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SPLIT REGULAR HOM-LEIBNIZ COLOR 3-ALGEBRAS

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Abstract. We introduce and describe the class of split regular Hom-Leibniz color 3-algebras as the natural extension of the class of split Lie algebras and superalgebras.

More precisely, we show that any such split regular Hom-Leibniz color 3-algebra T is of the form $T = \mathcal{U} + \sum_j I_j$ with $\mathcal{U}$ a subspace of the 0-root space T_0, and I_j an ideal of T such that for $j \neq k$,

$$[T, I_j, I_k] + [I_j, T, I_k] + [I_j, I_k, T] = 0.$$

Moreover, if T is of maximal length, we characterize the simplicity of T in terms of a connectivity property in its set of non-zero roots.

1. Introduction. n-Ary algebras have been applied in mathematical physics, in the study of supersymmetry, Bagger–Lambert theory or Nambu mechanics (see, for example, [AI10, T94]). On the other hand, there is a purely algebraic interest in the study of ternary algebras [CK10, CK14, K141, KP18, P09] and more generally n-ary Leibniz algebras [CL02, CI07, P03] and general n-ary algebras [K142, KP16].

In this paper we study the class of Hom-Leibniz color 3-algebras (of which the classes of Hom-Lie color 3-algebras and Hom-Lie triple color systems are subclasses) and Leibniz color 3-algebras with an automorphism.

The study of Hom-structures began in the paper of Hartwig, Larsson and Silvestrov [HL06]. The notion of Hom-Lie triple systems was introduced in [MS08]. Yau [Y09] gave a general method of constructing Hom-type algebras starting from usual algebras and a twisting self-map. For information on various types of Hom-algebras see [AM09, B17, BK19, MS08, SC15, WZ18, ZL17, GZ16, GW16, MS09, AE11, AS91, LS05, Y08].

In the present paper we study the structure of split regular Hom-Leibniz 3-algebras of arbitrary dimension and over an arbitrary base field K. Split structures appeared first in the classical theory of (finite-dimensional) Lie algebras but have been extended to more general settings like, for example,

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Leibniz algebras [CS12], Poisson algebras, Leibniz superalgebras [C131], regular Hom-Lie algebras [AC15], regular Hom-Lie superalgebras [AB18], regular Hom-Lie color algebras [C172], regular Hom-Poisson algebras [AC16], regular Hom-Leibniz algebras [CC18], regular BiHom-Lie algebras [CS16], regular BiHom-Lie superalgebras [ZC18]. As for the study of split ternary structures, see [C09, CF09] for Lie triple systems and twisted inner derivation triple systems, [CF11] for Lie 3-algebras, [CS17] for Leibniz 3-algebras and [CC15, C171] for Leibniz triple systems.

This paper is organized as follows. In Section 2 we recall the main definitions and results related to 3-algebras. For T a Hom-Leibniz color 3-algebra or a Leibniz color 3-algebra with an automorphism we construct its multiplication algebra $\mathfrak{L}$, which is a Lie color algebra with an automorphism, and its standard envelope, in terms of which we give the definition of a split Hom-Leibniz color 3-algebra and of a split Leibniz color 3-algebra with an automorphism, and show how these concepts are related.

In Section 3 we develop root connection techniques in this framework and apply it to get certain decomposition theorems for the algebras above (for root connection techniques see more in [BK18, BS19, CK18] and references therein). We prove that an arbitrary split regular Hom-Leibniz color 3-algebra T can be decomposed as $T = \mathcal{U} + \sum_j I_j$ where $\mathcal{U}$ is a subspace of the 0-root space T_0 and the I_j are ideals of T satisfying

$$[T, I_j, I_k] + [I_j, T, I_k] + [I_j, I_k, T] = 0 \quad \text{for } j \neq k.$$

Finally, in Section 4 we see how the above decomposition of T can be used to obtain a similar decomposition of $\mathfrak{L}$.

Remark. Note that this paper is the first attempt to extend the theory of split algebras to n-ary Hom-algebras ($n > 2$). Until now, only split Hom-algebras and split 3-algebras have been studied.

2. Preliminaries

2.1. Color algebras. In this subsection we recall the basic notions related to color algebras and maps on them.

Definition 2.1. Let $\mathbb{K}$ be a field and $\mathbb{G}$ be an abelian group. A map $\epsilon : \mathbb{G} \times \mathbb{G} \to \mathbb{K}^\times$ is called a *bicharacter* on $\mathbb{G}$ if for all $f, g, h \in \mathbb{G}$:

(1) $\epsilon(f, g + h) = \epsilon(f, g)\epsilon(f, h)$;
(2) $\epsilon(g + h, f) = \epsilon(g, f)\epsilon(h, f)$;
(3) $\epsilon(g, h)\epsilon(h, g) = 1$.

Definition 2.2. A $\mathbb{G}$-*graded color n-ary algebra* T is a vector space $T = \bigoplus_{g\in\mathbb{G}} T_g$ with an n-linear map $[\,\cdot\,, \dots, \cdot\,] : T \times \cdots \times T \to T$ satisfying

$$[T_{\theta_1}, \dots, T_{\theta_n}] \subseteq T_{\theta_1+\cdots+\theta_n}, \quad \theta_i \in \mathbb{G}.$$

Let $T = \bigoplus_{g\in\mathbb{G}} T_g$ be a color algebra. An element x is called *homogeneous of degree* $t \in \mathbb{G}$ if $x \in T_t$. We then write $\mathrm{hg}(x) = t$. The $\mathbb{G}$-grading on T induces a $\mathbb{G}$-grading on the space of linear maps on T: a linear map D on T is homogeneous of degree t, written $\mathrm{hg}(D) = t$, if $D(T_g) \subseteq T_{g+t}$ for all $g \in \mathbb{G}$; we then write. If D is homogeneous of degree $t = 0$ then D is said to be *even*. From now on, unless otherwise stated, we assume that all elements and maps are homogeneous.

Let ϵ be a bicharacter on $\mathbb{G}$. For homogeneous elements a and b we set $\epsilon(a, b) := \epsilon(\mathrm{hg}(a), \mathrm{hg}(b))$.

2.2. Color Hom**-algebras.** In this subsection we consider Hom-algebraic structures that are twisted versions of the original algebraic structures, and linear maps on them.

Definition 2.3. A *Hom-algebra* is an algebra A with a fixed linear map $\phi : A \to A$. A homomorphism of two Hom-algebras (A, ϕ) and (B, ψ) is an algebra homomorphism $\varphi : A \to B$ such that $\varphi\phi = \psi\varphi$.

Remark 2.4. The most general definition says that an n-ary Hom-algebra is an n-ary algebra A equipped with a certain family $\{\phi_i\}_{i\in I}$ of linear maps. However, it is hard to work in such a general context. Therefore, in this paper we will only consider the case where the maps ϕ_i are all equal to one map ϕ which is also a homomorphism of the underlying algebra structure of A. Such algebras are called *multiplicative n-ary Hom-algebras*. From now on we will work only with multiplicative Hom-algebras (the exact formulations will be given later). Also from now on we will assume that ϕ is an automorphism of A. In this case A is called a regular Hom-algebra.

The definitions of color and Hom-algebras can be unified into one:

Definition 2.5. A *color Hom-algebra* A is a color algebra (A, ϵ) with a fixed even linear map ϕ. A homomorphism of two color Hom-algebras $(A, \mathbb{G}, \epsilon_A, \phi)$ and $(B, \mathbb{G}, \epsilon_B, \psi)$ is an even map $\varphi : A \to B$ that is an algebra homomorphism and $\varphi\phi = \psi\varphi$.

2.3. Color Hom-Ω **algebras.** In this subsection we give definitions and study the basic properties of Hom-Leibniz color (binary and ternary) algebras.

First we briefly discuss color Hom-algebras defined by identities. Given a variety Ω of (n-ary) algebras defined by a family $\{f_i\}_{i\in I}$ of identities we may obtain the identities $\{(f_i)^{\mathrm{Hom}}_{\mathrm{co}}\}_{i\in I}$ of the corresponding variety of color Hom-Ω algebras by twisting the defining identities by homomorphisms, usually called the twisting maps. The details can be found in [BK19, CM15].

As an example, we give the definitions of Hom-Leibniz color (3-)algebras, and their important subclasses. As usual, let $\mathbb{G}$ be an abelian group and ϵ be a bicharacter on $\mathbb{G}$. From now on we only work with regular algebras.

DEFINITION 2.6. A *regular Hom-Leibniz color algebra* $(L, [\cdot,\cdot], \epsilon, \phi)$ is a $\mathbb{G}$-graded vector space L with even multiplication $[\cdot,\cdot]$ and an even automorphism ϕ satisfying

$$[\phi(x), [y, z]] = [[x, y], \phi(z)] + \epsilon(x, y)[\phi(y), [x, z]].$$

If additionally the identity

$$[x, y] = -\epsilon(x, y)[y, x]$$

holds in L (that is, multiplication in T is color-anticommutative), then L is called a *regular Hom-Lie color algebra.*

DEFINITION 2.7. A *regular Hom-Leibniz color* 3*-algebra* $(T, [\cdot,\cdot], \epsilon, \phi)$ is a $\mathbb{G}$-graded vector space T with a bicharacter ϵ, an even trilinear map $[\cdot,\cdot,\cdot]$ and an even automorphism ϕ satisfying

$$\begin{aligned}(2.1)\qquad &[\phi(x_1), \phi(x_2), [y_1, y_2, y_3]]\\ &= [[x_1, x_2, y_1], \phi(y_2), \phi(y_3)] + \epsilon(x_1 + x_2, y_1)[\phi(y_1), [x_1, x_2, y_2], \phi(y_3)]\\ &\quad + \epsilon(x_1 + x_2, y_1 + y_2)[\phi(y_1), \phi(y_2), [x_1, x_2, y_3]].\end{aligned}$$

If additionally the identities

$$[x_1, x_2, x_3] = -\epsilon(x_1, x_2)[x_2, x_1, x_3], \qquad [x_1, x_2, x_3] = -\epsilon(x_2, x_3)[x_1, x_3, x_2]$$

hold in T, then T is called a *regular Hom-Lie color 3-algebra*, and if the identities

$$[x_1, x_2, x_3] = -\epsilon(x_1, x_2)[x_2, x_1, x_3],$$
$$\epsilon(x_3, x_1)[x_1, x_2, x_3] + \epsilon(x_1, x_2)[x_2, x_3, x_1] + \epsilon(x_2, x_3)[x_3, x_1, x_2] = 0$$

hold in T, then T is called a *regular Hom-Lie triple color system.*

REMARK 2.8. In particular, if $\mathbb{G} = \mathbb{Z}_2$, we get the definition of a *Hom-Leibniz* (3-)*superalgebra*, and if $\mathbb{G}$ is the trivial group $\{1\}$ then we get a *Hom-Leibniz* (3-)*algebra*. If ϕ is the identity automorphism on T then we get the definition of a Leibniz color 3-algebra, as the defining identity turns into

$$\begin{aligned}(2.2)\qquad &[x_1, x_2, [y_1, y_2, y_3]] = [[x_1, x_2, y_1], y_2, y_3]\\ &+ \epsilon(x_1 + x_2, y_1)[y_1, [x_1, x_2, y_2], y_3] + \epsilon(x_1 + x_2, y_1 + y_2)[y_1, y_2, [x_1, x_2, y_3]].\end{aligned}$$

We recall that (2.2) is known as the *Leibniz identity.*

As an example, let T be any $\mathbb{G}$-graded space. We denote by $\mathrm{End}(T)$ the set of all linear maps on T. One easily checks that $\mathrm{End}(T)$ with the grading $\mathrm{End}(T) = \bigoplus_{g\in\mathbb{G}} \mathrm{End}_g(T)$ endowed with the color bracket

$$(2.3)\qquad [D_1, D_2] = D_1 D_2 - \epsilon(D_1, D_2) D_2 D_1$$

is a color Lie algebra.

Definition 2.9. Let T be a Hom-Leibniz color 3-algebra. An *ideal* of T is a ϕ-invariant subspace I such that

$$[I,T,T]+[T,I,T]+[T,T,I]\subseteq I.$$

The *annihilator* of T is the set

$$(T)=\{x\in T:[x,T,T]+[T,x,T]+[T,T,x]=0\}.$$

It is straightforward to check that (T) is an ideal of T.

So, our main objects will be regular Hom-Leibniz color 3-algebras. Any such algebra is given by the following data: $T=(T,[\cdot,\cdot,\cdot],\epsilon,\phi)$, where ϵ is a $\mathbb{G}$-bicharacter on T and ϕ is an automorphism such that (2.1) holds.

We also consider the category of Leibniz color 3-algebras with automorphisms. Any such algebra is given as $T=(T,[\cdot,\cdot,\cdot],\epsilon,\phi)$, where ϵ is a $\mathbb{G}$-bicharacter on T such that (2.2) holds and ϕ is an automorphism of T. The morphisms in this category are the maps respecting the algebra structure and grading, and commuting with automorphisms.

Note that the latter category is not a subcategory of the former: the defining relations are different! However, these categories are isomorphic. Indeed, given a Leibniz color 3-algebra $T=(T,[\cdot,\cdot,\cdot],\epsilon,\phi)$ with an automorphism, consider an algebra $T^{\phi}=(T,[\cdot,\cdot,\cdot]^{\phi},\phi,\epsilon)$, where

$$[x,y,z]^{\phi}=\phi([x,y,z]).$$

One can check that T^{ϕ} is a regular Hom-Leibniz color 3-algebra. Moreover, this construction is universal in the following sense: if $T=(T,[\cdot,\cdot,\cdot],\phi,\epsilon)$ is a regular Hom-Leibniz color 3-algebra, then one easily checks that $T^{\phi^{-1}}=(T,[\cdot,\cdot,\cdot]^{\phi^{-1}},\epsilon)$, where

$$[x,y,z]^{\phi^{-1}}=\phi^{-1}([x,y,z]),$$

is a Leibniz color 3-algebra with the automorphism ϕ, and this construction is clearly inverse to the one above. Hence, the category of regular Hom-Leibniz color 3-algebras is isomorphic to the category of Leibniz color 3-algebras with an automorphism. Occasionally, we will refer to the passage $T\to T^{\phi^{-1}}$ as *dehomification*.

The ideal J generated by the set

$$\big\{[x,y,z]+\epsilon(x,y)[y,x,z],\ [x,y,z]+\epsilon(y,z)[x,z,y]:x,y,z\in T\big\}$$

plays an important role in the study of Hom-Leibniz color 3-algebras, since T is a Hom-Lie color 3-algebra if and only if $J=\{0\}$.

Following the ideas of Abdykassymova and Dzhumadil'daev [AD01] for Leibniz algebras, and of Cao and Chen [CC15, C171] for Leibniz triple systems, we introduce the following notion.

Definition 2.10. We say that (T, ϕ) is a *simple Hom-Leibniz color 3-algebra* if its triple product is non-zero and its only ϕ-invariant ideals are $\{0\}$, J and T.

Note that this definition is consistent with the notion of a simple Hom-Lie 3-algebra since $J = \{0\}$ in that case.

2.4. Operator algebras. In this subsection, inspired by the ideas of [V17], we associate with a Hom-Leibniz color 3-algebra T a certain Lie color algebra $\mathfrak{L}$ of "derivations" of T, and analogously for a Leibniz color 3-algebra with an automorphism. This construction will be important in the following discussion.

Let $T = (T, [\cdot,\cdot,\cdot], \epsilon, \phi)$ be a Hom-Leibniz color 3-algebra. Consider the space $\mathfrak{L} = \mathrm{span}\{(x, y) : x, y \in T\}$, where $(x, y)(z) := [x, y, z]$, and let $\ell \in \mathfrak{L}$. Then the identity (2.1) implies that

$$(\ell \circ \phi^{-1})[y_1, y_2, y_3] = \tag{2.4}$$

$$[(\phi^{-1}\circ\ell)y_1, y_2, y_3] + \epsilon(\ell, y_1)[y_1, (\phi^{-1}\circ\ell)y_2, y_3] + \epsilon(\ell, y_1+y_2)[y_1, y_2, (\phi^{-1}\circ\ell)y_3].$$

Consider the algebra $\mathrm{End}(T)$ with multiplication

$$[D_1, D_2]_{\phi^{-1}} = D_1\phi^{-1}D_2 - \epsilon(D_1, D_2)D_2\phi^{-1}D_1.$$

This algebra is called the ϕ^{-1}-*homotope* of $\mathrm{End}(T)$, denoted by $\mathrm{End}(T)_{\phi^{-1}}$ and is in fact isomorphic to $\mathrm{End}(T)$ via the mapping φ sending $\ell \in \mathrm{End}(T)$ to $\phi^{-1}\ell$. In particular, $\mathrm{End}(T)_{\phi^{-1}}$ is a Lie color algebra.

Relation (2.4) implies that $\mathfrak{L}$ is a subalgebra of $\mathrm{End}(T)_{\phi^{-1}}$. In particular, multiplication in $\mathfrak{L}$ is given by

$$[(a, b),\ (c, d)] = ([a, b, c]^{\phi^{-1}}, d) + \epsilon(a + b, c)(c, [a, b, d]^{\phi^{-1}}). \tag{2.5}$$

Note that conjugation by ϕ remains an automorphism in $\mathrm{End}(T)_{\phi^{-1}}$:

$$[\phi D_1\phi^{-1}, \phi D_2\phi^{-1}]_{\phi^{-1}} = \phi[D_1, D_2]_{\phi^{-1}}\phi^{-1}.$$

Since ϕ is an automorphism of T, one can see that $\mathfrak{L}$ is invariant under this automorphism:

$$\phi(x, y)\phi^{-1} = (\phi(x), \phi(y)).$$

The algebra $\mathfrak{L}$ also has a $\mathbb{G}$-grading induced by the $\mathbb{G}$-grading of T. Hence, $(\mathfrak{L}, [\cdot,\cdot]_{\phi^{-1}}, \epsilon, \phi \cdot \phi^{-1})$ is a Lie color algebra with an automorphism. This algebra is called the *multiplication algebra* of T.

Note that since every element of $\mathfrak{L}$ is of the form $\sum(x_i, y_i)$, we have

$$[\ell, (a, b)] = (\phi^{-1}\ell a, b) + \epsilon(\ell, a)(a, \phi^{-1}\ell b) \tag{2.6}$$

for any $a, b \in T$ and $\ell \in \mathfrak{L}$.

Now we consider an analogous construction for a Leibniz color 3-algebra $T = (T, [\cdot,\cdot,\cdot], \epsilon, \phi)$ with an automorphism. The difference here is that the construction almost does not depend on the automorphism ϕ, so we may suppose that we are doing it for a Hom-Leibniz color 3-algebra with a trivial automorphism $(T, [\cdot,\cdot,\cdot], \epsilon, \mathrm{id})$, thus it is a particular case of the construction above. The only part of the new construction in which ϕ participates explicitly is the construction of the automorphism on $\mathfrak{L}$.

Recall that a linear mapping D on T is called a *derivation* if

$$D[a,b,c] = [D(a),b,c] + \epsilon(D,a)[a,D(b),c] + \epsilon(D,a+b)[a,b,D(c)]$$

for all $a,b,c \in T$. The identity (2.2) then says that the left multiplication operators (x,y), $x,y \in T$, are derivations.

Moreover, in this case the homotopy above is trivial and the bracket in $\mathfrak{L}$ coincides with the bracket (2.3):

$$\begin{aligned}[(a,b),(c,d)] &= (a,b)\circ(c,d) - \epsilon(a+b,c+d)(c,d)\circ(a,b) \\ &= ([a,b,c],d) + \epsilon(a+b,c)(c,[a,b,d]).\end{aligned} \tag{2.7}$$

Analogously to the construction above, $\mathfrak{L}$ has a $\mathbb{G}$-grading induced by $\mathbb{G}$-grading of T and conjugation by ϕ induces an automorphism in $\mathfrak{L}$. Thus, in this case $(\mathfrak{L}, [\cdot,\cdot], \epsilon, \phi\cdot\phi^{-1})$ is a Lie color algebra with an automorphism. Yet again we call it the *multiplication algebra* of T.

In this case the identity (2.6) takes the following simplified form:

$$[\ell,(a,b)] = (\ell a, b) + \epsilon(\ell,a)(a,\ell b) \tag{2.8}$$

for any $a,b \in T$ and $\ell \in \mathfrak{L}$.

2.5. The standard embedding. This subsection is inspired by [V17]. Here, for T a regular Hom-Leibniz color 3-algebra or a Leibniz color 3-algebra with an automorphism we extend the algebra $\mathfrak{L}$ constructed above by the space isomorphic to T and obtain a larger color (but not necessarily color Lie) 2-graded algebra $\mathcal{A}$ containing T in a certain sense. This algebra is called the standard envelope of T and plays a crucial role in our discussion. Finally, we prove that the standard envelopes of the algebra T and of its dehomification $T^{\phi^{-1}}$ are isomorphic, explicitly constructing an isomorphism between them.

Recall that an algebra A is said to be 2-*graded* if there exist linear subspaces A^0 and A^1 of A, called the *even part* and the *odd part* respectively, such that $A = A^0 \oplus A^1$ and $A^\alpha A^\beta \subset A^{\alpha+\beta}$ for all $\alpha,\beta \in \mathbb{Z}_2$.

Definition 2.11. The *standard embedding* of a regular Hom-Leibniz color 3-algebra $(T, [\cdot,\cdot,\cdot], \epsilon, \phi)$ is a color 2-graded algebra $(\mathcal{A}, \epsilon, \Phi)$ with an automorphism, where $\mathcal{A}^0 := \mathfrak{L}$, $\mathcal{A}^1 := T$, the product is given by

$$\begin{aligned}&((x,y),z)\cdot((u,v),w) \\ &\quad = \big(([x,y,u]^{\phi^{-1}},v) + \epsilon(x+y,u)(u,[x,y,v]^{\phi^{-1}}) + (z,w), \\ &\qquad\qquad [x,y,w]^{\phi^{-1}} - \epsilon(z,u+v)[u,v,z]^{\phi^{-1}}\big),\end{aligned} \tag{2.9}$$

the automorphism $\varPhi$ is

$$\varPhi : \begin{cases} x \mapsto \phi(x), \\ (x,y) \mapsto \phi(x,y)\phi^{-1} = (\phi(x), \phi(y)), \end{cases} \qquad \text{for } x, y \in T, \tag{2.10}$$

and the $\mathbb{G}$-grading is induced by the $\mathbb{G}$-gradings of T and $\mathfrak{L}$.

It can be easily verified that $\varPhi$ is indeed an automorphism of $\mathcal{A}$.

The name "embedding" is justified as follows. Introduce the structure of a Leibniz color 3-algebra in $\mathcal{A}$ by $[x, y, z] = [[x, y], z]$. Consider then the regular Hom-Leibniz color 3-algebra $\mathcal{A}^{\varPhi}$. One can easily see that the subspace $T \subset \mathcal{A}^{\varPhi}$ with the induced multiplication, automorphism and $\mathbb{G}$-grading is exactly the algebra $(T, [\cdot,\cdot,\cdot], \epsilon, \phi)$ (for details, see [V17]).

Definition 2.12. Let now $(T, [\cdot,\cdot,\cdot], \epsilon, \phi)$ be a Leibniz color 3-algebra with an automorphism. Then the standard embedding of T is a color 2-graded algebra $(\mathcal{A}, \epsilon, \varPhi)$ with an automorphism, where $\mathcal{A}^0 := \mathfrak{L}$, $\mathcal{A}^1 := T$, the product is given by

$$((x,y),z)\cdot((u,v),w) := \tag{2.11}$$

$$(([x,y,u],v) + \epsilon(x+y,u)(u,[x,y,v]) + (z,w),\ [x,y,w] - \epsilon(z,u+v)[u,v,z]).$$

and the automorphism $\varPhi$ and the $\mathbb{G}$-grading are as in the previous definition.

Remark 2.13. Although in both cases $\mathcal{A}'$ is a Lie color algebra, $\mathcal{A}$ is not, in general, a (2-graded) Lie color algebra. However, analogously to [V17] one can prove that if T is a Hom-Lie triple color system, then $\mathcal{A}$ is a Lie color algebra.

We will consider the relation between the standard embeddings of a Hom-Leibniz color 3-algebra $(T, [\cdot,\cdot,\cdot], \phi, \epsilon)$ and its dehomification $T^{\phi^{-1}}$ defined above. Note first that by the definition of $T^{\phi^{-1}}$ we have

$$_{T^{\phi^{-1}}}(x,y) = \phi_T^{-1}(x,y), \tag{2.12}$$

where $x, y \in T$ and $_T(x,y)$, $_{T^{\phi^{-1}}}(x,y)$ are the left multiplication operators in the corresponding algebras. Thus, we have

$$\mathfrak{L}(T^{\phi^{-1}}) = \phi^{-1}\mathfrak{L}(T),$$

and this allows us to define a linear map (recall the isomorphism between $\mathrm{End}(T)$ and its homotope $\mathrm{End}(T)_{\phi^{-1}}$ of the previous subsection) $\varphi : \mathcal{A}(T) \to \mathcal{A}(T^{\phi^{-1}})$ by

$$\varphi : \begin{cases} x \mapsto x, \\ _T(x,y) \mapsto_{T^{\phi^{-1}}} (x,y) = \phi_T^{-1}(x,y), \end{cases} \qquad \text{for } x, y \in T.$$

Proposition 2.14. *The map φ is an isomorphism of color 2-graded algebras with an automorphism.*

Proof. By construction, φ is bijective and respects the $\mathbb{G}$-grading and the 2-grading. So we only need to verify that φ respects multiplication and commutes with automorphisms, which is done by straightforward checking. Let $x, y, z, w \in T$. Using the definitions of multiplication in $\mathcal{A}(T)$ and $\mathcal{A}(T^{\phi^{-1}})$, we have

$$\begin{aligned}
\varphi({}_T(x,y)) \cdot_{\mathcal{A}(T^{\phi^{-1}})} \varphi({}_T(z,w)) &= {}_{T^{\phi^{-1}}}(x,y) \cdot_{\mathcal{A}(T^{\phi^{-1}})} {}_{T^{\phi^{-1}}}(z,w) \\
&= {}_{T^{\phi^{-1}}}([x,y,z]_{T^{\phi^{-1}}}, w) + \epsilon(x+y,z)_{T^{\phi^{-1}}}(z,[x,y,w])_{T^{\phi^{-1}}} \\
&= \phi^{-1}({}_T([x,y,z]_T^{\phi^{-1}}, w) + \epsilon(x+y,z){}_T(z,[x,y,w]_T^{\phi^{-1}})) \\
&= \varphi({}_T(x,y) \cdot_{\mathcal{A}(T)} {}_T(z,w)),
\end{aligned}$$

$$\varphi(z) \cdot_{\mathcal{A}(T^{\phi^{-1}})} \varphi(w) = z \cdot_{\mathcal{A}(T^{\phi^{-1}})} w = {}_{T^{\phi^{-1}}}(z,w) = \varphi({}_T(z,w)) = \varphi(z \cdot_{\mathcal{A}(T)} w),$$

$$\begin{aligned}
\varphi({}_T(x,y)) \cdot_{\mathcal{A}(T^{\phi^{-1}})} \varphi(w) &= {}_{T^{\phi^{-1}}}(x,y) \cdot_{\mathcal{A}(T^{\phi^{-1}})} w = [x,y,w]_{T^{\phi^{-1}}} \\
&= [x,y,w]_T^{\phi^{-1}} = \varphi({}_T(x,y) \cdot_{\mathcal{A}(T)} w).
\end{aligned}$$

Analogously one can check that φ preserves products of the type $z \cdot_{\mathcal{A}(T)} {}_T(x,y)$.

Now we check that φ commutes with automorphisms. Let Φ and Φ' be automorphisms of $\mathcal{A}(T)$ and $\mathcal{A}(T^{\phi^{-1}})$ respectively. Then

$$\begin{aligned}
\varphi(\Phi(x)) &= \phi(x) = \Phi'(\varphi(x)), \\
\varphi(\Phi({}_T(x,y))) &= \varphi({}_T(\phi(x),\phi(y))) = {}_{T^{\phi^{-1}}}(\phi(x),\phi(y)) \\
&= \Phi'({}_{T^{\phi^{-1}}}(x,y)) = \Phi'(\varphi({}_T(x,y))). \ \blacksquare
\end{aligned}$$

2.6. Split structures. In this subsection we introduce the class of split algebras in the framework of regular Hom-Leibniz color 3-algebras and Leibniz color 3-algebras with an automorphism. As in the previous papers on the subject (see [C09, CF11] and other papers on ternary split algebras), this is done by using the standard embedding algebra.

As in the previous section, let T be a regular Hom-Leibniz color 3-algebra (or a Leibniz color 3-algebra with an automorphism) and $\mathcal{A}$ its standard embedding. Observe that the product in $\mathcal{A}$ gives us a natural action:

$$\mathcal{A}^0 \times \mathcal{A}^1 \to \mathcal{A}^1, \qquad (x,y) \mapsto xy.$$

As we have seen earlier, $\mathcal{A}^0$ is a Lie color algebra. Moreover, the identity (2.1) shows that this action endows $\mathcal{A}^\infty$ with the structure of a color Lie module over $\mathcal{A}'$.

Definition 2.15. Let $T = (T, [\cdot,\cdot,\cdot], \epsilon, \phi)$ be a Hom-Leibniz color 3-algebra (or a Leibniz color 3-algebra with an automorphism) and let $\mathcal{A} = (\mathfrak{L} \oplus T, \cdot, \epsilon, \Phi)$ be its standard embedding. Let H be a maximal abelian subalgebra (briefly MASA) of $\mathfrak{L}_0$, the zeroth $\mathbb{G}$-component of $\mathfrak{L}$. The *root space* of T with respect to H associated to a linear functional $\alpha \in H^*$ is the subspace

$$T_\alpha := \{v \in T : h \cdot v = \alpha(h)v \text{ for any } h \in H\}.$$

The elements $\alpha \in H^*$ such that $T_\alpha \neq 0$ are called *roots* of T (with respect to H), and we write $\Lambda^T := \{\alpha \in H^* \setminus \{0\} : T_\alpha \neq 0\}$. Analogously, we denote by $\Lambda^{\mathfrak{L}}$ the set of all non-zero $\alpha \in H^*$ such that $\mathfrak{L}_\alpha \neq 0$, where

$$\mathfrak{L}_\alpha := \{e \in \mathfrak{L} : [h, e] = \alpha(h)e \text{ for any } h \in H\}$$

are root subspaces of $\mathfrak{L}$ with respect to T.

Remark 2.16. Despite the fact that we work in the context of Hom-algebras, our definition of the root spaces T_α does not contain the homomorphism ϕ explicitly (cf. [C172, CC18]). This is because we define the root spaces via multiplication in $\mathcal{A}(T)$, which is *not* a Hom-algebra (but an algebra with an automorphism). In fact, the definition of the product in the standard envelope $\mathcal{A}$ already contains the operator ϕ^{-1}, which "cancels out" with the operator ϕ which should be present in the "usual" definition of a split Hom-algebra (i.e. $(x, y) \cdot w = [x, y, w]^{\phi^{-1}} = \phi^{-1}([x, y, w])$). The material of the next subsection should clarify and justify this idea.

It is an easy exercise to prove that the root spaces T_α, $\alpha \in \Lambda^T$, and L_β, $\beta \in \Lambda^{\mathfrak{L}}$, are graded (see, for example, [C172]).

The following result collects some basic properties of the subspaces T_α and $\mathfrak{L}_\alpha$. The proof is based on the Jacobi identity (which holds in $\mathfrak{L}$) and on the identity (2.1). We omit the details since the argument is similar to the proof of [CF11, Lemma 2.1].

Lemma 2.17. *Let $(T, [\cdot,\cdot,\cdot], \phi, \epsilon)$ be a Leibniz color 3-algebra with an automorphism, $\mathcal{A} = \mathfrak{L} \oplus T$ its standard embedding and H a MASA of $\mathfrak{L}_0$. For any $\alpha, \beta, \gamma \in \Lambda^T \cup \{0\}$ and $\delta, \epsilon \in \Lambda^{\mathfrak{L}} \cup \{0\}$, the following assertions hold:*

(i) *If $(T_\alpha, T_\beta) \neq 0$ then $\alpha + \beta \in \Lambda^{\mathfrak{L}} \cup \{0\}$, and $(T_\alpha, T_\beta) \subset \mathfrak{L}_{\alpha+\beta}$.*
(ii) *If $\mathfrak{L}_\delta T_\alpha \neq 0$ then $\delta + \alpha \in \Lambda^T \cup \{0\}$ and $\mathfrak{L}_\delta T_\alpha \subset T_{\delta+\alpha}$.*
(iii) *If $T_\alpha \mathfrak{L}_\delta \neq 0$ then $\alpha + \delta \in \Lambda^T \cup \{0\}$ and $T_\alpha \mathfrak{L}_\delta \subset T_{\alpha+\delta}$.*
(iv) *If $[\mathfrak{L}_\delta, \mathfrak{L}_\epsilon] \neq 0$ then $\delta + \epsilon \in \Lambda^{\mathfrak{L}} \cup \{0\}$ and $[\mathfrak{L}_\delta, \mathfrak{L}_\epsilon] \subset \mathfrak{L}_{\delta+\epsilon}$.*
(v) *If $[T_\alpha, T_\beta, T_\gamma] \neq 0$ then $\alpha+\beta+\gamma \in \Lambda^T \cup \{0\}$ and $[T_\alpha, T_\beta, T_\gamma] \subset T_{\alpha+\beta+\gamma}$.*
(vi) *$\phi(T_\alpha) \subseteq T_{\alpha\Phi^{-1}}$, $\phi^{-1}(T_\alpha) \subseteq T_{\alpha\Phi}$ and $\alpha\Phi^k \in \Lambda^T$ for all $k \in \mathbb{Z}$.*
(vii) *$\Phi(\mathfrak{L}_\delta) \subseteq \mathfrak{L}_{\delta\Phi^{-1}}$, $\Phi^{-1}(\mathfrak{L}_\delta) \subseteq \mathfrak{L}_{\delta\Phi}$ and $\delta\Phi^k \in \Lambda^{\mathfrak{L}}$ for all $k \in \mathbb{Z}$.*

Remark 2.18. Note that the verbatim analogue of the above lemma can be proved for T a regular Hom-Leibniz color 3-algebra, except that in (v) we have $[T_\alpha, T_\beta, T_\gamma]^{\phi^{-1}} \subset T_{\alpha+\beta+\gamma}$.

Now we present the main definition of the paper.

Definition 2.19. Let T be a Hom-Leibniz color 3-algebra or a Leibniz color 3-algebra with an automorphism, and $\mathcal{A} = \mathfrak{L} \oplus T$ its standard embedding. Then T is said to be a *split algebra* if there exists a MASA H of $\mathfrak{L}_0$ such that

$$T = T_0 \oplus \bigoplus_{\alpha \in \Lambda^T} T_\alpha \tag{2.13}$$

in the sense of Definition 2.15. The set Λ^T is called the *root system* of T. We refer to (2.13) as the *root space decomposition* of T.

REMARK 2.20. In contrast to the previous papers considering split ternary algebras, in the color case we cannot guarantee that $[T_0, T_0, T_0] = 0$, since $(T_0, T_0) \nsubseteq H$.

2.7. Reduction to the color case. In this subsection we show that for a split regular Hom-Leibniz color 3-algebra T, the algebra $T^{\phi^{-1}}$ is also split, and vice versa. Therefore, we may consider only the case of Leibniz color 3-algebras with an automorphism.

Let $(T, [\cdot,\cdot,\cdot], \phi, \epsilon)$ be a regular Hom-Leibniz color 3-algebra, let H be a MASA of $\mathfrak{L}$, and $\alpha \in \Lambda$, $v \in T_\alpha$. Consider the Leibniz color 3-algebra $T^{\phi^{-1}}$. Recall that the algebras $\mathcal{A}(T)$ and $\mathcal{A}(T^{\phi^{-1}})$ are isomorphic (via the map φ) as 2-graded color algebras with an automorphism. Hence, the subalgebra $\varphi(H) = \phi^{-1}H$ is a MASA of $\mathfrak{L}(T^{\phi^{-1}})$. Now, $h = \varphi(h') \in \varphi(H)$ and since φ is an algebra isomorphism, we get

$$\begin{aligned} h \cdot_{\mathcal{A}(T^{\phi^{-1}})} v &= \varphi(h') \cdot_{\mathcal{A}(T^{\phi^{-1}})} \varphi(v) = \varphi(h' \cdot_{\mathcal{A}(T)} v) = \varphi(\alpha(h')v) \\ &= \alpha(h')v = \alpha(\varphi^{-1}(h))v = \alpha(\phi h)v. \end{aligned} \tag{2.14}$$

This clearly implies that

$$T_\alpha = (T^{\phi^{-1}})_{\alpha\circ\varphi^{-1}}, \tag{2.15}$$

and if T is a split Hom-Leibniz color 3-algebra (2.13), then

$$T^{\phi^{-1}} = T_0 \oplus \bigoplus_{\alpha \in \Lambda^T} T_{\alpha\circ\varphi^{-1}}.$$

Therefore, $T^{\phi^{-1}}$ is a split Leibniz color 3-algebra with the root system $\Lambda^{T^{\phi^{-1}}} = \Lambda^T \circ \varphi^{-1}$. Clearly, the converse also holds.

Hence, we need only prove our results in the case of split Leibniz color 3-algebras. Having obtained our main results for $T^{\phi^{-1}}$, we can go back to the Hom case by reversing the process above and obtain the analogous results. This will be done explicitly in the subsequent sections.

3. Split Leibniz color 3-algebras

3.1. Spaces associated to roots. In this subsection for any roots $\alpha, \beta \in \Lambda^T$ we introduce certain subspaces $T_{0,\alpha}$ and $T(\alpha, \beta)$ which are going to be important later.

Let $\alpha \in \Lambda^T$. We define the space $T_{0,\alpha}$ inductively: $T_{0,\alpha}^{(0)} = [T_\alpha, T_{-\alpha}, T_0]$, and $T_{0,\alpha}^{(k)} = [T_{0,\alpha}^{(k-1)}, T_0, T_0]$ for $k \geq 1$. Finally, we set

$$T_{0,\alpha} = \sum_{k\geq 0} T_{0,\alpha}^{(k)}. \tag{3.1}$$

The spaces $T_{0,\alpha}^{(k)}$ satisfy some relations:

LEMMA 3.1.

(1) $[T_0, T_0, T_{0,\alpha}^{(k)}] \subseteq T_{0,\alpha}^{(k)}$,
(2) $[T_0, T_{0,\alpha}^{(k)}, T_0] \subseteq T_{0,\alpha}^{(k)} + T_{0,\alpha}^{(k+1)}$.

The proof is a simple induction applying (2.2).

Now, let $\alpha, \beta \in \Lambda^T$. Consider the following subspace:

$$\begin{aligned} T(\alpha,\beta) = {} & ([T_\alpha, T_{-\alpha}, T_\beta])_{\circlearrowleft \alpha\leftrightarrow -\alpha, \beta\leftrightarrow -\beta, \alpha\leftrightarrow\beta} \\ & + ([T_{0,\alpha}, T_\beta, T_0] + [T_{0,\alpha}, T_{0,\beta}, T_0])_{\circlearrowleft S^3, \alpha\leftrightarrow -\alpha, \beta\leftrightarrow -\beta, \alpha\leftrightarrow\beta}, \end{aligned}$$

where the subscripts mean that we take the sum of all spaces obtained by changing the order in the triple product or interchanging α and $-\alpha$, β and $-\beta$ and α and β. Hence, by construction, this space is symmetric with respect to the symmetries above, i.e.,

$$T(\alpha,\beta) = T(-\alpha,\beta) = T(\alpha,-\beta) = T(\beta,\alpha) = \cdots .$$

3.2. Connections of roots. Our main tool is the so-called connections of roots. In this section we give the notion of connectivity of roots for Leibniz color 3-algebras with an automorphism. In what follows, T denotes a split Leibniz color 3-algebra with root space decomposition $T = T_0 \oplus \bigoplus_{\alpha\in\Lambda^T} T_\alpha$.

DEFINITION 3.2. The root system $\Lambda^T\,[\Lambda^{\mathfrak{L}}]$ is called *symmetric* if $\Lambda^T = -\Lambda^T$ $[\Lambda^{\mathfrak{L}} = -\Lambda^{\mathfrak{L}}]$, where for $\emptyset \neq \Upsilon \subset H^*$ the set $-\Upsilon$ is just $\{-\alpha : \alpha \in \Upsilon\}$.

From now on, we will suppose that both $\Lambda^T, \Lambda^{\mathfrak{L}}$ are symmetric.

For each $\alpha \in \Lambda^T$ introduce a new symbol θ_α, and let $\Theta = \{\theta_\alpha : \alpha \in \Lambda^T\}$. Define a new operation

$$\dot{+} : (\Lambda^T \cup \Lambda^{\mathfrak{L}} \cup \Theta_\Omega) \times (\Lambda^T \cup \{0\}) \to H^* \cup \Theta_\Omega$$

as follows:

- For $\alpha \in \Lambda^T \cup \Lambda^{\mathfrak{L}}$ and $\beta \in \Lambda^T \cup \{0\}$ with $\beta \neq -\alpha$, $\alpha \dot{+} \beta \in H^*$ is the usual sum of linear functionals.
- For $\alpha \in \Lambda^T$,

$$\alpha \dot{+} (-\alpha) = \theta_\alpha.$$

- For $\theta_\alpha \in \Theta$ and $\beta \in \Lambda^T$,

$$\theta_\alpha \dotplus \beta = \begin{cases} \beta & \text{if } \sum_{k\in\mathbb{Z}} T(\alpha\Phi^k, \beta\Phi^k) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

- For $\theta_\alpha \in \Theta$ and $0 : H \to K$, we define $\theta_\alpha \dotplus 0 = 0$.

REMARK 3.3. Note that $\alpha \dotplus \beta$ coincides with $\alpha + \beta$ for $\alpha \in \Lambda^T \cup \Lambda^{\mathfrak{L}}$ and $\beta \in \Lambda^T \cup \{0\}$, except for the case $\beta = -\alpha$, where $\alpha \dotplus -\alpha = \theta_\alpha$.

The following two lemmas collect basic properties of the operation $\dotplus$ which we will use later (see also [CF11, Lemmas 3.1–3.3]).

LEMMA 3.4. *For any $\alpha, \beta \in \Lambda^T$ such that $\theta_\alpha \dotplus \beta = \beta$ we have:*

(i) $\theta_\beta \dotplus \alpha = \alpha$.
(ii) $\theta_{-\alpha} \dotplus \beta = \beta$.
(iii) $\theta_{-\alpha} \dotplus (-\beta) = -\beta$.
(iv) $\theta_{\alpha\Phi^k} \dotplus \beta\Phi^k = \beta\Phi^k$ *for all* $k \in \mathbb{Z}$.

Proof. Just use the symmetry of $T(\alpha, \beta)$ with respect to $\alpha \leftrightarrow -\alpha, \beta \leftrightarrow -\beta, \alpha \leftrightarrow \beta$ and the definition of $\dotplus$. ∎

LEMMA 3.5. *Let $\alpha, \delta \in \Lambda^T$ and $\beta, \gamma \in \Lambda^T \cup \{0\}$, and let $\alpha \dotplus \beta \in \Lambda^{\mathfrak{L}} \cup \Theta$ with $(\alpha \dotplus \beta) \dotplus \gamma = \delta$. Then:*

(i) $\alpha\Phi^k \dotplus \beta\Phi^k \in \Lambda^{\mathfrak{L}} \cup \Theta$ *and* $(\alpha\Phi^k \dotplus \beta\Phi^k) \dotplus \gamma\Phi^k = \delta\Phi^k$ *for any* $k \in \mathbb{Z}$.
(ii) $(-\alpha) \dotplus (-\beta) \in \Lambda^{\mathfrak{L}} \cup \Theta$ *and* $((-\alpha) \dotplus (-\beta)) \dotplus (-\gamma) = -\delta$.
(iii) $\delta \dotplus (-\gamma) \in \Lambda^{\mathfrak{L}} \cup \Theta$ *and* $(\delta \dotplus (-\gamma)) \dotplus (-\beta) = \alpha$.

Proof. (i) Suppose first that $\alpha \dotplus \beta \in \Lambda^{\mathfrak{L}}$. Then by Remark 3.3, we have $\alpha \dotplus \beta = \alpha + \beta$ and $(\alpha \dotplus \beta) \dotplus \gamma = \alpha + \beta + \gamma = \delta$. Multiplying these equalities by Φ^k, we get $\alpha\Phi^k \dotplus \beta\Phi^k = \alpha\Phi^k + \beta\Phi^k = (\alpha + \beta)\Phi^k \in \Lambda^{\mathfrak{L}}$ and $(\alpha\Phi^k \dotplus \beta\Phi^k) \dotplus \gamma\Phi^k = \alpha\Phi^k + \beta\Phi^k + \gamma\Phi^k = \delta\Phi^k$.

Now suppose that $\alpha \dotplus \beta = \theta_\alpha$ and $\theta_\alpha \dotplus \gamma = \delta$. Then $\alpha + \beta = 0$, $\gamma = \delta$ and $\alpha\Phi^k \dotplus \beta\Phi^k = \theta_{\alpha\Phi^k}$. Since $\theta_\alpha \dotplus \gamma = \gamma$, Lemma 3.4(iv) implies that $\theta_{\alpha\Phi^k} \dotplus \gamma\Phi^k = \gamma\Phi^k = \delta\Phi^k$ for all $k \in \mathbb{Z}$.

(ii) is proved analogously: just use Lemma 3.4(iii).

(iii) Suppose first that $\alpha \dotplus \beta \in \Lambda^{\mathfrak{L}}$. Then again $\alpha \dotplus \beta = \alpha + \beta$ and $(\alpha \dotplus \beta) \dotplus \gamma = \alpha + \beta + \gamma = \delta$. Therefore, $\delta \dotplus (-\gamma) = \delta - \gamma = \alpha + \beta \in \Lambda^{\mathfrak{L}}$, $(\delta \dotplus (-\gamma)) \dotplus (-\beta) = \delta - \gamma - \beta = \alpha$.

Now suppose that $\alpha \dotplus \beta \in \Theta$. Then $\alpha + \beta = 0$, $\gamma = \delta$ and $\theta_\alpha \dotplus \gamma = \gamma$. Hence, $\delta \dotplus (-\gamma) = \theta_\delta \in \Theta$ and $(\delta \dotplus (-\gamma)) \dotplus (-\beta) = \theta_\gamma \dotplus \alpha = \alpha$ by Lemma 3.4(i). ∎

We are now ready to introduce the key tool in our study.

Definition 3.6. Let α and β be non-zero roots in Λ^T. We say that α is *connected* to β (written $\alpha \sim \beta$) if there exists an ordered set $\{\alpha_1, \dots, \alpha_{2n+1}\} \subset \Lambda^T \cup \{0\}$ satisfying the following conditions:

1. $\alpha_1 = \alpha$.
2. Consecutive odd-element $\dot{+}$-sums belong to $\pm\Lambda^T$:

$$\{\alpha_1, (\alpha_1 \dot{+} \alpha_2) \dot{+} \alpha_3, (((\alpha_1 \dot{+} \alpha_2) \dot{+} \alpha_3) \dot{+} \alpha_4) \dot{+} \alpha_5, \dots, \\ ((\dots((\alpha_1 \dot{+} \alpha_2) \dot{+} \alpha_3) \dot{+} \dots) \dot{+} \alpha_{2n}) \dot{+} \alpha_{2n+1}\} \subset \pm\Lambda^T.$$

3. Consecutive even-element $\dot{+}$-sums belong to $\pm\Lambda^{\mathfrak{L}}$ or Θ:

$$\{\alpha_1 \dot{+} \alpha_2, ((\alpha_1 \dot{+} \alpha_2) \dot{+} \alpha_3) \dot{+} \alpha_4, \dots, \\ ((\dots((\alpha_1 \dot{+} \alpha_2) \dot{+} \alpha_3) \dot{+} \dots) \dot{+} \alpha_{2n-1}) \dot{+} \alpha_{2n}\} \subset \pm\Lambda^{\mathfrak{L}} \cup \Theta.$$

4. $((\dots((\alpha_1 \dot{+} \alpha_2) \dot{+} \alpha_3) \dot{+} \dots) \dot{+} \alpha_{2n}) \dot{+} \alpha_{2n+1} = \pm\beta\Phi^k$, $k \in \mathbb{Z}$.

The family $\{\alpha_1, \dots, \alpha_{2n+1}\}$ is then called a *connection* from α to β.

By definition, for any $\alpha \in \Lambda^T$ the set $\{\alpha\}$ is a connection from α to any root $\pm\alpha\Phi^k, k \in \mathbb{Z}$. In particular, the relation $\sim$ is reflexive.

Let $\{\alpha_1, \dots, \alpha_{2n+1}\} \subset \pm\Lambda^T \cup \{0\}$ be a connection from α to β such that

$$((\dots((\alpha_1 \dot{+} \alpha_2) \dot{+} \alpha_3) \dot{+} \dots) \dot{+} \alpha_{2n}) \dot{+} \alpha_{2n+1} = \varepsilon\beta\Phi^k,$$

where $\varepsilon \in \{\pm 1\}$ and $k \in \mathbb{Z}$.

If $n = 0$, then $\alpha = \pm\beta\Phi^{-k}$. So $\{\beta\}$ is a connection from β to α. If $n \geq 1$, then Lemma 3.5(i, ii) implies that

$$((\dots((\varepsilon\alpha_1\Phi^{-k} \dot{+} \varepsilon\alpha_2\Phi^{-k}) \dot{+} \varepsilon\alpha_3\Phi^{-k}) \dot{+} \dots) \dot{+} \varepsilon\alpha_{2n}\Phi^{-k}) \dot{+} \varepsilon\alpha_{2n+1}\Phi^{-k} = \beta,$$

and Lemma 3.5(iii) implies that

$$\{\beta, -\varepsilon\alpha_{2n+1}\Phi^{-k}, -\varepsilon\alpha_{2n}\Phi^{-k}, \dots, -\varepsilon\alpha_3\Phi^{-k}, -\varepsilon\alpha_2\Phi^{-k}\}$$

is a connection from β to α. That is, $\sim$ is symmetric.

Now, let $\{\alpha_1, \dots, \alpha_{2n+1}\}$ and $\{\beta_1, \dots, \beta_{2m+1}\}$ be connections from α to β and from β to γ, respectively, such that

$$(\dots(\alpha_1 \dot{+} \alpha_2) \dot{+} \dots) \dot{+} \alpha_{2n+1} = \varepsilon_1\beta\Phi^k,$$
$$(\dots(\beta_1 \dot{+} \beta_2) \dot{+} \dots) \dot{+} \beta_{2m+1} = \varepsilon_2\gamma\Phi^l,$$

where $\varepsilon_1, \varepsilon_2 = \pm 1$ and $k, l \in \mathbb{Z}$. Again, Lemma 3.5(i, ii) implies that

$$(\dots(\varepsilon_1\beta_1\Phi^k \dot{+} \varepsilon_1\beta_2\Phi^k) \dot{+} \dots) \dot{+} \varepsilon_1\beta_{2m+1}\Phi^k = \varepsilon_1\varepsilon_2\gamma\Phi^{k+l}.$$

Thus, if $m = 0$, then $\beta = \varepsilon_2\gamma\Phi^l$ and $\{\alpha_1, \dots, \alpha_{2n+1}\}$ is a connection from α to γ, and if $m \geq 1$, then $\{\alpha_1, \dots, \alpha_{2n+1}, \varepsilon_1\beta_2\Phi^k, \dots, \epsilon_1\beta_{2m+1}\Phi^k\}$ is a connection from α to γ. Hence, the relation is transitive, and thus we have proved the following useful fact.

PROPOSITION 3.7. *Suppose that the root systems $\Lambda^T, \Lambda^{\mathfrak{L}}$ are symmetric. Then the relation $\sim$ on Λ^T is an equivalence relation.*

3.3. Decompositions. Let T be a split Leibniz color 3-algebra with an automorphism and $T = T_0 \oplus \bigoplus_{\alpha\in\Lambda^T} T_\alpha$ the corresponding root space decomposition. By Proposition 3.7 the connectivity relation is an equivalence relation in Λ^T, so we can consider the quotient set

$$\Lambda^T/{\sim} = \{[\alpha] : \alpha \in \Lambda^T\},$$

where $[\alpha]$ is the set of non-zero roots of T which are connected to α. By the definition of $\sim$, it is clear that if $\beta \in [\alpha]$ then $\pm\beta\Phi^k \in [\alpha]$ for all $k \in \mathbb{Z}$.

We list some important cases in which two roots are connected:

LEMMA 3.8. *Let $\alpha, \beta \in \Lambda^T$.*

(i) *If $\alpha + \beta \in \Lambda^{\mathfrak{L}} \cup \{0\}$, then $\alpha \sim \beta$.*
(ii) *If $T(\alpha, \beta) \neq 0$, then $\alpha \sim \beta$.*

Proof. (i) $\{\alpha, \beta, 0\}$ is a connection from α to β.
(ii) $\{\alpha, -\alpha, \beta\}$ is a connection from α to β. ∎

Our goal in this section is to associate to each $[\alpha]$ an ideal $T_{[\alpha]}$ of T. Given $\alpha \in \Lambda^T$, we start by defining a set $T_{0,[\alpha]} \subset T_0$ as follows:

$$T_{0,[\alpha]} = \operatorname{span}_K\big\{[T_\beta, T_\gamma, T_\delta] : \beta, \gamma, \delta \in [\alpha] \cup \{0\},\ \beta + \gamma + \delta = 0 \text{ and } \delta \neq 0\big\} + \sum_{\beta\in[\alpha]} T_{0,\beta}. \tag{3.2}$$

Next, we set

$$V_{[\alpha]} := \bigoplus_{\beta\in[\alpha]} T_\beta, \tag{3.3}$$

$$T_{[\alpha]} := T_{0,[\alpha]} \oplus V_{[\alpha]}. \tag{3.4}$$

REMARK 3.9. The space $T_{0,[\alpha]}$ collapses in special cases. For example, if the $\mathbb{G}$-grading is trivial, then for any $\alpha \in \Lambda^T$ we have $[T_\alpha, T_{-\alpha}, T_0] \subseteq HT_0 = 0$. If T is a Hom-Lie color 3-algebra, then $T_{0,\alpha}^{(k)} = [T_{0,\alpha}^{(k-1)}, T_0, T_0] = [T_0, T_0, T_{0,\alpha}^{(k-1)}] \subseteq T_{0,\alpha}^{(k-1)}$, and for any $\beta \in [\alpha]$ we have $[T_\beta, T_{-\beta}, T_0] = [0, T_\beta, T_{-\beta}]$. Hence, in this case

$$T_{0,[\alpha]} = \operatorname{span}_K\{[T_\beta, T_\gamma, T_\delta] : \beta, \gamma, \delta \in [\alpha] \cup \{0\},\ \beta + \gamma + \delta = 0 \text{ and } \delta \neq 0\}.$$

In fact, the same result holds for T a regular Hom-Lie triple system (for more details, see Lemma 3.11).

Our aim is to show that $T_{[\alpha]}$ is an ideal of T for any $[\alpha]$. We begin by considering products involving the spaces T_β, $\beta \in [\alpha]$.

Lemma 3.10. *Fix* $\alpha_0 \in \Lambda^T$ *and let* $\alpha \in [\alpha_0]$ *and* $\beta, \gamma \in \Lambda^T \cup \{0\}$.

(i) *If* $[T_\alpha, T_\beta, T_\gamma] \neq 0$, *then* $\beta, \gamma, \alpha + \beta + \gamma \in [\alpha_0] \cup \{0\}$ *and* $[T_\alpha, T_\beta, T_\gamma] \subseteq T_{[\alpha_0]}$.
(ii) *If* $[T_\beta, T_\alpha, T_\gamma] \neq 0$, *then* $\beta, \gamma, \beta + \alpha + \gamma \in [\alpha_0] \cup \{0\}$ *and* $[T_\beta, T_\alpha, T_\gamma] \subseteq T_{[\alpha_0]}$.
(iii) *If* $[T_\beta, T_\gamma, T_\alpha] \neq 0$, *then* $\beta, \gamma, \beta + \gamma + \alpha \in [\alpha_0] \cup \{0\}$ *and* $[T_\beta, T_\gamma, T_\alpha] \subseteq T_{[\alpha_0]}$.

Proof. This is proved analogously to [CS17, Lemma 4.4], so we only consider (i) (to gain insight into the definition of $\dotplus$ and $T(\alpha, \beta)$) and direct the interested reader to the paper mentioned above for other cases.

By Lemma 2.17(i) we have $\alpha + \beta \in \Lambda^{\mathfrak{L}} \cup \{0\}$, hence $\beta \in [\alpha_0] \cup \{0\}$ by Lemma 3.8(i). We consider two cases:

Suppose first $\alpha + \beta + \gamma = 0$. It remains to check that $\gamma \in [\alpha_0] \cup \{0\}$. If $\gamma \neq 0$, then since $-\gamma = \alpha + \beta \in \Lambda^{\mathfrak{L}}$, one can see that $\{\alpha, \beta, 0\}$ is a connection from α to γ.

Now suppose $\alpha + \beta + \gamma \neq 0$. We have $[T_\alpha, T_\beta, T_\gamma] \subseteq T_{0,[\alpha_0]}$ by the definition. If $\alpha + \beta \neq 0$, then $\alpha + \beta \in \Lambda^{\mathfrak{L}}$ and $\{\alpha, \beta, \gamma\}$ is a connection from α to $\alpha + \beta + \gamma$ and the whole product lies in $T_{\alpha+\beta+\gamma} \subseteq V_{[\alpha_0]}$. If $\gamma \neq 0$, then $\{\alpha, \beta, -\alpha - \beta - \gamma\}$ is a connection from α to γ. Finally, if $\alpha + \beta = 0$ then necessarily $\gamma \neq 0$ and $[T_\alpha, T_{-\alpha}, T_\gamma] \neq 0$. Therefore, $T(\alpha, \gamma) \neq 0$ and $\alpha \sim \gamma$ by Lemma 3.8(ii). ∎

In the next two lemmas we consider products involving $T_{0,[\alpha_0]}$.

Lemma 3.11. *Fix* $\alpha_0 \in \Lambda^T$. *Let* $\alpha, \beta \in [\alpha_0] \cup \{0\}$, $\gamma \in [\alpha_0]$ *with* $\alpha + \beta + \gamma = 0$ *and* $\delta, \epsilon \in \Lambda^T \cup \{0\}$.

(i) *If* $[[T_\alpha, T_\beta, T_\gamma], T_\delta, T_\epsilon] \neq 0$ *then*
$$\delta, \epsilon, \delta + \epsilon \in [\alpha_0] \cup \{0\} \quad \textit{and} \quad [[T_\alpha, T_\beta, T_\gamma], T_\delta, T_\epsilon] \subseteq T_{[\alpha_0]}.$$
(ii) *If* $[T_\delta, [T_\alpha, T_\beta, T_\gamma], T_\epsilon] \neq 0$ *then*
$$\delta, \epsilon, \delta + \epsilon \in [\alpha_0] \cup \{0\} \quad \textit{and} \quad [T_\delta, [T_\alpha, T_\beta, T_\gamma], T_\epsilon] \subseteq T_{[\alpha_0]}.$$
(iii) *If* $[T_\delta, T_\epsilon, [T_\alpha, T_\beta, T_\gamma]] \neq 0$ *then*
$$\delta, \epsilon, \delta + \epsilon \in [\alpha_0] \cup \{0\} \quad \textit{and} \quad [T_\delta, T_\epsilon, [T_\alpha, T_\beta, T_\gamma]] \subseteq T_{[\alpha_0]}.$$

Proof. We only prove (i); parts (ii) and (iii) are proved analogously.

By (2.2), we have
$$[[T_\alpha, T_\beta, T_\gamma], T_\delta, T_\epsilon] \\ \subseteq [T_\alpha, T_\beta, [T_\gamma, T_\delta, T_\epsilon]] + [T_\gamma, [T_\alpha, T_\beta, T_\delta], T_\epsilon] + [T_\gamma, T_\delta, [T_\alpha, T_\beta, T_\epsilon]],$$
so at least one of the spaces on the right side is non-zero.

Suppose that $[T_\alpha, T_\beta, [T_\gamma, T_\delta, T_\epsilon]] \neq 0$. Since $[T_\gamma, T_\delta, T_\epsilon] \neq 0$ and $\gamma \neq 0$, by Lemma 3.10 we have $\delta, \epsilon \in [\alpha_0] \cup \{0\}$. By Lemma 2.17, $[T_\alpha, T_\beta, [T_\gamma, T_\delta, T_\epsilon]] \subseteq$

$[T_\alpha, T_\beta, T_{\gamma+\delta+\epsilon}]$. Since at least one of α, β is non-zero, we may apply Lemma 3.10 once more to get $\alpha+\beta+\gamma+\delta+\epsilon = \delta+\epsilon \in [\alpha_0] \cup \{0\}$.

Moreover, if $\delta+\epsilon \neq 0$, then the whole product lies in $T_{\delta+\epsilon} \subseteq V_{[\alpha_0]}$, and if $\delta+\epsilon = 0$, then it lies in $[T_\alpha, T_\beta, T_\gamma] \subseteq T_{0,[\alpha_0]}$.

Now suppose $[T_\gamma, [T_\alpha, T_\beta, T_\delta], T_\epsilon] \neq 0$. Since $[T_\alpha, T_\beta, T_\delta]$ and at least one of α, β is non-zero, the previous lemma implies $\delta, \alpha+\beta+\delta \in [\alpha_0] \cup \{0\}$. By Lemma 2.17, $[T_\gamma, [T_\alpha, T_\beta, T_\delta], T_\epsilon] \subseteq [T_\gamma, T_{\alpha+\beta+\delta}, T_\epsilon]$. Since $\gamma \neq 0$, Lemma 3.10 implies that $\epsilon \in [\alpha_0] \cup \{0\}$ and $\alpha+\beta+\gamma+\delta+\epsilon = \delta+\epsilon \in [\alpha_0] \cup \{0\}$.

Moreover, if $\delta+\epsilon \neq 0$, then the whole product lies in $T_{\delta+\epsilon} \subseteq V_{[\alpha_0]}$. If $\delta = -\epsilon$, then it lies in $[T_\gamma, T_{\alpha+\beta+\delta}, T_{-\delta}] \subseteq T_{0,[\alpha_0]}$ since all roots indexing the spaces in the product are in $[\alpha_0] \cup \{0\}$ and $\gamma \neq 0$.

The case $[T_\gamma, T_\delta, [T_\alpha, T_\beta, T_\epsilon]] \neq 0$ is analogous to the previous one. ■

Lemma 3.12. *Let* $\alpha \in [\alpha_0]$, $\delta, \epsilon \in \Lambda^T \cup \{0\}$, *and* $k \in \mathbb{N}$.

(i) *If* $[T^{(k)}_{0,\alpha}, T_\delta, T_\epsilon] \neq 0$ *then* $\delta, \epsilon, \delta+\epsilon \in [\alpha_0] \cup \{0\}$ *and* $[T^{(k)}_{0,\alpha}, T_\delta, T_\epsilon] \subseteq T_{[\alpha_0]}$.

(ii) *If* $[T_\delta, T^{(k)}_{0,\alpha}, T_\epsilon] \neq 0$ *then* $\delta, \epsilon, \delta+\epsilon \in [\alpha_0] \cup \{0\}$ *and* $[T_\delta, T^{(k)}_{0,\alpha}, T_\epsilon] \subseteq T_{[\alpha_0]}$.

(iii) *If* $[T_\delta, T_\epsilon, T^{(k)}_{0,\alpha}] \neq 0$ *then* $\delta, \epsilon, \delta+\epsilon \in [\alpha_0] \cup \{0\}$ *and* $[T_\delta, T_\epsilon, T^{(k)}_{0,\alpha}] \subseteq T_{[\alpha_0]}$.

Proof. We use induction on k and begin with $k = 0$. Consider for example (i); let $\alpha \in [\alpha_0]$ and assume that $[[T_\alpha, T_{-\alpha}, T_0], T_\delta, T_\epsilon] \neq 0$. As in Lemma 3.11, apply (2.2) to this product:

$$[[T_\alpha, T_{-\alpha}, T_0], T_\delta, T_\epsilon] \subseteq [T_\alpha, T_{-\alpha}, [T_0, T_\delta, T_\epsilon]] + [T_0, [T_\alpha, T_{-\alpha}, T_\delta], T_\epsilon] + [T_0, T_\delta, [T_\alpha, T_{-\alpha}, T_\epsilon]],$$

and again consider three cases:

First suppose $[T_\alpha, T_{-\alpha}, [T_0, T_\delta, T_\epsilon]] \neq 0$. By Lemmas 2.17 and 3.10,

$$[T_\alpha, T_{-\alpha}, [T_0, T_\delta, T_\epsilon]] \subseteq [T_\alpha, T_{-\alpha}, T_{\delta+\epsilon}] \subseteq T_{\delta+\epsilon}$$

and $\delta+\epsilon \in [\alpha_0] \cup \{0\}$. If any of δ, ϵ is zero, then we are done. Suppose $\delta, \epsilon \neq 0$. Since $[T_0, T_\delta, T_\epsilon] \neq 0$, Lemma 3.10 implies $\delta \sim \epsilon$. If $\delta+\epsilon \neq 0$, then $\delta \sim \delta+\epsilon \sim \alpha_0$. If $\delta = -\epsilon$, then

$$0 \neq [T_\alpha, T_{-\alpha}, [T_0, T_\delta, T_{-\delta}]],$$

which implies $\alpha \sim \delta$ by Lemma 3.11(iii) and we are done.

Moreover, if $\delta+\epsilon \neq 0$, then the whole product lies in $T_{\delta+\epsilon} \subseteq V_{[\alpha_0]}$, and if $\delta+\epsilon = 0$, then it lies in $[T_\alpha, T_{-\alpha}, T_0] \subseteq T_{0,[\alpha_0]}$.

Now suppose that $[T_0, [T_\alpha, T_{-\alpha}, T_\delta], T_\epsilon] \neq 0$. This implies that

$$[T_\alpha, T_{-\alpha}, T_\delta] \neq 0$$

and by Lemma 3.10, $\delta \in [\alpha_0] \cup \{0\}$. By Lemma 2.17, $[T_0, T_\delta, T_\epsilon] \neq 0$. If $\delta \neq 0$, Lemma 3.10 implies $\epsilon, \delta+\epsilon \in [\delta] \cup \{0\} = [\alpha_0] \cup \{0\}$ and we are done. If $\delta = 0$ and $\epsilon \neq 0$, then the condition $[T_0, [T_\alpha, T_{-\alpha}, T_0], T_\epsilon] \neq 0$ implies that

$T(\alpha,\epsilon)\neq 0$ and $\alpha\sim\epsilon$ by Lemma 3.8. If $\delta=\epsilon=0$, then there is nothing to prove.

If at least one of δ,ϵ is non-zero, then the whole product lies in $[T_0,T_\delta,T_\epsilon]\subseteq T_{\delta+\epsilon}\subseteq V_{[\alpha_0]}$ if $\delta+\epsilon\neq 0$ or in $T_{0,[\alpha_0]}$ if $\delta+\epsilon=0$. If $\delta=\epsilon=0$ then the product lies in $[T_0,[T_\alpha,T_{-\alpha},T_0],T_0]\subseteq T^{(0)}_{0,\alpha}+T^{(1)}_{0,\alpha}\subseteq T_{0,[\alpha_0]}$ by Lemma 3.1.

The last case, where $[T_0,T_\delta,[T_\alpha,T_{-\alpha},T_\epsilon]]\neq 0$, is analogous to the previous one.

Now suppose that the conclusion holds for all $i\leq k$ and consider it for $k+1$. Consider, for example, case (iii): let $[T_\delta,T_\epsilon,T^{(k+1)}_{0,\alpha}]\neq 0$. Writing $T^{(k+1)}_{0,\alpha}$ as $[T^{(k)}_{0,\alpha},T_0,T_0]$ and applying (2.2) to the product, we get

$$[T_\delta,T_\epsilon,T^{(k+1)}_{0,\alpha}]$$
$$\subseteq [T^{(k)}_{0,\alpha},T_0,[T_\delta,T_\epsilon,T_0]]+[[T^{(k)}_{0,\alpha},T_0,T_\delta],T_\epsilon,T_0]+[T_\delta,[T^{(k)}_{0,\alpha},T_0,T_\epsilon],T_0].$$

Therefore, one of the spaces on the right hand side is non-zero. We can again analyze the relevant three cases separately. This analysis is very similar to the one in the base case of induction.

Suppose, for example, that $[T^{(k)}_{0,\alpha},T_0,[T_\delta,T_\epsilon,T_0]]\neq 0$. By applying Lemmas 2.17 and 3.10 we obtain

$$[T^{(k)}_{0,\alpha},T_0,[T_\delta,T_\epsilon,T_0]]\subseteq[T^{(k)}_{0,\alpha},T_0,T_{\delta+\epsilon}]\subseteq T_{\delta+\epsilon}$$

and by induction we have $\delta+\epsilon\subseteq[\alpha_0]\cup\{0\}$. If any of δ,ϵ is zero, then there is nothing more to prove. Suppose $\delta,\epsilon\neq 0$. Since $[T_\delta,T_\epsilon,T_0]\neq 0$, Lemma 3.10 implies $\delta\sim\epsilon$. If $\delta+\epsilon\neq 0$, then $\delta\sim\delta+\epsilon\sim\alpha_0$ and we are done. If $\delta=-\epsilon$, then

$$0\neq[T^{(k)}_{0,\alpha},T_0,[T_\delta,T_{-\delta},T_0]]=[T^{(k)}_{0,\alpha},T_0,T^{(1)}_{0,\delta}],$$

which implies that $T(\alpha,\delta)\neq 0$ and $\alpha\sim\delta$ by Lemma 3.8.

Moreover, if $\delta+\epsilon\neq 0$, then the whole product lies in $T_{\delta+\epsilon}\subseteq V_{[\alpha_0]}$, and if $\delta+\epsilon=0$, then it lies in $[T^{(k)}_{0,\alpha},T_0,T_0]=T^{(k+1)}_{0,\alpha}\subseteq T_{0,[\alpha_0]}$.

Other non-zero products can be analyzed similarly. ∎

Let us denote

$$T_{0,\Lambda^T}:=\sum_{\substack{\alpha+\beta+\gamma=0\\ \alpha,\beta\in\Lambda^T\cup\{0\}\\ \gamma\in\Lambda^T}}[T_\alpha,T_\beta,T_\gamma]+\sum_{\alpha\in\Lambda^T}T_{0,\alpha}.$$

Now we can prove the main results of this section:

THEOREM 3.13. *Let (T,ϕ) be a Leibniz color 3-algebra T with an automorphism and with root space decomposition $T=T_0\oplus\bigoplus_{\alpha\in\Lambda^T}T_\alpha$.*

(i) *For any $\alpha_0\in\Lambda^T$, the space $T_{[\alpha_0]}$ is an ideal of T.*

(ii) *If T is simple, then any two roots in Λ^T are connected.*

Proof. (i) The spaces $T_{[\alpha]}$ are $\mathbb{G}$-homogeneous by construction. Recall that by definition of connectivity, any root $\beta \in \Lambda^T$ is connected to all $\beta\Phi^k, k \in \mathbb{Z}$. Therefore, Lemma 2.17 implies that, given $\alpha \in [\alpha_0]$, we have

$$\phi^{\pm 1}(T_\alpha) = T_{\alpha\Phi^{\mp 1}} \subseteq V_{[\alpha_0]}, \tag{3.5}$$

thus $\phi^{\pm 1}(V_{[\alpha_0]}) = V_{[\alpha_0]}$. Moreover, (3.5) and the definition of $T_{0,[\alpha_0]}$ easily imply that $\phi(T_{0,[\alpha_0]}) = T_{0,[\alpha_0]}$. Hence, $\phi(T_{[\alpha_0]}) = T_{[\alpha_0]}$. The rest of the proof of (i) uses Lemmas 3.10–3.12 directly. Part (ii) is obvious. ∎

THEOREM 3.14. *If $\mathcal{U}$ is a vector space complement of T_{0,Λ^T}, then $T = \mathcal{U} \oplus \sum_{[\alpha]\in\Lambda^T/\sim} T_{[\alpha]}$. Moreover, $[T, T_{[\alpha]}, T_{[\beta]}] + [T_{[\alpha]}, T, T_{[\beta]}] + [T_{[\alpha]}, T_{[\beta]}, T] = 0$ whenever $[\alpha] \neq [\beta]$.*

Proof. The first statement is obvious, so we only prove the second. Let $\alpha_0, \beta_0 \in \Lambda^T$ be such that $\alpha_0 \nsim \beta_0$ and consider $[T, T_{[\alpha_0]}, T_{[\beta_0]}]$. Lemmas 3.10–3.12 imply

$$[T, V_{[\alpha_0]}, V_{[\beta_0]}] + [T, T_{0,[\alpha_0]}, V_{[\beta_0]}] + [T, V_{[\alpha_0]}, T_{0,[\beta_0]}] = 0,$$

so we only need to prove that $[T, T_{0,[\alpha]}, T_{0,[\beta]}] = 0$. Moreover, Lemmas 3.11 and 3.12 imply $[T_\gamma, T_{0,[\alpha]}, T_{0,[\beta]}] = 0$ for any $\gamma \in \Lambda^T$, so it suffices to prove that $[T_0, T_{0,[\alpha]}, T_{0,[\beta]}] = 0$.

Let $\alpha, \beta, \gamma \in [\alpha_0] \cup \{0\}$ with $\alpha + \beta + \gamma = 0$ and $\delta, \epsilon, \chi \in [\beta_0] \cup \{0\}$ with $\delta + \epsilon + \chi = 0$. Suppose $\gamma \neq 0$ (hence at least one of α, β is $\neq 0$) By (2.2), we have

$$\begin{aligned}&[T_0, [T_\alpha, T_\beta, T_\gamma], T_{0,[\beta_0]}] \subseteq \\ &[T_\alpha, T_\beta, [T_0, T_\gamma, T_{0,[\beta_0]}]] + [[T_\alpha, T_\beta, T_0], T_\gamma, T_{0,[\beta_0]}] + [T_0, T_\gamma, [T_\alpha, T_\beta, T_{0,[\beta_0]}]] = 0\end{aligned}$$

by Lemmas 3.11 and 3.12. Analogously, we can show that if $\chi \neq 0$, then

$$[T_0, T_{0,[\alpha]}, [T_\delta, T_\epsilon, T_\chi]] = 0.$$

Therefore, it suffices to show that $[T_0, T_{0,\alpha}^{(k)}, T_{0,\beta}^{(l)}] = 0$ for all $\alpha \in [\alpha_0]$, $\beta \in [\beta_0]$ and $k, l \in \mathbb{N}$. But if any such product is non-zero, then $T(\alpha, \beta) \neq 0$ and $\alpha \sim \beta$ by Lemma 3.8, a contradiction.

The proof for the remaining products is analogous. ∎

As an obvious corollary, we get the following result:

COROLLARY 3.15. *If $T_0 = T_{0,\Lambda^T}$ and $(T) = 0$, then $T = \bigoplus_{[\alpha]\in\Lambda^T/\sim} T_{[\alpha]}$.*

3.4. The Hom case. Now let us go back to the original setting. That is, let (T, ϕ) be a split regular Hom-Leibniz color 3-algebra. The results of the previous section imply that $T^{\phi^{-1}}$ is a split Leibniz color 3-algebra with the root system $\Lambda^{T^{\phi^{-1}}} = \Lambda^T \circ \varphi^{-1}$, where $\varphi : \mathcal{A}(T) \to \mathcal{A}(T^{\phi^{-1}})$ is an isomorphism of the canonical envelopes which restricts to an isomorphism of $\mathfrak{L}(T)$ and $\mathfrak{L}(T^{\phi^{-1}})$. The main results of the last subsection assert that there exists an equivalence relation $\sim_{T^{\phi^{-1}}}$ on $\Lambda^{T^{\phi^{-1}}}$ such that $T^{\phi^{-1}}$ is the direct sum of a subspace $\mathcal{U} \subseteq T_0$ and the sum of ϕ-invariant ideals $T^{\phi^{-1}}_{[\alpha]}$ indexed by equivalence classes of $\sim_{T^{\phi^{-1}}}$ such that

$$[T^{\phi^{-1}}, T^{\phi^{-1}}_{[\alpha]}, T^{\phi^{-1}}_{[\beta]}]_{T^{\phi^{-1}}} + [T^{\phi^{-1}}_{[\alpha]}, T^{\phi^{-1}}, T^{\phi^{-1}}_{[\beta]}]_{T^{\phi^{-1}}} + [T^{\phi^{-1}}_{[\alpha]}, T^{\phi^{-1}}_{[\beta]}, T^{\phi^{-1}}]_{T^{\phi^{-1}}} = 0$$

whenever $[\alpha] \neq [\beta]$.

Now we obtain an analogous decomposition for T. First, the equation relating multiplication in T and $T^{\phi^{-1}}$ implies immediately that the ideals $T^{\phi^{-1}}_{[\alpha]}$, $\alpha \in \Lambda^{T^{\phi^{-1}}}$, are ϕ-invariant ideals in T, and

$$[T, T^{\phi^{-1}}_{[\alpha]}, T^{\phi^{-1}}_{[\beta]}]_T + [T^{\phi^{-1}}_{[\alpha]}, T, T^{\phi^{-1}}_{[\beta]}]_T + [T^{\phi^{-1}}_{[\alpha]}, T^{\phi^{-1}}_{[\beta]}, T]_T = 0.$$

Having in mind (2.15), let us introduce an equivalence relation $\sim_T$ in Λ^T by declaring that $\alpha \sim_T \beta$ for $\alpha, \beta \in \Lambda^T$ if and only if $\alpha \circ \varphi^{-1} \sim_{T^{\phi^{-1}}} \beta\varphi^{-1}$. Then, for $\alpha \in \Lambda^{T^{\phi^{-1}}}$ we have $[\alpha \circ \varphi]_\sim = [\alpha]_{\sim_{T^{\phi^{-1}}}} \circ \varphi$ and

$$V_{[\alpha]} = \bigoplus_{\beta \in [\alpha]_{\sim_{T^{\phi^{-1}}}}} T^{\phi^{-1}}_\beta = \bigoplus_{\beta \in [\alpha]_{\sim_{T^{\phi^{-1}}}}} T_{\beta\circ\varphi} = \bigoplus_{\beta' \in [\alpha\circ\varphi]_\sim} T_{\beta'}.$$

Analogously one can show that

$$\begin{aligned} &T_{0,[\alpha]} \\ &= \operatorname{span}_K\{[T^{\phi^{-1}}_\beta, T^{\phi^{-1}}_\gamma, T^{\phi^{-1}}_\delta]^{\phi^{-1}} : \beta, \gamma, \delta \in [\alpha]_{\sim_{T^{\phi^{-1}}}} \cup \{0\}, \beta+\gamma+\delta = 0, \delta \neq 0\} \\ &\quad + \sum_{\beta \in [\alpha]_{\sim_{T^{\phi^{-1}}}}} T^{\phi^{-1}}_{0,\beta} \\ &= \operatorname{span}_K\{[T_{\beta'}, T_{\gamma'}, T_{\delta'}] : \beta', \gamma', \delta' \in [\alpha\circ\varphi]_{\sim_T} \cup \{0\}, \beta'+\gamma'+\delta' = 0, \delta' \neq 0\} \\ &\quad + \sum_{\beta' \in [\alpha\circ\varphi]_{\sim_T}} T_{0,\beta'}, \end{aligned}$$

where the spaces $T_{0,\beta'}$ are constructed as in (3.1). Therefore, for any $\alpha \in \Lambda^T$ we may define the ideal $T_{[\alpha]_{\sim_T}}$ by the same equations (3.2)–(3.4). Moreover, $T_{[\alpha]_{\sim_T}} = T_{[\alpha\circ\varphi^{-1}]_{\sim_{T^{\phi^{-1}}}}}$.

We summarize our discussion in the following theorem.

THEOREM 3.16. *Let (T, ϕ) be a split Hom-Leibniz color 3-algebra with multiplication algebra $\mathfrak{L}$. Suppose that the root systems Λ^T and $\Lambda^{\mathfrak{L}}$ are symmetric. Then there exists an equivalence relation $\sim$ on Λ^T and a subspace $\mathcal{U} \subseteq T_0$ such that $T = \mathcal{U} \oplus \sum_{[\alpha]\in\Lambda^T/\sim} T_{[\alpha]}$. Moreover,*

$$[T, T_{[\alpha]}, T_{[\beta]}] + [T_{[\alpha]}, T, T_{[\beta]}] + [T_{[\alpha]}, T_{[\beta]}, T] = 0$$

whenever $[\alpha] \neq [\beta]$.

Finally, note that the relation $\sim_T$ can be introduced in a manner completely analogous to the definition of $\sim_{T^{\phi^{-1}}}$, that is, using connections via chains of roots in Λ^T (that is, it can be interpreted in terms of the original algebra T).

3.5. The passage of the split structure to $\mathfrak{L}$. Let T be a split Hom-Leibniz color 3-algebra or a split Leibniz color 3-algebra with an automorphism. In this subsection we show that the decomposition provided by Theorems 3.14 and 3.16 induces a similar decomposition for the multiplication algebra $\mathfrak{L}(T)$, and that these two decompositions are related in a certain way. Moreover, we study how this decomposition is related to the usual decomposition of Lie color algebras obtained by usual root connectivity techniques.

Let (T, ϕ) be a split Leibniz color 3-algebra with an automorphism, and $(\mathfrak{L}, \Phi)$ its multiplication algebra (the case of a split Hom-Leibniz color 3-algebra is completely analogous, so we will not consider it here). Again, suppose that both roots systems Λ^T and $\Lambda^{\mathfrak{L}}$ are symmetric and hence the conditions required in Theorem 3.14 are satisfied.

For any equivalence class $[\alpha] \subseteq \Lambda^T$ consider the space $(T_{[\alpha]}) = (T_{[\alpha]}, T) + (T, T_{[\alpha]}) \subseteq \mathfrak{L}$. Let $\Lambda^{\mathfrak{L}}_{[\alpha]}$ be the set of roots $\gamma \in \Lambda^{\mathfrak{L}}$ such that $\mathfrak{L}_\gamma \cap (T_{[\alpha]}) \neq 0$.

THEOREM 3.17. *Let $T, \mathfrak{L}, \Lambda^T, \Lambda^{\mathfrak{L}}$ be as above. Then:*

(1) *$\Lambda^{\mathfrak{L}}$ is the disjoint union of the sets $\Lambda^{\mathfrak{L}}_{[\alpha]}$, $[\alpha] \in \Lambda^T/\sim$, and there exists an equivalence relation $\approx$ in $\Lambda^{\mathfrak{L}}$ such that $\Lambda^T/\sim = \Lambda^{\mathfrak{L}}/\approx$.*
(2) *If $\gamma \in \Lambda^{\mathfrak{L}}_{[\alpha_0]}$, then $\mathfrak{L}_\gamma \subseteq (T_{[\alpha_0]})$. In other words, for any $[\alpha_0] \in \Lambda^T$,*

$$(T_{[\alpha]}) = ((T_{[\alpha]}) \cap \mathfrak{L}_0) \oplus \bigoplus_{\gamma \in \Lambda^{\mathfrak{L}}_{[\alpha]}} \mathfrak{L}_\gamma.$$

(3) *For any equivalence class $[\alpha] \in \Lambda^T/\sim$ the space $(T_{[\alpha]})$ is a Φ-invariant ideal of $\mathfrak{L}$.*
(4) *$[(T_{[\alpha]}), (T_{[\beta]})] = 0$ for $[\alpha] \neq [\beta]$.*
(5) *There exists a subspace $\mathcal{U}' \subseteq \mathfrak{L}_0$ such that $\mathfrak{L} = \sum_{[\alpha]} (T_{[\alpha]}) + \mathcal{U}'$.*
(6) *$(T_{[\alpha]}) \cdot T_{[\beta]} = 0$ for $[\alpha] \neq [\beta]$.*

Proof. (1) We recall that, by Theorem 3.14,

$$T = \mathcal{U} \oplus \sum_{[\alpha] \in \Lambda^T/\sim} T_{[\alpha]}$$

with $\mathcal{U} \subseteq T_0$ and $[T_{[\alpha]}, T_{[\beta]}, T] = 0$ whenever $[\alpha] \neq [\beta]$. Hence, for any $\alpha \in \Lambda^T$,

$$(T_{[\alpha]}) = (T_{[\alpha]}, T_{[\alpha]}) + (T_{[\alpha]}, \mathcal{U}) + (\mathcal{U}, T_{[\alpha]}) \tag{3.6}$$

and we obtain the following decomposition for $\mathfrak{L}$:

$$\mathfrak{L} = (T,T) = \sum_{[\alpha]} (T_{[\alpha]}) + (\mathcal{U},\mathcal{U}). \tag{3.7}$$

Since the ideals $T_{[\alpha]}$ are graded with respect to Λ^T, according Lemma 2.17 the spaces $(T_{[\alpha]}) \subseteq \mathfrak{L}$ are graded with respect to $\Lambda^{\mathfrak{L}}$:

$$(T_{[\alpha]}) = \bigoplus_{\gamma \in \Lambda^{\mathfrak{L}} \cup \{0\}} ((T_{[\alpha]}) \cap \mathfrak{L}_\gamma). \tag{3.8}$$

Hence the decomposition (3.7) and the fact that $(\mathcal{U},\mathcal{U}) \subseteq \mathfrak{L}_0$ imply that $\Lambda^{\mathfrak{L}} = \bigcup_{[\alpha]\subseteq \Lambda^T} \Lambda^{\mathfrak{L}}_{[\alpha]}$. Moreover, (3.6) yields

$$\Lambda^{\mathfrak{L}}_{[\alpha_0]} = \{\alpha + \beta : \alpha, \beta \in [\alpha_0] \cup \{0\},\ \alpha + \beta \neq 0,\ (T_\alpha, T_\beta) + (T_\beta, T_\alpha) \neq 0\}$$

Suppose now that $\gamma \in \Lambda^{\mathfrak{L}}_{[\alpha_0]} \cap \Lambda^{\mathfrak{L}}_{[\beta_0]}$, that is, $\gamma = \alpha_1 + \alpha_2 = \beta_1 + \beta_2$ where $\alpha_1, \alpha_2, \beta_1, \beta_2$ satisfy the conditions above. It is clear that one of α_1, α_2 is non-zero; say $\alpha_1 \neq 0$. Analogously let $\beta_1 \neq 0$. Then one can easily see that $\{\alpha_1, \alpha_2, -\beta_2\}$ is a connection from α_1 to β_1, thus $[\alpha_0] = [\beta_0]$. Therefore, any root $\gamma \in \Lambda_{\mathfrak{L}}$ belongs to a unique set $\Lambda^{\mathfrak{L}}_{[\alpha]}$ with $[\alpha] \subseteq \Lambda^T$ and so $\Lambda^{\mathfrak{L}} = \bigsqcup_{[\alpha]\subseteq\Lambda^T} \Lambda^{\mathfrak{L}}_{[\alpha]}$.

This decomposition induces an equivalence relation $\approx$ in $\Lambda^{\mathfrak{L}}$ (two elements are equivalent if they lie in the same set $\Lambda^{\mathfrak{L}}_{[\alpha]}$) and hence (1) follows.

(2) The first claim is just a restatement of the assertion that the sets $\Lambda^{\mathfrak{L}}_{[\alpha]}$, $[\alpha] \in \Lambda^T/\sim$, have empty intersections, and the second is a consequence of the first and the $\Lambda^{\mathfrak{L}}$-homogeneity of $(T_{[\alpha]})$ (see (3.8)).

(3) The Φ-invariance is a consequence of the definition of Φ and the ϕ-invariance of $T_{[\alpha]}$. The rest is a simple application of (2.7), e.g.

$$[(T_{[\alpha]},T),(T,T)] = ([T_{[\alpha]},T,T],T) + (T,[T_{[\alpha]},T,T]) \subseteq (T_{[\alpha]}).$$

The proof for the remaining products is analogous.

(4) Let $[\beta] \neq [\alpha]$. Again, by (2.7),

$$[(T,T_{[\alpha]}),(T,T_{[\beta]})] = ([T,T_{[\alpha]},T],T_{[\beta]}) + (T,[T,T_{[\alpha]},T_{[\beta]}]) = 0.$$

Other products are considered analogously.

(5) In the decomposition (3.7) we cannot guarantee that $(\mathcal{U},\mathcal{U}) \cap \sum_{[\alpha]}(T_{[\alpha]}) = 0$, so just taking another subspace $\mathcal{U}' \subseteq \mathfrak{L}_0$ complementing $(\sum_{[\alpha]}(T_{[\alpha]})) \cap \mathfrak{L}_0$, we get

$$\mathfrak{L} = \sum_{[\alpha]} (T_{[\alpha]}) + \mathcal{U}'.$$

(6) Apply Theorem 3.14. ∎

REMARK 3.18. Note that the multiplication algebra $\mathfrak{L}$ (see Subsection 2.4) has a decomposition very similar to the one in Theorems 3.14 and 3.16. The only difference is that there does not seem to be an easy way to express $(T_{[\alpha]}) \cap \mathfrak{L}_0$ in terms of $\mathfrak{L}_\gamma, \gamma \in \Lambda^{\mathfrak{L}}_{[\alpha]}$. Another decomposition (by classical root connection methods) of split Hom-Lie color algebras was obtained in [C172].

REMARK 3.19. Recall that $\mathfrak{L}$ is a split Hom-Lie color algebra, and that T is a weight $\mathfrak{L}$-module. Weight modules over split Lie algebras were considered in [C132], in which the authors obtained related decompositions of a split Lie algebra L and its weight module M.

Note, however, that our decomposition only holds for split Lie color algebras with an automorphism which are the multiplication algebras of Leibniz color 3-algebras with an automorphism. But this class is relatively large, as the following result shows.

PROPOSITION 3.20. *Let $(\mathfrak{L}, \Phi)$ be a split Lie color algebra with an automorphism such that $\mathfrak{L} = \mathfrak{L}^2$. Then there exists a split Leibniz color 3-algebra T with an automorphism ϕ such that its multiplication algebra is $\mathfrak{L}$.*

Proof. Recall that any split Leibniz color 3-algebra T with an automorphism ϕ is a weight Lie color module over its multiplication algebra $\mathfrak{L}$, and (2.2) is equivalent to the map $T^{\otimes 2} \to \mathfrak{L}$ being a homomorphism of $\mathfrak{L}$-modules. Clearly, by the construction of $\mathfrak{L}$, this homomorphism must be surjective. Moreover, the automorphisms ϕ and Φ are compatible in the sense that $\Phi(\ell) \cdot \phi(x) = \phi(\ell \cdot x)$ for all $\ell \in \mathfrak{L}$ and $x \in T$. In fact, one can easily check that the converse also holds: to construct the desired algebra T, it suffices to give a weight $\mathfrak{L}$-module T with a surjective module homomorphism $T^{\otimes 2} \to \mathfrak{L}$ and an automorphism ϕ of T compatible with Φ.

So, we take $T = \mathfrak{L}$ with the regular module action, the automorphism $\phi = \Phi$ and the map : $x \otimes y \mapsto [x, y]$, where $[\cdot, \cdot]$ is multiplication in $\mathfrak{L}$. This system satisfies the conditions above and is therefore a split Leibniz color 3-algebra T (in fact, a Lie triple system) with an automorphism, with the multiplication algebra $\mathfrak{L}$. ■

Given a split Lie color algebra L with an automorphism φ and the (symmetric) root system Λ^L one can introduce the notion of root connectivity in L (see, for example, [CS12] and other papers on binary split algebras):

DEFINITION 3.21. Two roots $\alpha, \beta \in \Lambda^L$ are defined to be *connected* (written $\alpha \sim \beta$) if there exists a chain $\{\alpha_1, \dots, \alpha_n\} \subseteq \Lambda^L$ such that

- $\alpha_1 = \alpha$.
- $\alpha_1 + \cdots + \alpha_k \in \Lambda^L$ for any $k = 1, \dots, n$.
- $\alpha_1 + \cdots + \alpha_n = \pm\beta\varphi^k$ for some $k \in \mathbb{Z}$.

One can see that $\sim$ is an equivalence relation on Λ^L, and there exists a subspace $\mathcal{V} \subseteq L_0$ such that $L = V \oplus \bigoplus_{[\gamma]\in\Lambda^L/\sim} L_{[\gamma]}$, where the $L_{[\gamma]}$ are φ-invariant ideals such that $[L_{[\gamma]}, L_{[\delta]}] = 0$ for $[\gamma] \neq [\delta]$.

Now let us return to the algebra $\mathfrak{L}$. There are now two equivalence relations on the root system $\Lambda^{\mathfrak{L}}$: the usual (intrinsic) root connectivity given by the definition above and the relation $\approx$ obtained by "descending" the decomposition of T to $\mathfrak{L}$. It would be interesting to see how these two equivalence relations are related. For this purpose we introduce the following definition.

Definition 3.22. Let L be a split Lie color algebra L with an automorphism φ and the root system Λ^L. We say that L is *root-multiplicative* if for any two roots $\alpha, \beta \in \Lambda^L$ such that $\alpha + \beta \in \Lambda^L \cup \{0\}$ we have $[L_\alpha, L_\beta] \neq 0$.

Proposition 3.23. *Let $\mathfrak{L}$ be root-multiplicative and let $\gamma_1, \gamma_2 \in \Lambda^{\mathfrak{L}}$ be two connected roots. Then $\gamma_1 \approx \gamma_2$.*

Proof. We proceed by induction. Let first $\gamma_2 = \pm\gamma_1\Phi^k$ for some $k \in \mathbb{Z}$. By definition of $\mathfrak{L}$, we have

$$\gamma_i = \alpha_i + \beta_i, \quad \text{where} \quad (T_{\alpha_i}, T_{\beta_i}) + (T_{\beta_i}, T_{\alpha_i}) \neq 0,\ \alpha_i, \beta_i \in \Lambda^T \cup \{0\}.$$

Without loss of generality we can assume that α_1, α_2 are non-zero. Then $\{\alpha_1, \beta_1, \mp\beta_2\Phi^k\}$ is a connection from α_1 to α_2, and $\gamma_1 \approx \gamma_2$.

Now, let $\gamma_1 + \gamma_2 = \pm\gamma_3\Phi^k$, where $\gamma_3 = \alpha_3 + \beta_3$ with $\alpha_3, \beta_3 \in \Lambda^T \cup \{0\}$. By root-multiplicativity of $\mathfrak{L}$, we have $[\mathfrak{L}_{\gamma_1}, \mathfrak{L}_{\gamma_2}] \neq 0$, so

$$[(T_{\alpha_1}, T_{\beta_1}), (T_{\alpha_2}, T_{\beta_2})] \neq 0$$

for some $\alpha_i, \beta_i \in \Lambda^T \cup \{0\}$ such that $\alpha_i + \beta_i = \gamma_i$. By (2.7), we get

$$0 \neq [(T_{\alpha_1}, T_{\beta_1}), (T_{\alpha_2}, T_{\beta_2})] \subseteq ([T_{\alpha_1}, T_{\beta_1}, T_{\alpha_2}], T_{\beta_2}) + ([T_{\alpha_1}, T_{\beta_1}, T_{\beta_2}], T_{\alpha_2}),$$

thus $[T_{\alpha_1}, T_{\beta_1}, T_{\alpha_2}] + [T_{\alpha_1}, T_{\beta_1}, T_{\beta_2}] \neq 0$. Without loss of generality we can suppose that $[T_{\alpha_1}, T_{\beta_1}, T_{\alpha_2}] \neq 0$, which implies that $\alpha_1 + \beta_1 + \alpha_2 \in \Lambda^T \cup \{0\}$. Moreover, we can assume that one of α_1, β_1 is non-zero, say $\alpha_1 \neq 0$.

Suppose that $\alpha_1 + \beta_1 + \alpha_2 \in \Lambda^T$. It is easy to see that the set

$$\{\alpha_1, \beta_1, \alpha_2, \beta_2, \mp\beta_3\Phi^k\}$$

is a connection from α_1 to α_3, and $\{\alpha_1, \beta_1, -(\alpha_1 + \beta_1 + \alpha_2)\}$ is a connection from α_1 to α_2.

Now, if $\alpha_1+\beta_1+\alpha_2 = 0$, then $\alpha_1+\beta_2 = -\alpha_2 \in \Lambda^T \cap \Lambda^{\mathfrak{L}}$ and $\{\alpha_1, \beta_1, 0\}$ is a connection from α_1 to α_2. Moreover, $\beta_2 = \pm(\alpha_3+\beta_3)\Phi^k = \pm\gamma_3\Phi^k \in \Lambda^T \cap \Lambda^{\mathfrak{L}}$ and $\{\beta_2, 0, \mp\beta_3\Phi^k\}$ is a connection from β_2 to α_3. Therefore, all non-zero roots α_i, β_i are connected and $\gamma_1 \approx \gamma_2 \approx \gamma_3$. ∎

Remark 3.24. It would be interesting to express the relation $\approx$ in terms of strings of roots in $\Lambda^T, \Lambda^{\mathfrak{L}}$, as in Definition 3.21. Unfortunately, up to now the authors have not been able to do it.

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