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Measures of finite pluricomplex energy

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Abstract. We study a complex Monge–Ampère type equation of the form

$$(dd^c u)^n = \frac{ke^{-u}dV}{\int e^{-u}dV}.$$

1. Introduction. This paper is a revised version of [C5]. We study a complex Monge–Ampère type equation of the form

$$(dd^c u)^n = \frac{ke^{-u}\,dV}{\int e^{-u}\,dV},$$

where dV denotes the Lebesgue measure in $\mathbb{C}^n$. We denote by $\Omega \subseteq \mathbb{C}^n$, $n \geq 1$, a bounded, connected, open, and hyperconvex set. Let $\mathcal{E}_0$, $\mathcal{F}_1$ and $\mathcal{F}$ be the energy classes introduced in [C2, C3] (see Section 2 for details). For more information the reader may wish to consult [BB], [BT2], [CZ1], [GKY], [Ko1] and [Ko2].

2. Preliminaries. We denote by $\mathcal{E}_0$ the family of all bounded plurisubharmonic functions φ defined on Ω such that

$$\lim_{z\to\xi} \varphi(z) = 0 \quad \text{for every } \xi \in \partial\Omega, \quad \text{and} \quad \int_\Omega (dd^c\varphi)^n < \infty,$$

where $(dd^c\,\cdot)^n$ is the complex Monge–Ampère operator, normalized so that $dd^c = (i/\pi)\partial\bar\partial$. Assume now u that is a function such that there exists a decreasing sequence $\{u_j\} \subset \mathcal{E}_0$ that converges pointwise to u on Ω as $j \to \infty$. For $p \geq 1$, we say that

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- $u \in \mathcal{F}_p$ if

$$\sup_{j\geq 1} \int_{\Omega} ((-u_j)^p + 1)(dd^c u_j)^n < \infty;$$

- $u \in \mathcal{E}_p$ if

$$\sup_{j\geq 1} \int_{\Omega} (-u_j)^p (dd^c u_j)^n < \infty;$$

- $u \in \mathcal{F}$ if

$$\sup_{j\geq 1} \int_{\Omega} (dd^c u_j)^n < \infty.$$

The complex Monge–Ampère operator is well-defined on these classes. See e.g. [C2, C3, CZ1, Ko1] for further information about these energy classes.

THEOREM 2.1. *Let $p \geq 1$ and $n \geq 2$. Then there exists a constant $D(n,p) \geq 1$, depending only on n and p, such that for any $u_0, u_1, \dots, u_n$ in $\mathcal{E}_p$,*

$$\int_{\Omega} (-u_0)^p dd^c u_1 \wedge \dots \wedge dd^c u_n \\ \leq D(n,p) \Big(\int_{\Omega} (-u_0)^p (dd^c u_0)^n \Big)^{p/(n+p)} \cdots \Big(\int_{\Omega} (-u_n)^p (dd^c u_n)^n \Big)^{1/(n+p)}.$$

Furthermore, $D(n,1) = 1$ and $D(n,p) \geq 1$ for $p \neq 1$.

Proof. This is Theorem 3.4 in [P] (see also [AC1, ACP, C2, CP2]). ∎

It was proved in [AC1] (see also [AC2]) that for $p \neq 1$ the constant $D(n,p)$ in Theorem 2.1 is strictly greater than 1. The following variant of Theorem 2.1 is proved in [C3].

THEOREM 2.2. *Let $n \geq 2$. For any $u_0, u_1, \dots, u_n \in \mathcal{F}$,*

$$\int_{\Omega} (-u_0) dd^c u_1 \wedge \dots \wedge dd^c u_n \\ \leq \Big(\int_{\Omega} (-u_0)(dd^c u_1)^n \Big)^{1/n} \cdots \Big(\int_{\Omega} (-u_0)(dd^c u_n)^n \Big)^{1/n}.$$

3. Mean field equation. We consider $\mathcal{F}$ as a convex cone in $L^1(\Omega, dV)$. Theorems 2.1 and 2.2 show that for $u, v \in \mathcal{F}$,

$$\Big(\int_{\Omega} (dd^c(u+v))^n \Big)^{1/n} \leq \Big(\int_{\Omega} (dd^c u)^n \Big)^{1/n} + \Big(\int_{\Omega} (dd^c v)^n \Big)^{1/n},$$

and for $u, v \in \mathcal{E}_1$,

$$(3.1)\qquad \Big(\int_\Omega -(u+v)(dd^c(u+v))^n\Big)^{1/(n+1)} \le \Big(\int_\Omega -u(dd^c u)^n\Big)^{1/(n+1)} + \Big(\int_\Omega -v(dd^c v)^n\Big)^{1/(n+1)}.$$

To show (3.1), we use Theorem 2.1 to get

$$\begin{aligned}\int_\Omega -(u+v)(dd^c(u+v))^n &= \int_\Omega (-u)(dd^c(u+v))^n + \int_\Omega (-v)(dd^c(u+v))^n \\ &\le \Big(\int_\Omega -u(dd^c u)^n\Big)^{1/(n+1)} \Big(\int_\Omega -(u+v)(dd^c(u+v))^n\Big)^{n/(n+1)} \\ &\quad + \Big(\int_\Omega -v(dd^c v)^n\Big)^{1/(n+1)} \Big(\int_\Omega -(u+v)(dd^c(u+v))^n\Big)^{n/(n+1)} \\ &= \Big(\Big(\int_\Omega -u(dd^c u)^n\Big)^{1/(n+1)} + \Big(\int_\Omega -v(dd^c v)^n\Big)^{1/(n+1)}\Big) \\ &\quad \cdot \Big(\int_\Omega -(u+v)(dd^c(u+v))^n\Big)^{n/(n+1)}.\end{aligned}$$

Hence, $\mathcal{E}_1$ is convex. Define

$$\begin{aligned} A_q &= \Big\{u \in \mathcal{F} : \Big(\int_\Omega (dd^c u)^n\Big)^{1/n} \le q\Big\} \\ B_q &= \Big\{u \in \mathcal{E}_1 : \Big(\int_\Omega -u(dd^c u)^n\Big)^{1/(n+1)} \le q\Big\} \\ C_q &= \Big\{u \in \mathcal{F}_1 : \Big(\int_\Omega -(u+1)(dd^c u)^n\Big)^{1/(n+1)} \le q\Big\}. \end{aligned}$$

Then A_q and C_q are compact (by the Banach–Alaoglu theorem) and convex. However, B_q is convex but not compact (cf. [C2, Example 3.11]).

LEMMA 3.1. *Assume $u_j, u \in \mathcal{E}_1$ and $\sup \int_\Omega -u_j(dd^c u_j)^n < \infty$. If $u_j \to u$ as distributions when $j \to \infty$, then $u_j \to u$ in $L^1((dd^c w)^n)$ for every $w \in \mathcal{E}_1$.*

Proof. Let $m > 0$. Then

$$\begin{aligned}|u - u_j| &\le |u - \max(u, mw) + \max(u, mw) - \max(u_j, mw) + \max(u_j, mw) - u_j| \\ &\le \max(u, mw) - u + |\max(u, mw) - \max(u_j, w)| + \max(u_j, mw) - u_j,\end{aligned}$$

so

$$\int_\Omega |u-u_j|(dd^cw)^n \le \int_\Omega (\max(u,mw)-u)(dd^cw)^n$$
$$+\int_\Omega |\max(u,mw)-\max(u_j,w)|(dd^cw)^n + \int_\Omega (\max(u_j,mw)-u_j)(dd^cw)^n.$$

On the right hand side, when $m\to\infty$, the first integral tends to 0 by monotone convergence; the second tends to 0 as $j\to\infty$ by [CK, Lemma 1.4]. We use Theorems 4.1 below and 2.1 to estimate the third term:

$$\int_\Omega (\max(u_j,mw)-u_j)(dd^cw)^n$$
$$\le \int_{\{u_j<mw\}} -u_j(dd^cw)^n = \int_\Omega -u_j\chi_{\{u_j<mw\}}(dd^cw)^n$$
$$\le \Big(\int_\Omega -u_j(dd^cu_j)^n\Big)^{1/(n+1)}\Big(\int_\Omega -w\frac{u_j}{mw}(dd^cw)^n\Big)^{n/(n+1)} \le \frac{\text{const}}{m^{n/(n+1)}}\to 0$$

as $m\to\infty$. ∎

Remark 3.2. In connection with Lemma 3.1 recall that it may happen that

$$\int_\Omega -u_j(dd^cu_j)^n = 1$$

but $u_j\to 0$ as distributions when $j\to\infty$.

Theorem 3.3 ([CP1]). *There exists a constant C such that*

$$\sup|u-v| \le C\|g-h\|_2^{1/n}$$

for all $u,v\in\mathcal{F}$, where $(dd^cu)^n = gdV, (dd^cv)^n = hdV$ and $f,g\in L^2(dV)$.

Theorem 3.4 ([ACKPZ]). *There exists a constant $a_n>0$, depending only on n, such that for any $0\le\mu<n$ and $u\in\mathcal{F}_1$ such that $\int_\Omega(dd^cu)^n\le\mu^n$, we have*

$$\int_\Omega e^{-2u}\,dV \le \Big(\pi^n + a_n\frac{\mu}{(n-\mu)^n}\Big)\delta_\Omega^{2n}, \tag{3.2}$$

where δ_Ω is the diameter of Ω.

Theorem 3.5. *For every $b>1/(2n)^n$ there exists a constant $B>0$ such that*

$$\int_\Omega \exp(-u)\,dV \le B\exp\Big(b\int_\Omega(-u)(dd^cu)^n\Big) \quad \textit{for all } u\in\mathcal{E}_1.$$

Proof. Set

$$a = \int_\Omega (-u)(dd^cu)^n.$$

Then

$$\begin{aligned}(dd^c u)^n &= \chi_{\{u>-ab\}}(dd^c u)^n + \chi_{\{u\le -ab\}}(dd^c u)^n\\ &= \chi_{\{u>-ab\}}(dd^c \max(u,-ab))^n + \chi_{\{u\le -ab\}}(dd^c u)^n\\ &\le (dd^c \max(u,-ab))^n + \chi_{\{u\le -ab\}}(dd^c u)^n.\end{aligned}$$

Solve $(dd^c w)^n = \chi_{\{u\le -ab\}}(dd^c u)^n$ for $w\in\mathcal{F}$. By [C2, Theorem 4.5],

$$u \ge \max(u,-ab) + w,$$

and since

$$\int_\Omega (dd^c w)^n \le \frac{a}{ab} < (2n)^n$$

it follows from Theorem 3.4 that there exists a constant $B>0$ such that

$$\int_\Omega \exp(-u)dV \le B\exp(ab)$$

and the proof is complete. ∎

Remark 3.6. A stronger version of Theorem 3.5 is proved in [BB].

Recall Schauder's fixed point theorem [S] in our setting: Suppose A is a convex and compact subset of $\mathcal{F}$. If $T: A\to A$ is a continuous map, then there is $u\in A$ with $u=T(u)$.

Theorem 3.7. *For every $k<(2n)^n$ there is a function $u\in\mathcal{E}_0\cap\mathcal{C}$ with*

$$(dd^c u)^n = \frac{ke^{-u}\,dV}{\int_\Omega e^{-u}\,dV}.$$

Proof. We define

$$A = \Big\{u\in\mathcal{F} : \int_\Omega (dd^c u)^n \le k\Big\}.$$

Note that using Theorems 3.3 and 3.4 we can consider the map $A\ni u\mapsto T(u)\in A$ where $T(u)$ is the unique function in $\mathcal{F}$ with

$$(dd^c T(u))^n = \frac{ke^{-u}\,dV}{\int_\Omega e^{-u}\,dV}.$$

Choose m such that $\big(\frac{m}{m-1}\big)^n k < (2n)^n$. By Theorem 3.4, there is a constant c such that

$$\begin{aligned}\int_\Omega -T(u)(dd^c T(u))^n &= \frac{k\int_\Omega -T(u)e^{-u}\,dV}{\int_\Omega e^{-u}\,dV}\\ &\le k\Big(\int_\Omega (-T(u))^m\,dV\Big)^{1/m}\Big(\int_\Omega e^{-\frac{m}{m-1}u}\,dV\Big)^{(m-1)/m} \le (m!)^{1/m}kc\end{aligned}$$

for all $u\in A$. Hence $T(u)\in\mathcal{F}_1$, and Theorems 3.3 and 3.4 now imply that $T(u)\in\mathcal{E}_0\cap\mathcal{C}$.

It remains to show that T is continuous on A. It is enough by Theorem 3.3 to prove that if $u_j, u \in A$ and $u_j \to u$ as distributions when $j \to \infty$, then

$$\frac{e^{-u_j}}{\int_\Omega e^{-u_j}\, dV} \to \frac{e^{-u}}{\int_\Omega e^{-u} dV} \quad \text{in } L^2(dV) \text{ as } j \to \infty.$$

Choose $t, p > 1$ so that $kt^n < n^n$ and $1/t + 1/p = 1$ and define $w_j = (\sup_{k\geq j} u_k)^*$. Now,

$$\int_\Omega |e^{-u_j} - e^{-u}|^2\, dV \leq 2\int_\Omega |e^{-u_j} - e^{-w_j}|^2 dV + 2\int_\Omega |e^{-u} - e^{-w_j}|^2\, dV$$

and

$$\begin{aligned}\int_\Omega |e^{-u_j} - e^{-w_j}|^2\, dV &= \int_\Omega e^{-2u_j}|1 - e^{u_j - w_j}|^2\, dV \\ &\leq \Big(\int_\Omega e^{-2tu_j}\, dV\Big)^{1/t}\Big(\int_\Omega |1 - e^{u_j-w_j}|^{2p}\, dV\Big)^{1/p} \\ &\leq C\Big(2p\int_\Omega (w_j - u_j)\, dV\Big)^{1/p} \to 0 \quad \text{as } j \to \infty\end{aligned}$$

by Lemma 3.1. The constant C can be estimated using Theorem 3.5. ∎

4. Dirichlet's problem. In this section we study $\mathcal{F}_1$. We say that a non-negative Radon measure μ belongs to $\mathcal{M}_1$ if there exists a constant A such that

$$\int_\Omega (-u)\, d\mu \leq A\Big(\int_\Omega (-u)(dd^c u)^n\Big)^{1/(n+1)} \quad \text{for all } u \in \mathcal{E}_0.$$

Note that $\mathcal{M}_1$ with fixed A is convex, and $\mu(P)$ is zero for every pluripolar P and every $\mu \in \mathcal{M}_1$. The next theorem is a special case of a theorem in [C2].

THEOREM 4.1. *Let μ be a non-negative Radon measure. Then the following conditions are equivalent:*

(1) *there exists a function $u \in \mathcal{F}_1$ such that $(dd^c u)^n = \mu$;*
(2) *$\mu \in \mathcal{M}_1$.*

Theorem 4.1 gives a complete characterization of measures for which there exists a solution of the Dirichlet problem for the complex Monge–Ampère operator in the class $\mathcal{F}_1$. The solutions of the Dirichlet problem in Theorem 4.1 are always unique (see [C2, Theorem 4.5]).

Proof of Theorem 4.1. It follows from Theorem 2.2 that (1) implies (2). To prove the opposite implication, we construct a convergent sequence.

It is no loss of generality to assume that μ has compact support. So let $\mu \in \mathcal{M}_1$ with compact support. Let Q_j be a standard smoothing kernel (see e.g. [K]) and put $\mu_j = Q_j * \mu$, which is a well-defined non-negative compactly supported smooth function. Solve, using [BT1], $(dd^c u_j)^n = \mu_j$ for $u_j \in \mathcal{E}_0$. We show that this sequence converges to the solution of the Dirichlet problem, i.e.

$$u = \lim_{j\to\infty} \Big(\sup_{k\geq j} u_k\Big)^* \in \mathcal{F}_1 \quad \text{and} \quad (dd^c u)^n = \mu.$$

Choose a weak*-convergent subsequence, again denoted by u_j, converging weak* to u. Then by the construction of u_j we have

$$\int_\Omega -u_j (dd^c u_j)^n \leq \int_\Omega -u_j \, d\mu \leq A\Big(\int_\Omega (-u_j)(dd^c u_j)^n\Big)^{1/(n+1)},$$

so it follows from integration by parts that

$$\int_\Omega -\Big(\sup_{k\geq j} u_k\Big)^* \Big(dd^c \Big(\sup_{k\geq j} u_k\Big)^*\Big)^n \leq \int_\Omega -u_j (dd^c u_j)^n \leq A^{(n+1)/n}.$$

Note that integration by parts also gives the inequality

$$\int_\Omega v \, d\mu \leq \int v (dd^c u)^n$$

for all negative plurisubharmonic functions v (see [C4]). Theorem 2.1 gives

$$\begin{aligned}\int_\Omega -u \, d\mu &= \lim_{j\to\infty} \int_\Omega -u (dd^c u_j)^n \\ &\leq \Big(\int_\Omega -u (dd^c u)^n\Big)^{1/(n+1)} \lim_{j\to\infty} \Big(\int_\Omega -u_j (dd^c u_j)^n\Big)^{n/(n+1)}.\end{aligned}$$

We will show that

$$\lim_{j\to\infty} \int_\Omega -u_j (dd^c u_j)^n \leq \int_\Omega -u \, d\mu,$$

which implies that $\int_\Omega u \, d\mu = \int_\Omega u (dd^c u)^n$. Let $j \leq k$. Then

$$\begin{aligned}\int_\Omega -u_j (dd^c u_j)^n &\leq \int_\Omega -u_j (dd^c u_k)^n \\ &\leq \Big(\int_\Omega -u_j (dd^c u_j)^n\Big)^{1/(n+1)} \Big(\int_\Omega -u_k (dd^c u_k)^n\Big)^{n/(n+1)}\end{aligned}$$

so $\int_\Omega -u_j (dd^c u_j)^n$ is increasing to some $c < \infty$. Theorem 2.2 gives

$$\begin{aligned}\int_\Omega -u_j (dd^c u_j)^n &\leq \int_\Omega -u_k dd^c u_j \wedge (dd^c u_k)^{n-1} \\ &\leq \Big(\int_\Omega -u_k (dd^c u_j)^n\Big)^{1/n} \Big(\int_\Omega -u_k (dd^c u_k)^n\Big)^{(n-1)/n}.\end{aligned}$$

Hence,

$$\int_\Omega -u_j(dd^cu_j)^n \le \Big(\int_\Omega -u_k(dd^cu_j)^n\Big)^{1/n} c^{(n-1)/n}.$$

Now

$$\lim_{k\to\infty}\int_\Omega -u_k(dd^cu_j)^n = \int_\Omega Q_j * u\,d\mu = \int_\Omega -u(dd^cu_j)^n$$

since $Q_j * u_k$ tends uniformly to $Q_j * u$ on the support of μ as $k\to\infty$ and we have $\int_\Omega u\,d\mu = \int_\Omega u(dd^cu)^n$ since $\lim_{j\to\infty}\int_\Omega -u(dd^cu_j)^n = \int_\Omega -u\,d\mu$ by construction.

Let now $v \in \mathcal{E}_0$ be given. For $t \ge 0$ we have

$$\begin{aligned}\int_\Omega -(u+tv)\mu &= \lim_{j\to\infty}\int_\Omega -(u+tv)(dd^cu_j)^n\\ &\le \Big(\int_\Omega -(u+tv)(dd^cu+tv)^n\Big)^{1/(n+1)} \lim_{j\to\infty}\Big(\int_\Omega -u_j(dd^cu_j)^n\Big)^{n/(n+1)}\\ &= \Big(\int_\Omega -(u+tv)(dd^cu+tv)^n\Big)^{1/(n+1)} \Big(\int_\Omega -u(dd^cu)^n\Big)^{n/(n+1)}\\ &\le \frac{1}{n+1}\int_\Omega -(u+tv)(dd^cu+tv)^n + \frac{n}{n+1}\int_\Omega -u(dd^cu)^n.\end{aligned}$$

This is an inequality between two polynomials in t. The polynomials are equal at $t=0$ so the coefficients for their first order terms satisfy the same inequality. The coefficient for the left hand side is $\int_\Omega -v\,d\mu$ and for the right hand side $\int_\Omega -v(dd^cu)^n$. Therefore $\int_\Omega -v\,d\mu \le \int_\Omega -v(dd^cu)^n$ for every $v\in\mathcal{E}_0$ and since we already know the opposite inequality we have proved that $\mu = (dd^cu)^n$, by [C3]. The function u is uniquely determined, so the original sequence was already weak*-convergent to u. ∎

Remark 4.2. It follows from Rainwater's lemma [R] that every probability measure μ can be decomposed as $\mu = \mu_1 + \mu_2$, where $\mu_1 = f(dd^cv)^n$ for a $v \in \mathcal{F}$ so the solutions u_j to $(dd^cu_j)^n = \min(f,j)(dd^cv)^n$ decrease to u as $j\to\infty$, and μ_2 is concentrated on a pluripolar set.

Remark 4.3. Let μ be a positive measure with $\mu(\Omega) < \infty$ and $\mu(P) = 0$ for every pluripolar set P. Then there is a unique function $u \in \mathcal{F}$ with $(dd^cu)^n = \mu$. It is interesting to note that by [CZ2] the equation $-w(dd^cw)^n = (dd^cu)^n$ has a unique solution $w \in \mathcal{E}_1$.

In this paper, we have only considered measures that had no mass on a pluripolar set. What about measures with pluripolar support?

5. The variational method. We end this paper with a quick look at a classical method to solve the Dirichlet problem. See [ACC], [BB], [BBGZ] or [GKY] and [C2, C3] for calculations.

Let $u, v \in \mathcal{E}_1$, and assume that v is continuous. For $t < 0$, put

$$P(u+tv) = \sup\{w \in \mathcal{E}_1 : w \le u + tv\}.$$

Then $P(u+tv) \in \mathcal{E}_1$ (see [ACC]). Write

$$e(u) = \int_\Omega -u(dd^c u)^n,$$

and let $J : \mathcal{E}_1 \to \mathbb{R}$ be a continuous functional on $\mathcal{E}_1$. Put

$$F(u) = \frac{1}{n+1} e(u) + J(u).$$

If

$$\liminf_{t\to 0^-} \frac{J(P(u+tv)) - J(u)}{t} \ge \liminf_{t\to 0^+} \frac{J(u+tv) - J(u)}{t}$$

for all $v \in \mathcal{E}_0 \cap C$ and if u_m is a minimum point of F, then

$$\int_\Omega -v(dd^c u_m)^n + J'(u_m + tv)|_{t=0} = 0 \quad \text{for all } v \in \mathcal{E}_0 \cap C.$$

EXAMPLE 5.1. Take $J(u) = -\log \int_\Omega e^{-u}\, dV$. Then J is defined on $\mathcal{E}_1$ and Theorem 3.3 shows that we only have to minimize over a compact convex subset. A calculation shows that

$$\liminf_{t\to 0^-} \frac{J(P(u+tv)) - J(u)}{t} \ge \lim_{t\to 0^+} \frac{J(u+tv) - J(u)}{t} = \frac{\int ve^{-u} dV}{\int e^{-u} dV}$$

for all $v \in \mathcal{E}_0 \cap C$, so

$$-(dd^c u_m)^n + \frac{e^{-u_m}\, dV}{\int e^{-u_m}\, dV} = 0.$$

Let $v \in \mathcal{E}_0 \cap \mathcal{C}$. Then for $t < 0$ we have

$$\begin{aligned}
\frac{J(P(u+tv)) - J(u)}{t} &= \frac{\log \frac{\int_\Omega e^{-P(u+tv)}\, dV}{\int_\Omega e^{-u}\, dV}}{-t} = \frac{\log\left(1 + \frac{\int_\Omega (e^{-P(u+tv)} - e^{-u}\, dV}{\int_\Omega e^{-u})\, dV}\right)}{-t} \\
&\ge \frac{\int_\Omega (e^{-P(u+tv)} - e^{-u})\, dV}{\int_\Omega e^{-u}\, dV} = \frac{\int_\Omega e^{-u}(e^{-P(u+tv)+u} - 1)\, dV}{\int_\Omega e^{-u}\, dV} \\
&\ge \frac{\log\left(1 + \frac{\int_\Omega e^{-u}(-P(u+tv)+u)\, dV}{\int_\Omega e^{-u}\, dV}\right)}{-t} \\
&= \frac{\log\left(1 + \frac{\int_\Omega e^{-u}(-P(u+tv)+u+tv)\, dV}{\int_\Omega e^{-u}\, dV} - \frac{\int_\Omega e^{-u} tv\, dV}{\int_\Omega e^{-u}\, dV}\right)}{-t} \\
&\ge \frac{\log\left(1 - \frac{\int_\Omega e^{-u} tv\, dV}{\int_\Omega e^{-u}\, dV}\right)}{-t} \to \frac{\int_\Omega e^{-u} v\, dV}{\int_\Omega e^{-u}\, dV} \quad \text{as } t \to 0. \ \blacksquare
\end{aligned}$$

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