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NOTES ON DRINFELD TWISTS OF MULTIPLIER HOPF ALGEBRAS

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Abstract. This paper determines how the integral changes under a Drinfeld twist in multiplier Hopf algebras. For a multiplier Hopf algebra A with a Drinfeld twist J, we construct a new multiplier Hopf algebra A^J. If A is quasitriangular, then so is A^J. Finally, for a counimodular algebraic quantum group A, A^J is an algebraic quantum group, and as an application we give a formula for integrals of H^J, where H is an infinite-dimensional counimodular coFrobenius Hopf algebra.

1. Introduction. Hopf algebras in symmetric monoidal categories arise naturally in the deformation quantization of triangular solutions of the classical Yang–Baxter equations. This deformation quantization in [10] was achieved by means of a Drinfeld twist.

Drinfeld twists play an important role in Hopf algebra theory [9]. In [1], the authors studied the properties of Drinfeld twists of finite-dimensional Hopf algebras. Given a Hopf algebra or a quasitriangular Hopf algebra H with a Drinfeld twist J, they constructed another Hopf algebra or quasitriangular Hopf algebra H^J and also determined how the integral of the dual to a "finite-dimensional" unimodular Hopf algebra changes under a twist.

Therefore the following question naturally arises:

(Q) *How the integral changes under a Drinfeld twist in the infinite-dimensional Hopf algebra and in the multiplier Hopf algebra cases?*

As is known, multiplier Hopf algebras can be considered as a generalization of Hopf algebras, and give a nice substitute for the dual of an infinite-dimensional Hopf algebra. In this paper, we consider Drinfeld twists in this more general case, and provide an answer to (Q) for the dual of an infinite-dimensional Hopf algebra.

The paper is organized in the following way. In Section 2, we recall some notions which we will use in the following, such as multiplier Hopf alge-

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bras, algebraic quantum groups and their duality, and pairing and actions of multiplier Hopf algebras.

In Section 3, we first give the definitions of twists of multiplier Hopf algebras, which generalize the Drinfeld twist of Hopf algebras. Let (A, Δ) be a regular multiplier Hopf algebra. Then (A^J, Δ^J) is also a regular multiplier Hopf algebra with the same algebra structure and counit as (A, Δ), and the comultiplication and antipode are given by $\Delta^J(a) = J^{-1}\Delta(a)J$ and $S^J(a) = Q_J^{-1}S(a)Q_J$ for all $a \in A$. Then we consider two kinds of A^J-module algebras $(X_\star, \blacktriangleright)$ and $(X, \blacktriangleright_J)$, and show that there is an algebra isomorphism $D_J : X_\star \to X$, which is also an isomorphism of A^J-module algebras, i.e., D_J intertwines the A^J-actions $\blacktriangleright$ and $\blacktriangleright_J$, which generalizes the main results of [2] to multiplier Hopf algebras.

In Section 4, we consider the quasitriangular structure and integrals under a twist, and give an answer to the question (Q). More precisely, we show in Theorem 4.1 that if A is quasitriangular with quasitriangular structure $\mathcal{R}$, then A^J is also quasitriangular and its generalized R-matrix is given by $\mathcal{R}^J = J_{21}^{-1}\mathcal{R}J$. We also show in Theorem 4.3 that for a counimodular algebraic quantum group A, A^J is an algebraic quantum group and the elements $\varphi^J = u_J \rightharpoonup \varphi$ and $\psi^J = \psi \leftharpoonup u_J^{-1}$ are non-zero left and right integrals on (A^J, Δ^J) respectively.

2. Preliminaries. In this section, we will recall from [4], [12] and [13] some of the basic notions and results in the theory of algebraic quantum groups, which will be used in this paper.

2.1. Multiplier Hopf algebras. Throughout, all spaces we consider are over a fixed field k. Algebras may or may not have units, but should be always non-degenerate. For an algebra A, the *multiplier algebra* $M(A)$ of A is defined as the largest algebra with unit in which A is a two-sided dense ideal (for more details see [12, appendix]).

Now, we recall the definition of a multiplier Hopf algebra (see [12] for details). A *comultiplication* on an algebra A is a homomorphism $\Delta : A \to M(A \otimes A)$ such that $\Delta(a)(1 \otimes b)$ and $(a \otimes 1)\Delta(b)$ belong to $A \otimes A$ for all $a, b \in A$. We require Δ to be coassociative in the sense that

$$(a \otimes 1 \otimes 1)(\Delta \otimes \iota)(\Delta(b)(1 \otimes c)) = (\iota \otimes \Delta)((a \otimes 1)\Delta(b))(1 \otimes 1 \otimes c)$$

for all $a, b, c \in A$ (where ι denotes the identity map).

A pair (A, Δ) of an algebra A with a non-degenerate product and a comultiplication Δ on A is called a *multiplier Hopf algebra* if the linear maps T_1, T_2 defined by

$$T_1(a \otimes b) = \Delta(a)(1 \otimes b), \quad T_2(a \otimes b) = (a \otimes 1)\Delta(b) \tag{2.1}$$

are bijective. The bijectivity is equivalent to the existence of a counit and an antipode S satisfying (and defined by)

$$(\varepsilon \otimes \iota)(\Delta(a)(1 \otimes b)) = ab, \quad m(S \otimes \iota)(\Delta(a)(1 \otimes b)) = \varepsilon(a)b, \tag{2.2}$$

$$(\iota \otimes \varepsilon)((a \otimes 1)\Delta(b)) = ab, \quad m(\iota \otimes S)((a \otimes 1)\Delta(b)) = \varepsilon(b)a, \tag{2.3}$$

where $\varepsilon : A \to k$ is a homomorphism, $S : A \to M(A)$ is an anti-homomorphism and m is the multiplication map, considered as a linear map from $A \otimes A$ to A and extended to $M(A) \otimes A$ and $A \otimes M(A)$.

A multiplier Hopf algebra (A, Δ) is called *regular* if $(A, \Delta^{\mathrm{cop}})$ is also a multiplier Hopf algebra, where Δ^{cop} denotes the co-opposite comultiplication defined as $\Delta^{\mathrm{cop}} = \tau \circ \Delta$ with τ the usual flip map from $A \otimes A$ to itself (and extended to $M(A \otimes A)$). In this case, $\Delta(a)(b \otimes 1), (1 \otimes a)\Delta(b) \in A \otimes A$ for all $a, b \in A$. By [13, Proposition 2.9], a multiplier Hopf algebra (A, Δ) is regular if and only if the antipode S is bijective from A to A.

Throughout this paper we freely use the coalgebra, Hopf algebra and multiplier Hopf algebra terminology introduced in [3, 8, 11, 12, 13]. We will use the adapted Sweedler notation (see [14]) for multiplier Hopf algebras, e.g., we write $a_{(1)} \otimes a_{(2)}b$ for $\Delta(a)(1 \otimes b)$ and $ab_{(1)} \otimes b_{(2)}$ for $(a \otimes 1)\Delta(b)$.

2.2. Algebraic quantum groups and their dualities. Assume in what follows that (A, Δ) is a regular multiplier Hopf algebra. A linear functional φ on A is called *left invariant* if $(\iota \otimes \varphi)\Delta(a) = \varphi(a)1$ in $M(A)$ for all $a \in A$. A non-zero left invariant functional φ is called a *left integral* on A. A right integral ψ can be defined similarly.

In general, left and right integrals are unique up to a scalar if they exist. And if a left integral φ exists, a right integral also exists, namely $\varphi \circ S$. Although possibly different, left and right integrals are related. For a left integral φ, there is a unique group-like element (modular element) $\delta \in M(A)$ such that $\varphi(S(a)) = \varphi(a\delta)$ for all $a \in A$.

For an algebraic quantum group (A, Δ) with integrals, define $\widehat{A}$ as the space of linear functionals on A of the form $\varphi(\cdot\, a)$, where $a \in A$. Then $\widehat{A}$ can be made into a regular multiplier Hopf algebra with a product (resp. coproduct $\widehat{\Delta}$ on $\widehat{A}$) dual to the coproduct Δ on A (resp. product of A). It is called the *dual* of (A, Δ). The various objects associated with $(\widehat{A}, \widehat{\Delta})$ are denoted as for (A, Δ) but with a hat. However, we use ε and S also for the counit and antipode on the dual. The dual $(\widehat{A}, \widehat{\Delta})$ also has integrals, i.e., the dual is also an algebraic quantum group. A right integral $\widehat{\psi}$ on $\widehat{A}$ is defined by $\widehat{\psi}(\varphi(\cdot\, a)) = \varepsilon(a)$ for all $a \in A$. Repeating the procedure, i.e., taking the dual of $(\widehat{A}, \widehat{\Delta})$, we can get $\widehat{\widehat{A}} \cong A$ (see [13, Theorem (Biduality) 4.12]).

From [4, Definition 1.5], an algebraic quantum group A is called *counimodular* if the dual multiplier Hopf algebra $\widehat{A}$ is unimodular integral, i.e.,

$\widehat{\delta} = 1$ in $M(A)$. For a counimodular algebraic quantum group, we have $\varphi(ab) = \varphi(bS^2(a))$ for all $a, b \in A$ (see [4, Proposition 1.6]).

3. Drinfeld twists for multiplier Hopf algebras. In [1], the properties of Drinfeld twists for finite-dimensional Hopf algebras were studied. For a given Hopf algebra or quasitriangular Hopf algebra, one gets another such structure by twisting it with a Drinfeld twist, and the authors also determine how the integral of the dual to a finite-dimensional unimodular Hopf algebra changes under a twist (see [1, Theorem 3.4]).

However, the following question remains open:

(Q1) *Does* [1, *Theorem* 3.4] *remain valid for any infinite-dimensional Hopf algebra or any multiplier Hopf algebra?*

To answer this question, we first define a twist of a multiplier Hopf algebra. Let (A, Δ) be a regular multiplier Hopf algebra. We first generalize Drinfeld twists to the multiplier Hopf algebra case, and then construct some new multiplier Hopf algebras by using twists.

Definition 3.1. A *twist* of a regular multiplier Hopf algebra A is an invertible element $J \in M(A \otimes A)$ which satisfies

$$(\Delta \otimes \iota)(J)(J \otimes 1) = (\iota \otimes \Delta)(J)(1 \otimes J). \tag{3.1}$$

Remark. (1) Taking the inverse of (3.1), we get the equivalent equality

$$(J^{-1} \otimes 1)(\Delta \otimes \iota)(J^{-1}) = (1 \otimes J^{-1})(\iota \otimes \Delta)(J^{-1}).$$

Let $\mathcal{R}$ be a quasitriangular structure for a regular multiplier Hopf algebra (A, Δ) (the definition will be recalled in Section 3). Then from [15, Proposition 3] we can see that $\mathcal{R}^{-1}$ is a Drinfeld twist.

(2) Applying $\iota \otimes \varepsilon \otimes \iota$ to (2.1), one sees that as in the Hopf case, $c = (\varepsilon \otimes \iota)(J) = (\iota \otimes \varepsilon)(J)$ is a non-zero scalar for the twist J. One can always replace J by $c^{-1}J$ to normalize the twist in such a way that

$$(\varepsilon \otimes \iota)(J) = (\iota \otimes \varepsilon)(J) = 1. \tag{3.2}$$

In the following, we will always assume that J is normalized in this way.

(3) Let $x \in M(A)$ be an invertible element such that $\varepsilon(x) = 1$. If J is a twist of A, then so is $J^x := \Delta(x)J(x^{-1} \otimes x^{-1})$. Indeed, this is similar to the Hopf algebra case except that we should take the (unique) extension for the homomorphism $\iota \otimes \Delta$ and $\Delta \otimes \iota$ from $A \otimes A$ to $M(A \otimes A)$. The twists J and J^x are said to be *gauge equivalent.*

Example 3.2. (1) Let A and B be regular multiplier Hopf algebras and $\langle A, B \rangle$ be a multiplier Hopf algebra pairing. Let $W \in M(A \otimes B)$ be the canonical element defined in [5, Definition 4.1]. Then it is straightforward to check that

$$J = W_{14} = (\iota \otimes \iota \otimes \tau)(\iota \otimes \tau \otimes \iota)(W \otimes 1 \otimes 1)$$

is a twist for the multiplier Hopf algebra $A^{\mathrm{op}}\otimes B$. Indeed, $(\Delta\otimes\iota)(J)(J\otimes 1) = W_{14}W_{16}W_{36} = (\iota\otimes\Delta)(J)(1\otimes J)$. Moreover, if A is an algebraic quantum group, $\widehat{A}$ is its dual, and $W\in M(\widehat{A}\otimes A)$ is the canonical element, then $J = W_{14}$ is a twist for the multiplier Hopf algebra $\widehat{A}^{\mathrm{op}}\otimes A$.

(2) Let G be an infinite group. Denote by $B = kG$ the group algebra and by $A = k(G)$ the classical multiplier Hopf algebra. Then $\langle k(G), kG\rangle$ is a multiplier Hopf algebra pairing, and

$$J = \sum_{g\in G}(\delta_g\otimes e)\otimes(1\otimes g)$$

is a twist for the multiplier Hopf algebra $k(G)^{\mathrm{op}}\otimes kG$, where e is the group unit and $1 = \sum_{g\in G}\delta_g\in M(A)$.

(3) Let H be a coFrobenius Hopf algebra with a left integral φ and a left cointegral $t\in H$ satisfying $\varphi(t) = 1$. Let $A = \varphi(\cdot\, H)$ be the dual multiplier Hopf algebra. Then by [15, Lemma 9] the element $J = \sum(\varphi(\cdot\, t_{(2)})\otimes 1)\otimes(\varepsilon\otimes t_{(1)})$ is a twist for the multiplier Hopf algebra $A^{\mathrm{op}}\otimes H$.

(4) Let H be a Hopf algebra with a twist J, and A be a multiplier Hopf algebra. Then $H\otimes A$ is a multiplier Hopf algebra with the product, coproduct, counit and antipode defined as follows:

$$\begin{aligned}
&(h\otimes a)(h'\otimes a') = hh'\otimes aa', &&\Delta(h\otimes a) = (\iota\otimes\tau\otimes\iota)(\Delta(h)\otimes\Delta(a)),\\
&\varepsilon(h\otimes a) = \varepsilon(h)\varepsilon(a), &&S(h\otimes a) = S(h)\otimes S(a).
\end{aligned}$$

In this case, there is a twist J_{13} on $H\otimes A$, where $J_{13} = (\iota\otimes\tau\otimes\iota)(J\otimes 1\otimes 1)$.

Furthermore, suppose that $J(1\otimes a), (1\otimes a)J\in M(A)\otimes A$ and $J(a\otimes 1)$, $(a\otimes 1)J\in A\otimes M(A)$ for all $a\in A$. Then the same also holds with J replaced by J^{-1}, and we call J the *Drinfeld twist* for A. We denote $(a\otimes 1)J = aJ^{(1)}\otimes J^{(2)}$ and $J(a\otimes 1) = J^{(1)}a\otimes J^{(2)}$ in $A\otimes M(A)$.

We remark here the above assumptions are reasonable. Take the non-trival Example 3.2(1). The canonical elements W in many specific examples (e.g. Example 3.2(2) and [5, Example 4.9]) satisfy the condition $W(1\otimes b)$, $(1\otimes b)W\in M(A)\otimes B$ and $W(a\otimes 1), (a\otimes 1)W\in A\otimes M(B)$ for all $a\in A$ and $b\in B$. Hence the above assumption holds.

Under the above assumption, we can define a multiplier $Q_J = S(J^{(1)})J^{(2)}\in M(A)$ by

$$\begin{aligned}
Q_J a &= m(S\otimes\iota)(J(1\otimes a))\in A,\\
aQ_J &= m(S\otimes\iota)(J(S^{-1}(a)\otimes 1))\in A
\end{aligned}$$

for all $a\in A$. The multiplier Q_J is invertible with inverse $Q_J^{-1} = J^{-(1)}S(J^{-(2)})$.

Indeed, for any $a, b \in A$,

$$aQ_J^{-1}Q_Jb = aJ^{-(1)}S(J^{-(2)})S(J^{(1)})J^{(2)}b = aJ^{-(1)}S(J^{(1)}J^{-(2)})J^{(2)}b$$

$$= aJ'^{(1)}J^{-(1)}S(J^{(1)}J^{-(2)})\varepsilon(J'^{(2)})J^{(2)}b = aJ'^{(1)}J^{-(1)}S(J'^{(2)}_{(1)}J^{(1)}J^{-(2)})J'^{(2)}_{(2)}J^{(2)}b$$

$$= aJ^{(1)}_{(1)}J'^{(1)}J^{-(1)}S(J^{(1)}_{(2)}J'^{(2)}J^{-(2)})J^{(2)}b = aJ^{(1)}_{(1)}S(J^{(1)}_{(2)})J^{(2)}b = ab$$

and similarly $aQ_J^{-1}Q_Jb = ab$.

The element Q_J satisfies

$$\Delta(Q_J) = (S \otimes S)(J_{21}^{-1})(Q_J \otimes Q_J)J^{-1}, \tag{3.3}$$

where $J_{21}^{-1} = \tau J^{-1}$. These equations make sense because (anti-) homomorphisms can be extended (see [12, Proposition A.5]).

By using the Drinfeld twist J, we can get a new multiplier Hopf algebra as follows; this generalizes the result in [1], and gives a positive answer to the question at the beginning of this section.

Proposition 3.3. *Let (A, Δ) be a regular multiplier Hopf algebra. Then (A^J, Δ^J) is also a regular multiplier Hopf algebra with the same algebra structure and counit as (A, Δ), and the comultiplication and antipode are given by*

$$\Delta^J(a) = J^{-1}\Delta(a)J, \qquad S^J(a) = Q_J^{-1}S(a)Q_J$$

for all $a \in A$.

Proof. It is sufficient to check the equivalent definition, i.e., equalities (2.2) and (2.3). Firstly, it is easy to check that Δ^J is a homomorphism and S^J is a bijective anti-homomorphism. In the following, we only check (2.2), as (2.3) is similar. We have

$$\begin{aligned}(\varepsilon \otimes \iota)(\Delta^J(a)(1 \otimes b)) &= (\varepsilon \otimes \iota)(J^{-1}\Delta(a)J(1 \otimes b))\\ &= (\varepsilon \otimes \iota)(J^{-1})(\varepsilon \otimes \iota)(\Delta(a)(\varepsilon \otimes \iota)(J)b = ab,\end{aligned}$$

where the second equality holds because ε is a homomorphism. Moreover,

$$\begin{aligned}m(S^J \otimes \iota)(\Delta^J(a)(1 \otimes b)) &= m(S^J \otimes \iota)(J^{-1}\Delta(a)J(1 \otimes b))\\ &= m(S^J \otimes \iota)(J^{-(1)}a_{(1)}J'^{(1)} \otimes J^{-(2)}a_{(2)}J'^{(2)}b)\\ &= Q_J^{-1}S(J^{-(1)}a_{(1)}J'^{(1)})Q_JJ^{-(2)}a_{(2)}J'^{(2)}b\\ &= Q_J^{-1}S(J'^{(1)})S(a_{(1)})S(J^{-(1)})Q_JJ^{-(2)}a_{(2)}J'^{(2)}b\\ &= \varepsilon(a)b.\end{aligned}$$

The equality $J^{-1}\Delta(a)J(1 \otimes b) = J^{-(1)}a_{(1)}J'^{(1)} \otimes J^{-(2)}a_{(2)}J'^{(2)}b$ makes sense. Since $J(1 \otimes b) \in A \otimes A$, we denote it as $J'^{(1)} \otimes J'^{(2)}b$, and then we have $\Delta(a)(J'^{(1)} \otimes J'^{(2)}b) \in A \otimes A$ by the "cover" technique of [14]. ∎

Remark. The regular multiplier Hopf algebra A^J admits the Drinfeld twist J^{-1}, because $(\Delta^J \otimes \iota)(J^{-1})(J^{-1} \otimes 1) = (\iota \otimes \Delta^J)(J^{-1})(1 \otimes J^{-1})$ is equivalent to (3.1). It follows from Proposition 3.4 that the regular multiplier Hopf algebra $(A^J)^{J^{-1}}$ is canonically isomorphic to A.

Proposition 3.4. *Let J be a Drinfeld twist for a multiplier Hopf algebra A. Then for any $a, b \in A$,*

$$aS(J^{(1)})J^{(2)}_{(1)} \otimes bJ^{(2)}_{(2)} = (a \otimes b)(Q_J \otimes 1)J^{-1},$$
$$aJ^{-(1)}S(J^{-(2)}_{(1)}) \otimes bS(J^{-(2)}_{(2)}) = (aQ_J^{-1} \otimes b)(S \otimes S)(J).$$

Proof. We only check the first equation, and the second one is similar.

$$\begin{aligned}\big(aS(J^{(1)})J^{(2)}_{(1)} \otimes bJ^{(2)}_{(2)}\big)J &= aS(J^{(1)})J^{(2)}_{(1)}\bar{J}^{(1)} \otimes bJ^{(2)}_{(2)}\bar{J}^{(2)} \\ &\overset{(3.1)}{=} aS(J^{(1)}_{(1)}\bar{J}^{(1)})J^{(1)}_{(2)}\bar{J}^{(2)} \otimes bJ^{(2)} \\ &= aS(\bar{J}^{(1)})S(J^{(1)}_{(1)})J^{(1)}_{(2)}\bar{J}^{(2)} \otimes bJ^{(2)} \\ &= aS(\bar{J}^{(1)})\bar{J}^{(2)} \otimes b = aQ_J \otimes b,\end{aligned}$$

so $(aS(J^{(1)})J^{(2)}_{(1)} \otimes bJ^{(2)}_{(2)})J = aQ_J \otimes b$ for any $a, b \in A$. From the non-degeneracy of the product we have $(S(J^{(1)})J^{(2)}_{(1)} \otimes J^{(2)}_{(2)})J = Q_J \otimes 1$, and by multiplying by J^{-1} on the right we get the assertion. ∎

Recall from [7] that if A is a regular multiplier Hopf algebra, then an algebra X is called an A-*module algebra* if X is a unital A-module (we denote the A-action by $\triangleright$, then $A \triangleright X = X$) and $a \triangleright (xx') = (a_{(1)} \triangleright x)(a_{(2)} \triangleright x')$ for any $a \in A$ and $x, x' \in X$. In the following, we consider two kinds of A^J-module algebras and their relations.

Proposition 3.5. *Let A be a regular multiplier Hopf algebra with a Drinfeld twist J, and X an A-module algebra (not necessarily with unit). Then there exists an A^J-module algebra $X_\star$, which has the same K-module structure as A and the action of A^J on $X_\star$ is that of A on X. The product of $X_\star$ is defined by*

$$x \star y = m(J \triangleright (x \otimes y)) \tag{3.4}$$

for any $x, y \in X$.

Proof. First it is easy to show the product is well-defined. Then we check the associativity of this new product: for all $x, y, z \in X$,

$$\begin{aligned}(x \star y) \star z = m(J \triangleright (x \otimes y)) \star z &= m\big(J \triangleright (m(J \triangleright (x \otimes y)) \otimes z)\big) \\ &= m(m \otimes \iota)\big((\Delta \otimes \iota)(J)(J \otimes 1) \triangleright (x \otimes y \otimes z)\big) \\ &= m(\iota \otimes m)\big((\iota \otimes \Delta)(J)(1 \otimes J) \triangleright (x \otimes y \otimes z)\big) \\ &= m\big(J \triangleright \big(x \otimes m(J \triangleright (y \otimes z))\big)\big) = x \star (y \star z).\end{aligned}$$

Finally, we need to prove that the product in $A_\star$ is compatible with the multiplier Hopf algebra structure of A^J. For all $a \in A^J$ and $x, y \in X$,

$$\begin{aligned} a \triangleright (x \star y) &= m(\Delta(a)J \triangleright (x \otimes y)) = m(J\Delta^J(a) \triangleright (x \otimes y)) \\ &= m(J \triangleright (a^J_{(1)} \triangleright x \otimes a^J_{(2)} \triangleright y)) = (a^J_{(1)} \triangleright x) \star (a^J_{(2)} \triangleright y), \end{aligned}$$

where $\Delta^J(a)(1 \otimes b) = a^J_{(1)} \otimes a^J_{(2)} b$. ∎

We recall from [2] that an (X, Y)-*bimodule* is a left X-module and a right Y-module V satisfying the compatibility condition: $(x \cdot v) \cdot y = x \cdot (v \cdot y)$ for $x \in X$, $y \in Y$ and $v \in V$. Now, we consider a unital (X, Y)-bimodule V (i.e., $X \cdot V = V$ and $V \cdot Y = V$), where X and Y are A-module algebras and V is also a left unital A-module. Compatibility between the structure of A and the (X, Y)-bimodule structure leads to the following covariance requirement.

Definition 3.6. Let A be a regular multiplier Hopf algebra, and X, Y be A-module algebras. A *left A-module (X, Y)-bimodule* (or ${}_{A,X}\mathcal{M}_Y$*-module*) is an (X, Y)-bimodule V which is also a unital left A-module such that for any $a \in A$, $x \in X$, $y \in Y$ and $v \in V$,

$$a \triangleright (x \cdot v) = (a_{(1)} \triangleright x) \cdot (a_{(2)} \triangleright v), \tag{3.5}$$

$$a \triangleright (v \cdot y) = (a_{(1)} \triangleright v) \cdot (a_{(2)} \triangleright y). \tag{3.6}$$

An algebra E is a *left A-module (X, Y)-bimodule algebra* (or ${}_{A,X}\mathcal{M}_Y$*-algebra*) if E is an ${}_{A,X}\mathcal{M}_Y$-module and also an A-module algebra.

Proposition 3.7. *Let A be a regular multiplier Hopf algebra with a Drinfeld twist J, and let X and Y be A-module algebras. Given an ${}_{A,X}\mathcal{M}_Y$-module V, there exists an ${}_{A^J,X_\star}\mathcal{M}_{Y_\star}$-module $V_\star$. Here $V_\star = V$ as vector spaces and the left action of A^J on $V_\star$ is that of A on V. The $X_\star$- and $Y_\star$-actions on $V_\star$ are respectively given by*

$$x \star v = \cdot \circ J \triangleright (x \otimes v), \tag{3.7}$$

$$v \star y = \cdot \circ J \triangleright (v \otimes y). \tag{3.8}$$

If $V = E$ is further an ${}_{A,X}\mathcal{M}_Y$-algebra, then $E_\star$ is an ${}_{A^J,X_\star}\mathcal{M}_{Y_\star}$-algebra with the product given in Proposition 3.5.

Proof. First, we need to check that the two actions on $V_\star$ are well-defined, i.e., $(xx') \star v = x \star (x' \star v)$ and $v \star (yy') = (v \star y) \star y'$. Here we only check the first equation, the second one is similar:

$$\begin{aligned} (xx') \star v &= (m(J \triangleright (x \otimes y))) \star v = \cdot \circ J \triangleright (m(J \triangleright (x \otimes y)) \otimes v), \\ &= \cdot \circ (m \otimes \iota)\big((\Delta \otimes \iota)(J)(J \otimes 1) \triangleright (x \otimes x' \otimes v)\big) \\ &= \cdot \circ (\iota \otimes \cdot)\big((\iota \otimes \Delta)(J)(1 \otimes J) \triangleright (x \otimes x' \otimes v)\big) \\ &= \cdot \circ J \triangleright (x \otimes \cdot \circ J \triangleright (x' \otimes v)) = \cdot \circ J \triangleright (x \otimes x' \star v) = x \star (x' \star v). \end{aligned}$$

Then we check the compatibility of the left A^J-action and the left $X_\star$-action. For any $a \in A^J$, $x \in X$ and $v \in V$,

$$\begin{aligned} a \triangleright (x \star v) &= a \triangleright (\cdot \circ J \triangleright (x \otimes v)) = \cdot \circ (\Delta(a)J) \triangleright (x \otimes v) \\ &= \cdot \circ (J\Delta^J(a)) \triangleright (x \otimes v) = \cdot \circ J \triangleright \circ \Delta^J(a) \triangleright (x \otimes v) \\ &= (a^J_{(1)} \triangleright x) \star (a^J_{(2)} \triangleright v). \end{aligned}$$

Compatibility of the left A^J-action and the right $Y_\star$-action is proved similarly.

Finally, if $V = E$ is an A-module algebra, then by Proposition 3.5, $E_\star$ is an A^J-module algebra. Combining the above results, we can easily see that $E_\star$ is an ${}_{A^J,X_\star}\mathcal{M}_{Y_\star}$-algebra. ∎

REMARK. The equations (3.7) and (3.8) are well-defined. Indeed, e.g. for (3.7), because X and V are unital A-modules, there exists $a_i, b_j \in A$, $x_i \in X$ and $v_i \in V$ such that $x = \sum_i a_i \triangleright x_i$ and $v = \sum_j b_j \triangleright v_j$. Then

$$\cdot \circ J \triangleright (x \otimes v) = \sum_{i,j} \cdot \circ J(a_i \otimes b_j) \triangleright (x_i \otimes v_j) = \sum_{i,j} (J^{(1)} a_i \triangleright x_i) \cdot (J^{(2)} b_j \triangleright v_j),$$

where $\sum_{i,j} J^{(1)} a_i \otimes J^{(2)} b_j \in A \otimes A$. Therefore the equation (3.7) is reasonable, and (3.8) is similar.

Given a unital A-bimodule V, we can consider the adjoint action: for $a \in A$ and $v \in V$, we have $a \blacktriangleright v = a_{(1)} \cdot v \cdot S(a_{(2)}) := a_{(1)} v S(a_{(2)})$. In the following, we will consider the algebra isomorphism between X and $X_\star$.

PROPOSITION 3.8. *Consider a regular multiplier Hopf algebra A and an A-bimodule X that is also an algebra (not necessarily with unit). If for all $a \in A$ and $x, x' \in X$, the "generalized associativity" conditions*

$$(xx')a = x(x'a), (xa)x' = x(ax'), a(xx') = (ax)x'$$

hold, then the adjoint action makes X an A-module algebra. Given a twist J of the regular multiplier Hopf algebra A, the twist deformed algebra $X_\star$ is isomorphic to X via the map

$$D_J : X_\star \to X, \qquad x \mapsto D_J(x) := (J^{(1)} \blacktriangleright x) J^{(2)}.$$

Proof. Firstly, we check that X is an A-module algebra, i.e., $a \blacktriangleright (xx') = (a_{(1)} \blacktriangleright x)(a_{(2)} \blacktriangleright x')$. Indeed,

$$\begin{aligned} (a_{(1)} \blacktriangleright x)(a_{(2)} \blacktriangleright x') &= (a_{(1)} x S(a_{(2)}))(a_{(3)} x' S(a_{(4)})) \\ &= a_{(1)} \big((xS(a_{(2)}))(a_{(3)} x' S(a_{(4)}))\big) \\ &= a_{(1)} \big(x[S(a_{(2)})(a_{(3)} x' S(a_{(4)}))]\big) \\ &= a_{(1)} \big(x[S(a_{(2)}) a_{(3)} (x' S(a_{(4)}))]\big) \\ &= a_{(1)} \big(x(x' S(a_{(2)}))\big) = a_{(1)} (xx') S(a_{(2)}) = a \blacktriangleright (xx'). \end{aligned}$$

Next, we need to show that D_J is an isomorphism. Obviously D_J is a K-linear map. It is also an algebra homomorphism, since for $x, x' \in X$,

$$\begin{aligned} D_J(x \star x') &= D_J\big((J^{(1)} \blacktriangleright x)(J^{(2)} \blacktriangleright x')\big) \\ &= \bar{J}^{(1)} \blacktriangleright \big((J^{(1)} \blacktriangleright x)(J^{(2)} \blacktriangleright x')\big)\bar{J}^{(2)} \\ &= \big((\bar{J}^{(1)}_{(1)} J^{(1)} \blacktriangleright x)(\bar{J}^{(1)}_{(2)} J^{(2)} \blacktriangleright x')\big)\bar{J}^{(2)} \\ &\overset{(3.1)}{=} (\bar{J}^{(1)} \blacktriangleright x)(\bar{J}^{(2)}_{(1)} J^{(1)} \blacktriangleright x')\bar{J}^{(2)}_{(2)} J^{(2)} \\ &= (\bar{J}^{(1)} \blacktriangleright x)\big(\bar{J}^{(2)}_{(1)} \blacktriangleright (J^{(1)} \blacktriangleright x')\big)\bar{J}^{(2)}_{(2)} J^{(2)} \\ &= (\bar{J}^{(1)} \blacktriangleright x)\big(\bar{J}^{(2)}_{(1)} (J^{(1)} \blacktriangleright x') S(\bar{J}^{(2)}_{(2)})\big)\bar{J}^{(2)}_{(3)} J^{(2)} \\ &= (\bar{J}^{(1)} \blacktriangleright x)\bar{J}^{(2)}(J^{(1)} \blacktriangleright x')J^{(2)} = D_J(x)D_J(x'). \end{aligned}$$

Finally, we need to check that D_J is invertible. In fact, for $x \in X$, the inverse is given by $D_J^{-1}(x) = J^{(1)} x J^{-(1)} S(J^{(2)} J^{-(2)}) = J^{(1)} x Q_J^{-1} S(J^{(2)})$. ∎

Remark. Using (3.1) we can easily see that

$$\begin{aligned} D_J(x) &= (J^{(1)} \blacktriangleright x)J^{(2)} = J^{(1)}_{(1)} x S(J^{(1)}_{(2)})J^{(2)} \\ &= J^{(1)} J^{(-1)} x S(J^{(2)}_{(1)} J'^{(1)} J^{(-2)}) J^{(2)}_{(2)} J'^{(2)} = J^{(-1)} x S(J^{(-2)}) Q_J, \end{aligned}$$

and $D_{J^{-1}}(x) = D_J^{-1}(x)$.

Example 3.9. Given a unital left A-module algebra X (not necessarily with unit), we consider the smash product $A \# X$. By definition $A \# X = A \otimes X$ as a vector space, and the product is given by $(a \# x)(a' \# x') = a(a'_{(1)} \cdot x) \# a'_{(2)} x'$, which we simply rewrite as

$$a x a' x' = a(a'_{(1)} \cdot x) a'_{(2)} x'.$$

The algebra $A \# X$ is an A-module algebra with the action $a \blacktriangleright (x a') = (a_{(1)} \cdot x)(a_{(2)} \blacktriangleright a')$. The right A-module structure is given by $(xa)a' = x(aa')$. Then by the above proposition the deformed algebra $(A \# X)_\star$ is isomorphic to $A \# X$.

Under the hypotheses of Proposition 3.8, the algebra X has an A^J-module algebra structure given by the A^J-adjoint action: for $a \in A^J$ and $x \in X$,

$$a \blacktriangleright_J x := a^J_{(1)} x S^J(a^J_{(2)}). \tag{3.9}$$

We denote this A^J-module algebra by $(X, \blacktriangleright_J)$. Then we have the following result, which generalizes [2, Theorem 3.10].

Theorem 3.10. *The algebra isomorphism $D_J : X_\star \to X$ is also an isomorphism between the A^J-module algebras $(X_\star, \blacktriangleright)$ and $(X, \blacktriangleright_J)$, i.e., D_J intertwines the A^J-actions $\blacktriangleright$ and $\blacktriangleright_J$: for any $a \in A^J$ and $x \in X$,*

$$D_J(a \blacktriangleright x) = a \blacktriangleright_J D_J(x). \tag{3.10}$$

Proof. For any $a \in A^J$ and $x \in X$,

$$\begin{aligned} D_J(a \blacktriangleright x) &= J^{(-1)}(a \blacktriangleright x)S(J^{(-2)})Q_J = J^{(-1)}a_{(1)}xS(a_{(2)})S(J^{(-2)})Q_J \\ &= J^{(-1)}a_{(1)}J^{(1)}J'^{-(1)}xS(J^{(-2)}a_{(2)}J^{(2)}J'^{-(2)})Q_J \\ &= a^J_{(1)}J'^{-(1)}xS(a^J_{(2)}J'^{-(2)})Q_J = a^J_{(1)}J'^{-(1)}xS(J'^{-(2)})S(a^J_{(2)})Q_J \\ &= a^J_{(1)}J'^{-(1)}xS(J'^{-(2)})Q_JQ_J^{-1}S(a^J_{(2)})Q_J = a^J_{(1)}D_J(x)S^J(a^J_{(2)}) \\ &= a \blacktriangleright_J D_J(x). \ \blacksquare \end{aligned}$$

4. Quasitriangular structure and integral under a twist. Recall from [15] a regular multiplier Hopf algebra (A, Δ) is called *quasitriangular* if there exists an invertible multiplier $\mathcal{R}$ in $M(A \otimes A)$ which satisfies

(1) $(\Delta \otimes \iota)(\mathcal{R}) = \mathcal{R}^{13}\mathcal{R}^{23}$, $(\iota \otimes \Delta)(\mathcal{R}) = \mathcal{R}^{13}\mathcal{R}^{12}$,
(2) $\mathcal{R}\Delta(a) = \Delta^{\mathrm{cop}}(a)\mathcal{R}$ for all $a \in A$,
(3) $(\iota \otimes \varepsilon)(\mathcal{R}) = 1 = (\varepsilon \otimes \iota)(\mathcal{R})$.

Let (A, Δ) be a multiplier Hopf algebra. By Proposition 3.4, (A^J, Δ^J) is also a multiplier Hopf algebra. When (A, Δ) is quasitriangular, how about (A^J, Δ^J)?

THEOREM 4.1. *Let (A, Δ) be a quasitriangular multiplier Hopf algebra with generalized R-matrix $\mathcal{R}$. Then (A^J, Δ^J) is also quasitriangular, and the quasitriangular structure is given by*

$$\mathcal{R}^J = J_{21}^{-1}\mathcal{R}J.$$

Proof. It is sufficient to check the conditions (1)–(3) in the definition of quasitriangularity. Firstly, we check $\mathcal{R}^J\Delta^J(a) = (\Delta^J)^{\mathrm{cop}}(a)\mathcal{R}^J$. In fact,

$$\begin{aligned} \mathcal{R}^J\Delta^J(a) &= J_{21}^{-1}\mathcal{R}JJ^{-1}\Delta(a)J = J_{21}^{-1}\mathcal{R}\Delta(a)J, \\ (\Delta^J)^{\mathrm{cop}}(a)\mathcal{R}^J &= \tau(J^{-1}\Delta(a)J)J_{21}^{-1}\mathcal{R}J = J_{21}^{-1}\Delta^{\mathrm{cop}}(a)J_{21}J_{21}^{-1}\mathcal{R}J \\ &= J_{21}^{-1}\Delta^{\mathrm{cop}}(a)\mathcal{R}J. \end{aligned}$$

Because (A, Δ) is quasitriangular, we have $\mathcal{R}\Delta(a) = \Delta^{\mathrm{cop}}(a)\mathcal{R}$, so the desired equality holds.

Secondly, by extending the homomorphism ε, it is easy to check that $(\iota \otimes \varepsilon)(\mathcal{R}^J) = 1 = (\varepsilon \otimes \iota)(\mathcal{R}^J)$.

Finally, we need to check $(\Delta^J \otimes \iota)\mathcal{R}^J = \mathcal{R}^{J^{13}}\mathcal{R}^{J^{23}}$ and $(\iota \otimes \Delta^J)\mathcal{R}^J = \mathcal{R}^{J^{13}}\mathcal{R}^{J^{12}}$. We only prove the first equality; the verification of the second is similar.

For any $x \otimes y \in A \otimes A$, we have $x \otimes y = \sum_i \Delta^J(a_i)(1 \otimes b_i)$ by Proposition 4.3. Thus

$$
\begin{aligned}
&(\Delta^J \otimes \iota)(\mathcal{R}^J)(x \otimes y \otimes z) = (\Delta^J \otimes \iota)(\mathcal{R}^J)\Big(\sum_i \Delta^J(a_i)(1 \otimes b_i) \otimes z\Big) \\
&\quad = \sum_i (\Delta^J \otimes \iota)(\mathcal{R}^J(a_i \otimes z))(1 \otimes b_i \otimes 1) \\
&\quad = \sum_i (J^{-1} \otimes 1)(\Delta \otimes \iota)(J_{21}^{-1}\mathcal{R}J(a_i \otimes z))(J \otimes 1)(1 \otimes b_i \otimes 1) \\
&\quad = \sum_i (J^{-1} \otimes 1)(\Delta \otimes \iota)(J_{21}^{-1})(\Delta \otimes \iota)(\mathcal{R})(\Delta \otimes \iota)(J(a_i \otimes z)) \\
&\qquad \cdot (J \otimes 1)(1 \otimes b_i \otimes 1) \\
&\quad = \sum_i (J^{-1} \otimes 1)(\Delta \otimes \iota)(J_{21}^{-1})\mathcal{R}^{13}\mathcal{R}^{23}(\Delta \otimes \iota)(J)(\Delta \otimes \iota)(a_i \otimes z) \\
&\qquad \cdot (J \otimes 1)(1 \otimes b_i \otimes 1) \\
&\quad = \sum_i (J^{-1} \otimes 1)(\Delta \otimes \iota)(J_{21}^{-1})\mathcal{R}^{13}\mathcal{R}^{23}(\Delta \otimes \iota)(J)(J \otimes 1) \\
&\qquad \cdot (J^{-1} \otimes 1)(\Delta \otimes \iota)(a_i \otimes z)(J \otimes 1)(1 \otimes b_i \otimes 1) \\
&\quad = \sum_i (J^{-1} \otimes 1)(\Delta \otimes \iota)(J_{21}^{-1})\mathcal{R}^{13}\mathcal{R}^{23}(\iota \otimes \Delta)(J)(1 \otimes J) \\
&\qquad \cdot (J^{-1}\Delta(a_i)J \otimes 1)(1 \otimes b_i \otimes z) \\
&\quad = (\dot{J}^{-(1)}J_{(1)}^{-(2)} \otimes \dot{J}^{-(2)}J_{(2)}^{-(2)} \otimes J^{-(1)})\mathcal{R}^{13}\mathcal{R}^{23}(\iota \otimes \Delta)(J)(1 \otimes J)(x \otimes y \otimes z) \\
&\quad = \underline{\dot{J}^{-(1)}J_{(1)}^{-(2)}}\mathcal{R}^{(1)}J'^{(1)}x \otimes \underline{\dot{J}^{-(2)}J_{(2)}^{-(2)}}\mathcal{R}'^{(1)}J'^{(2)}_{(1)}\dot{J}'^{(1)}y \\
&\qquad \otimes \underline{J^{-(1)}}\mathcal{R}^{(2)}\mathcal{R}'^{(2)}J'^{(2)}_{(2)}\dot{J}'^{(2)}z \\
&\quad = J^{-(2)}\dot{J}_{(2)}^{-(1)}\mathcal{R}^{(1)}J'^{(1)}x \otimes \dot{J}^{-(2)}\underline{\mathcal{R}'^{(1)}J'^{(2)}_{(1)}}\dot{J}'^{(1)}y \\
&\qquad \otimes J^{-(1)}\dot{J}_{(1)}^{-(1)}\mathcal{R}^{(2)}\underline{\mathcal{R}'^{(2)}J'^{(2)}_{(2)}}\dot{J}'^{(2)}z \\
&\quad = J^{-(2)}\underline{\dot{J}_{(2)}^{-(1)}\mathcal{R}^{(1)}}J'^{(1)}x \otimes \dot{J}^{-(2)}J'^{(2)}_{(2)}\mathcal{R}'^{(1)}\dot{J}'^{(1)}y \\
&\qquad \otimes J^{-(1)}\underline{\dot{J}_{(1)}^{-(1)}\mathcal{R}^{(2)}}J'^{(2)}_{(1)}\mathcal{R}'^{(2)}\dot{J}'^{(2)}z \\
&\quad = J^{-(2)}\mathcal{R}^{(1)}\underline{\dot{J}_{(1)}^{-(1)}J'^{(1)}}x \otimes \underline{\dot{J}^{-(2)}J'^{(2)}_{(2)}}\mathcal{R}'^{(1)}\dot{J}'^{(1)}y \\
&\qquad \otimes J^{-(1)}\mathcal{R}^{(2)}\underline{\dot{J}_{(2)}^{-(1)}J'^{(2)}_{(1)}}\mathcal{R}'^{(2)}\dot{J}'^{(2)}z \\
&\quad = J^{-(2)}\mathcal{R}^{(1)}J^{(1)}x \otimes \dot{J}^{-(2)}\mathcal{R}'^{(1)}\dot{J}'^{(1)}y \otimes J^{-(1)}\mathcal{R}^{(2)}J^{(2)}J'^{-(1)}\mathcal{R}'^{(2)}\dot{J}'^{(2)}z \\
&\quad = \mathcal{R}^{J^{13}}\mathcal{R}^{J^{23}}(x \otimes y \otimes z),
\end{aligned}
$$

where the penultimate equation holds because of $(\Delta \otimes \iota)(J^{-1})(\iota \otimes \Delta)(J) = (J \otimes 1)(1 \otimes J)$. ∎

From [6, Theorem 4.6 and Proposition 2.6], we can easily get the following result.

PROPOSITION 4.2. *Let (A,Δ) be a quasitriangular multiplier Hopf algebra and J a twist. Then for all $a\in A$,*

$$(S^J)^4(a)=gag^{-1},$$

where $g=\mu S(\mu)^{-1}$ and $\mu=S^J(\mathcal{R}^{J(2)})\mathcal{R}^{J(1)}$.

Because the integral plays an important role in Pontryagin duality, then the following question naturally arises:

(Q2) *For a regular multiplier Hopf algebra (A,Δ) with an integral, does (A^J,Δ^J) also admit an integral?*

Before answering this question, we first consider a multiplier $u_J=Q_J^{-1}S(Q_J)$ in $M(A)$. By (3.3), we have

$$\begin{aligned}\Delta(u_J)&=\Delta(Q_J^{-1}S(Q_J))=J(Q_J^{-1}S(Q_J)\otimes Q_J^{-1}S(Q_J))(S^2\otimes S^2)(J^{-1})\\&=J(u_J\otimes u_J)(S^2\otimes S^2)(J^{-1}).\end{aligned}$$

THEOREM 4.3. *Let (A,Δ) be a counimodular algebraic quantum group with a non-zero left (resp. right) integral φ (resp. ψ) and J be a Drinfeld twist. Then the elements $\varphi^J=u_J\rightharpoonup\varphi$ and $\psi^J=\psi\leftharpoonup u_J^{-1}$ are non-zero left and right integrals on (A^J,Δ^J) respectively.*

Proof. We need to check that $(\iota\otimes\varphi^J)\Delta^J(a)=\varphi^J(a)1$, equivalently $S(\iota\otimes\varphi^J)\Delta^J(a)=\varphi^J(a)1$. Indeed, for any $x\in A$, there exists $b\in A$ such that $x=S(b)$. Thus

$$\begin{aligned}&S(\iota\otimes\varphi^J)(\Delta^J(a))x=S(\iota\otimes u_J\rightharpoonup\varphi)(J^{-1}\Delta(a)J)x\\&=S(\iota\otimes\varphi)(J^{-1}\Delta(a)J(1\otimes u_J))S(b)=S(\iota\otimes\varphi)((b\otimes 1)J^{-1}\Delta(a)J(1\otimes u_J))\\&=S(\iota\otimes\varphi)(bJ^{-(1)}a_{(1)}\dot{J}^{(1)}\otimes J^{-(2)}a_{(2)}\dot{J}^{(2)}u_J)\\&=S(bJ^{-(1)}a_{(1)}\dot{J}^{(1)})\varphi(J^{-(2)}a_{(2)}\dot{J}^{(2)}u_J)\\&=S(\dot{J}^{(1)})\underline{S(a_{(1)})}S(J^{-(1)})S(b)\underline{\varphi(a_{(2)}\dot{J}^{(2)}u_JS^2(J^{-(2)}))}\\&\overset{(1)}{=}S(\dot{J}^{(1)})\dot{J}^{(2)}_{(1)}u_{J(1)}S^2(J^{-(2)}_{(1)})S(J^{-(1)})S(b)\varphi(a\dot{J}^{(2)}_{(2)}u_{J(2)}S^2(J^{-(2)}_{(2)}))\\&=(\iota\otimes\varphi)\Big((1\otimes a)(\underline{S(\dot{J}^{(1)})\dot{J}^{(2)}_{(1)}\otimes\dot{J}^{(2)}_{(2)}})\Delta(u_J)\\&\quad\cdot\big((S\otimes S)(b\underline{\underline{J^{-(1)}S(J^{-(2)}_{(1)})\otimes S(J^{-(2)}_{(2)})}}))\big)\Big)\\&\overset{(2)}{=}(\iota\otimes\varphi)\Big((1\otimes a)((Q_J\otimes 1)J^{-1})\Delta(u_J)\big((S\otimes S)((bQ_{J^{-1}}\otimes 1)(S\otimes S)(J))\big)\Big)\end{aligned}$$

$$
\begin{aligned}
&= (\iota\otimes\varphi)\Big((1\otimes a)((Q_J\otimes 1)J^{-1})(J(u_J\otimes u_J)(S^2\otimes S^2)(J^{-1}))\\
&\qquad\cdot\big((S^2\otimes S^2)(J)(S(Q_{J^{-1}})\otimes 1)(S(b)\otimes 1)\big)\Big)\\
&= (\iota\otimes\varphi)\big((1\otimes a)(Q_J\otimes 1)(u_J\otimes u_J)(S(Q_{J^{-1}})\otimes 1)(S(b)\otimes 1)\big)\\
&= (\iota\otimes\varphi)\big((1\otimes a)(1\otimes u_J)(S(b)\otimes 1)\big) = \varphi^J(a)x,
\end{aligned}
$$

where (1) holds because $S(\iota\otimes\varphi)(\Delta(a)(1\otimes b)) = (\iota\otimes\varphi)((1\otimes a)\Delta(b))$ and (2) holds because of Proposition 2.4. ∎

Note that Theorem 4.3 contains an answer to the question (Q) in the Introduction. From Theorem 4.3 we can easily get the following result in the Hopf algebra case.

COROLLARY 4.4. *Let (H,Δ) be an infinite-dimensional counimodular coFrobenius Hopf algebra. Suppose φ (resp. ψ) is the non-zero left (resp. right) integral on H and $J\in H\otimes H$ is a twist. Then the elements $\varphi^J = u_J \rightharpoonup \varphi$ and $\psi^J = \psi \leftharpoonup u_J^{-1}$ are non-zero left and right integrals on (H^J,Δ^J) respectively.*

EXAMPLE 4.5. Let G be an (infinite) abelian group, let kG be its group algebra with coefficients in a field k, and let $k(G)$ be the dual multiplier Hopf algebra. Then $D(G) = k(G)^{\mathrm{cop}}\bowtie kG$ is the Drinfeld double with the following quasitriangular multiplier Hopf algebra structure. For any $g,h,p,q\in G$,

$$
\begin{aligned}
&(\delta_g\bowtie p)(\delta_h\bowtie q) = \delta_g\delta_{php^{-1}}\bowtie pq, \qquad && \varepsilon(\delta_g\bowtie p) = \delta_{g,e},\\
&\Delta(\delta_g\bowtie p) = \sum_{s\in G}(\delta_{s^{-1}g}\bowtie p)\otimes(\delta_s\bowtie p), && S(\delta_g\bowtie p) = \delta_{p^{-1}g^{-1}p}\bowtie p^{-1},\\
&\mathcal{R} = \sum_{g\in G}(1\bowtie g)\otimes(\delta_g\bowtie e).
\end{aligned}
$$

In this case, the Drinfeld twist J equals $\mathcal{R}^{-1} = \sum_{g\in G}(1\bowtie g)\otimes(\delta_{g^{-1}}\bowtie e)$, where $1=\sum_{g\in G}\delta_g$. By Theorem 4.1 the quasitriangular structure in $D(G)^J$ is given by

$$
\mathcal{R}^J = J_{21}^{-1}\mathcal{R}J = \sum_{g\in G}(\delta_g\bowtie e)\otimes(1\bowtie g).
$$

Since $Q_J = S(J^{(1)})J^{(2)} = \sum_{g\in G}\delta_g\bowtie g^{-1}$ and $u_J = Q_J^{-1}S(Q_J) = 1\bowtie e$, by Theorem 4.3 the left and right integrals on $(D(G)^J,\Delta^J)$ are given by $\varphi^J = \psi^J = f\otimes\delta_e$, where f maps every δ_g, $g\in G$, to 1.

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