $M \in [N]$ such that

$$\lim_{n\to\infty} \lim_{m\in M,\, m\to\infty} x_{nm} \quad \text{exists},$$

where $x_n =: (x_{nm})_{m\in\mathbb{N}}$. Obviously, for every operator $T\colon l_1 \to l_\infty$ there is $M \in [\mathbb{N}]$ such that for every $n \in \mathbb{N}$ the limit $\lim_{m\in M,\, m\to\infty} y_{nm}$ exists, where $Te_n =: (y_{nm})_{m\in\mathbb{N}}$ and e_n is the nth unit vector in l_1. Hence, $(P_M Te_n)$ is a weakly conditionally compact sequence and $P_M \circ T \in \mathcal{R}$. This means that $\mathcal{R}$ is not injectively-l_∞-determined although it is an injective closed operator ideal.

As $\mathcal{R}(l_1, l_\infty) = \mathcal{S}(l_1, l_\infty)$ (because every closed infinite-dimensional subspace of l_1 contains an isomorphic copy of l_1), $\mathcal{S}$ is not injectively-l_∞-determined either. By duality, we can show that for every operator $T\colon l_1 \to c_0$ there is $M \in [\mathbb{N}]$ such that $T|_{l_1(M)} \in \mathcal{C}$. In particular, $\mathcal{C}$ is not surjectively-l_∞-determined. ∎

REMARK 10.4. The ideal of operators with separable range is an example of a surjective and injective closed operator ideal whose components are neither surjectively- nor injectively-l_∞-determined.

PROPOSITION 10.5.

(a) *In any category* $\mathcal{E}$ *containing* l_1 *and consisting of* SC *spaces, the ideal* $\mathcal{SR}$ *is the largest surjectively-*l_∞*-determined operator ideal.*
(b) *In any category* $\mathcal{E}$ *containing* l_∞ *and consisting of* QC *spaces, the ideal* $\mathcal{SBP}$ *is the largest injectively-*l_∞*-determined operator ideal.*

Proof. (a): From Proposition 10.3 we already know that $\mathcal{SR}$ is surjectively-l_∞-determined. Now, it is clear that if $\mathcal{A}$ is a surjectively-l_∞-determined operator ideal on $\mathcal{E}$ and $\mathcal{A} \setminus \mathcal{SR} \neq \emptyset$, then $\mathrm{id}_{l_1} \in \mathcal{A}$. Since $\mathcal{A}$ is surjectively-l_∞-determined, we get $\mathcal{A} = \mathcal{L}$.

(b): We use the description of $\mathcal{SBP}$ given in Subsubsection 10A(iii). The details, similar to those in (a), are left to the reader. ■

COROLLARY 10.6. *Let* E *and* F *be quasi-complete infinite-dimensional* LCS*s, and assume* E *is* $\aleph_0$*-barrelled.*

(1) *Each of the spaces* $\mathcal{K}(E,F)$, $\mathcal{W}(E,F)$, $\mathcal{L}_{\mathrm{ltd}}(E,F)$, $\mathcal{R}(E,F)$ *is injective iff* E' *and* F *are injective and either* $E \simeq \mathbb{K}^{(I)}$ *or* $F \simeq \mathbb{K}^J$.
(2) *Each of the spaces* $\mathcal{BP}(E,F)$, $\mathcal{SBP}(E,F)$, $\mathcal{SR}(E,F)$ *is injective iff* E' *and* F *are injective.*
(3) $\mathcal{V}(E,F)$ *is injective iff* E' *and* F *are injective and either* E *has the Schur property or* $F \simeq \mathbb{K}^I$.
(4) $\mathcal{WV}(E,F)$ *is injective iff* E' *and* F *are injective and either* E *is weakly* SC *or* $F \simeq \mathbb{K}^I$.

Proof. Essentially, it is enough to apply Corollary 9.8 (together with its proof) and Proposition 10.3. In addition:

For the proofs of (1), (3), and (4) observe that $\mathbb{K}^I$ are the only injective LCSs having the Schur property or being weakly SC (by a result from [20] and [57] quoted in the proof of Remark 6.3(a)(14)).

For the proof of (2) observe that if E' and F are injective, then $\mathcal{BP}(E,F) = \mathcal{SBP}(E,F) = \mathcal{L}(E,F)$. Indeed, if an operator $T\colon E \to F$ fixes a copy of c_0, then so does $T''\colon E'' \to F''$, in particular, E'' contains a copy of c_0. Since E' is injective, it is also barrelled and complete. It follows (see the proof of Corollary 9.8) that E' contains a complemented copy of l_1, contradicting the injectivity of E'.

Similarly, $\mathrm{id}_F \in \mathcal{SR}$ for any injective space F. ■

REMARKS 10.7. (a) Although it seems that Corollary 10.6(1) for the ideal of compact operators was never explicitly stated, nonetheless this fact was essentially known at least in the category of Banach spaces. Namely, F. Thorp [75] proved the following:

(∗) *If X and Y are Banach spaces, X contains a complemented copy of l_1, and Y contains a copy of l_∞, then $\mathcal{K}(X,Y)$ is uncomplemented in $\mathcal{L}(X,Y)$.*

If we look at the proof of Corollary 9.8 we realize that Corollary 10.6(1) for $\mathcal{K}$ is easily implied by Thorp's result. Besides Thorp's elementary Thorp's proof, there are many other proofs of (∗) and its generalizations (see [2, Ths. 1(a) and 2(a)], [47, Lemma 3], [35, Cor. 1]). A quite different proof of Corollary 10.6(1) for $\mathcal{K}$ follows from [36, Th. 2.3] with the use of [2, Lemma 3] (remembering that $\mathcal{K}(l_1,l_\infty)=l_\infty\tilde{\otimes}_\varepsilon l_\infty$).

For the space of weakly compact operators an analogue of Thorp's result (∗) was also known before [2, Th. 1(c) and Lemma 3].

For E, F Banach spaces, $\mathcal{S}(E,F)$ is injective iff E' and F are injective and either E or F is finite-dimensional. Indeed, by the proof of Corollary 9.8 it is enough to show that if E contains a complemented copy of l_1 and F contains a copy of l_∞, then $\mathcal{S}(E,F)$ is not injective. It is easily seen that $\mathcal{S}(l_1,l_\infty)$ is isomorphic to a complemented subspace of $\mathcal{S}(E,F)$. On the other hand, $\mathcal{S}(l_1,l_\infty)=\mathcal{R}(l_1,l_\infty)$, and the latter space is not injective by Corollary 10.6.

Applying Theorem 9.6 and Proposition 10.3 we get:

Corollary 10.8.

(a) *Let E and F be quasi-complete and assume that E contains a complemented copy of l_1. If $\mathcal{K}(E,F)$ (resp., $\mathcal{W}(E,F)$, $\mathcal{L}_{\rm ltd}(E,F)$, $\mathcal{R}(E,F)$, $\mathcal{SR}(E,F)$) is complemented in the dual to a bornological space, then F is semi-Montel (resp., semi-reflexive, semi-Montel, contains no copy of l_1, contains no complemented copy of l_1).*

(b) *Let E and F be quasi-complete and assume that F contains a copy of c_0. If $\mathcal{K}(E,F)$ (resp., $\mathcal{W}(E,F)$, $\mathcal{V}(E,F)$, $\mathcal{BP}(E,F)$, $\mathcal{WV}(E,F)$, $\mathcal{SBP}(E,F)$) is complemented in the dual to a bornological space, then E is semi-Montel (resp., semi-reflexive, has the Schur property, contains no copy of c_0, is weakly* SC*, contains no copy of l_∞).*

Remarks 10.9. (a) If a Fréchet space contains a complemented copy of c_0, then it is not complemented in a dual Fréchet space. In view of this, Corollary 10.8(b) for the space of compact operators, when E or F has the approximation property and F is a Banach space, follows from [36, Th. 2.3] (see also [70] and [71]). Similarly, if E and F are Banach spaces, then Corollary 10.8(a) for $\mathcal{K}$ follows from [30, Cor. 1] or [27, Cor. 2].

(b) The results of Corollary 10.8(a, b) for $\mathcal{K}$ are also connected with the question when $\mathcal{K}(E,F)$ is reflexive (see [40], [41], [42] and [69]). M. Feder [34, proof of Th. 5] proved that if E and F are reflexive Banach spaces, and E' or F are subspaces of Banach spaces with unconditional bases, then $\mathcal{K}(E,F)$ is either reflexive or else uncomplemented in a dual Banach space.

(c) There is also a strong connection between Corollary 10.8 for $\mathcal{K}$ and Corollaries 10.10(4) and 10.11(3) (see [45, Prop. 1]).

(d) For weakly compact operators much less was known. In [27, Cor. 3] or [30, Cor. 2] it was recently proved that if E and F are Banach spaces such that both E and F' contain copies of c_0, then $\mathcal{W}(E',F)$ contains a complemented copy of c_0, hence obviously is not complemented in a dual space.

(e) For Banach spaces E containing a copy of c_0, some other closed operator ideals $\mathcal{A}$ have been shown in [30] to have the property that $\mathcal{A}(E',F)$ contains a complemented copy of c_0, and so is not complemented in a dual space. This, and the similar result from the previous remark, are particular cases of Corollary 10.8.

We can also obtain some results on the complementability of the components of one operator ideal in the corresponding components of another. These sample results are consequences of Corollary 9.3, and Propositions 9.5 and 10.3.

COROLLARY 10.10. *Let E and F be quasi-complete and assume that E contains a complemented copy of l_1.*

(1) *$\mathcal{K}(E,F)$ is complemented in $\mathcal{W}(E,F)$ iff $\mathcal{K}(E,F)=\mathcal{W}(E,F)$ iff F has the Schur property.*
(2) *$\mathcal{K}(E,F)$ is complemented in $\mathcal{R}(E,F)$ iff $\mathcal{K}(E,F)=\mathcal{R}(E,F)$; for F Fréchet: iff F has the Schur property.*
(3) *$\mathcal{K}(E,F)$ is complemented in $\mathcal{L}_{\mathrm{ltd}}(E,F)$ iff $\mathcal{K}(E,F)=\mathcal{L}_{\mathrm{ltd}}(E,F)$ iff F has the Gelfand–Phillips property.*
(4) *$\mathcal{K}(E,F)$ is complemented in $\mathcal{L}(E,F)$ iff $\mathcal{K}(E,F)=\mathcal{L}(E,F)$ iff F is semi-Montel.*
(5) *$\mathcal{W}(E,F)$ is complemented in $\mathcal{R}(E,F)$ iff $\mathcal{W}(E,F)=\mathcal{R}(E,F)$; for F Fréchet: iff F is weakly* SC.
(6) *$\mathcal{W}(E,F)$ is complemented in $\mathcal{L}(E,F)$ iff $\mathcal{W}(E,F)=\mathcal{L}(E,F)$ iff F is semireflexive.*
(7) *$\mathcal{R}(E,F)$ is complemented in $\mathcal{L}(E,F)$ iff $\mathcal{R}(E,F)=\mathcal{L}(E,F)$ iff F contains no copy of l_1.*
(8) *$\mathcal{L}_{\mathrm{ltd}}(E,F)$ is complemented in $\mathcal{L}(E,F)$ iff $\mathcal{L}_{\mathrm{ltd}}(E,F)=\mathcal{L}(E,F)$; for F Fréchet or quasi-normable: iff F is semi-Montel.*
(9) *$\mathcal{K}(E,F)$ is complemented in $\mathcal{SP}(E,F)$ iff $\mathcal{K}(E,F)=\mathcal{SP}(E,F)$ iff F is semi-Montel.*

Proof. By Corollary 9.3, in each case the first two conditions are equivalent, and also equivalent to the second condition with l_1 replacing E. Now, identifying $\mathcal{L}(l_1,F)$ with $l_\infty(F)$ via the mapping $T\mapsto(Te_n)$, we also have $\mathcal{K}(l_1,F)=\kappa(F)$, $\mathcal{W}(l_1,F)=\kappa(F,\sigma')$, $\mathcal{L}_{\mathrm{ltd}}(l_1,F)=l(F)$ and $\mathcal{R}(l_1,F)=r(F)$ (in the last case in virtue of the Rosenthal l_1-theorem). We conclude by applying appropriate parts from Remarks 6.3. ∎

Corollary 10.11. *Let E and F be quasi-complete and assume E is $\aleph_0$-barrelled and F contains a copy of c_0.*

(1) *$\mathcal{K}(E,F)$ is complemented in $\mathcal{W}(E,F)$ iff $\mathcal{K}(E,F)=\mathcal{W}(E,F)$ iff E' has the sequential Schur property.*
(2) *For E Fréchet or quasi-normable: $\mathcal{K}(E,F)$ is complemented in $\mathcal{R}(E,F)$ iff $\mathcal{K}(E,F)=\mathcal{R}(E,F)$ iff E is semi-Montel.*
(3) *For E Fréchet or quasi-normable: $\mathcal{K}(E,F)$ is complemented in $\mathcal{L}(E,F)$ iff $\mathcal{K}(E,F)=\mathcal{L}(E,F)$ iff E is semi-Montel.*
(4) *For E Banach: $\mathcal{K}(E,F)$ is complemented in $\mathcal{V}(E,F)$ iff $\mathcal{K}(E,F)=\mathcal{V}(E,F)$ iff E contains no copy of l_1.*

Proof. If $\mathcal{A}(E,F)$ and $\mathcal{B}(E,F)$ stand for the pair of spaces in (1)–(4) and if $\mathcal{A}(E,F)$ is complemented in $\mathcal{B}(E,F)$ then, by Proposition 9.5, $\mathcal{A}(E,c_0)=\mathcal{B}(E,c_0)$. Expressing this equality in each case in terms of the corresponding (representing) spaces of sequences, we obtain the following equalities.

(1): $c_0(E')=\kappa(E',\sigma)$;

(2) and (3): $c_0(E')=c_0(E',\sigma')$;

(4): $c_0(E')=c_0(E',\mu')$ (here consult also Subsubsection 10A(iii)).

By Remarks 6.3(a)(1, 10, 13), the required properties of E follow. The argument for the other direction is rather obvious. ∎

Corollary 10.12. *Let E and F be quasi-complete and assume F contains a copy of l_∞.*

(1) *$\mathcal{W}(E,F)$ is complemented in $\mathcal{L}(E,F)$ iff $\mathcal{W}(E,F)=\mathcal{L}(E,F)$ iff E is semireflexive.*
(2) *$\mathcal{V}(E,F)$ is complemented in $\mathcal{L}(E,F)$ iff $\mathcal{V}(E,F)=\mathcal{L}(E,F)$ iff E has the Schur property.*
(3) *$\mathcal{BP}(E,F)$ is complemented in $\mathcal{L}(E,F)$ iff $\mathcal{BP}(E,F)=\mathcal{L}(E,F)$ iff E contains no copy of c_0.*
(4) *$\mathcal{WV}(E,F)$ is complemented in $\mathcal{L}(E,F)$ iff $\mathcal{WV}(E,F)=\mathcal{L}(E,F)$ iff E is weakly* SC.

Proof. By Corollary 9.3, in each case the first condition is equivalent to the second one, and also equivalent to the second condition with l_∞ replacing F. The rest is obvious. ∎

Corollary 10.13. *Let E and F be quasi-complete and let $\mathcal{A}$ be an operator ideal in a category containing E, F and all Banach spaces.*

(a) *If $\mathcal{SR}(E,F)\subset\mathcal{A}(E,F)$, then $\mathcal{SR}(E,F)$ is a complemented subspace of $\mathcal{A}(E,F)$ iff $\mathcal{SR}(E,F)=\mathcal{A}(E,F)$.*

(b) *If* $\mathcal{SBP}(E,F) \subset \mathcal{A}(E,F)$, *then* $\mathcal{SBP}(E,F)$ *is a complemented subspace of* $\mathcal{A}(E,F)$ *iff* $\mathcal{SBP}(E,F) = \mathcal{A}(E,F)$.

Proof. (a): If E contains no complemented copy of l_1, then $\mathcal{SR}(E,F) = \mathcal{L}(E,F)$. Otherwise, apply Proposition 9.5. The proof of (b) is similar. ∎

REMARKS 10.14. Here, we collect a number of comments about the position of our results within the existing literature.

(a) There is an extensive literature on the problem of complementability of $\mathcal{K}(E,F)$ in $\mathcal{L}(E,F)$ for Banach spaces E and F: [2], [31], [32], [34], [35], [44], [45], [47], [75], [76] and [77]. Not surprisingly, it led to the long standing conjecture that $\mathcal{K}(E,F)$ is complemented in $\mathcal{L}(E,F)$ iff both spaces coincide. In this generality, as already mentioned in the Introduction, the problem remained open until the 2011 work of Argyros and Haydon [1] who solved it in the negative by constructing a Banach space E whose endomorphisms are all of the form $a\,\mathrm{id}_E + K$, K being a compact operator in E. Naturally, for some particular cases it has been solved earlier. In the non-locally-convex setting the conjecture was known to be trivially false: the only compact endomorphism on $L_p(0,1)$ for $p < 1$ is the zero operator [48]. The result contained in Corollaries 10.10(4) and 10.11(3) is known [47, Lemma 3], [35, Cor. 1]. Somewhat stronger results can be found in [31] and [44].

(b) The analogue of the above mentioned conjecture for the weakly compact operators is false. Indeed, every endomorphism T of the James space J [52, Ex. 1.d.2] is of the form $T = a\,\mathrm{id}_J + U$, where U is a weakly compact operator. Therefore $\mathcal{W}(J)$ is of codimension one in $\mathcal{L}(J)$ and, of course, complemented.

Corollary 10.10(6) in the case of Banach spaces is known [2, Th. 1(c)]. The corresponding Corollary 10.12(1) was proved in [2, Th. 2(c)] only for separable E. For other results of a similar type (partially covered by our Corollary 10.12), see [29].

(c) For Banach spaces Corollary 10.10(1) is also known [2, Th. 1(b)]. For other results of a similar type, see [33, Th. 5], [2, Th. 2(b)] and [50].

(d) Corollaries 10.11 and 10.12(4) cover particular cases proved earlier in [33, Thms. 2 and 3]. For additional results see [29, Th. 1].

(e) Finally, it is worth recalling two results of comparable importance and ingenuity which preceded that of Argyros and Haydon. First, Shelah [73] constructed (under the set-theoretical axiom of constructability) a nonseparable Banach space E_1 such that every $T \in \mathcal{L}(E_1)$ is of the form $a\,\mathrm{id} + S$, where a is a scalar and $S \in \mathcal{L}(E_1)$ has a separable range. Then Gowers and Maurey [38] constructed a Banach space E_2 such that every $T \in \mathcal{L}(E_2)$ is of the same form as above but with S strictly singular. Clearly, the spaces $\mathcal{SP}(E_1)$ and $\mathcal{S}(E_2)$ are complemented in $\mathcal{L}(E_1)$ and $\mathcal{L}(E_2)$, respectively.

Acknowledgements. The authors are grateful to Professors Jose Bonet and Przemysław Wojtaszczyk for some remarks and information concerning the problems considered in the paper. In 2019 thanks go again to J. Bonet for the very careful reading of the paper and encouraging its publication. Finally, the suggestions made by the referee were very helpful in improving the paper.

Much of the paper was written while the first author held the Alexander von Humboldt Fellowship at the University of Wuppertal in 1996. He was greatly indebted to Humboldt's Foundation for hospitality and financial support, and particularly to Professor Dietmar Vogt for his friendship and arranging the stay in Wuppertal. In addition, the research of both authors was supported by Komitet Badań Naukowych (State Committee for Scientific Research), Poland, grant no. 2 1083 91 01.

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