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ON CERTAIN ASPECTS OF THE MÖBIUS RANDOMNESS PRINCIPLE

BY

DAVIT KARAGULYAN (Stockholm)

Abstract. We study different aspects of the Möbius randomness principle. We rephrase the Chowla and Sarnak conjectures and the Riemann hypothesis for abstract sequences and study their relationship. In particular, we show that in this setting the Chowla and Sarnak conjectures do not imply the Riemann hypothesis. We also study the connections between the multiplicative and additive van der Corput criteria.

1. Introduction.

Let μ denote the *Möbius function*, i.e.

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n = p_1 \cdots p_k \text{ for distinct primes } p_i, \\ 0 & \text{otherwise.} \end{cases}$$

It is of great importance in number theory because of its connection with the Riemann ζ-function via the formulas

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s} = \frac{\zeta(s)}{\zeta(2s)}, \qquad \text{for } \mathrm{Re}(s) > 1.$$

We also recall the definition of the *Liouville function*

$$\lambda(n) = (-1)^{\Omega(n)},$$

where $\Omega(n)$ is the number of prime factors of n, counted with multiplicity. Furthermore, the estimate

$$\Big| \sum_{n=1}^{N} \mu(n) \Big| = O(N^{1/2+\epsilon}), \qquad \forall \epsilon > 0, \tag{1.1}$$

is equivalent to the Riemann hypothesis [21, p. 315]. We also note that the property

$$\frac{1}{N} \sum_{n=1}^{N} \mu(n) = o(1) \qquad \text{as } N \to \infty \tag{1.2}$$

is equivalent to the prime number theorem (see e.g. [5, p. 91]).

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The general *Möbius pseudorandomness heuristics* suggests that the sign of μ behaves so randomly (or pseudorandomly) that one should expect a substantial amount of cancellation in sums that involve the sign fluctuation of the Möbius function in a nontrivial fashion, comparable to the amount that an analogous random sum would provide. There are a number of ways to make this heuristic precise. As already mentioned, the Riemann hypothesis (1.1) can be considered one manifestation of this heuristic. Another instance is the following conjecture of Sarnak [18], [17].

A sequence $f : \mathbb{Z} \to \mathbb{C}$ is said to be *deterministic* if it is of the form

$$f(n) = F(T^n x)$$

for all n and some topological dynamical system (Y, T) with zero topological entropy $h(Y, T) = 0$, a base point $x \in Y$, and a continuous function $F : Y \to \mathbb{C}$.

Conjecture 1 (Sarnak, [18]). *Let $f : \mathbb{N} \to \mathbb{C}$ be a deterministic sequence. Then*

$$S_n(T(x), f) = \frac{1}{n}\sum_{k=1}^{n} \mu(k) f(k) = o(1) \quad \text{as } n \to \infty. \tag{1.3}$$

We note that (1.2) is an instance of Conjecture 1. The orthogonality of the Möbius function to any sequence arising from a rotation dynamical system (Y is the circle $\mathbb{T}$ and $T(x) = x + \alpha \bmod 1$, $\alpha \in \mathbb{T}$) follows from an old inequality of Davenport [9].

The conjecture is also known to be true for many other instances of zero entropy dynamical systems (see e.g. [8], [11], [7], [12], [4]).

Another instance of the Möbius pseudorandomness principle is the following old conjecture of Chowla.

Conjecture 2 (Chowla). *For $k \geq 0$, let $0 = a_0 < a_1 < \cdots < a_k$ be natural numbers and let $\epsilon_0, \epsilon_1, \ldots, \epsilon_k \in \{1, 2\}$ not all 2. Then as $N \to \infty$,*

$$\sum_{n=1}^{N} \mu^{\epsilon_0}(n + a_0)\mu^{\epsilon_1}(n + a_1) \cdots \mu^{\epsilon_k}(n + a_k) = o(N). \tag{1.4}$$

The conjecture is still open. However, recent progress is due to K. Matomäki, M. Radziwiłł and T. Tao [16], who build on the breakthrough paper by Matomäki and Radziwiłł [15]. In their paper they formulate and prove an averaged version of the Chowla conjecture.

We mention the following fundamental observation of P. Sarnak [18]:

$$\text{Chowla conjecture} \implies \text{Sarnak's conjecture}. \tag{1.5}$$

In the proof, however, it is not used that the Möbius function is multiplicative. It is a purely combinatorial observation. Proofs of this statement can also be found in Tao's blog [20] and in [3].

We note that all the conjectures mentioned above are still open.

El Abdalaoni et al. [3] the authors rephrase the Chowla and Sarnak conjectures in an abstract setting, that is, for sequences in $\{-1,0,1\}^{\mathbb{N}^*}$, and study the relationships between these conjectures, and also their dynamical and ergodic properties. More specifically, they consider sequences satisfying the following properties:

Definition 1.1. We say that a sequence $\{z_n\}_{n=1}^{\infty}$, $z_n \in \{-1,0,1\}$, satisfies the *Chowla property*, or the property (Chw) for short, if for all $k \geq 0$, and nonnegative numbers $0 = a_0 < a_1 < \cdots < a_k$ and $\epsilon_0, \epsilon_1, \ldots, \epsilon_k \in \{1,2\}$ not all 2, we have

$$\sum_{n=1}^{N} z_n^{\epsilon_0} z_{n+a_1}^{\epsilon_1} \cdots z_{n+a_k}^{\epsilon_k} = o(N) \quad \text{as } N \to \infty. \tag{1.6}$$

Similarly, inspired by Conjecture 1 we introduce:

Definition 1.2. We say that a sequence $\{z_n\}_{n=1}^{\infty}$, $z_n \in \{0,-1,1\}$, satisfies the *Sarnak property*, or the property (S) for short, if

$$\frac{1}{N}\sum_{n=1}^{N} z_n f(n) = o(1) \quad \text{as } N \to \infty \tag{1.7}$$

for any deterministic sequence $f : \mathbb{N} \to \mathbb{C}$.

We will continue this line of research and add the Riemann hypothesis to this list. As mentioned above, the Riemann hypothesis is equivalent to the property (1.1) of the Möbius function. Motivated by this we give the following definition:

Definition 1.3. We say that a sequence $\{z_n\}_{n=1}^{\infty}$, $z_n \in \{0,-1,1\}$, satisfies the *Riemann property*, or the property (R) for short, if for any positive $\epsilon > 0$,

$$\sum_{n=1}^{N} z_n = O(N^{1/2+\epsilon}) \quad \text{as } N \to \infty.$$

Our first goal is to investigate the relation of the property (R) to the other two. We essentially show that the property (R) is independent of (Chw) and (S) and that the only relation between the three properties is (Chw) $\Rightarrow$ (S). We also show that there is a sequence $\{z_n\}_{n\geq 1}$ which fulfills (R) and (S), but for which all the correlation averages in (1.6) fail to converge to zero (Theorem 2.5).

We will also be interested in a consequence of one of the main tools in the study of Sarnak's conjecture, namely the following result proved by Kátai [13] and Bourgain–Sarnak–Ziegler [8]:

THEOREM 1.4 ([8], [13]). *Let $(a_n)_{n\geq 1}$ be a bounded sequence of complex numbers such that for any prime numbers $p \neq q$,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} a_{np}\bar{a}_{nq} = 0. \tag{1.8}$$

Then for any bounded, multiplicative function ν,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \nu(n) a_n = 0. \tag{1.9}$$

This criterion was originally formulated more quantitatively, but here we will only use the present form.

We note that the Bourgain–Sarnak–Ziegler–Kátai criterion is of great importance in the study of Sarnak's conjecture. It allows one to eliminate the Möbius function in the problem and reduce the study to intrinsic properties of the underlying dynamical system, i.e. to relationships between different prime powers of the dynamical system, where one can use methods from joining theory, spectral theory, etc.

One can draw a simple, more specific corollary from this criterion: if we take $\nu(n) = 1$ for all $n \in \mathbb{N}$, then the condition (1.9) reads

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} a_n = 0. \tag{1.10}$$

From this we get the following criterion for equidistribution of sequences. If for some sequence $\{b_n\}_{n=1}^{\infty}$ all the differences $\{b_{np} - b_{nq}\}_{n\geq 1}$ are equidistributed modulo 1, then so is the sequence $\{b_n\}_{n=1}^{\infty}$. Indeed, if for all pairs (p, q) the sequence $\{b_{np} - b_{nq}\}_{n=1}^{\infty}$ is equidistributed, then according to Weyl's criterion for any $k \in \mathbb{Z}$ we will have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k(b_{np}-b_{nq})} = 0,$$

which is equivalent to the condition (1.8) for the sequence $c_n = e^{2\pi i k b_n}$. Therefore we will have (1.10) for b_n in place of a_n, which in turn will yield the equidistribution of $\{b_n\}_{n=1}^{\infty}$. This criterion is also known as the *multiplicative van der Corput criterion.*

In the second part of the present paper we will be interested in the comparison of this criterion with the classical van der Corput criterion.

2. Comparison of properties. In this section we study the combinatorial relations between sequences satisfying the properties (Chw), (S) or (R). Recall that for the Möbius function the fulfillment of (R), (Chw) or (S) is equivalent to the Riemann hypothesis, the Chowla conjecture and the Sarnak conjecture respectively.

Consider a pair of sets

$$\{0 = a_0 < a_1 < \cdots < a_k\} \quad \text{and} \quad \{\epsilon_0, \epsilon_1, \ldots, \epsilon_k\},$$

where $k \geq 0$ and $a_k \in \mathbb{N}$, $\epsilon_k \in \{1, 2\}$. We will write this pair briefly as $(\boldsymbol{a}, \boldsymbol{\epsilon})$. We will call $(\boldsymbol{a}, \boldsymbol{\epsilon})$ *admissible* if $\epsilon_r = 1$ for at least one $0 \leq r \leq k$. For a sequence $\{z_k\}_{k=1}^{\infty}$ of real numbers, we will write

$$z_n^{(\boldsymbol{a},\boldsymbol{\epsilon})} = z_n^{\epsilon_0} z_{n+1}^{\epsilon_1} \cdots z_{n+a_k}^{\epsilon_k}.$$

As already mentioned in the introduction, the Chowla property implies the Sarnak property for the Möbius function; a proof can be found in Tao's blog [20]. We recall a basic inequality from that blog, which implies Sarnak's conjecture.

Proposition 2.1. *Assume that the Möbius function $\{\mu(n)\}_{n=1}^{\infty}$ satisfies the Chowla property. Then for any $m \geq 1$, any $\epsilon > 0$ and any coefficients $c_1, \ldots, c_m$ such that $|c_j| \leq 1$ for $1 \leq j \leq m$, we have*

$$\mathbf{P}\left(\left|\frac{1}{m}\sum_{i=1}^{m} c_i \mu(n+i)\right| \geq \epsilon\right) \leq C \exp(-\epsilon^2 m/C) + o_{x\to\infty;m,\epsilon}(1), \tag{2.1}$$

where C is an absolute constant and $o_{x\to\infty;m,\epsilon}(1)$ goes to zero as $x \to \infty$ for fixed m, ϵ (uniformly in the choice of coefficients $c_1, \ldots, c_m$), and n is uniformly distributed on $[1, x]$.

As noted by Tao, "The argument does not use any number-theoretic properties of the Möbius function; one could replace μ in both conjectures (the Chowla and Sarnak conjectures) by any other function from the natural numbers to $\{-1, 0, +1\}$ and obtain the same implication". This means that any sequence $\{z_n\}_{n\geq 1}$ which satisfies the property (Chw) also satisfies the property (S). Thus we have the implication

$$\text{(Chw)} \Rightarrow \text{(S)}. \tag{2.2}$$

First, we note that

$$\text{(R)} \not\Rightarrow \text{(S)};$$

from this and (2.2) it will automatically follow that

$$\text{(R)} \not\Rightarrow \text{(Chw)}.$$

As an example witnessing this, one can consider the sequence $z_n = (-1)^n$, $n = 1, 2, \ldots$. Indeed, $\{z_n\}_{n\geq 1}$ is periodic and hence of zero entropy, so the property (S) is not fulfilled. However, it is easy to see that the sum $\sum_{k=1}^{N} z_n$ is either 0 or -1, which clearly implies (R).

We now construct a large class of sequences which satisfy (Chw) and hence (S).

PROPOSITION 2.2. *Let* $\{X_n\}_{n=1}^{\infty}$ *be a sequence of independent random variables taking values in* $\{-1,0,1\}$*, with* $\mathrm{E}[X_n]\to 0$ *as* $n\to\infty$*. Then the sequence* $\{X_n(\omega)\}_{n=1}^{\infty}$ *satisfies the properties* (Chw) *and* (S) *for almost all* $\omega\in\Omega$*.*

Proof. Recall the following form of the strong law of large numbers (see e.g. [19, Theorem 2.3.10]). Let $Y_1, Y_2, \dots$ be a sequence of independent random variables with

$$\sum_{k=1}^{\infty}\frac{\operatorname{Var}Y_k}{k^2}<\infty. \tag{2.3}$$

Then

$$\frac{Y_1+\cdots+Y_n}{n}-\mathrm{E}\left[\frac{Y_1+\cdots+Y_n}{n}\right]\to 0 \tag{2.4}$$

almost surely.

Let now $(\boldsymbol{a},\boldsymbol{\epsilon})$ be an admissible pair of indices. Then

$$\sum_{n=1}^{N}X_n^{(\boldsymbol{a},\boldsymbol{\epsilon})}=\sum_{s=0}^{a_k}\sum_{n\equiv s\,(\mathrm{mod}\,(a_k+1)),\,n\le N}X_n^{(\boldsymbol{a},\boldsymbol{\epsilon})}.$$

Consider the sum

$$\sum_{n\equiv s\,(\mathrm{mod}\,(a_k+1)),\,n\le N}X_n^{(a,\epsilon)}=\sum_{n\equiv s\,(\mathrm{mod}\,(a_k+1)),\,n\le N}X_n^{\epsilon_0}X_{n+a_1}^{\epsilon_1}\cdots X_{n+a_k}^{\epsilon_k}. \tag{2.5}$$

Note that for fixed a_k and $0\le s\le a_k$, $\{X_n^{(\boldsymbol{a},\boldsymbol{\epsilon})}\}_{n\equiv s\,(\mathrm{mod}\,(a_k+1))}$ is a sequence of independent random variables, since $\epsilon_r=1$ for some $0\le r\le k$ and the two blocks of indices

$$[n,n+a_k]\quad\text{and}\quad[m,m+a_k]$$

are disjoint for any distinct $n,m\equiv s$ (mod (a_k+1)).

For this sequence we now verify the condition (2.3). We have

$$|X_n^{(\boldsymbol{a},\boldsymbol{\epsilon})}|\le 1.$$

Therefore

$$\operatorname{Var}[X_n^{(\boldsymbol{a},\boldsymbol{\epsilon})}]\le 1-(\mathrm{E}[X_{n+a_1}^{\epsilon_1}\cdots X_{n+a_k}^{\epsilon_k}])^2\le 1. \tag{2.6}$$

So (2.3) is fulfilled.

It remains to show that the second term in (2.4) converges to zero. For this, note that

$$\mathrm{E}[X_n^{(\boldsymbol{a},\boldsymbol{\epsilon})}]=\mathrm{E}[X_n^{\epsilon_0}X_{n+a_1}^{\epsilon_1}\cdots Y_{n+a_k}^{\epsilon_k}]=\mathrm{E}[X_n^{\epsilon_0}]\,\mathrm{E}[X_{n+a_1}^{\epsilon_1}]\cdots\mathrm{E}[Y_{n+a_k}^{\epsilon_k}]\to 0,$$

since $\mathrm{E}[X_{n+a_r}^{\epsilon_r}]=\mathrm{E}[X_{n+a_r}]\to 0$ as $n\to\infty$. Therefore, from the general law of large numbers mentioned above, for the sum (2.5) we will almost surely have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n\equiv s\,(\mathrm{mod}\,(a_k+1)),\,n\le N} X_n^{(a,\epsilon)}=0.$$

This, in view of (2.2), completes the proof. ■

As a corollary we get another proof of Theorem 2.10 (Main result 4) in [2]. The authors introduce a random model of the Möbius function defined as follows:

$$\mu_{\mathrm{rand}}(n)=\begin{cases}\epsilon_n & \text{if } n=p_1\cdots p_k \text{ for distinct primes } p_k,\\ 0 & \text{otherwise,}\end{cases}$$

where $\{\epsilon_n\}_{n=1}^{\infty}$ is a sequence of independent and identically distributed random variables with

$$\mathrm{P}(\epsilon_n=1)=1/2 \quad\text{and}\quad \mathrm{P}(\epsilon_n=-1)=1/2. \tag{2.7}$$

In [2] the authors show that the sequence

$$\{\mu_{\mathrm{rand}}(n)\}_{n=1}^{\infty}$$

satisfies the property (S) almost surely. Observe that if in Proposition 2.2 we take X_n identically equal to 0 for square-free numbers n, i.e. when $n=p_1\cdots p_k$ for distinct primes p_i, then the theorem stated above will automatically follow.

Theorem 2.3. *The* (Chw) *and* (S) *properties do not imply* (R).

Proof. It is enough to construct a sequence which satisfies the Chowla property but fails (R). Define a sequence $\{X_k\}_{k=1}^{\infty}$ of independent random variables as follows:

$$\mathrm{P}(X_k=1)=\frac{1+\frac{1}{\log(k+1)}}{2},\qquad \mathrm{P}(X_k=-1)=\frac{1-\frac{1}{\log(k+1)}}{2}.$$

We are going to show that this sequence is as desired almost surely. We have

$$\mathrm{E}[X_k]=\frac{1+\frac{1}{\log(k+1)}}{2}-\frac{1-\frac{1}{\log(k+1)}}{2}=\frac{1}{\log(k+1)},$$

$$\mathrm{Var}[X_k]=\mathrm{E}[X_k^2]-(\mathrm{E}[X_k])^2=1-\frac{1}{\log^2(k+1)}.$$

Denote

$$S_n=X_1+\cdots+X_n.$$

Let $0<\epsilon<1/2$. Then

$$\mathrm{E}\left[\frac{S_n}{n^{1-\epsilon}}\right]=\frac{1}{n^{1-\epsilon}}\sum_{k=1}^{n}\mathrm{E}[X_k]=\frac{1}{n^{1-\epsilon}}\sum_{k=1}^{n}\frac{1}{\log(k+1)}.$$

But

$$\frac{1}{n^{1-\epsilon}}\sum_{k=1}^{n}\frac{1}{\log(k+1)} \geq \frac{1}{n^{1-\epsilon}}\,\frac{n}{\log(n+1)} = \frac{n^{\epsilon}}{\log(n+1)},$$

and the last expression tends to infinity as $n \to \infty$. Hence

$$\mathrm{E}\left[\frac{S_n}{n^{1-\epsilon}}\right] \to \infty \quad \text{as } n \to \infty. \tag{2.8}$$

We now compute

$$\mathrm{Var}\left[\frac{S_n}{n^{1-\epsilon}}\right] = \frac{1}{n^{2-2\epsilon}}\,\mathrm{Var}[S_n].$$

Since the random variables $X_1, X_2, \ldots$ are independent,

$$\begin{aligned} \frac{1}{n^{2-2\epsilon}}\,\mathrm{Var}[S_n] &= \frac{1}{n^{2-2\epsilon}}\sum_{k=1}^{n}\mathrm{Var}[X_k] \\ &= \frac{1}{n^{2-2\epsilon}}\sum_{k=1}^{n}\left(1-\frac{1}{\log^2(k+1)}\right) \leq \frac{1}{n^{2-2\epsilon}}n = \frac{1}{n^{1-2\epsilon}}. \end{aligned} \tag{2.9}$$

Therefore, if $\epsilon < 1/2$, then $\mathrm{Var}[S_n/n^{1-2\epsilon}] \to 0$ as $n \to \infty$. This means that the sequence of random variables $S_n/n^{1-\epsilon}$ has high concentration around the mean. Therefore from Chebyshev's inequality, for $t > 0$ we have

$$\mathrm{P}\left(\left|\frac{S_n}{n^{1-\epsilon}} - \mathrm{E}\left[\frac{S_n}{n^{1-\epsilon}}\right]\right| \geq t\right) \leq \frac{\mathrm{Var}[\frac{S_n}{n^{1-\epsilon}}]}{t^2} \leq \frac{1}{n^{1-2\epsilon}t^2}.$$

Hence, if we put n^2 instead of n, then

$$\mathrm{P}\left(\left|\frac{S_{n^2}}{n^{2(1-\epsilon)}} - \mathrm{E}\left[\frac{S_{n^2}}{n^{2-2\epsilon}}\right]\right| \geq t\right) \leq \frac{1}{n^{2(1-2\epsilon)}t^2} = \frac{1}{n^{2-4\epsilon}t^2}.$$

If $\epsilon < 1/4$, then $\sum_{n=1}^{\infty}\frac{1}{n^{2-4\epsilon}} < \infty$. Therefore, if

$$E_n = \left\{\omega : \left|\frac{S_{n^2}(\omega)}{n^{2-2\epsilon}} - \mathrm{E}\left[\frac{S_{n^2}}{n^{2-2\epsilon}}\right]\right| \geq t\right\},$$

then the probability that infinitely many of the events E_n will occur is zero. This means that for almost all ω, for infinitely many values of n we have

$$\left|\frac{S_{n^2}(\omega)}{n^{2-2\epsilon}} - \mathrm{E}\left[\frac{S_{n^2}}{n^{2-2\epsilon}}\right]\right| < t,$$

which, in view of (2.8), implies that

$$\limsup_{n\to\infty}\left|\frac{S_{n^2}(\omega)}{n^{2-2\epsilon}}\right| = \limsup_{n\to\infty}\left|\frac{S_n(\omega)}{n^{1-\epsilon}}\right| = \infty.$$

Thus we have shown that $\{X_k\}_{k=1}^{\infty}$ fails the property (R) almost surely for $\epsilon < 1/4$.

To show (Chw), we verify the requirements of Proposition 2.2. First, the random variables $\{X_k\}_{k=1}^{\infty}$ take values in $\{-1,1\}$, and second,

$$\mathrm{E}[X_k] = \frac{1}{\log(k+1)} \to 0 \quad \text{as } k \to \infty.$$

So the property (Chw) for almost all realizations of $\{X_k\}_{k\geq 1}$ follows from Proposition 2.2. ∎

Remark 2.4. We remark that the result proved above contradicts [1], where it is claimed that for the Liouville function the Chowla conjecture implies the Riemann hypothesis. However, the multiplicativity property of the Liouville function is not used in the proof. But this cannot be true as from the above argument it follows that without the multiplicativity condition the Riemann hyphothesis cannot be obtained from the Chowla property.

We note that in the preceding theorem we have used the property (Chw) to ensure the property (S). It is a natural question whether there is a sequence which satisfies (S), but fails (Chw). As a simple example of such a sequence one can consider

$$X_1X_1X_2X_2\dots X_nX_n\dots, \tag{2.10}$$

where $\{X_n\}_{n=1}^{\infty}$ is a sequence of i.i.d ± 1 Bernoulli-like random variables with parameter $1/2$. Denote the sequence (2.10) by $\{Y_n\}_{n=1}^{\infty}$. Observe that each of the two sequences

$$X_1 0 X_2 0 \dots \quad \text{and} \quad 0 X_1 0 X_2 0 \dots$$

satisfies the property (S) almost surely according to Proposition 2.2. Therefore so does their sum (2.10). However,

$$\sum_{n=1}^{N} Y_nY_{n+1} = \sum_{n=1}^{[N/2]} X_n^2 + \sum_{n=1}^{[N/2]} X_nX_{n+1} + o(N) = \left[\frac{N}{2}\right] + \sum_{n=1}^{[N/2]} X_nX_{n+1} + o(N),$$

where for the second term on the right hand side we have

$$\lim_{N\to\infty} \frac{2}{N} \sum_{n=1}^{[N/2]} X_kX_{k+1} = \mathrm{E}[X_1X_2] = \mathrm{E}[X_1]\mathrm{E}[X_2] = 0.$$

So

$$\frac{1}{N}\sum_{n=1}^{N} Y_nY_{n+1} = \frac{1}{2} + o(N).$$

Thus the property (Chw) does not hold. We note further that the sequence (2.10) also satisfies the concentration inequality in (2.1). Since the two sequences mentioned above each satisfy (Chw), they also satisfy (2.1). Thus the sequence (2.10) also satisfies (2.1).

In [3] the authors construct an example of a sequence for which all the correlations in (1.6), where $1 < k \leq m$, fail to converge to zero for any fixed m. But the properties of higher order correlations are not clear. However, as the example constructed above shows, in order to have the concentration inequality in (2.1), which implies Sarnak's conjecture, one does not need the convergence to 0 for all the admissible multi-indices (a, ϵ) in Definition 1.1. Note that from the definition of (S) it follows that

$$\lim_{n\to\infty} \frac{1}{N}\sum_{n=1}^{N} z_n = 0,$$

which is (1.6) for $k = 0$.

We now construct a sequence for which the property (Chw) fails for any $k \geq 1$ and any admissible pair (a, ϵ), but which satisfies both (S) and (R).

Theorem 2.5. *There is a sequence* $\{z_k\}_{k=1}^{\infty}$, $z_k \in \{0, 1, -1\}$, *such that for any* $k \geq 1$ *and for any pair*

$$\{0 = a_0 < a_1 < \cdots < a_k\} \quad \textit{and} \quad \{\epsilon_0, \epsilon_1, \epsilon_2, \ldots, \epsilon_k\},$$

where the ϵ_k*s are not all* 2, *we have*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} z_n^{(a,\epsilon)} \neq 0,$$

but the properties (S) *and* (R) *are fulfilled.*

The idea of the proof is the following. First, for each pair admissible $(\boldsymbol{a}, \boldsymbol{\epsilon})$, we construct a sequence of random variables for which the corresponding averages fail to converge to zero and are such that the sequence can be decomposed into two subsequences of independent random variables; then we use Proposition 2 for each subsequence and hence obtain the properties (S) and (R).

We begin the proof of Theorem 2.5 by taking a sequence $\{X_k\}_{k\geq 1}$ of independent and identically distributed random variables with

$$\mathrm{P}(X_k = 1) = 1/2 \quad \text{and} \quad \mathrm{P}(X_k = -1) = 1/2. \tag{2.11}$$

Let $\boldsymbol{a} = \{0 = a_0 < a_1 < \cdots < a_k\}$. Denote $d = a_k + 1$. We will distinguish two kinds of admissible pairs $(\boldsymbol{a}, \boldsymbol{\epsilon})$ and define two corresponding sequences of random variables. For an admissible pair $(\boldsymbol{a}, \boldsymbol{\epsilon})$ with $\epsilon_r = 1$ for at least two different indices r, we define a sequence $\{Z_n\}_{n=1}^{\infty}$ of random variables as follows:

$$Z_n = \begin{cases} X_n & \text{for } n \in [1 + 2ds, d + 2ds] \setminus \{a_r + 2ds + 1\}, \\ \prod_{l=0,\, l\neq r}^{k} X_{2ds+a_l+1}^{\epsilon_l} & \text{for } n = a_r + 2ds + 1, \\ 0 & \text{otherwise,} \end{cases} \tag{2.12}$$

for $s = 0, 1, \ldots$. Observe that if we had $\epsilon_0 = \cdots = \epsilon_{r-1} = \epsilon_{r+1} = \cdots = \epsilon_k = 2$, then the random variable defined for $n = a_r + 2ds + 1$ above would be identically 1.

For a random variable X we define a new random variable Y satisfying

$$\mathrm{E}[XY^2] = 1/2, \tag{2.13}$$

$$\mathrm{E}[Y] = 0. \tag{2.14}$$

Let $\{\widetilde{X}_n\}_{n=1}^{\infty}$ be another sequence of i.i.d. random variables similar to $\{X_n\}_{n=1}^{\infty}$ and independent of it. Define

$$Y_n = \frac{X_n + 1}{2} \widetilde{X}_n.$$

Observe, that X_n and Y_n satisfy (2.13) and (2.14) for all $n \geq 1$ and $\{Y_n\}_{n=1}^{\infty}$ is an i.i.d. sequence.

Let now $(\boldsymbol{a}, \boldsymbol{\epsilon})$ be a pair with $\epsilon_l = 1$ for all $l = 0, \ldots, k$, except for $l = r$. By assumption $k \geq 1$. Let h be any index different from r. Define a sequence of random variables as follows:

$$Z_n = \begin{cases} X_n & \text{for } n \notin [1 + 2ds, d + 2ds] \setminus \{a_h + 2ds + 1\}, \\ Y_{2ds + a_r + 1} & \text{for } n = a_h + 2ds + 1, \\ 0 & \text{otherwise}, \end{cases} \tag{2.15}$$

for $s = 0, 1, \ldots$.

Claim 1. *Both for* (2.12) *and* (2.15) *we have*

$$\mathrm{E}[Z_n] = 0 \quad \textit{and} \quad \|Z_n\|_\infty \leq 1, \qquad \textit{for all } n \geq 1.$$

Proof. By definition $\mathrm{E}[X_n] = 0$ and $\mathrm{E}[Y_n] = 0$ for $n \geq 1$. So one only needs to check the claim when $n = a_r + 2ds + 1$ for the sequence (2.12). But in this case we have

$$\mathrm{E}[X_{2ds+1}^{\epsilon_0} \cdots X_{2as+a_{r-1}+1}^{\epsilon_{r-1}} X_{2as+a_{r+1}+1}^{\epsilon_{r+1}} \cdots X_{2as+a_k+1}^{\epsilon_k}] \\ = \mathrm{E}[X_{2ds+1}^{\epsilon_0}] \cdots \mathrm{E}[X_{2ds+a_k+1}^{\epsilon_k}] = 0,$$

since by assumption at least one of the numbers $\epsilon_0, \epsilon_1, \ldots, \epsilon_{r-1}, \epsilon_{r+1}, \ldots, \epsilon_k$ is different from 2. The uniform boundedness of $\{Z_n\}_{n\geq 1}$ follows from the boundedness of X_n and Y_n. ∎

Claim 2. *For each admissible pair* $(\boldsymbol{a}, \boldsymbol{\epsilon})$ *and the corresponding sequence of random variables there exist two sequences of* 0*'s and* 1*'s,* $\{\zeta(n)\}_{n=1}^{\infty}$ *and* $\{\nu(n)\}_{n=1}^{\infty}$*, such that*

$$\{Z_n \nu(n)\}_{n=1}^{\infty} \quad \textit{and} \quad \{Z_n \zeta(n)\}_{n=1}^{\infty}$$

are sequences of independent random variables and

$$1 = \zeta(n) + \nu(n) \qquad \textit{for all } n \in \mathbb{N}.$$

Proof. To see this for the sequence (2.12), it is enough to note that for any two numbers $s_1 \neq s_2$ the random variables in each block $[1+2ds_1, d+2ds_1]$ and $[1+2ds_2, d+2ds_2]$ are independent and within the blocks $[1+2ds, d+2ds]$ all random variables are independent of each other, except for the one at $n = a_r + 2sd$, so we will define $\zeta(n) = 1$ for $n = a_r + 2sd, s = 0, 1, \dots,$ and $\zeta(n) = 0$ otherwise, and we define $\nu(n) = 1 - \zeta(n)$ for $n \geq 1$.

As for the sequence (2.15), within the blocks $[1+2ds_1, d+2ds_1]$ all random variables, except for X_n and Y_n, are independent. Therefore, if we define $\zeta(n) = 1$ if Z_n is not of type Y_n and zero otherwise, and $\nu(n) = 1 - \zeta(n)$, the requirements of Claim 2 will follow. ■

CLAIM 3. *For any admissible pair $(\boldsymbol{a}, \boldsymbol{\epsilon})$ the corresponding sequence satisfies*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} Z_n^{\epsilon_0} Z_{n+a_1}^{\epsilon_1} \cdots Z_{n+a_k}^{\epsilon_k} \geq C_{(\boldsymbol{a},\boldsymbol{\epsilon})} > 0 \tag{2.16}$$

almost surely.

Proof. Consider the sum

$$\sum_{n=1}^{N} Z_n^{\epsilon_0} Z_{n+a_1}^{\epsilon_1} \cdots Z_{n+a_k}^{\epsilon_k} = \sum_{n=1}^{N} Z_n^{(\boldsymbol{a},\boldsymbol{\epsilon})}.$$

Both in (2.12) and (2.15), in the case of $n \notin [1 + 2ds, d + 2ds]$, we have $Z_n \equiv 0$. Therefore, for $n \neq 1 + 2ds$ with $s = 0, 1, \dots,$ one has

$$Z_n^{(\boldsymbol{a},\boldsymbol{\epsilon})} = Z_n^{\epsilon_0} Z_{n+a_1}^{\epsilon_1} \cdots Z_{n+a_k}^{\epsilon_k} = 0,$$

as one of the two numbers n and $n + a_k$ will be outside of these intervals. When $n = 1 + 2ds$, $s = 0, 1, \dots,$ for the sequence (2.12) we have

$$\begin{aligned} Z_{1+2ds}^{(\boldsymbol{a},\boldsymbol{\epsilon})} &= X_{1+2ds}^{\epsilon_0} \cdots (X_{1+2ds}^{\epsilon_0} \cdots X_{2ds+a_{r-1}}^{\epsilon_{r-1}} X_{2ds+a_{r+1}+1}^{\epsilon_{r+1}} \cdots X_{d+2ds}^{\epsilon_k}) \cdots X_{d+2ds}^{\epsilon_k} \\ &= (X_{2ds+1}^{2\epsilon_0} X_{2ds+a_1+1}^{2\epsilon_1} \cdots X_{d+2ds+1}^{2\epsilon_k} \cdots X_{d+2ds+1}^{2\epsilon_k}) = 1. \end{aligned}$$

So

$$\sum_{n=1}^{N} Z_n^{(\boldsymbol{a},\boldsymbol{\epsilon})} = \sum_{n\equiv 1 \,(\mathrm{mod}\, 2d),\, n\leq N} 1,$$

which implies (2.16).

As for the sequence (2.15), we have

$$Z_{1+2ds}^{(\boldsymbol{a},\boldsymbol{\epsilon})} = Z_{1+2ds}^2 \cdots Z_{1+2ds+a_r} \cdots Z_{1+2ds+a_k}^2 = X_{1+2ds+a_r} Y_{1+2ds+a_r}^2.$$

From this it follows that

$$\sum_{n=1}^{N} Z_n^{(\boldsymbol{a},\boldsymbol{\epsilon})} = \sum_{n\equiv 1 \pmod{2d},\, n\le N} X_{n+a_r} Y_{n+a_r}^2 .$$

Since the random variables on the right hand side are independent, identically distributed and $\mathrm{E}[X_{n+a_r} Y_{n+a_r}^2] = 1/2$, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\equiv 1 \pmod{2d},\, n\le N} X_{n+a_r} Y_{n+a_r}^2 = \frac{1}{4d}$$

almost surely. If we now set $C_{(\boldsymbol{a},\boldsymbol{\epsilon})} = (4d)^{-1}$, then (2.16) will follow, completing the proof of Claim 3. ∎

Proof of Theorem 2.5. We now turn to the construction of a sequence which fails the property (Chw) and satisfies (S) and (R). Enumerate all possible admissible pairs

$$\{a_0 = 0 < a_1 < \cdots < a_k\} \quad \text{and} \quad \{\epsilon_0, \epsilon_1, \ldots, \epsilon_k\} \tag{2.17}$$

for $k \ge 1$ as

$$\{(\boldsymbol{a}_k, \boldsymbol{\epsilon}_k)\}_{k=1}^{\infty} \tag{2.18}$$

in such a way that any pair appears in the enumeration infinitely often.

We now construct blocks $\{A_k\}_{k=1}^{\infty}$ of random variables and sets $\{B_k\}_{k=1}^{\infty}$ by induction. For the first pair $(\boldsymbol{a}_1, \boldsymbol{\epsilon}_1)$ from (2.18) we choose a number N_1 such that

$$\frac{1}{N_1} \sum_{n=1}^{N_1} Z_n^{(\boldsymbol{a}_1,\boldsymbol{\epsilon}_1)} > C_{(a_1,\epsilon_1)} > 0$$

on a set B_1 with measure $|B_1| > 1/2$. The existence of such an N_1 follows from Claim 3. We define the first block A_1 to be

$$\{Z_k\}_{k=1}^{N_1}.$$

In the second step, we consider the pair $(\boldsymbol{a}_2, \boldsymbol{\epsilon}_2)$ from (2.18) and construct another block $A_2 = \{Z_k\}_{k=N_1+1}^{N_2}$ of $N_2 - N_1$ random variables, where N_2 is chosen in such a way that

$$\frac{1}{N_2}\Big(\sum_{n=1}^{N_1} Z_n^{(\boldsymbol{a}_2,\boldsymbol{\epsilon}_2)} + \sum_{n=N_1+1}^{N_2} Z_n^{(\boldsymbol{a}_2,\boldsymbol{\epsilon}_2)} \Big) > C_{(\boldsymbol{a}_2,\boldsymbol{\epsilon}_2)} > 0$$

on a set B_2 with $|B_2| > 1 - 1/4$. The existence of N_2 follows from (2.16) and the uniform estimate

$$\Big| \frac{1}{N_2} \sum_{n=1}^{N_1} Z_n^{(\boldsymbol{a}_2,\boldsymbol{\epsilon}_2)} \Big| \le \frac{N_1}{N_2} \to 0 \quad \text{as } N_2 \to \infty.$$

In a similar way in the mth step we consider the pair $(\boldsymbol{a}_m, \boldsymbol{\epsilon}_m)$ and construct a block A_m of $N_m - N_{m-1}$ random variables such that

$$\frac{1}{N_m}\Big(\sum_{n=1}^{N_1} Z_n^{(\boldsymbol{a}_m,\boldsymbol{\epsilon}_m)} + \cdots + \sum_{n=N_{m-1}+1}^{N_m} Z_n^{(\boldsymbol{a}_m,\boldsymbol{\epsilon}_m)}\Big) > C_{(\boldsymbol{a}_m,\boldsymbol{\epsilon}_m)} > 0 \tag{2.19}$$

on a set B_m with $|B_m| > 1 - 1/2^m$. Thus we get blocks $\{A_k\}_{k=1}^{\infty}$ of random variables and sets $\{B_k\}_{k=1}^{\infty}$. To finish the construction, we will put all the blocks of random variables constructed above together, i.e. we form

$$A_1 A_2 \dots A_k \dots,$$

and consider the resulting sequence $\{Z_k\}_{k=1}^{\infty}$. We are going to show that this sequence has the required properties almost surely. Since $|B_k| > 1 - 1/2^k$, we can find a set B of full measure for which

$$B = \liminf_{k\to\infty} B_k,$$

i.e. any point from $\omega \in B$ falls into all sets B_k for $k \geq k_0(\omega)$. Let us show that for any pair $(\boldsymbol{a}, \boldsymbol{\epsilon})$ we almost surely have

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} Z_n^{(\boldsymbol{a},\boldsymbol{\epsilon})} > 0.$$

Since in the enumeration (2.18) any pair $(\boldsymbol{a}, \boldsymbol{\epsilon})$ occurs infinitely often, there exists a sequence $\{N_{k_m}\}_{m=1}^{\infty}$ for which

$$\frac{1}{N_{k_m}} \sum_{n=1}^{N_{k_m}} Z_n^{(\boldsymbol{a},\boldsymbol{\epsilon})} > C_{(\boldsymbol{a},\boldsymbol{\epsilon})} \quad \text{on } B_{k_m}.$$

Therefore this estimate will eventually hold for any point in B.

We will now show that $\{Z_k\}_{k=1}^{\infty}$ satisfies the property (S). For this consider the sequences

$$\{\zeta(n)\}_{n=1}^{\infty} \quad \text{and} \quad \{\nu(n)\}_{n=1}^{\infty}, \tag{2.20}$$

each being the union of the corresponding ζ and ν sequences for each block B_k constructed in Claim 2. Therefore, according to Claim 2,

$$Z_n\nu(n) + Z_n\zeta(n) = Z_n \quad \text{for any } n \geq 1, \tag{2.21}$$

the random variables in each of the sequences

$$\{Z_n\nu(n)\}_{n=1}^{\infty} \quad \text{and} \quad \{Z_n\zeta(n)\}_{n=1}^{\infty} \tag{2.22}$$

are independent, and from Claim 1,

$$\mathrm{E}[Z_n] = 0 \quad \text{for any } n \in \mathbb{N}.$$

From this, (2.21) and Proposition 2.2 we see that the two sequences in (2.22) separately satisfy (S). Therefore it is also satisfied by $\{Z_n\}_{n=1}^{\infty}$.

Since each sequence in (2.20) is a sequence of independent and bounded random variables, the property (R) follows from the central limit theorem. ∎

REMARK 2.6. In the discussion above we have not assumed that the sequence $\{z(n)\}_{n=1}^{\infty}$ is multiplicative.

Of course the assumption of multiplicativity will change the situation substantially; however, the sole assumption of multiplicativity, of uniform boundedness and even of convergence to zero of Cesàro averages does not guarantee the fulfillment of any of the properties (R), (S) or (Chw).

Consider the Dirichlet character $\chi_1(n)$ modulo 3, i.e.

$$\chi_1(n) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 3, \\ 1 & \text{if } n \equiv 1 \pmod 3, \\ -1 & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Clearly

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_1(n) = 0,$$

but

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_1(3n+1) = 1.$$

Therefore the properties (S), (R) and (Chw) are not satisfied. To the best of the author's knowledge there are not any known examples of nontrivial multiplicative sequences which satisfy (S) or (Chw). However, there are nontrivial multiplicative sequences which satisfy (R). Consider

$$f(n) = \begin{cases} \prod_{p|n} \epsilon_p & \text{if } n \text{ is squarefree}, \\ 0 & \text{otherwise}, \end{cases}$$

where $\{\epsilon_n\}_{n=1}^{\infty}$ is as in (2.7). It is shown in [22] that this sequence satisfies (R) almost surely. In this context it also important to mention a result of Frantzikinakis [10]: if the Liouville function λ is ergodic, then it satisfies the Chowla conjecture. However, as mentioned in [10], it is not known whether this system is of positive entropy, weakly mixing, or even ergodic.

3. Comparison of criteria. We recall the multiplicative van der Corput criterion: if for some sequence $\{b_n\}_{n=1}^{\infty}$ the sequence $\{b_{np}-b_{nq}\}_{n=1}^{\infty}$ is uniformly distributed modulo 1 for any prime numbers p, q, then so is $\{b_n\}_{n=1}^{\infty}$. Therefore one gets an equidistribution criterion for sequences which may be interpreted as a multiplicative version of the classical van der Corput criterion. We now recall the latter:

Theorem 3.1. *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that for any $h \in \mathbb{N}$ all the differences $\{a_{n+h} - a_n\}_{n=1}^{\infty}$ are uniformly distributed modulo* 1*. Then the sequence $\{a_n\}_{n=1}^{\infty}$ is also uniformly distributed modulo* 1*.*

In this section we want to investigate the connection between the two criteria mentioned above. If, for instance, the additive van der Corput criterion yielded the convergence of any of the differences appearing in the additive van der Corput criterion, then it might give another tool for studying Sarnak's conjecture. However, as will be shown next, there is no connection between the criteria. However, it is worth mentioning the following result of Korobov and Postnikov [14].

Theorem 3.2. *Assume that for the sequence $\{a_n\}_{n=1}^{\infty}$ the differences $\{a_{n+h} - a_n\}_{n\geq 1}$ are uniformly distributed modulo* 1 *for any $h \in \mathbb{N}$. Then one also has equidistribution for the sequences $\{a_{np+l}\}_{n\geq 1}$ for $l, p \in \mathbb{N}$.*

In particular, the sequence $\{a_{np}\}_{n=1}^{\infty}$ is equidistributed for any $p \geq 1$.

We observe that it is not difficult to construct a sequence which satisfies the multiplicative criterion, but fails the additive one. As a simple example, one can consider the sequence $\{n\theta \pmod 1\}_{n\in\mathbb{N}}$ for any irrational θ. We see that for any $h \in \mathbb{N}$ there are $h_1, h_2 \in \mathbb{Z}$ such that

$$\{(n+h)\theta\} - \{n\theta\} = ((n+h)\theta + h_1) - (n\theta + h_2) = h\theta + h_1 - h_2,$$

which is apparently not equidistributed; however, one can check from Weyl's criterion that the differences $\{np\theta\} - \{nq\theta\} = n(p-q)\theta + m$, $m \in \mathbb{Z}$, are equidistributed modulo 1 for any distinct $p, q \in \mathbb{Z}$.

We now investigate the other direction of the question. We will construct an example of a uniformly distributed sequence for which all the van der Corput differences are equidistributed, while none of the multiplicative differences are.

We will need a few lemmas.

Lemma 3.3. *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of numbers from $[0, 1)$ for which the set $\{n : a_n = 0\}$ has positive upper density. Then $\{a_n\}_{n=1}^{\infty}$ is not uniformly distributed modulo* 1*.*

Proof. Let the upper Banach density of $\{n : a_n = 0\}$ be τ. Consider an interval which contains the origin 0 in its interior and whose length is strictly smaller than τ. If $\{a_n\}_{n=1}^{\infty}$ is uniformly distributed, then the density of the members falling inside this interval will be smaller than τ, a contradiction. ∎

We now enumerate all pairs (p, q) of distinct prime numbers $p > q$ as

$$\{(p_k, q_k)\}_{k=1}^{\infty} \tag{3.1}$$

in such a way that every pair occurs in (3.1) infinitely often. For each k we also define

$$A_k = \{(p_k q_k l + 1)p_k : l \in \mathbb{N}\}, \qquad B_k = \{(p_k q_k l + 1)q_k : l \in \mathbb{N}\}.$$

These sets are disjoint for each k. Indeed, if

$$(sp_k q_k + 1)q_k = (lp_k q_k + 1)p_k$$

for some $s, l \in \mathbb{N}$, then

$$p_k q_k (sq_k - lp_k) = p_k - q_k,$$

but since $p_k - q_k \neq 0$ and

$$|p_k - q_k| < p_k \quad \text{or} \quad |p_k - q_k| < q_k,$$

the equality above cannot be fulfilled. For $n \in B_k$, i.e. for n of the form $(p_k q_k l + 1)p_k$, we define

$$\bar{n} = (p_k q_k l + 1)q_k.$$

We now construct a sequence $\{d_k\}_{k=1}^{\infty} \subset \mathbb{Z}$ by induction, together with another sequence $\{N_k\}_{k=1}^{\infty}$. Let $\{c_n\}_{n\geq 1}$ be a monotone sequence of positive integers with $c_{n+1} - c_n \to \infty$ as $n \to \infty$. For the base step $k = 1$ we define $N_1 = (p_1 q_1 + 1)p_1$ and

$$d_n = \begin{cases} c_n & \text{for } n \leq N_1 \text{ and } n \notin B_1, \\ c_{\bar{n}} & \text{for } n \leq N_1 \text{ and } n \in B_1. \end{cases}$$

For the definition to be correct, we first define d_n for $n \leq N_1$ with $n \notin B_1$, and then for other $n \leq N_1$. One can check that $d_{(p_1 q_1+1)p_1} - d_{(p_1 q_1+1)q_1} = 0$. Now assume we have defined d_n for $n \leq N_{k-1}$. Let l be the smallest integer for which

$$(p_k q_k l + 1)q_k > N_{k-1},$$

or the smallest number from the set B_{k-1} which is larger than N_{k-1}. We define $N_k = (p_k q_k (N_{k-1} + l - 1) + 1)p_k$. We now define

$$d_n = \begin{cases} c_n & \text{for } N_{k-1} < n \leq N_k \text{ and } n \notin B_k, \\ c_{\bar{n}} & \text{for } N_{k-1} < n \leq N_k \text{ and } n \in B_k. \end{cases}$$

One can see that

$$\frac{N_k}{N_{k-1}} < (2p_k q_k + 1)p_k. \tag{3.2}$$

This finishes the induction.

We now make a few observations about the construction.

LEMMA 3.4. *For any $h \in \mathbb{N}$, let*

$$d_{n+h} - d_n = c_n^{(h)}.$$

Then $c_n^{(h)} \in \mathbb{Z}$ and $|c_n^{(h)}| \to \infty$ as $n \to \infty$.

Proof. Since $c_n \in \mathbb{N}$, clearly $c_n^{(h)} \in \mathbb{Z}$. Let

$$d_{n+h} = c_{n_1} \quad \text{and} \quad d_n = c_{n_2}. \tag{3.3}$$

From the construction of $\{d_n\}_{n=1}^{\infty}$ it follows that if $n > N_k$ for some $k \geq 0$, then $n_2, n_1 > N_k$ too. Hence, if $n_2 \neq n_1$, then

$$|d_{n+h} - d_n| = |c_{n_1} - c_{n_2}| > |c_{n_1} - c_{n_1-1}| \to \infty$$

whenever $n \to \infty$. It remains to see that $n_2 \neq n_1$ when n is sufficiently large. But $n_2 = n_1$ can only happen if $n + h = s$ and $n = \bar{s}$ for some $s > 0$, or

$$n + h = (p_m q_m l + 1)p_m \quad \text{and} \quad n = (p_m q_m l + 1)q_m,$$

for some m, l, p_m and q_m. But then

$$h = (p_m q_m l + 1)p_m - (p_m q_m l + 1)q_m = (p_m q_m l + 1)(p_m - q_m).$$

The right hand side achieves its minimum for $p_m = 3$ and $q_m = 2$. Hence

$$h \geq 6l + 1.$$

But since $l \to \infty$ as $n \to \infty$, this cannot hold, so $n_1 \neq n_2$ for sufficiently large n. ∎

Lemma 3.5. *For any (p, q),*

$$\limsup_{N\to\infty} \frac{\#\{n : d_{np} - d_{nq} = 0,\, n \leq N\}}{N} > 0. \tag{3.4}$$

Proof. By construction any pair (p, q) of primes occurs in the enumeration (3.1) infinitely often. Let $\{m_k\}_{k=1}^{\infty}$ be the positions where one encounters (p, q). By construction, if $np \in B_{m_k}$, then $d_{np} - d_{nq} = 0$ and

$$\#\{n : N_{m_k} < n \leq N_{m_{k+1}},\, n \in B_{m_k}\} = N_{m_k}.$$

From this and (3.2), we have

$$\frac{\#\{n : d_{np} - d_{nq} = 0,\, n \leq N_{m_k+1}\}}{N_{m_k+1}} \geq \frac{N_{m_k}}{N_{m_k+1}} > \frac{1}{(2pq+1)p} > 0.$$

From this we get (3.4). ∎

We now recall the following theorem from the theory of uniformly distributed sequences:

Theorem 3.6 ([6]). *Let $\{a_n\}_{n\geq 1}$ be a sequence of integers with $|a_n| \to \infty$ as $n \to \infty$. Then for almost all $\theta \in [0, 1)$ the sequence $\{a_n\theta\}_{n=1}^{\infty}$ is uniformly distributed modulo* 1.

We are now ready to construct an example of a sequence that satisfies the classical van der Corput criterion, but fails the multiplicative one. Consider the sequence $\{d_n\theta\}_{n=1}^{\infty}$. From Lemma 2, $\{(d_{n+h} - d_n)\theta\} = \{c_n^{(h)}\theta\}$, where $|c_n^{(h)}| \to \infty$, hence Theorem 3.6 shows that $\{(d_{n+h} - d_n)\theta\}_{n=1}^{\infty}$ is uniformly distributed for almost all $\theta \in [0,1)$. On the other hand, for any pair (p,q), from Lemma 3.5 we know that

$$\limsup_{N\to\infty} \frac{\#\{n : d_{np}\theta - d_{nq}\theta = 0,\, n \le N\}}{N} > 0.$$

Hence $\{(d_{np}-d_{nq})\theta\}_{n=1}^{\infty}$ cannot be uniformly distributed for any $\theta \in [0,1)$ according to Lemma 3.3. Thus, for almost all $\theta \in [0,1)$ the sequence $\{d_n\theta\}_{n=1}^{\infty}$ is as required.

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Davit Karagulyan
Department of Mathematics
Royal Institute of Technology
Lindstedtsvägen 25
S-100 44 Stockholm, Sweden
E-mail: davitk@kth.se