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Parallel almost paracontact structures on affine hypersurfaces

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Abstract. Let $\widetilde{J}$ be the canonical paracomplex structure on $\mathbb{R}^{2n+2} \simeq \widetilde{\mathbb{C}}^{n+1}$. We study real affine hypersurfaces $f\colon M \to \widetilde{\mathbb{C}}^{n+1}$ with a $\widetilde{J}$-tangent transversal vector field. Such a vector field induces in a natural way an almost paracontact structure (φ, ξ, η) on M as well as an affine connection ∇. In this paper we give a classification of hypersurfaces with the property that φ or η is parallel relative to the connection ∇. Moreover, we show that if $\nabla\varphi = 0$ (respectively $\nabla\eta = 0$) then around each point of M there exists a parallel almost paracontact structure. We illustrate the results with appropriate examples.

1. Introduction. Paracomplex and paracontact structures have been studied by many authors. These structures play an important role in pseudo-Riemannian geometry as well as in modern mathematical physics. In particular, some recent results related to paracontact geometry can be found in [15, 1, 6]. Moreover, recently some relations between paracomplex and affine differential geometry were also studied (see [7, 2] and [14]).

If we denote by $\widetilde{J}$ the canonical paracomplex structure on $\mathbb{R}^{2n+2} \simeq \widetilde{\mathbb{C}}^{n+1}$ then, much as in the complex case [3, 9, 13], one may consider affine hypersurfaces $f\colon M \to \widetilde{\mathbb{C}}^{n+1}$ with a $\widetilde{J}$-tangent transversal vector field. Some recent results for affine hypersurfaces with a $\widetilde{J}$-tangent transversal vector field can be found in [10, 11].

In [12] the present author studied real affine hypersurfaces of the complex space $\mathbb{C}^{n+1}$ with a J-tangent transversal vector field and an induced almost contact structure (φ, ξ, η) such that φ or η is parallel relative to the induced affine connection. Now, it is natural to ask what happens in the case of paracomplex structures. It is worth highlighting that in this case we do not

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have the canonical $\widetilde{J}$-tangent transversal vector field (the Riemannian normal field is not $\widetilde{J}$-tangent in general), so the situation is more complicated.

In Section 2 we briefly recall basic formulas of affine differential geometry.

In Section 3 we recall the definition of an almost paracontact structure introduced in [5]. We recall the notion of a $\widetilde{J}$-tangent transversal vector field and a $\widetilde{J}$-invariant distribution. We also recall some results obtained in [10, 11] for induced almost paracontact structures which will be used in the next section.

Section 4 contains the main results of this paper. First we show some basic relations among induced objects under an additional condition that either φ or η is ∇-parallel. Next we show that if any of the above is satisfied for some $\widetilde{J}$-tangent transversal vector field then we can always find (at least locally) another $\widetilde{J}$-tangent transversal vector field such that the induced almost paracontact structure is parallel. Finally, we provide a full local classification of the above mentioned hypersurfaces. In order to illustrate the results some examples are also given. In particular, we show that (in contrast with the case when $\nabla\eta = 0$ or $\nabla\varphi = 0$) the condition $\nabla\xi = 0$ is much weaker.

2. Preliminaries. We briefly recall the basic formulas of affine differential geometry. For more details we refer to [8]. Let $f\colon M \to \mathbb{R}^{n+1}$ be an orientable, connected differentiable n-dimensional hypersurface immersed in the affine space $\mathbb{R}^{n+1}$ equipped with its usual flat connection D. Then for any transversal vector field C we have

$$\mathrm{D}_X f_*Y = f_*(\nabla_X Y) + h(X,Y)C, \tag{2.1}$$

$$\mathrm{D}_X C = -f_*(SX) + \tau(X)C, \tag{2.2}$$

where X, Y are vector fields tangent to M. The formulas (2.1) and (2.2) are called the formulas of Gauss and Weingarten, respectively. For any transversal vector field C, ∇ is a torsion-free connection, h is a symmetric bilinear form on M, called the *second fundamental form*, S is a tensor of type $(1,1)$, called the *shape operator*, and τ is a 1-form, called the *transversal connection form*.

We shall now consider a change of a transversal vector field for a given immersion f. We have

THEOREM 2.1 ([8]). *Suppose we change a transversal vector field C to*

$$\overline{C} = \Phi C + f_*(Z),$$

where Z is a tangent vector field on M and Φ is a nowhere vanishing function on M. Then the affine fundamental form, the induced connection, the *transversal connection form, and the affine shape operator change as follows:*

$$\begin{aligned}
&\overline{h} = \frac{1}{\Phi}h,\\
&\overline{\nabla}_X Y = \nabla_X Y - \frac{1}{\Phi}h(X,Y)Z,\\
&\overline{\tau} = \tau + \frac{1}{\Phi}h(Z,\cdot) + d\ln|\Phi|,\\
&\overline{S} = \Phi S - \nabla.Z + \overline{\tau}(\cdot)Z.
\end{aligned}$$

If h is nondegenerate, then we say that the hypersurface or the hypersurface immersion is *nondegenerate.*

For an arbitrary transversal vector field C the induced connection ∇, the second fundamental form h, the shape operator S, and the 1-form τ satisfy the following equations:

$$R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY, \tag{2.3}$$

$$(\nabla_X h)(Y,Z) + \tau(X)h(Y,Z) = (\nabla_Y h)(X,Z) + \tau(Y)h(X,Z), \tag{2.4}$$

$$(\nabla_X S)(Y) - \tau(X)SY = (\nabla_Y S)(X) - \tau(Y)SX, \tag{2.5}$$

$$h(X,SY) - h(SX,Y) = 2d\tau(X,Y). \tag{2.6}$$

The equations (2.3), (2.4), (2.5), and (2.6) are called the equations of Gauss, Codazzi for h, Codazzi for S, and Ricci, respectively. The equations (2.3)–(2.6) are sometimes called the *fundamental equations* (see [8]).

For a hypersurface immersion $f\colon M \to \mathbb{R}^{n+1}$ a transversal vector field C is said to be *equiaffine* (resp. *locally equiaffine*) if $\tau = 0$ (resp. $d\tau = 0$).

3. Almost paracontact structures. A $(2n+1)$-dimensional manifold M is said to have an *almost paracontact structure* if there exist on M a tensor field φ of type (1,1), a vector field ξ and a 1-form η which satisfy

$$\varphi^2(X) = X - \eta(X)\xi, \tag{3.1}$$

$$\eta(\xi) = 1 \tag{3.2}$$

for every $X \in TM$ and the tensor field φ induces an almost paracomplex structure on the distribution $\mathcal{D} = \ker\eta$, that is, the eigendistributions $\mathcal{D}^+, \mathcal{D}^-$ corresponding to the eigenvalues $1, -1$ of φ have equal dimension n. Let ∇ be a connection on M. We say that an almost paracontact structure (φ,ξ,η) is ∇-*parallel* if $\nabla\varphi = 0$, $\nabla\eta = 0$ and $\nabla\xi = 0$.

We denote by $\widetilde{\mathbb{C}}$ the real algebra of paracomplex numbers; then the free $\widetilde{\mathbb{C}}$-module $\widetilde{\mathbb{C}}^{n+1}$ is a paracomplex vector space. We always assume that $\mathbb{R}^{2n+2} \cong \widetilde{\mathbb{C}}^{n+1}$ is endowed with the standard paracomplex structure $\widetilde{J}$ given by

$$\widetilde{J}(x_1,\ldots,x_{n+1},y_1,\ldots,y_{n+1}) = (y_1,\ldots,y_{n+1},x_1,\ldots,x_{n+1}).$$

For more details on paracomplex structures we refer to [4, 7].

Let $\dim M = 2n+1$ and $f\colon M \to \mathbb{R}^{2n+2}$ be an affine hypersurface. Let C be a transversal vector field on M. We say that C is *$\widetilde{J}$-tangent* if $\widetilde{J}C_x \in f_*(T_xM)$ for every $x \in M$. We also define a distribution $\mathcal{D}$ on M as the biggest $\widetilde{J}$-invariant distribution on M, that is,

$$\mathcal{D}_x := f_*^{-1}(f_*(T_xM) \cap \widetilde{J}(f_*(T_xM)))$$

for every $x \in M$. We have $\dim \mathcal{D}_x \geq 2n$. If for some x $\dim \mathcal{D}_x = 2n+1$ then $\mathcal{D}_x = T_xM$ and it is not possible to find a $\widetilde{J}$-tangent transversal vector field in a neighbourhood of x. Since we only study hypersurfaces with a $\widetilde{J}$-tangent transversal vector field, we always have $\dim \mathcal{D} = 2n$. The distribution $\mathcal{D}$ is smooth as an intersection of two smooth distributions and because $\dim \mathcal{D}$ is constant. A vector field X is called a *$\mathcal{D}$-field* if $X_x \in \mathcal{D}_x$ for every x from its domain. We use the notation $X \in \mathcal{D}$ for vectors as well as for $\mathcal{D}$-fields.

To simplify the writing, we will omit f_* in front of vector fields in most cases.

Let $f\colon M \to \mathbb{R}^{2n+2}$ be an affine hypersurface with a $\widetilde{J}$-tangent transversal vector field C. Then we can define a vector field ξ, a 1-form η and a tensor field φ of type (1,1) as follows:

$$\xi := \widetilde{J}C, \tag{3.3}$$

$$\eta|_{\mathcal{D}} = 0 \quad \text{and} \quad \eta(\xi) = 1, \tag{3.4}$$

$$\varphi|_{\mathcal{D}} = \widetilde{J}|_{\mathcal{D}} \quad \text{and} \quad \varphi(\xi) = 0. \tag{3.5}$$

It is easy to see that (φ, ξ, η) is an almost paracontact structure on M. This structure is called the *induced almost paracontact structure*. Using Theorem 2.1 one may prove the following:

Lemma 3.1 ([10]). *Let C be a $\widetilde{J}$-tangent transversal vector field. Then any other $\widetilde{J}$-tangent transversal vector field $\overline{C}$ has the form*

$$\overline{C} = \Phi C + f_*Z,$$

where $\Phi \neq 0$ and $Z \in \mathcal{D}$. Moreover, if (φ, ξ, η) is an almost paracontact structure induced by C, then $\overline{C}$ induces an almost paracontact structure $(\overline{\varphi}, \overline{\xi}, \overline{\eta})$, where

$$\begin{cases} \overline{\xi} = \Phi\xi + \varphi Z, \\ \overline{\eta} = \frac{1}{\Phi}\eta, \\ \overline{\varphi} = \varphi - \eta(\cdot)\frac{1}{\Phi}Z. \end{cases}$$

For an induced almost paracontact structure we have the following theorem:

Theorem 3.2 ([11]). *Let $f\colon M \to \mathbb{R}^{2n+2}$ be an affine hypersurface with a $\widetilde{J}$-tangent transversal vector field C. If (φ, ξ, η) is an induced almost para-*

contact structure on M then the following equalities hold:

$$\eta(\nabla_X Y) = h(X, \varphi Y) + X(\eta(Y)) + \eta(Y)\tau(X), \tag{3.6}$$

$$\varphi(\nabla_X Y) = \nabla_X \varphi Y - \eta(Y)SX - h(X, Y)\xi, \tag{3.7}$$

$$\eta([X, Y]) = h(X, \varphi Y) - h(Y, \varphi X) + X(\eta(Y)) - Y(\eta(X)) + \eta(Y)\tau(X) - \eta(X)\tau(Y), \tag{3.8}$$

$$\varphi([X, Y]) = \nabla_X \varphi Y - \nabla_Y \varphi X + \eta(X)SY - \eta(Y)SX, \tag{3.9}$$

$$\eta(\nabla_X \xi) = \tau(X), \tag{3.10}$$

$$\eta(SX) = -h(X, \xi) \tag{3.11}$$

for all $X, Y \in \mathcal{X}(M)$.

4. Parallel induced almost paracontact structures. In this section we assume that (φ, ξ, η) is an almost paracontact structure induced by a $\widetilde{J}$-tangent transversal vector field C, and ∇, h, S, τ are affine objects induced by C as well. Sometimes we denote a transversal vector field by $\overline{C}$ or $\overline{\overline{C}}$. In such cases we use the corresponding (i.e. barred) notation for induced objects.

For ∇-parallel φ we have the following

LEMMA 4.1. *Let (φ, ξ, η) be an induced almost paracontact structure such that $\nabla\varphi = 0$. Then*

$$h|_{\mathcal{D}\times\mathcal{D}} = 0, \tag{4.1}$$

$$h(\xi, X) = h(X, \xi) = 0 \quad \text{for all } X \in \mathcal{D}, \tag{4.2}$$

$$S|_{\mathcal{D}} = 0, \tag{4.3}$$

$$S\xi = -h(\xi, \xi)\xi, \tag{4.4}$$

$$d\tau = 0, \tag{4.5}$$

$$R = 0. \tag{4.6}$$

Proof. From the formula (3.7) we have

$$(\nabla_X \varphi)(Y) = \eta(Y)SX + h(X, Y)\xi$$

for all $X, Y \in \mathcal{X}(M)$. Since $\nabla\varphi = 0$ we get $h(X, Y) = 0$ and $h(\xi, Y) = 0$ for all $X, Y \in \mathcal{D}$. Now, taking $X \in \mathcal{D}$ and $Y = \xi$ we have $SX = 0$. Taking $X = Y = \xi$ we easily get $S\xi = -h(\xi, \xi)\xi$. The equation (4.5) follows immediately from the Ricci equation (2.6). The last equation is an immediate consequence of the Gauss equation and (4.1)–(4.3). ∎

By Lemma 4.1 the transversal vector field C is locally equiaffine, that is, there exists (at least locally) a non-vanishing function Φ, such that $\overline{C} = \Phi C$ is equiaffine. Of course $\overline{C}$ is $\widetilde{J}$-tangent. Now, using Theorem 2.1 and Lemma 3.1 we get the following corollary:

Corollary 4.2. *Let C be a $\widetilde{J}$-tangent transversal vector field such that $\nabla\varphi = 0$ and let Φ be a nowhere vanishing function on M. If $\overline{C} = \Phi C$, then $\overline{\nabla}\overline{\varphi} = 0$ (actually $\overline{\nabla} = \nabla$ and $\overline{\varphi} = \varphi$). This means that the condition $\nabla\varphi = 0$ is a property depending only on the direction of C. In particular, locally we can always choose C equiaffine.*

We shall prove

Lemma 4.3. *Let (φ, ξ, η) be an induced almost paracontact structure such that $\nabla\eta = 0$. Then*

$$h|_{\mathcal{D}\times\mathcal{D}} = 0, \tag{4.7}$$

$$h(\xi, X) = h(X, \xi) = 0 \quad \text{for every } X \in \mathcal{D}, \tag{4.8}$$

$$\tau = 0, \tag{4.9}$$

$$\nabla_X Y \in \mathcal{D} \quad \text{for all } X, Y \in \mathcal{D}, \tag{4.10}$$

$$\nabla_X \xi \in \mathcal{D} \quad \text{for every } X \in \mathcal{X}(M), \tag{4.11}$$

$$\nabla_\xi X \in \mathcal{D} \quad \text{for every } X \in \mathcal{D}, \tag{4.12}$$

$$X(h(\xi, \xi)) = 0 \quad \text{for every } X \in \mathcal{D}. \tag{4.13}$$

Proof. Since $\nabla\eta = 0$, we have

$$\eta(\nabla_X Y) = X(\eta(Y)) \tag{4.14}$$

for all $X, Y \in \mathcal{X}(M)$. Now, using the formula (3.6) we get

$$h(X, \varphi Y) = -\eta(Y)\tau(X) \tag{4.15}$$

for all $X, Y \in \mathcal{X}(M)$. Hence, if $X, Y \in \mathcal{D}$, then $h(X, \varphi Y) = 0$, which proves (4.7). Taking $X = \xi$ and $Y \in \mathcal{D}$ in (4.15) we easily get (4.8). On the other hand, taking $Y = \xi$ we get $\tau(X) = 0$, that is, (4.9). The formulas (4.10)–(4.12) can be directly obtained from (4.14). To prove (4.13) let us note that from the Codazzi equation for h (and using (4.9)) we obtain

$$(\nabla_X h)(\xi, \xi) = (\nabla_\xi h)(X, \xi) = \xi(h(X, \xi)) - h(\nabla_\xi X, \xi) - h(X, \nabla_\xi \xi).$$

Now, if we take $X \in \mathcal{D}$ and use (4.7)–(4.8), we get $h(X, \xi) = 0$ and $h(X, \nabla_\xi \xi) = 0$, whereas (4.12) implies that we also have $h(\nabla_\xi X, \xi) = 0$. Thus, we obtain

$$0 = (\nabla_X h)(\xi, \xi) = X(h(\xi, \xi)) - 2h(\nabla_X \xi, \xi)$$

for every $X \in \mathcal{D}$. Now, applying (4.11) in the above formula we get

$$X(h(\xi, \xi)) = 0$$

for every $X \in \mathcal{D}$. This finishes the proof of (4.13). ∎

Lemma 4.4. *If $\nabla\varphi = 0$ then $(\nabla_X \eta)Y = -\eta(Y)\tau(X)$.*

Proof. If $\nabla\varphi = 0$ then from (3.7) we get $h(X, \varphi Y) = 0$ for all $X, Y \in \mathcal{X}(M)$. So from (3.6) we obtain $(\nabla_X \eta)Y = -\eta(Y)\tau(X)$. ∎

LEMMA 4.5. *If* $\nabla\varphi = 0$ *or* $\nabla\eta = 0$ *then*

$$\operatorname{rank} h \leq 1, \tag{4.16}$$

$$\mathcal{D}, \mathcal{D}^+, \mathcal{D}^- \quad \textit{are } \nabla\textit{-parallel}, \tag{4.17}$$

$$\mathcal{D}, \mathcal{D}^+, \mathcal{D}^- \quad \textit{are involutive}. \tag{4.18}$$

Proof. The property (4.16) is immediate from Lemma 4.1 or Lemma 4.3. By (3.6) we have

$$\eta(\nabla_X Y) = h(X, \varphi Y)$$

for all $X \in \mathcal{X}(M)$ and $Y \in \mathcal{D}$. Now (4.1)–(4.2) (or (4.7)–(4.8)) imply that $\eta(\nabla_X Y) = 0$ for $X \in \mathcal{X}(M)$ and $Y \in \mathcal{D}$, that is, $\mathcal{D}$ is ∇-parallel. Using (3.7) we get

$$\varphi(\nabla_X Y) = \nabla_X \varphi Y = \nabla_X Y$$

for $X \in \mathcal{X}(M)$ and $Y \in \mathcal{D}^+$, so $\nabla_X Y \in \mathcal{D}^+$. Similarly, $\nabla_X Y \in \mathcal{D}^-$ for $X \in \mathcal{X}(M)$ and $Y \in \mathcal{D}^-$, that is, both $\mathcal{D}^+$ and $\mathcal{D}^-$ are ∇-parallel. Now (4.18) easily follows from (4.17) and the fact that ∇ is torsion-free. ∎

EXAMPLE 4.6. Let $f\colon \mathbb{R}^3 \to \mathbb{R}^4$ be given by the formula

$$f(x, y, z) := \begin{bmatrix} x + y \\ \sinh z \\ x - y \\ \cosh z \end{bmatrix}.$$

Let $\{\partial_x, \partial_y, \partial_z\}$ be the canonical frame field on $\mathbb{R}^3$ generated by the coordinate system (x, y, z). It easily follows from the definition that f is an immersion and $\widetilde{J} f_x = f_x$, $\widetilde{J} f_y = -f_y$. Let

$$C\colon \mathbb{R}^3 \ni (x, y, z) \mapsto \begin{bmatrix} x \\ \sinh z \\ x \\ \cosh z \end{bmatrix} \in \mathbb{R}^4.$$

Since $C = \widetilde{J} f_z + x f_x$ we infer that C is a $\widetilde{J}$-tangent transversal vector field. Moreover,

$$\tau = 0, \qquad S(\partial_x) = -\partial_x, \qquad S(\partial_y) = 0, \qquad S(\partial_z) = -\partial_z$$

and

$$h = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Using now the formula (3.6) and the fact that $\partial_x \in \mathcal{D}^+$, $\partial_y \in \mathcal{D}^-$ and $\xi = \partial_z + x\partial_x$, we obtain $\nabla\eta = 0$, since $\varphi(\partial_z) = -x\partial_x$. However, $\nabla\varphi \neq 0$. Indeed, from (3.7) we get

$$(\nabla_{\partial_z}\varphi)\partial_z = \eta(\partial_z)S\partial_z + h(\partial_z,\partial_z)\xi = -\partial_z + \xi = x\partial_x.$$

We also have

$$R(\partial_x,\partial_z)\partial_z = h(\partial_z,\partial_z)S\partial_x - h(\partial_x,\partial_z)S\partial_z = -\partial_x,$$

so ∇ is not flat.

Now, let us consider f with the transversal vector field

$$\overline{C}\colon \mathbb{R}^3 \ni (x,y,z) \mapsto \begin{bmatrix} 0 \\ \sinh z \\ 0 \\ \cosh z \end{bmatrix} \in \mathbb{R}^4.$$

Of course $\overline{C}$ is $\widetilde{J}$-tangent. As above we compute that the almost paracontact structure $(\overline{\varphi},\overline{\xi},\overline{\eta})$ induced by $\overline{C}$ is $\overline{\nabla}$-parallel. In particular, $\overline{\nabla}$ is flat.

Finally, let us define

$$N\colon \mathbb{R}^3 \ni (x,y,z) \mapsto \begin{bmatrix} 0 \\ \frac{\sinh z}{\sqrt{\cosh^2 z + \sinh^2 z}} \\ 0 \\ -\frac{\cosh z}{\sqrt{\cosh^2 z + \sinh^2 z}} \end{bmatrix} \in \mathbb{R}^4.$$

One sees that N is the normal field (in the sense of Riemannian geometry) for f, since it is orthogonal to f_x, f_y and f_z and $\langle N,N\rangle = 1$. Moreover $\widetilde{J}N$ is not tangent (if only $z \neq 0$), that is, N is not $\widetilde{J}$-tangent.

The above example shows that in general the condition $\nabla\eta = 0$ is weaker than $\nabla\varphi = 0$. However we shall show that if $\nabla\eta = 0$ for some $\widetilde{J}$-tangent transversal vector field C, we can (at least locally) find another equiaffine $\widetilde{J}$-tangent transversal vector field $\overline{C}$ such that the whole structure $(\overline{\varphi},\overline{\xi},\overline{\eta})$ is $\overline{\nabla}$-parallel. To prove this we will need three lemmas.

Lemma 4.7. *If $\nabla\eta = 0$ then there exists a vector field $W \in \mathcal{X}(M)$ such that the connection $\overline{\nabla}$ defined by*

$$\overline{\nabla}_X Y := \nabla_X Y + h(X,Y)W$$

is flat.

Proof. Let us denote by $\overline{C}$ the normal vector field in the sense of Riemannian geometry. From Theorem 2.1 there exist a non-vanishing function

Φ and a vector field $Z_0 \in \mathcal{X}(M)$ such that

$$\overline{C} = \Phi C + f_* Z_0.$$

Let $\overline{\nabla}$, $\overline{h}$, $\overline{S}$ and $\overline{\tau} \equiv 0$ be affine objects induced by $\overline{C}$ and let g be the first fundamental form on M (i.e. Riemannian metric on M induced from the canonical inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^{2n+2}$). We have

$$g(\overline{S}X, Y) = \overline{h}(X, Y) = \frac{1}{\Phi} h(X, Y) \tag{4.19}$$

for all $X, Y \in \mathcal{X}(M)$. Since $h(X, Y) = 0$ for all $X \in \mathcal{D}$ and $Y \in \mathcal{X}(M)$, and g is the Riemannian metric on M, (4.19) implies that $\overline{S} = 0$ on $\mathcal{D}$. Now the Gauss equation

$$\overline{R}(X, Y)Z = \overline{h}(Y, Z)\overline{S}X - \overline{h}(X, Z)\overline{S}Y$$

implies that $\overline{R} = 0$, that is, $\overline{\nabla}$ is flat. Since ∇ and $\overline{\nabla}$ are related (see Theorem 2.1) by

$$\overline{\nabla}_X Y = \nabla_X Y - \frac{1}{\Phi} h(X, Y) Z_0,$$

it is sufficient to take $W := -\frac{1}{\Phi} Z_0$. ∎

LEMMA 4.8. *Let $f\colon M \to \mathbb{R}^{2n+2}$ be an affine hypersurface with a $\widetilde{J}$-tangent transversal vector field C and let (φ, ξ, η) be an induced almost paracontact structure on M. If $\nabla\eta = 0$ then for every $p \in M$ there exist a neighbourhood U of p and a local basis $\{\partial_1, \dots, \partial_{2n+1}\}$ on U such that $\{\partial_1, \dots, \partial_n\}$ span the distribution $\mathcal{D}^+$, $\{\partial_{n+1}, \dots, \partial_{2n}\}$ span the distribution $\mathcal{D}^-$ and $\nabla_{\partial_i}\partial_j = 0$ for $i, j = 1, \dots, 2n$, $\nabla_{\partial_i}\partial_{2n+1} = \nabla_{\partial_{2n+1}}\partial_i = 0$ for $i = 1, \dots, 2n$.*

Proof. Let $p \in M$ and let $\overline{\nabla}$ be the connection from Lemma 4.7. Since $\overline{\nabla}$ is flat, in some neighbourhood U of p there exists a basis $\partial_1, \dots, \partial_{2n+1}$ such that $\overline{\nabla}_{\partial_i}\partial_j = 0$ for $i, j = 1, \dots, 2n+1$. In particular, $\overline{\nabla}_X \partial_i = 0$ for $i = 1, \dots, 2n+1$ and $X \in \mathcal{X}(U)$. Without loss of generality (shrinking U if needed) we may assume that $\partial_{2n+1} \notin \mathcal{D}$. Then for $i = 1, \dots, 2n$ we have the decomposition

$$\partial_i = \partial_i^+ + \partial_i^- + \alpha_i \partial_{2n+1}$$

where $\partial_i^+ \in \mathcal{D}^+$, $\partial_i^- \in \mathcal{D}^-$ and α_i is a smooth function on U. Now, for any $X \in \mathcal{X}(U)$ we have

$$\begin{aligned} 0 = \overline{\nabla}_X \partial_i &= \overline{\nabla}_X \partial_i^+ + \overline{\nabla}_X \partial_i^- + \overline{\nabla}_X(\alpha_i \partial_{2n+1}) \\ &= \overline{\nabla}_X \partial_i^+ + \overline{\nabla}_X \partial_i^- + X(\alpha_i)\partial_{2n+1} \\ &= \nabla_X \partial_i^+ + \nabla_X \partial_i^- + X(\alpha_i)\partial_{2n+1}, \end{aligned}$$

where the last equality is an immediate consequence of the fact that $\overline{\nabla}_X Y = \nabla_X Y$ for all $X \in \mathcal{X}(M)$ and $Y \in \mathcal{D}$. Since $\mathcal{D}^+$ and $\mathcal{D}^-$ are ∇-parallel, we obtain $\nabla_X \partial_i^+ = 0$, and $\nabla_X \partial_i^- = 0$ for any $X \in \mathcal{X}(U)$ and $\alpha_i = \mathrm{const}$ for $i = 1, \ldots, 2n$. In particular,

$$\nabla_{\partial_i^+} \partial_j^+ = 0, \quad \nabla_{\partial_i^-} \partial_j^- = 0, \quad \nabla_{\partial_i^+} \partial_j^- = 0, \quad \nabla_{\partial_i^-} \partial_j^+ = 0$$

for $i, j = 1, \ldots, 2n$. Let us consider

$$\operatorname{span}\{\partial_1^+, \ldots, \partial_{2n}^+\} \subset \mathcal{D}^+ \quad \text{and} \quad \operatorname{span}\{\partial_1^-, \ldots, \partial_{2n}^-\} \subset \mathcal{D}^-.$$

Since every ∂_i is a linear combination of elements from $\{\partial_1^+, \ldots, \partial_{2n}^+\}$, $\{\partial_1^-, \ldots, \partial_{2n}^-\}$ and ∂_{2n+1}, we have

$$\operatorname{span}\{\partial_1^+, \ldots, \partial_{2n}^+\} \oplus \operatorname{span}\{\partial_1^-, \ldots, \partial_{2n}^-\} \oplus \operatorname{span}\{\partial_{2n+1}\} = TM.$$

In particular $\dim \operatorname{span}\{\partial_1^+, \ldots, \partial_{2n}^+\} = \dim \operatorname{span}\{\partial_1^-, \ldots, \partial_{2n}^-\} = n$.

Now we can choose $2n$ linearly independent vector fields $\{\partial'_1, \ldots, \partial'_{2n}\}$ around p such that $\partial'_i \in \{\partial_1^+, \ldots, \partial_{2n}^+\}$ and $\partial'_{i+n} \in \{\partial_1^-, \ldots, \partial_{2n}^-\}$ for $i = 1, \ldots, n$. We have

$$\nabla_{\partial'_i} \partial'_j = 0$$

for $i, j = 1, \ldots, 2n$. Note also that $\nabla_{\partial'_i} \partial_{2n+1} = \bar{\nabla}_{\partial'_i} \partial_{2n+1} = 0$ and $\nabla_{\partial_{2n+1}} \partial'_i = 0$ for $i = 1, \ldots, 2n$. Finally, we see that the basis $\{\partial'_1, \ldots, \partial'_{2n}, \partial_{2n+1}\}$ has the required properties. ∎

Lemma 4.9. *Let $f \colon M \to \mathbb{R}^{2n+2}$ be an affine hypersurface with a $\widetilde{J}$-tangent transversal vector field C and let (φ, ξ, η) be an induced almost paracontact structure on M. If $\nabla \eta = 0$ then for every $p \in M$ there exist a neighbourhood U of p and a $\widetilde{J}$-tangent transversal vector field $\overline{C}$ defined on U and a local coordinate system $(x_1, \ldots, x_{2n}, y)$ on U such that $\partial_{x_1}, \ldots, \partial_{x_n} \in \mathcal{D}^+$, $\partial_{x_{n+1}}, \ldots, \partial_{x_{2n}} \in \mathcal{D}^-$ and the following conditions are satisfied:*

$$\bar{\xi} = \partial_y, \tag{4.20}$$

$$\overline{\nabla} \bar{\eta} = 0, \tag{4.21}$$

$$\overline{\nabla}_{\partial_{x_i}} \partial_{x_j} = 0 \qquad \text{for } i, j = 1, \ldots, 2n, \tag{4.22}$$

$$\overline{\nabla}_{\partial_{x_i}} \partial_y = \overline{\nabla}_{\partial_y} \partial_{x_i} = 0 \qquad \text{for } i = 1, \ldots, 2n, \tag{4.23}$$

where $(\overline{\varphi}, \bar{\xi}, \bar{\eta})$ and $\overline{\nabla}$ are the almost paracontact structure and the affine connection induced by $\overline{C}$, respectively.

Proof. By Lemma 4.8 for any $p \in M$ there exist a neighbourhood U of p and a local frame $\{\partial_1, \ldots, \partial_{2n+1}\}$ on U with properties described in Lemma 4.8. Of course $\eta(\partial_{2n+1}) \neq 0$. Since $\nabla_{\partial_i} \partial_{2n+1} = 0$ for $i = 1, \ldots, 2n$, using (3.6) we obtain

$$0 = \eta(\nabla_{\partial_i} \partial_{2n+1}) = \partial_i(\eta(\partial_{2n+1})) \tag{4.24}$$

for $i=1,\dots,2n$. Let us define

$$Y := \frac{1}{\eta(\partial_{2n+1})}\cdot \partial_{2n+1}.$$

Thanks to (4.24) we get

$$\nabla_{\partial_i} Y = \nabla_Y \partial_i = 0$$

for $i=1,\dots,2n$. In particular, $[\partial_i, Y]=0$ for $i=1,\dots,2n$ and there exists a local coordinate system $(x_1,\dots,x_{2n},y)$ around p such that $\partial_i=\partial_{x_i}$ for $i=1,\dots,2n$ and $Y=\partial_y$. We also have $\partial_{x_i}\in\mathcal{D}^+$ and $\partial_{x_{n+i}}\in\mathcal{D}^-$ for $i=1,\dots,n$. We now have

$$\eta(\partial_y)=\eta(Y)=1=\eta(\xi),$$

that is, there exists $Z\in\mathcal{D}$ such that $\partial_y=\xi+Z$. Let us define

$$\overline{C} := C + f_*(\varphi Z).$$

By Theorem 2.1 and Lemma 3.1 we have

$$\overline{\xi}=\xi+Z=\partial_y, \quad \overline{\eta}=\eta$$

and

$$\overline{\nabla}_X Y = \nabla_X Y - h(X,Y)\varphi Z \quad \text{for } X,Y\in\mathcal{X}(M).$$

In consequence,

$$\overline{\nabla}_{\partial_{x_i}}\partial_{x_j} = \nabla_{\partial_{x_i}}\partial_{x_j}=0$$

and

$$\overline{\nabla}_{\partial_{x_i}}\partial_y=\overline{\nabla}_{\partial_y}\partial_{x_i}=\nabla_{\partial_y}\partial_{x_i}=\nabla_{\partial_{x_i}}\partial_y=0$$

for $i,j=1,\dots,2n$. Finally, we obtain

$$\begin{aligned}(\overline{\nabla}_X\overline{\eta})(Y) &= X(\overline{\eta}(Y))-\overline{\eta}(\overline{\nabla}_X Y) = X(\eta(Y))-\eta(\nabla_X Y - h(X,Y)\varphi Z)\\ &= (\nabla_X\eta)(Y)+h(X,Y)\eta(\varphi Z)=(\nabla_X\eta)(Y)=0. \ \blacksquare\end{aligned}$$

Now we can prove the following

THEOREM 4.10. *Let $f\colon M\to\mathbb{R}^{2n+2}$ be an affine hypersurface with a $\widetilde{J}$-tangent transversal vector field C and let (φ,ξ,η) be an induced almost paracontact structure on M. If $\nabla\eta=0$ then for every $p\in M$ there exist a neighbourhood U of p and a $\widetilde{J}$-tangent transversal vector field $\overline{\overline{C}}$ defined on U such that the induced almost paracontact structure $(\overline{\overline{\varphi}},\overline{\overline{\xi}},\overline{\overline{\eta}})$ is $\overline{\overline{\nabla}}$-parallel. Moreover, around p there exists a local coordinate system $(x_1,\dots,x_{2n},x_{2n+1})$ such that $\partial_{x_1},\dots,\partial_{x_n}\in\mathcal{D}^+$, $\partial_{x_{n+1}},\dots,\partial_{x_{2n}}\in\mathcal{D}^-$, $\nabla_{\partial_{x_i}}\partial_{x_j}=0$ for $i,j=1,\dots,2n+1$ and $\partial_{x_{2n+1}}=\overline{\overline{\xi}}$.*

Proof. By Lemma 4.9 for any $p\in M$ there exist a neighbourhood U of p, local coordinates $(x_1,\dots,x_{2n},y)$ on U and a $\widetilde{J}$-tangent transversal vector field $\overline{C}$ defined on U such that (4.20)–(4.23) are satisfied. Using (3.7), for $i = 1, \dots, 2n$ we have

$$\overline{S}\partial_{x_i} = -\overline{\varphi}(\overline{\nabla}_{\partial_{x_i}}\overline{\xi}) = -\overline{\varphi}(\overline{\nabla}_{\partial_{x_i}}\partial_y) = 0,$$

where $\overline{S}$ is the shape operator induced by $\overline{C}$. In particular, $\overline{S}|\mathcal{D} = 0$. From (3.6) we get $\overline{\eta}(\overline{\nabla}_{\overline{\xi}}\overline{\xi}) = 0$, that is, $\overline{\nabla}_{\overline{\xi}}\overline{\xi} \in \mathcal{D}$. Therefore there exist smooth functions p_i ($i = 1, \dots, 2n$) such that

$$\overline{\nabla}_{\overline{\xi}}\overline{\xi} = \sum_{i=1}^{2n} p_i\partial_{x_i}.$$

Now, from the Gauss equation we deduce that for every $i = 1, \dots, 2n$,

$$0 = \overline{R}(\partial_{x_i}, \partial_y)\partial_y = \overline{\nabla}_{\partial_{x_i}}\overline{\nabla}_{\partial_y}\partial_y = \overline{\nabla}_{\partial_{x_i}}\sum_{j=1}^{2n} p_j\partial_{x_j} = \sum_{j=1}^{2n} \partial_{x_i}(p_j)\partial_{x_j},$$

and hence p_j depends only on y for $j = 1, \dots, 2n$. Let us define

$$Z := \sum_{i=1}^{2n} a_i\partial_{x_i},$$

where the a_i are smooth functions defined by

$$a_i := -e^{\int \overline{h}(\partial_y,\partial_y)\,dy} \cdot \int p_i e^{-\int \overline{h}(\partial_y,\partial_y)\,dy}\,dy, \tag{4.25}$$

$$a_{i+n} := -e^{-\int \overline{h}(\partial_y,\partial_y)\,dy} \cdot \int p_{i+n} e^{\int \overline{h}(\partial_y,\partial_y)\,dy}\,dy \tag{4.26}$$

for $i = 1, \dots, n$. By (4.13) we have $\partial_{x_i}(\overline{h}(\partial_y, \partial_y)) = 0$ for $i = 1, \dots, 2n$, that is, $\overline{h}(\partial_y, \partial_y)$ depends only on y, and hence a_i depends only on y for $i = 1, \dots, 2n$. Now we can define another transversal vector field $\overline{\overline{C}} := \overline{C} + f_*\varphi Z$. Since $Z \in \mathcal{D}$, we see that $\overline{\overline{C}}$ is $\widetilde{J}$-tangent. First note that

$$\overline{\overline{\nabla}}_{\overline{\overline{\xi}}}\partial_{x_i} = 0, \quad \overline{\overline{\nabla}}_{\partial_{x_i}}\overline{\overline{\xi}} = 0, \quad \overline{\overline{\nabla}}_{\partial_{x_i}}\partial_{x_j} = 0 \tag{4.27}$$

for $i, j = 1, \dots, 2n$. Indeed, (4.27) follows immediately from the fact that $\overline{\overline{\xi}} = \overline{\xi} + Z = \partial_y + Z$, each a_i depends only on y and $\overline{\overline{\nabla}}_X Y = \overline{\nabla}_X Y$ if only $X \in \mathcal{D}$ or $Y \in \mathcal{D}$. Let us denote $a_i' := \frac{\partial a_i}{\partial y}$. One computes

$$\begin{aligned}
\overline{\overline{\nabla}}_{\overline{\overline{\xi}}}\overline{\overline{\xi}} &= \overline{\nabla}_{\overline{\overline{\xi}}}\overline{\overline{\xi}} - \overline{h}(\overline{\overline{\xi}}, \overline{\overline{\xi}})\varphi Z = \overline{\nabla}_{\overline{\xi}+Z}(\overline{\xi} + Z) - \overline{h}(\overline{\xi} + Z, \overline{\xi} + Z)\varphi Z \\
&= \overline{\nabla}_{\overline{\xi}}\overline{\xi} + \overline{\nabla}_{\overline{\xi}}Z + \overline{\nabla}_Z\overline{\xi} + \overline{\nabla}_Z Z - \overline{h}(\overline{\xi}, \overline{\xi})\varphi Z \\
&= \overline{\nabla}_{\partial_y}\partial_y + \overline{\nabla}_{\partial_y}Z + \overline{\nabla}_Z Z - \overline{h}(\partial_y, \partial_y)\varphi Z \\
&= \sum_{i=1}^{2n} p_i\partial_{x_i} + \overline{\nabla}_{\partial_y}\Big(\sum_{i=1}^{2n} a_i\partial_{x_i}\Big) - \overline{h}(\partial_y, \partial_y)\varphi\Big(\sum_{i=1}^{2n} a_i\partial_{x_i}\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{2n} p_i \partial_{x_i} + \sum_{i=1}^{2n} a_i' \partial_{x_i} - \overline{h}(\partial_y, \partial_y) \sum_{i=1}^{n} a_i \partial_{x_i} + \overline{h}(\partial_y, \partial_y) \sum_{i=1}^{n} a_{i+n} \partial_{x_{i+n}} \\
&= \sum_{i=1}^{n} p_i \partial_{x_i} + \sum_{i=1}^{n} a_i' \partial_{x_i} - \overline{h}(\partial_y, \partial_y) \sum_{i=1}^{n} a_i \partial_{x_i} \\
&\quad + \sum_{i=1}^{n} p_{i+n} \partial_{x_{i+n}} + \sum_{i=1}^{n} a_{i+n}' \partial_{x_{i+n}} + \overline{h}(\partial_y, \partial_y) \sum_{i=1}^{n} a_{i+n} \partial_{x_{i+n}} = 0,
\end{aligned}
$$

where the last equality easily follows from (4.25) and (4.26). Now, using (4.27) and the fact that $\overline{\overline{\nabla}}_{\overline{\overline{\xi}}}\overline{\overline{\xi}} = 0$ we have

$$
\begin{aligned}
(\overline{\overline{\nabla}}_{\overline{\overline{\xi}}}\overline{\overline{\varphi}})\overline{\overline{\xi}} &= -\overline{\overline{\varphi}}(\overline{\overline{\nabla}}_{\overline{\overline{\xi}}}\overline{\overline{\xi}}) = 0, \\
(\overline{\overline{\nabla}}_{\overline{\overline{\xi}}}\overline{\overline{\varphi}})(\partial_{x_i}) = \overline{\overline{\nabla}}_{\overline{\overline{\xi}}}(\overline{\overline{\varphi}}(\partial_{x_i})) - \overline{\overline{\varphi}}(\overline{\overline{\nabla}}_{\overline{\overline{\xi}}}\partial_{x_i}) &= 0, \\
(\overline{\overline{\nabla}}_{\partial_{x_i}}\overline{\overline{\varphi}})(\overline{\overline{\xi}}) = -\overline{\overline{\varphi}}(\overline{\overline{\nabla}}_{\partial_{x_i}}\overline{\overline{\xi}}) &= 0, \\
(\overline{\overline{\nabla}}_{\partial_{x_i}}\overline{\overline{\varphi}})(\partial_{x_j}) &= 0,
\end{aligned}
$$

that is, $\overline{\overline{\nabla}}\,\overline{\overline{\varphi}} = 0$. Since $\overline{\overline{\nabla}}_{\overline{\overline{\xi}}}\overline{\overline{\xi}} = 0$ and $\overline{\overline{\nabla}}_{\partial_{x_i}}\overline{\overline{\xi}} = 0$ for $i = 1, \ldots, 2n$, we find that $\overline{\overline{\nabla}}_X\overline{\overline{\xi}} = 0$ for $X \in \mathcal{X}(U)$, that is, $\overline{\overline{\nabla}}\,\overline{\overline{\xi}} = 0$. Since $\overline{\tau} = 0$, Theorem 2.1 implies that $\overline{\overline{\tau}} = 0$ and now Lemma 4.4 implies that $\overline{\overline{\nabla}}\,\overline{\overline{\eta}} = 0$.

Finally, note that for $\partial_{x_1}, \ldots, \partial_{x_{2n}}, \partial_{x_{2n+1}} := \overline{\overline{\xi}}$ we have $[\partial_{x_i}, \partial_{x_j}] = 0$ for $i, j = 1, \ldots, 2n+1$, so there exists a local coordinate system $(x_1, \ldots, x_{2n}, x_{2n+1})$ with the desired properties. ∎

Now we can state the following classification theorem.

Theorem 4.11. *Let $f\colon M \to \mathbb{R}^{2n+2}$ be an affine hypersurface with a $\widetilde{J}$-tangent transversal vector field C and let (φ, ξ, η) be an induced almost paracontact structure on M. If φ or η is ∇-parallel then for every $p \in M$ there exists a neighbourhood U of p and a coordinate system $(x_1, \ldots, x_{2n}, y)$ such that f can be locally expressed in the form*

$$
\begin{aligned}
f(x_1, \ldots, x_{2n}, y) &= x_1 b_1 + \cdots + x_{2n} b_{2n} \\
&\quad + \widetilde{J}v \int \cosh\alpha(y)\,dy + v \int \sinh\alpha(y)\,dy,
\end{aligned} \tag{4.28}
$$

where $b_1, \ldots, b_{2n}, v \in \mathbb{R}^{2n+2}$, $b_1, \ldots, b_{2n}, v, \widetilde{J}v$ are linearly independent, $\widetilde{J}b_i = b_i$ for $i = 1, \ldots, n$, $\widetilde{J}b_i = -b_i$ for $i = n+1, \ldots, 2n$, and α is some smooth function. Moreover, the converse is true in the sense that for the function (4.28) there exists a global $\widetilde{J}$-tangent equiaffine transversal vector field C such that (φ, ξ, η) is ∇-parallel.

Proof. Let $p \in M$. Thanks to Theorem 4.10, Corollary 4.2 and Lemma 4.4, without loss of generality we may assume that (φ, ξ, η) is ∇-parallel around p. Let $(x_1, \ldots, x_{2n}, x_{2n+1})$ be the local coordinate system from Theorem 4.10. Set $x_{2n+1} = y$. Now, by the Gauss formula we have the following system of differential equations:

$$f_{x_i x_j} = 0, \tag{4.29}$$

$$f_{x_i y} = 0, \tag{4.30}$$

$$f_{yy} = h(\xi, \xi)C = h(\xi, \xi)\widetilde{J} f_y \tag{4.31}$$

for $i = 1, \ldots, 2n$. Solving (4.29) and (4.30) we obtain

$$f(x_1, \ldots, x_{2n}, y) = \sum_{i=1}^{2n} x_i b_i + A(y),$$

where $b_1, \ldots, b_{2n} \in \mathbb{R}^{2n+2}$, $b_1, \ldots, b_{2n}$ are linearly independent, $\widetilde{J} b_i = b_i$ for $i = 1, \ldots, n$, $\widetilde{J} b_i = -b_i$ for $i = n + 1, \ldots, 2n$, and A is some smooth function depending only on y with values in $\mathbb{R}^{2n+2}$. Now (4.31) takes the form

$$A'' = \beta \widetilde{J} A', \tag{4.32}$$

where $\beta := h(\xi, \xi)$. Substituting $G := A'$ we get

$$G' = \beta \widetilde{J} G. \tag{4.33}$$

Let $\overline{\beta}$ be any integral of β. First note that for every $v \in \mathbb{R}^{2n+2}$ the function

$$G(y) = \widetilde{J} v \cosh \overline{\beta}(y) + v \sinh \overline{\beta}(y) \tag{4.34}$$

is a solution of (4.33). On the other hand, since (4.33) is a first order ordinary differential equation all its solutions have the form (4.34). The above arguments imply that all solutions of (4.32) have the form

$$A(y) = \widetilde{J} v \int \cosh \overline{\beta}(y)\, dy + v \int \sinh \overline{\beta}(y)\, dy.$$

Since f is an immersion and C is transversal, we deduce that $b_1, \ldots, b_{2n}$, $f_y = G(y)$ and $C = \widetilde{J} f_y$ are linearly independent. In particular, $b_1, \ldots,$ $b_{2n}, v, \widetilde{J} v$ are linearly independent too. Indeed, if we assume that

$$\sum_{i=1}^{2n} a_i b_i + a_{2n+1} v + a_{2n+2} \widetilde{J} v = 0$$

for some functions $a_1, \ldots, a_{2n+2}$, we get

$$\begin{aligned} 0 &= \sum_{i=1}^{2n} a_i b_i + a_{2n+1} v + a_{2n+2} \widetilde{J} v \\ &= \sum_{i=1}^{2n} a_i b_i + (a_{2n+2} \cosh \overline{\beta}(y) - a_{2n+1} \sinh \overline{\beta}(y)) f_y \\ &\quad + (a_{2n+1} \cosh \overline{\beta}(y) - a_{2n+2} \sinh \overline{\beta}(y)) \widetilde{J} f_y. \end{aligned}$$

Now, since $\{b_1, \dots, b_{2n}, f_y, \widetilde{J} f_y\}$ are linearly independent, we obtain $a_1 = \dots = a_{2n} = 0$ and

$$a_{2n+2} \cosh \overline{\beta}(y) - a_{2n+1} \sinh \overline{\beta}(y) = a_{2n+1} \cosh \overline{\beta}(y) - a_{2n+2} \sinh \overline{\beta}(y) = 0.$$

The above implies that $a_{2n+1} = a_{2n+2} = 0$. Therefore f can be locally expressed in the form

$$\begin{aligned} f(x_1, \dots, x_{2n}, y) &= x_1 b_1 + \dots + x_{2n} b_{2n} \\ &\quad + \widetilde{J} v \int \cosh \overline{\beta}(y)\, dy + v \int \sinh \overline{\beta}(y)\, dy. \end{aligned}$$

Now denoting $\alpha := \overline{\beta}$ we get the assertion.

In order to prove the "moreover" part note that since $b_1, \dots, b_{2n}, v, \widetilde{J} v$ are linearly independent, the function f given by (4.28) is an immersion and $C := \widetilde{J} f_y$ is transversal and $\widetilde{J}$-tangent. Let (φ, ξ, η) be an almost paracontact structure induced by C. Since $C_{x_i} = 0$ and $C_y = \alpha' f_y$, we have $\tau = 0$, $S|_{\mathcal{D}} = 0$ and $S\xi = -\alpha' \xi$. Since $f_{x_i x_j} = 0$, $f_{x_i y} = 0$ and $f_{yy} = \alpha' C$, we have $h(X, Y) = 0$ if only $X \in \mathcal{D}$ or $Y \in \mathcal{D}$ and $h(\xi, \xi) = \alpha'$. Now using (3.7) we easily obtain $\nabla \varphi = 0$. Since C is equiaffine, we also have $\nabla \eta = 0$ and $\nabla \xi = 0$. ∎

Corollary 4.12. *If* $\nabla \varphi = 0$ *or* $\nabla \eta = 0$ *and* $\operatorname{rank} h = 0$ *on* M *then* f *is a piece of a hyperplane.*

Proof. By Theorem 4.11, f can be locally expressed in the form (4.28). In particular

$$f_{yy} = \alpha'(y) \cdot (\widetilde{J} v \sinh \alpha(y) + v \cosh \alpha(y)) = \alpha'(y) \widetilde{J} f_y.$$

On the other hand f_{yy} is tangent, since $h = 0$. Since $\widetilde{J} f_y$ is transversal, we obtain $\alpha' = 0$, that is, $\alpha = \text{const}$, and so f is a piece of a hyperplane. ∎

We conclude with an example showing that the condition $\nabla \xi = 0$ is much weaker than $\nabla \varphi = 0$ or $\nabla \eta = 0$.

Example 4.13. Let $f\colon (0, \infty)^2 \times \mathbb{R} \to \mathbb{R}^4$ be an affine immersion given by

$$f(x, y, z) := \begin{bmatrix} \frac{1}{2}(x^2 + y^2) \\ \sinh z \\ \frac{1}{2}(x^2 - y^2) \\ \frac{1}{3}(x^3 + y^3) + \cosh z \end{bmatrix}.$$

It is easy to verify that

$$C\colon (0, \infty)^2 \times \mathbb{R} \ni (x, y, z) \mapsto \begin{bmatrix} 0 \\ \sinh z \\ 0 \\ \cosh z \end{bmatrix} \in \mathbb{R}^4$$

is a $\widetilde{J}$-tangent transversal vector field for f. Let $\{\partial_x, \partial_y, \partial_z\}$ be the canonical frame field on $(0,\infty)^2 \times \mathbb{R}$ generated by the coordinate system (x, y, z) and let (φ, ξ, η) be an almost paracontact structure induced by C. Note that $\xi = \partial_z$. By straightforward computations we obtain

$$h = \begin{bmatrix} x\cosh z & 0 & 0 \\ 0 & y\cosh z & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

so in particular f is nondegenerate, that is, $\operatorname{rank} h = 3$. Now by Lemma 4.5 it is not possible to find a $\widetilde{J}$-tangent transversal vector field such that $\nabla\varphi = 0$ or $\nabla\eta = 0$. However, by the Gauss formula we have

$$\nabla_{\partial_x}\partial_z = \nabla_{\partial_y}\partial_z = \nabla_{\partial_z}\partial_z = 0,$$

that is, $\nabla\xi = 0$.

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References

[1] B. Cappelletti Montano, I. Küpeli Erken and C. Murathan, *Nullity conditions in paracontact geometry*, Differential Geom. Appl. 30 (2012), 665–693.

[2] V. Cortés, M. A. Lawn and L. Schäfer, *Affine hyperspheres associated to special para-Kähler manifolds*, Int. J. Geom. Methods Modern Phys. 3 (2006), 995–1009.

[3] V. Cruceanu, *Real hypersurfaces in complex centro-affine spaces*, Results Math. 13 (1988), 224–234.

[4] V. Cruceanu, P. Fortuny and P. M. Gadea, *A survey on para-complex geometry*, Rocky Mountain J. Math. 26 (1996), 83–115.

[5] S. Kaneyuki and F. L. Williams, *Almost paracontact and parahodge structures on manifolds*, Nagoya Math. J. 99 (1985), 173–187.

[6] I. Küpeli Erken and C. Murathan, *A study of three-dimensional paracontact $(\tilde{k}, \tilde{\mu}, \tilde{\nu})$-spaces*, Int. J. Geom. Methods Modern Phys. 14 (2017), no. 7, 1750106, 35 pp.

[7] M. A. Lawn and L. Schäfer, *Decompositions of para-complex vector bundles and para-complex affine immersions*, Results Math. 48 (2005), 246–274.

[8] K. Nomizu and T. Sasaki, *Affine Differential Geometry*, Cambridge Univ. Press, 1994.

[9] M. Szancer and Z. Szancer, *Real hypersurfaces with an induced almost contact structure*, Colloq. Math. 114 (2009), 41–51.

[10] Z. Szancer, *On 3-dimensional $\widetilde{J}$-tangent centro-affine hypersurfaces and $\widetilde{J}$-tangent affine hyperspheres with some null-directions*, Turk. J. Math. 42 (2018), 2779–2797.

[11] Z. Szancer, *$\widetilde{J}$-tangent affine hypersurfaces with an induced almost paracontact structure*, arXiv:1710.10488 (2017).

[12] Z. Szancer, *Real hypersurfaces with parallel induced almost contact structures*, Ann. Polon. Math. 104 (2012), 203–215.

[13] Z. Szancer, *J-tangent affine hyperspheres with an involutive contact distribution*, Publ. Math. Debrecen 89 (2016), 399–413.

[14] Z. Szancer, *On para-complex affine hyperspheres*, Results Math. 72 (2017), 491–513.

[15] S. Zamkovoy, *Canonical connections on paracontact manifolds*, Ann. Glob. Anal. Geom. 36 (2009), 37–60.

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