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A CONSTRUCTION OF RELATIVE LEFT DERIVED FUNCTORS SIMILAR TO Tor

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Abstract. We use the notion of $\mathcal{S}$-purity and methods of relative homological algebra to construct balanced left derived functors similar to Tor.

1. Introduction. Throughout this paper, R will denote an associative ring with identity and all modules will be assumed unitary. The notion of purity was introduced by P. M. Cohn [3] for left R-modules and by J. Łoś [15] for abelian groups; see also J.-M. Maranda [16]. In 1967, R. Kiełpiński [12] introduced the notion of relative Γ-purity. Two years later, R. B. Warfield [22] introduced the notion of $\mathcal{S}$-purity for any class $\mathcal{S}$ of R-modules. The notion of purity has been gradually developed in various directions with important applications to algebra and model theory. For nice results on various notions of purity, the reader is referred to [3], [4], [6]–[8], [10]–[23] and [25]. In particular, using pure projective resolutions and pure injective resolutions of R-modules M and N (respectively r.d.-projective resolutions of an R-module M), the pure-extension groups $\mathrm{Pext}^n_R(M,N)$ or $\mathrm{Pext}^n(M,N)$ (respectively the r.d. pure-extension groups $\mathrm{Pext}^n_{\mathrm{rd}}(M,N)$), the pure projective dimension, r.d.-projective dimension, pure-left global dimension and r.d.-left global dimension of R were defined and applied in [8], [11], [13] and [19]–[21]. Moreover, pure projective dimension, pure injective dimension and pure global dimension were studied in the framework of functor categories in [19]–[21].

Let $\mathcal{S}$ be a class of finitely presented left R-modules containing R. Assume in addition that $\mathcal{S}$ contains a set $\mathcal{S}^*$ such that every module in $\mathcal{S}$ is isomorphic to a module in $\mathcal{S}^*$. In that situation, the $\mathcal{S}$-pure projective dimension, $\mathcal{S}$-pure injective dimension and left global $\mathcal{S}$-pure projective dimension of R (equal to the left global $\mathcal{S}$-pure injective dimension of R) were defined in [25].

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For a given class $\mathcal{S}$ of left (respectively right) R-modules, a short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ of left (respectively right) R-modules and left (respectively right) R-homomorphisms is called *$\mathcal{S}$-pure exact* if the induced homomorphism $\mathrm{Hom}_R(U, B) \to \mathrm{Hom}_R(U, C)$ is surjective for all $U \in \mathcal{S}$.

Let $\mathcal{S}$ be a class of left R-modules. A left R-module M is called *$\mathcal{S}$-pure projective* (respectively *$\mathcal{S}$-pure injective*) if the functor $\mathrm{Hom}_R(M, -)$ (respectively $\mathrm{Hom}_R(-, M)$) leaves any $\mathcal{S}$-pure exact sequence of left R-modules and left R-homomorphisms exact. Also, a right R-module M is called *$\mathcal{S}$-pure flat* if the functor $M \otimes_R -$ leaves any $\mathcal{S}$-pure exact sequence of left R-modules and left R-homomorphisms exact.

For a given class $\mathcal{S}$ of right R-modules, a right R-module M is called *$\mathcal{S}$-pure projective* (respectively *$\mathcal{S}$-pure injective*) if the functor $\mathrm{Hom}_R(M, -)$ (respectively $\mathrm{Hom}_R(-, M)$) leaves any $\mathcal{S}$-pure exact sequence of right R-modules and right R-homomorphisms exact. A left R-module M is called *$\mathcal{S}$-pure flat* if the functor $- \otimes_R M$ leaves any $\mathcal{S}$-pure exact sequence of right R-modules and right R-homomorphisms exact.

Let $\mathcal{S}$ be an arbitrary class of left R-modules. The classes of all $\mathcal{S}$-pure projective left R-modules, of all $\mathcal{S}$-pure injective left R-modules and of all $\mathcal{S}$-pure flat right R-modules will be denoted by $\mathcal{SPL}$, $\mathcal{SIL}$ and $\mathcal{SFR}$, respectively. Also, for a given class $\mathcal{S}$ of right R-modules, the classes of all $\mathcal{S}$-pure projective right R-modules, of all $\mathcal{S}$-pure injective right R-modules and of all $\mathcal{S}$-pure flat left R-modules will be denoted by $\mathcal{SPR}$, $\mathcal{SIR}$ and $\mathcal{SFL}$, respectively.

In this paper, our aim is to construct some left derived functors similar to $\mathrm{Tor}_n^R(-, \sim)$. Assume that $\mathcal{S}$ is a class of (R, R)-bimodules with the property that every module in $\mathcal{S}$ is also an $\mathcal{S}$-pure flat left and right R-module. Let M be a right R-module and N be a left R-module. Let $\mathbf{F}_\circ$ and $\mathbf{F}'_\circ$ denote the deleted complexes corresponding to the minimal left $\mathcal{SFR}$-resolution of M and the minimal left $\mathcal{SFL}$-resolution of N, respectively. We show that $- \otimes_R \sim$ is left balanced by $\mathcal{SFR} \times \mathcal{SFL}$ and $\mathrm{H}_n(\mathbf{F}_\circ \otimes_R N) \cong \mathrm{H}_n(M \otimes_R \mathbf{F}'_\circ)$ for all non-negative integers n (Proposition 3.2 and Theorem 3.4). Assume in addition that $\mathcal{SPL}$ is a precovering class contained in $\mathcal{SFL}$, and $\mathcal{SPR}$ is a precovering class contained in $\mathcal{SFR}$. Let $\mathbf{P}_\circ$ and $\mathbf{P}'_\circ$ denote the deleted complexes corresponding to a left $\mathcal{SPR}$-resolution of M and a left $\mathcal{SPL}$-resolution of N, respectively. Then, in Proposition 3.3 and Theorem 3.5, we prove that $- \otimes_R \sim$ is left balanced by $\mathcal{SPR} \times \mathcal{SPL}$ and $\mathrm{H}_n(\mathbf{P}_\circ \otimes_R N) \cong \mathrm{H}_n(M \otimes_R \mathbf{P}'_\circ)$ for all non-negative integers n (see also Examples 3.6 and 3.7). For all non-negative integers n we will denote $\mathrm{H}_n(\mathbf{F}_\circ \otimes_R \sim)$, $\mathrm{H}_n(- \otimes_R \mathbf{F}'_\circ)$, $\mathrm{H}_n(\mathbf{P}_\circ \otimes_R \sim)$ and $\mathrm{H}_n(- \otimes_R \mathbf{P}'_\circ)$ by $\mathrm{Tor}_n^{\mathcal{SF}}(M, \sim)$, $\mathrm{Tor}_n^{\mathcal{SF}}(-, N)$, $\mathrm{Tor}_n^{\mathcal{SP}}(M, \sim)$ and $\mathrm{Tor}_n^{\mathcal{SP}}(-, N)$, respectively. In Theorem 4.10, we link the $\mathcal{S}$-pure flat dimension of a module to $\operatorname{Tor}_n^{\mathcal{SF}}(-,\sim)$. It seems that, in general, $\operatorname{Tor}_n^{\mathcal{SF}}(-,\sim)$ is similar to $\operatorname{Tor}_n^R(-,\sim)$, while $\operatorname{Tor}_n^{\mathcal{SP}}(-,\sim)$ is different from $\operatorname{Tor}_n^R(-,\sim)$ (see also Remark 4.13).

As an application, assume that R is a virtually Gorenstein ring [24, Definition 3.9] and $\mathcal{GP}$, $\mathcal{GI}$ and $\mathcal{GF}$ denote the classes of all Gorenstein projective R-modules, of all Gorenstein injective R-modules and of all Gorenstein flat R-modules, respectively. Then, for $\mathcal{S} := \mathcal{GP}$, $\mathcal{GP}$ coincides with the class of all $\mathcal{S}$-pure projective R-modules, $\mathcal{GI}$ coincides with the class of all $\mathcal{S}$-pure injective R-modules, and $\mathcal{GF}$ coincides with the class of all $\mathcal{S}$-pure flat R-modules. Moreover, $\mathcal{GP}$ is a precovering class contained in $\mathcal{GF}$. Hence the functor $-\otimes_R \sim$ is left balanced by $\mathcal{GF}\times\mathcal{GF}$ and $\mathcal{GP}\times\mathcal{GP}$, and we can define balanced left derived functors $\operatorname{Tor}_n^{\mathcal{GF}}(-,\sim)$ and $\operatorname{Tor}_n^{\mathcal{GP}}(-,\sim)$ (see Proposition 3.13, Theorems 3.14 and 3.15 and Definition 4.2). Thus we link the Gorenstein flat dimension of an R-module to $\operatorname{Tor}_n^{\mathcal{GF}}(-,\sim)$ in Corollary 4.11. Furthermore, over a virtually Gorenstein Artinian ring R, we link the Gorenstein projective dimension of an R-module to $\operatorname{Tor}_n^{\mathcal{GP}}(-,\sim)$ (see Theorem 4.15).

2. Left $\mathcal{S}$-pure projective resolutions and minimal left $\mathcal{S}$-pure flat resolutions. Throughout, we will denote the Pontryagin duality functor $\operatorname{Hom}_{\mathbb{Z}}(-,\mathbb{Q}/\mathbb{Z})$ by $(-)^+$.

Definition 2.1. Let $\mathcal{S}$ be a class of left (respectively right) R-modules. A short exact sequence $0\to A\to B\to C\to 0$ of left (respectively right) R-modules and left (respectively right) R-homomorphisms is called $\mathcal{S}$-*copure exact* if the induced homomorphism $\operatorname{Hom}_R(B,U)\to\operatorname{Hom}_R(A,U)$ is surjective for all $U\in\mathcal{S}$.

As a starting point, we want to define $\mathcal{S}$-pure exact and $\mathcal{S}$-copure exact complexes. To prepare the ground for this, we present the following lemma which has a straightforward proof. Throughout, ι (possibly with subscripts) stands for various inclusion maps.

Lemma 2.2. *Let*

$$\mathbf{X}:\cdots\xrightarrow{d_{i+2}}X_{i+1}\xrightarrow{d_{i+1}}X_i\xrightarrow{d_i}X_{i-1}\xrightarrow{d_{i-1}}\cdots$$

be an exact complex of left (*respectively right*) *R-modules and left* (*respectively right*) *R-homomorphisms. For each $i\in\mathbb{Z}$, let $\mathbf{X}_i$ denote the short exact sequence* $0\to\operatorname{Im}d_{i+1}\overset{\iota}{\hookrightarrow}X_i\xrightarrow{\bar{d}_i}\operatorname{Im}d_i\to 0$ *where $\bar{d}_i : X_i\to\operatorname{Im}d_i$ is induced by d_i. Then for a given left* (*respectively right*) *R-module U, the functor* $\operatorname{Hom}_R(U,-)$ *leaves* $\mathbf{X}$ *exact if and only if it leaves* $\mathbf{X}_i$ *exact for all* $i\in\mathbb{Z}$. *Also,* $\operatorname{Hom}_R(-,U)$ *leaves* $\mathbf{X}$ *exact if and only if it leaves* $\mathbf{X}_i$ *exact for all* $i\in\mathbb{Z}$.

Now, we are ready to define $\mathcal{S}$-pure exact complexes: if we break a complex down into short exact sequences, all of them must be $\mathcal{S}$-pure exact.

Definition 2.3. Let $\mathcal{S}$ be a class of left (respectively right) R-modules and let the situation and notation be as in Lemma 2.2. Then $\mathbf{X}$ is called *$\mathcal{S}$-pure exact* if $\mathbf{X}_i$ is $\mathcal{S}$-pure exact for all $i \in \mathbb{Z}$. Also, $\mathbf{X}$ is called *$\mathcal{S}$-copure exact* if $\mathbf{X}_i$ is $\mathcal{S}$-copure exact for all $i \in \mathbb{Z}$.

Definition 2.4. Let $\mathcal{F}$ be a class of left (respectively right) R-modules and M be a left (respectively right) R-module. A left (respectively right) R-homomorphism $\phi : F \to M$ where $F \in \mathcal{F}$ is called an *$\mathcal{F}$-precover* of M if the induced homomorphism $\mathrm{Hom}_R(F', F) \to \mathrm{Hom}_R(F', M)$ is surjective for all $F' \in \mathcal{F}$. We note that an $\mathcal{F}$-precover is not surjective in general, but it is certainly surjective when $\mathcal{F}$ contains R. If $\phi : F \to M$ is an $\mathcal{F}$-precover of M and every left (respectively right) R-homomorphism $f : F \to F$ such that $\phi f = \phi$ is an automorphism, then ϕ is called an *$\mathcal{F}$-cover* of M; it is unique up to isomorphism (if it exists). The class $\mathcal{F}$ is said to be *(pre)covering* if every R-module admits an $\mathcal{F}$-(pre)cover.

Remark 2.5. For a given class $\mathcal{S}$ of left (respectively right) R-modules, it follows from [25, Corollary 2.3] that $\mathcal{SFR}$ (respectively $\mathcal{SFL}$) is covering. Let M be a right (respectively left) R-module and $\varphi : F_0 \to M$ be the $\mathcal{S}$-pure flat cover of M. Then the short exact sequence $0 \to \operatorname{Ker}\varphi \overset{\iota}{\hookrightarrow} F_0 \overset{\varphi}{\to} M \to 0$ is $\mathcal{SFR}$-pure (respectively $\mathcal{SFL}$-pure) exact. Now, assume that $\varphi_1 : F_1 \to \operatorname{Ker}\varphi$ is the $\mathcal{S}$-pure flat cover of $\operatorname{Ker}\varphi$ and $d_1 := \iota \circ \varphi_1$. Then $0 \to \operatorname{Ker}\varphi_1 \overset{\iota_1}{\hookrightarrow} F_1 \overset{\varphi_1}{\to} \operatorname{Ker}\varphi \to 0$ is $\mathcal{SFR}$-pure (respectively $\mathcal{SFL}$-pure) exact. Proceeding in this way, we can construct the $\mathcal{SFR}$-pure (respectively $\mathcal{SFL}$-pure) exact complex

$$
\begin{array}{l}
\mathbf{F}_\bullet : \cdots \xrightarrow{d_{n+1}} F_n \xrightarrow{\qquad d_n \qquad} F_{n-1} \xrightarrow{d_{n-1}} \cdots \\
\qquad\qquad\qquad F_n \xrightarrow{\varphi_n} \operatorname{Ker} d_{n-1} \overset{\iota_{n-1}}{\hookrightarrow} F_{n-1} \\
\qquad\qquad\qquad\qquad \cdots \xrightarrow{d_2} F_1 \xrightarrow{\qquad d_1 \qquad} F_0 \xrightarrow{\varphi} M \to 0 \\
\qquad\qquad\qquad\qquad\qquad F_1 \xrightarrow{\varphi_1} \operatorname{Ker}\varphi \overset{\iota}{\hookrightarrow} F_0
\end{array}
$$

where $\varphi_n : F_n \to \operatorname{Ker} d_{n-1}$ is the $\mathcal{S}$-pure flat cover of $\operatorname{Ker} d_{n-1}$ and $d_n := \iota_{n-1} \circ \varphi_n$ for all integers $n \geq 2$. The complex $\mathbf{F}_\bullet$ is called the *minimal left $\mathcal{SFR}$-resolution* (respectively *$\mathcal{SFL}$-resolution*) of M; it is unique up to isomorphism. For a given right (respectively left) R-module M, we define the

$\mathcal{S}$*-pure flat dimension* of M as the length of the minimal left $\mathcal{SFR}$-resolution (respectively the minimal left $\mathcal{SFL}$-resolution) of M.

Definition 2.6. Let $\mathcal{S}$ be a class of left (respectively right) R-modules, M be a right (respectively left) R-module and

$$\mathbf{F}_\bullet : \cdots \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\varphi} M \to 0$$

be the minimal left $\mathcal{SFR}$-resolution (respectively the minimal left $\mathcal{SFL}$-resolution) of M. Then $K_0 := M$ is called the $0th$ $\mathcal{SFR}$*-syzygy* (respectively the $0th$ $\mathcal{SFL}$*-syzygy*) of M, $K_1 := \operatorname{Ker}\varphi$ is called the first $\mathcal{SFR}$*-syzygy* (respectively the first $\mathcal{SFL}$*-syzygy*) of M and $K_n := \operatorname{Ker} d_{n-1}$ is called the nth $\mathcal{SFR}$*-syzygy* (respectively the nth $\mathcal{SFL}$*-syzygy*) of M for all $n \geq 2$.

Remark 2.7. Let $\mathcal{S}$ be a class of left (respectively right) R-modules with the property that $\mathcal{SPL}$ (respectively $\mathcal{SPR}$) is precovering. For a given left (respectively right) R-module M, by hypothesis, there exists an $\mathcal{SPL}$-pure (respectively $\mathcal{SPR}$-pure) exact or equivalently $\mathcal{S}$-pure exact sequence $0 \to \operatorname{Ker}\psi \overset{\iota}{\hookrightarrow} P_0 \overset{\psi}{\to} M \to 0$ where ψ is an $\mathcal{S}$-pure projective precover of M. Assume that $\psi_1 : P_1 \to \operatorname{Ker}\psi$ is an $\mathcal{S}$-pure projective precover of $\operatorname{Ker}\psi$ and $d_1 := \iota\circ\psi_1$. Then $0 \to \operatorname{Ker}\psi_1 \overset{\iota_1}{\hookrightarrow} P_1 \overset{\psi_1}{\to} \operatorname{Ker}\psi \to 0$ is $\mathcal{SPL}$-pure (respectively $\mathcal{SPR}$-pure) exact or equivalently $\mathcal{S}$-pure exact. Proceeding in this way, we can construct the $\mathcal{SPL}$-pure (respectively $\mathcal{SPR}$-pure) exact or equivalently $\mathcal{S}$-pure exact complex

$$\begin{array}{ccccccc} \mathbf{P}_\bullet : \cdots \xrightarrow{d_{n+1}} & P_n & & \xrightarrow{\quad d_n \quad} & & P_{n-1} & \xrightarrow{d_{n-1}} \cdots \\ & & \searrow^{\psi_n} & & \nearrow_{\iota_{n-1}} & & \\ & & & \operatorname{Ker} d_{n-1} & & & \end{array}$$

$$\begin{array}{ccccccc} \cdots \xrightarrow{d_2} & P_1 & & \xrightarrow{\quad d_1 \quad} & & P_0 & \xrightarrow{\psi} M \to 0 \\ & & \searrow^{\psi_1} & & \nearrow_{\iota} & & \\ & & & \operatorname{Ker}\psi & & & \end{array}$$

where ψ_n is an $\mathcal{S}$-pure projective precover of $\operatorname{Ker} d_{n-1}$ and $d_n := \iota_{n-1}\circ\psi_n$ for all $n \geq 2$. The complex $\mathbf{P}_\bullet$ is called a *left* $\mathcal{SPL}$*-resolution* (respectively a *left* $\mathcal{SPR}$*-resolution*) of M; it is unique up to homotopy. We define the $\mathcal{S}$*-pure projective dimension* of M as the infimum of the lengths of such complexes.

Definition 2.8. Assume that $\mathcal{S}$ is a class of left (respectively right) R-modules with the property that $\mathcal{SPL}$ (respectively $\mathcal{SPR}$) is precovering. For a given left (respectively right) R-module M, assume that

$$\mathbf{P}_\bullet : \cdots \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\psi} M \to 0$$

is a left $\mathcal{SPL}$-resolution (respectively a left $\mathcal{SPR}$-resolution) of M. Then $K_0 := M$ is called the $0th$ *$\mathcal{SPL}$-syzygy* (respectively the $0th$ *$\mathcal{SPR}$-syzygy*) of M, $K_1 := \operatorname{Ker}\psi$ is called a first *$\mathcal{SPL}$-syzygy* (respectively a first *$\mathcal{SPR}$-syzygy*) of M and $K_n := \operatorname{Ker} d_{n-1}$ is called an nth *$\mathcal{SPL}$-syzygy* (respectively an nth *$\mathcal{SPR}$-syzygy*) of M for all $n \geq 2$.

For further information pertaining to Remarks 2.5 and 2.7 and Definitions 2.6 and 2.8, the reader is referred to [5, Section 8.1].

3. Balance. In this section, we prepare the ground for introducing our desired left derived functors.

Definition 3.1. Let $\mathcal{F}$ and $\mathcal{G}$ denote a class of right R-modules and a class of left R-modules, respectively. We say that a not necessarily exact complex $\mathbf{X}$ is $\mathcal{F}\otimes_R \sim$ (respectively $-\otimes_R \mathcal{G}$) *exact* if the functor $F\otimes_R \sim$ (respectively $-\otimes_R G$) makes $\mathbf{X}$ exact for all modules $F \in \mathcal{F}$ (respectively $G \in \mathcal{G}$). Then $-\otimes_R \sim$ is called *left balanced by* $\mathcal{F}\times\mathcal{G}$ if for every left R-module M, there exists an $\mathcal{F}\otimes_R \sim$ exact complex $\cdots \to G_1 \to G_0 \to M \to 0$ where $G_n \in \mathcal{G}$ for all non-negative n and for every right R-module N, there exists a $-\otimes_R \mathcal{G}$ exact complex $\cdots \to F_1 \to F_0 \to N \to 0$ where $F_n \in \mathcal{F}$ for all non-negative n.

See also [5, Sections 8.1 and 8.2] for more information related to Definition 3.1.

Proposition 3.2. *Let $\mathcal{S}$ be a class of (R,R)-bimodules such that every module in $\mathcal{S}$ is also an $\mathcal{S}$-pure flat left and right R-module. Then $-\otimes_R \sim$ is left balanced by $\mathcal{SFR}\times\mathcal{SFL}$.*

Proof. Suppose that M is a left (respectively right) R-module and

$$\mathbf{F}_\bullet : \cdots \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\varphi} M \to 0$$

is the minimal left $\mathcal{SFL}$-resolution (respectively the minimal left $\mathcal{SFR}$-resolution) of M. Since $\mathcal{S}$ is contained in $\mathcal{SFL}$ (respectively $\mathcal{SFR}$), we conclude from $\mathcal{SFL}$-pure (respectively $\mathcal{SFR}$-pure) exactness of $\mathbf{F}_\bullet$ that it is also $\mathcal{S}$-pure exact. Hence $F'\otimes_R \mathbf{F}_\bullet$ (respectively $\mathbf{F}_\bullet\otimes_R F'$) is exact for all $\mathcal{S}$-pure flat right (respectively left) R-modules F'. ■

Proposition 3.3. *Let $\mathcal{S}$ be a class of (R,R)-bimodules such that $\mathcal{SPL}$ is a precovering class contained in $\mathcal{SFL}$, and $\mathcal{SPR}$ is a precovering class contained in $\mathcal{SFR}$. Then $-\otimes_R \sim$ is left balanced by $\mathcal{SPR}\times\mathcal{SPL}$.*

Proof. Let M be a left (respectively right) R-module and

$$\mathbf{P}_\bullet : \cdots \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\psi} M \to 0$$

be a left $\mathcal{SPL}$-resolution (respectively a left $\mathcal{SPR}$-resolution) of M. Then since $\mathbf{P}_\bullet$ is $\mathcal{SPL}$-pure (respectively $\mathcal{SPR}$-pure) exact, it is also $\mathcal{S}$-pure exact.

Therefore, by hypothesis, $P' \otimes_R \mathbf{P}_\bullet$ (respectively $\mathbf{P}_\bullet \otimes_R P'$) is exact for all $\mathcal{S}$-pure projective right (respectively left) R-modules P'. ∎

THEOREM 3.4. *Let $\mathcal{S}$ be a class of (R,R)-bimodules with the property that every module in $\mathcal{S}$ is also an $\mathcal{S}$-pure flat left and right R-module. Assume that M is a right R-module, N is a left R-module and*

$$\mathbf{F}_\bullet : \cdots \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\alpha} M \to 0$$

and

$$\mathbf{F}'_\bullet : \cdots \xrightarrow{d'_{n+1}} F'_n \xrightarrow{d'_n} \cdots \xrightarrow{d'_2} F'_1 \xrightarrow{d'_1} F'_0 \xrightarrow{\beta} N \to 0$$

are the minimal left $\mathcal{SFR}$-resolution of M and the minimal left $\mathcal{SFL}$-resolution of N, respectively. Let

$$\mathbf{F}_\circ : \cdots \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to 0$$

and

$$\mathbf{F}'_\circ : \cdots \xrightarrow{d'_{n+1}} F'_n \xrightarrow{d'_n} \cdots \xrightarrow{d'_2} F'_1 \xrightarrow{d'_1} F'_0 \to 0$$

denote the deleted complexes corresponding to $\mathbf{F}_\bullet$ and $\mathbf{F}'_\bullet$, respectively. Then $\mathrm{H}_n(\mathbf{F}_\circ \otimes_R N) \cong \mathrm{H}_n(M \otimes_R \mathbf{F}'_\circ)$ *for all non-negative integers n.*

Proof. When $n = 0$, both $\mathrm{H}_0(\mathbf{F}_\circ \otimes_R N)$ and $\mathrm{H}_0(M \otimes_R \mathbf{F}'_\circ)$ are isomorphic to $M \otimes_R N$. For a given positive n, let K_n and K'_n denote the nth $\mathcal{SFR}$-syzygy of M and the nth $\mathcal{SFL}$-syzygy of N, respectively. Also, suppose that $\bar{d_n} : F_n \to K_n$ and $\bar{d'_n} : F'_n \to K'_n$ denote the right R-homomorphism and the left R-homomorphism induced by d_n and d'_n, respectively. Then, by construction,

$$\mathbf{F}^n_\bullet : \cdots \xrightarrow{d_{n+3}} F_{n+2} \xrightarrow{d_{n+2}} F_{n+1} \xrightarrow{d_{n+1}} F_n \xrightarrow{\bar{d_n}} K_n \to 0$$

and

$$\mathbf{F}'^n_\bullet : \cdots \xrightarrow{d'_{n+3}} F_{n+2} \xrightarrow{d'_{n+2}} F'_{n+1} \xrightarrow{d'_{n+1}} F'_n \xrightarrow{\bar{d'_n}} K'_n \to 0$$

are the minimal left $\mathcal{SFR}$-resolution of K_n and the minimal left $\mathcal{SFL}$-resolution of K'_n, respectively. We denote the corresponding deleted complexes by

$$\mathbf{F}^n_\circ : \cdots \xrightarrow{d_{n+3}} F_{n+2} \xrightarrow{d_{n+2}} F_{n+1} \xrightarrow{d_{n+1}} F_n \to 0$$

and

$$\mathbf{F}'^n_\circ : \cdots \xrightarrow{d'_{n+3}} F_{n+2} \xrightarrow{d'_{n+2}} F'_{n+1} \xrightarrow{d'_{n+1}} F'_n \to 0.$$

Consider the short exact sequences

$$(\star) \qquad 0 \to K_1 \xhookrightarrow{\iota} F_0 \xrightarrow{\alpha} M \to 0$$

and

$$(\sharp) \qquad 0 \to K'_1 \overset{\kappa}{\hookrightarrow} F'_0 \overset{\beta}{\to} N \to 0$$

where ι and κ are the inclusion maps. By hypothesis, $(\star)$ is $\mathcal{SFR}$-pure exact and $(\sharp)$ is $\mathcal{SFL}$-pure exact, and so they are both $\mathcal{S}$-pure exact. Hence $(\star)$ is $-\otimes_R \mathcal{SFL}$ exact and $(\sharp)$ is $\mathcal{SFR}\otimes_R \sim$ exact. Thus, [5, Theorem 8.2.3(1)] yields the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & & & 0 & & \\
 & & \downarrow & & & & \downarrow & & \\
 & & \mathrm{H}_1(\mathbf{F}_\circ \otimes_R K'_1) & & 0 & & \mathrm{H}_1(\mathbf{F}_\circ \otimes_R N) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to \mathrm{H}_1(K_1 \otimes_R \mathbf{F}'_\circ) & \longrightarrow & K_1 \otimes_R K'_1 & \longrightarrow & K_1 \otimes_R F'_0 & \longrightarrow & K_1 \otimes_R N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 \to F_0 \otimes_R K'_1 & \longrightarrow & F_0 \otimes_R F'_0 & \longrightarrow & F_0 \otimes_R N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to \mathrm{H}_1(M \otimes_R \mathbf{F}'_\circ) & \longrightarrow & M \otimes_R K'_1 & \longrightarrow & M \otimes_R F'_0 & \longrightarrow & M \otimes_R N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where all rows and all columns are exact. Hence, by elementary homological algebra,

$$(1) \qquad \mathrm{H}_1(\mathbf{F}_\circ \otimes_R K'_1) \cong \mathrm{H}_1(K_1 \otimes_R \mathbf{F}'_\circ),$$

$$(2) \qquad \mathrm{H}_1(\mathbf{F}_\circ \otimes_R N) \cong \mathrm{H}_1(M \otimes_R \mathbf{F}'_\circ).$$

For a given positive integer n, clearly K_{n+1} and K'_{n+1} are the first $\mathcal{SFR}$-syzygy of K_n and the first $\mathcal{SFL}$-syzygy of K'_n, respectively. We therefore have the $\mathcal{S}$-pure exact sequences

$$0 \to K_{n+1} \overset{\iota_n}{\hookrightarrow} F_n \overset{d_n}{\longrightarrow} K_n \to 0$$

and

$$0 \to K'_{n+1} \overset{\kappa_n}{\hookrightarrow} F'_n \overset{d'_n}{\longrightarrow} K'_n \to 0$$

where ι_n and κ_n are the inclusion maps.

Now, assume that $n > 1$. Then, in view of the above, isomorphisms (1) and (2) imply that $\mathrm{H}_1(\mathbf{F}^{n-m}_\circ \otimes_R K'_{m-1}) \cong \mathrm{H}_1(\mathbf{F}^{n-m-1}_\circ \otimes_R K'_m)$ for $0 < m < n$ ($K'_0 = N$). In particular, $\mathrm{H}_1(\mathbf{F}^1_\circ \otimes_R K'_{n-2}) \cong \mathrm{H}_1(\mathbf{F}_\circ \otimes_R K'_{n-1})$. Hence we deduce that $\mathrm{H}_1(\mathbf{F}^{n-1}_\circ \otimes_R N) \cong \mathrm{H}_1(M \otimes_R \mathbf{F}'^{n-1}_\circ)$. Thus $\mathrm{H}_n(\mathbf{F}_\circ \otimes_R N) \cong \mathrm{H}_n(M \otimes_R \mathbf{F}'_\circ)$. ∎

When $\mathcal{S}$ is a class of (R,R)-bimodules such that $\mathcal{SPL}$ is a precovering class contained in $\mathcal{SFL}$, and $\mathcal{SPR}$ is a precovering class contained in $\mathcal{SFR}$, in the proof of Theorem 3.4, minimal left $\mathcal{SFR}$-resolutions can be replaced by left $\mathcal{SPR}$-resolutions and minimal left $\mathcal{SFL}$-resolutions can be replaced by left $\mathcal{SPL}$-resolutions. This yields the following theorem:

THEOREM 3.5. *Let $\mathcal{S}$ be a class of (R,R)-bimodules such that $\mathcal{SPL}$ is a precovering class contained in $\mathcal{SFL}$, and $\mathcal{SPR}$ is a precovering class contained in $\mathcal{SFR}$. For a given right R-module M and a given left R-module N, let*

$$\mathbf{P}_\bullet : \cdots \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\alpha} M \to 0$$

and

$$\mathbf{P}'_\bullet : \cdots \xrightarrow{d'_{n+1}} P'_n \xrightarrow{d'_n} \cdots \xrightarrow{d'_2} P'_1 \xrightarrow{d'_1} P'_0 \xrightarrow{\beta} N \to 0$$

be a left $\mathcal{SPR}$-resolution of M and a left $\mathcal{SPL}$-resolution of N, respectively. Let

$$\mathbf{P}_\circ : \cdots \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to 0$$

and

$$\mathbf{P}'_\circ : \cdots \xrightarrow{d'_{n+1}} P'_n \xrightarrow{d'_n} \cdots \xrightarrow{d'_2} P'_1 \xrightarrow{d'_1} P'_0 \to 0$$

be the corresponding deleted complexes. Then $\mathrm{H}_n(\mathbf{P}_\circ \otimes_R N) \cong \mathrm{H}_n(M \otimes_R \mathbf{P}'_\circ)$ for all non-negative integers n.

The following examples show classes of R-modules satisfying the conditions of Propositions 3.2 and 3.3 and Theorems 3.4 and 3.5.

EXAMPLE 3.6. Let R be a commutative ring and $\mathcal{S}$ be the class of all finitely presented R-modules. In this case, $\mathcal{S}$-purity is called just *purity*. Since, by [22, Propositions 3], $\mathcal{SF}$ is the class of all R-modules, $\mathcal{S}$ satisfies the conditions of Proposition 3.2 and Theorem 3.4. Moreover, there exists a set $\mathcal{S}^*$ of R-modules of the form R^n/K where n is a positive integer and K is a submodule of R^n with the property that for a given R-module $U \in \mathcal{S}$ there exists a module in $\mathcal{S}^*$ isomorphic to U. Hence it follows from [22, Propositions 1 and 3] that $\mathcal{SP}$ is a precovering class contained in $\mathcal{SF}$. Thus $\mathcal{S}$ also satisfies the conditions of Proposition 3.3 and Theorem 3.5.

EXAMPLE 3.7. Let R be a commutative ring and $\mathcal{S}$ be the class of all R-modules of the form R/Rr where $r \in R$. In this case, $\mathcal{S}$-purity is called *RD*-. By [22, Proposition 2], every module in $\mathcal{S}$ is *RD*-flat. We conclude that $\mathcal{S}$ satisfies the conditions of Proposition 3.2 and Theorem 3.4. There exists a set $\mathcal{S}^*$ of R-modules of the form R/I where I is an ideal of R such that for a given R-module $U \in \mathcal{S}$, there exists a module in $\mathcal{S}^*$ isomorphic to U. So [22, Propositions 1 and 2] show that $\mathcal{S}$ also satisfies the conditions of Proposition 3.3 and Theorem 3.5.

Definition 3.8. Let R be a commutative ring and $\mathcal{P}$ be the class of all projective R-modules. An R-module M is called *Gorenstein projective* if there exists a $\mathcal{P}$-copure exact complex

$$\cdots \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d^0} P^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} P^n \xrightarrow{d^n} \cdots$$

of projective R-modules such that $M \cong \operatorname{Coker} d_1$.

Definition 3.9. Let R be a commutative ring and $\mathcal{I}$ be the class of all injective R-modules. An R-module M is called *Gorenstein injective* if there exists an $\mathcal{I}$-pure exact complex

$$\cdots \xrightarrow{d_{n+1}} I_n \xrightarrow{d_n} \cdots \xrightarrow{d_2} I_1 \xrightarrow{d_1} I_0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} I^n \xrightarrow{d^n} \cdots$$

of injective R-modules such that $M \cong \operatorname{Ker} d^0$.

Definition 3.10. Let R be a commutative ring and $\mathcal{I}$ be the class of all injective R-modules. An R-module M is called *Gorenstein flat* if there exists an exact and $\mathcal{I} \otimes_R -$ exact complex

$$\cdots \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d^0} F^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} F^n \xrightarrow{d^n} \cdots$$

of flat R-modules such that $M \cong \operatorname{Coker} d_1$.

For a given commutative ring R, we will denote the classes of all Gorenstein projective R-modules, of all Gorenstein injective R-modules and of all Gorenstein flat R-modules by $\mathcal{GP}$, $\mathcal{GI}$ and $\mathcal{GF}$, respectively.

Definition 3.11. Let R be a commutative Noetherian ring of finite Krull dimension and let $\mathcal{GP}^{\perp}$ (respectively ${}^{\perp}\mathcal{GI}$) denote the class of all R-modules with $\operatorname{Ext}^n_R(Q, M) = 0$ (respectively $\operatorname{Ext}^n_R(M, E) = 0$) for all R-modules $Q \in \mathcal{GP}$ (respectively $E \in \mathcal{GI}$) and all integers $n \geq 1$. R is called *virtually Gorenstein* if $\mathcal{GP}^{\perp} = {}^{\perp}\mathcal{GI}$.

Example 3.12. Suppose that R is a virtually Gorenstein ring and $\mathcal{S}$ is the class of all Gorenstein projective R-modules. Then, by [24, Theorem 3.12], $\mathcal{SP} = \mathcal{GP}$, $\mathcal{SI} = \mathcal{GI}$ and $\mathcal{SF} = \mathcal{GF}$. Therefore by [17, Theorem A.1] and [2, Proposition 3.7], $\mathcal{SP}$ is a precovering class contained in $\mathcal{SF}$. Hence, over a virtually Gorenstein ring R, the class $\mathcal{S} := \mathcal{GP}$ satisfies the conditions of Propositions 3.2 and 3.3 and Theorems 3.4 and 3.5. Moreover, by [9, Theorems 2.10 and 3.23], the $\mathcal{S}$-pure projective dimension and the $\mathcal{S}$-pure flat dimension of an R-module are equal to its Gorenstein projective dimension and its Gorenstein flat dimension, respectively.

In view of Example 3.12, we present the results of this section once again in a different style:

Proposition 3.13. *Let R be a virtually Gorenstein ring. Then $- \otimes_R \sim$ is left balanced by $\mathcal{GF} \times \mathcal{GF}$ and $\mathcal{GP} \times \mathcal{GP}$.*

THEOREM 3.14. *Assume that R is a virtually Gorenstein ring, M and N are R-modules and*

$$\mathbf{F}_\bullet : \cdots \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\alpha} M \to 0$$

and

$$\mathbf{F}'_\bullet : \cdots \xrightarrow{d'_{n+1}} F'_n \xrightarrow{d'_n} \cdots \xrightarrow{d'_2} F'_1 \xrightarrow{d'_1} F'_0 \xrightarrow{\beta} N \to 0$$

are the minimal left $\mathcal{GF}$-resolutions of M and N, respectively. Let

$$\mathbf{F}_\circ : \cdots \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to 0$$

and

$$\mathbf{F}'_\circ : \cdots \xrightarrow{d'_{n+1}} F'_n \xrightarrow{d'_n} \cdots \xrightarrow{d'_2} F'_1 \xrightarrow{d'_1} F'_0 \to 0$$

denote the corresponding deleted complexes. Then $\mathrm{H}_n(\mathbf{F}_\circ \otimes_R N) \cong \mathrm{H}_n(M \otimes_R \mathbf{F}'_\circ)$ for all non-negative integers n.

THEOREM 3.15. *Assume that R is a virtually Gorenstein ring. For two given R-modules M and N, let*

$$\mathbf{P}_\bullet : \cdots \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\alpha} M \to 0$$

and

$$\mathbf{P}'_\bullet : \cdots \xrightarrow{d'_{n+1}} P'_n \xrightarrow{d'_n} \cdots \xrightarrow{d'_2} P'_1 \xrightarrow{d'_1} P'_0 \xrightarrow{\beta} N \to 0$$

be left $\mathcal{GP}$-resolutions of M and N, respectively. Let

$$\mathbf{P}_\circ : \cdots \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to 0$$

and

$$\mathbf{P}'_\circ : \cdots \xrightarrow{d'_{n+1}} P'_n \xrightarrow{d'_n} \cdots \xrightarrow{d'_2} P'_1 \xrightarrow{d'_1} P'_0 \to 0$$

denote the corresponding deleted complexes. Then $\mathrm{H}_n(\mathbf{P}_\circ \otimes_R N) \cong \mathrm{H}_n(M \otimes_R \mathbf{P}'_\circ)$ for all non-negative integers n.

4. $\mathrm{Tor}_n^{\mathcal{SF}}(-,\sim)$ and $\mathrm{Tor}_n^{\mathcal{SP}}(-,\sim)$. Throughout this section, $\mathcal{S}$ will denote a class of (R,R)-bimodules such that $\mathcal{SPL}$ is a precovering class contained in $\mathcal{SFL}$, and $\mathcal{SPR}$ is a precovering class contained in $\mathcal{SFR}$.

In view of Propositions 3.2 and 3.3 and Theorems 3.4 and 3.5, we can make the following definition:

DEFINITION 4.1. Let M be a right (respectively left) R-module and

$$\mathbf{F}_\bullet : \cdots \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\varphi} M \to 0$$

be the minimal left $\mathcal{SFR}$-resolution (respectively the minimal left $\mathcal{SFL}$-resolution) of M. Also, let

$$\mathbf{P}_\bullet : \cdots \xrightarrow{d'_{n+1}} P_n \xrightarrow{d'_n} \cdots \xrightarrow{d'_2} P_1 \xrightarrow{d'_1} P_0 \xrightarrow{\psi} M \to 0$$

be a left $\mathcal{SPR}$-resolution (respectively a left $\mathcal{SPL}$-resolution) of M. Let

$$\mathbf{F}_\circ : \cdots \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to 0$$

and

$$\mathbf{P}_\circ : \cdots \xrightarrow{d'_{n+1}} P_n \xrightarrow{d'_n} \cdots \xrightarrow{d'_2} P_1 \xrightarrow{d'_1} P_0 \to 0$$

denote the corresponding deleted complexes. Then $\mathrm{H}_n(\mathbf{F}_\circ \otimes_R \sim)$ (respectively $\mathrm{H}_n(- \otimes_R \mathbf{F}_\circ))$ will be denoted by $\mathrm{Tor}_n^{\mathcal{SF}}(M, \sim)$ (respectively $\mathrm{Tor}_n^{\mathcal{SF}}(-, M))$, and $\mathrm{H}_n(\mathbf{P}_\circ \otimes_R \sim)$ (respectively $\mathrm{H}_n(- \otimes_R \mathbf{P}_\circ))$ will be denoted by $\mathrm{Tor}_n^{\mathcal{SP}}(M, \sim)$ (respectively $\mathrm{Tor}_n^{\mathcal{SP}}(-, M))$, for all $n \geq 0$.

Proposition 3.13 and Theorems 3.14 and 3.15 lead to the following definition:

Definition 4.2. Let R be a virtually Gorenstein ring, M be an R-module and

$$\mathbf{F}_\bullet : \cdots \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\varphi} M \to 0$$

and

$$\mathbf{P}_\bullet : \cdots \xrightarrow{d'_{n+1}} P_n \xrightarrow{d'_n} \cdots \xrightarrow{d'_2} P_1 \xrightarrow{d'_1} P_0 \xrightarrow{\psi} M \to 0$$

be the minimal left $\mathcal{GF}$-resolution and a left $\mathcal{GP}$-resolution of M, respectively. Let

$$\mathbf{F}_\circ : \cdots \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to 0$$

and

$$\mathbf{P}_\circ : \cdots \xrightarrow{d'_{n+1}} P_n \xrightarrow{d'_n} \cdots \xrightarrow{d'_2} P_1 \xrightarrow{d'_1} P_0 \to 0$$

denote the corresponding deleted complexes. Then for all $n \geq 0$, $\mathrm{H}_n(\mathbf{F}_\circ \otimes_R \sim)$, $\mathrm{H}_n(- \otimes_R \mathbf{F}_\circ)$, $\mathrm{H}_n(\mathbf{P}_\circ \otimes_R \sim)$ and $\mathrm{H}_n(- \otimes_R \mathbf{P}_\circ)$ will be denoted by $\mathrm{Tor}_n^{\mathcal{GF}}(M, \sim)$, $\mathrm{Tor}_n^{\mathcal{GF}}(-, M)$, $\mathrm{Tor}_n^{\mathcal{GP}}(M, \sim)$ and $\mathrm{Tor}_n^{\mathcal{GP}}(-, M)$, respectively.

Lemma 4.3. *Let R be a commutative ring, and M and N be R-modules. Then* $\mathrm{Tor}_n^{\mathcal{SF}}(M, N) \cong \mathrm{Tor}_n^{\mathcal{SF}}(N, M)$ *and* $\mathrm{Tor}_n^{\mathcal{SP}}(M, N) \cong \mathrm{Tor}_n^{\mathcal{SP}}(N, M)$ *for all non-negative integers n.*

Lemma 4.4. *Assume that R is a virtually Gorenstein ring. For given R-modules M and N,* $\mathrm{Tor}_n^{\mathcal{GF}}(M, N) \cong \mathrm{Tor}_n^{\mathcal{GF}}(N, M)$ *and* $\mathrm{Tor}_n^{\mathcal{GP}}(M, N) \cong \mathrm{Tor}_n^{\mathcal{GP}}(N, M)$ *for all non-negative integers n.*

Bearing in mind that a short exact sequence is $\mathcal{S}$-pure exact if and only if it is $\mathcal{SP}$-pure exact, the following two theorems are immediate consequences of [5, Theorem 8.2.3(1)].

Theorem 4.5. *Let N be a right R-module and $0 \to K \to L \to M \to 0$ be a short $\mathcal{SFL}$-pure (respectively $\mathcal{S}$-pure) exact sequence of left R-modules. Then we have the following exact complexes:*

$$\begin{aligned} \cdots \to \mathrm{Tor}_{n+1}^{\mathcal{SF}}(N,M) \to \mathrm{Tor}_{n}^{\mathcal{SF}}(N,K) \to \mathrm{Tor}_{n}^{\mathcal{SF}}(N,L) \\ \to \mathrm{Tor}_{n}^{\mathcal{SF}}(N,M) \to \mathrm{Tor}_{n-1}^{\mathcal{SF}}(N,K) \to \cdots \to \mathrm{Tor}_{1}^{\mathcal{SF}}(N,M) \\ \to N \otimes_R K \to N \otimes_R L \to N \otimes_R M \to 0 \end{aligned}$$

and respectively

$$\begin{aligned} \cdots \to \mathrm{Tor}_{n+1}^{\mathcal{SP}}(N,M) \to \mathrm{Tor}_{n}^{\mathcal{SP}}(N,K) \to \mathrm{Tor}_{n}^{\mathcal{SP}}(N,L) \\ \to \mathrm{Tor}_{n}^{\mathcal{SP}}(N,M) \to \mathrm{Tor}_{n-1}^{\mathcal{SP}}(N,K) \to \cdots \to \mathrm{Tor}_{1}^{\mathcal{SP}}(N,M) \\ \to N \otimes_R K \to N \otimes_R L \to N \otimes_R M \to 0. \end{aligned}$$

Theorem 4.6. *Let N be a left R-module and $0 \to K \to L \to M \to 0$ be a short $\mathcal{SFR}$-pure (respectively $\mathcal{S}$-pure) exact sequence of right R-modules. Then we have the following exact complexes:*

$$\begin{aligned} \cdots \to \mathrm{Tor}_{n+1}^{\mathcal{SF}}(M,N) \to \mathrm{Tor}_{n}^{\mathcal{SF}}(K,N) \to \mathrm{Tor}_{n}^{\mathcal{SF}}(L,N) \\ \to \mathrm{Tor}_{n}^{\mathcal{SF}}(M,N) \to \mathrm{Tor}_{n-1}^{\mathcal{SF}}(K,N) \to \cdots \to \mathrm{Tor}_{1}^{\mathcal{SF}}(M,N) \\ \to K \otimes_R N \to L \otimes_R N \to M \otimes_R N \to 0 \end{aligned}$$

and respectively

$$\begin{aligned} \cdots \to \mathrm{Tor}_{n+1}^{\mathcal{SP}}(M,N) \to \mathrm{Tor}_{n}^{\mathcal{SP}}(K,N) \to \mathrm{Tor}_{n}^{\mathcal{SP}}(L,N) \\ \to \mathrm{Tor}_{n}^{\mathcal{SP}}(M,N) \to \mathrm{Tor}_{n-1}^{\mathcal{SP}}(K,N) \to \cdots \to \mathrm{Tor}_{1}^{\mathcal{SP}}(M,N) \\ \to K \otimes_R N \to L \otimes_R N \to M \otimes_R N \to 0. \end{aligned}$$

Corollary 4.7. *Assume that R is a virtually Gorenstein ring, N is an R-module and $0 \to K \to L \to M \to 0$ is a short $\mathcal{GF}$-pure (respectively $\mathcal{GP}$-pure) exact sequence of R-modules. Then we have the following exact complexes:*

$$\begin{aligned} \cdots \to \mathrm{Tor}_{n+1}^{\mathcal{GF}}(N,M) \to \mathrm{Tor}_{n}^{\mathcal{GF}}(N,K) \to \mathrm{Tor}_{n}^{\mathcal{GF}}(N,L) \\ \to \mathrm{Tor}_{n}^{\mathcal{GF}}(N,M) \to \mathrm{Tor}_{n-1}^{\mathcal{GF}}(N,K) \to \cdots \to \mathrm{Tor}_{1}^{\mathcal{GF}}(N,M) \\ \to N \otimes_R K \to N \otimes_R L \to N \otimes_R M \to 0 \end{aligned}$$

and respectively

$$\begin{aligned} \cdots \to \mathrm{Tor}_{n+1}^{\mathcal{GP}}(N,M) \to \mathrm{Tor}_{n}^{\mathcal{GP}}(N,K) \to \mathrm{Tor}_{n}^{\mathcal{GP}}(N,L) \\ \to \mathrm{Tor}_{n}^{\mathcal{GP}}(N,M) \to \mathrm{Tor}_{n-1}^{\mathcal{GP}}(N,K) \to \cdots \to \mathrm{Tor}_{1}^{\mathcal{GP}}(N,M) \\ \to N \otimes_R K \to N \otimes_R L \to N \otimes_R M \to 0. \end{aligned}$$

Theorem 4.8. *For a given right (respectively left) R-module M, the following conditions hold:*

(i) *If M is $\mathcal{S}$-pure flat, then $\operatorname{Tor}_n^{\mathcal{SF}}(M,-)$ (respectively $\operatorname{Tor}_n^{\mathcal{SF}}(-,M)$) vanishes for all $n \geq 1$.*

(ii) *If M is $\mathcal{S}$-pure projective, then both $\operatorname{Tor}_n^{\mathcal{SP}}(M,-)$ and $\operatorname{Tor}_n^{\mathcal{SF}}(M,-)$ (respectively both $\operatorname{Tor}_n^{\mathcal{SP}}(-,M)$ and $\operatorname{Tor}_n^{\mathcal{SF}}(-,M)$) vanish for all $n \geq 1$.*

(iii) *If either $\operatorname{Tor}_1^{\mathcal{SF}}(M,-)$ or $\operatorname{Tor}_1^{\mathcal{SP}}(M,-)$ (respectively $\operatorname{Tor}_1^{\mathcal{SF}}(-,M)$ or $\operatorname{Tor}_1^{\mathcal{SP}}(-,M)$) vanishes, then M is $\mathcal{S}$-pure flat.*

Proof. We may assume that M is a right R-module.

(i), (ii) Let M be an $\mathcal{S}$-pure flat (respectively $\mathcal{S}$-pure projective) right R-module, N be a left R-module and

$$\mathbf{F}_\bullet : \cdots \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\varphi} N \to 0$$

respectively

$$\mathbf{P}_\bullet : \cdots \xrightarrow{d'_{n+1}} P_n \xrightarrow{d'_n} \cdots \xrightarrow{d'_2} P_1 \xrightarrow{d'_1} P_0 \xrightarrow{\psi} N \to 0$$

be the minimal left $\mathcal{SFL}$-resolution (respectively a left $\mathcal{SPL}$-resolution) of N. Then, by Proposition 3.2 (respectively Proposition 3.3), $M \otimes_R F_\bullet$ (respectively $M \otimes_R P_\bullet$) is exact. We conclude that $\operatorname{Tor}_n^{\mathcal{SF}}(M,N) = 0$ (respectively $\operatorname{Tor}_n^{\mathcal{SP}}(M,N) = 0$) for all $n \geq 1$. Now, for a given $\mathcal{S}$-pure projective right R-module M, since M is $\mathcal{S}$-pure flat, we have $\operatorname{Tor}_n^{\mathcal{SF}}(M,N) = 0$ for all $n \geq 1$ too.

(iii) Assume that $\operatorname{Tor}_1^{\mathcal{SP}}(M,-)$ vanishes. Then Theorem 4.5 shows that the functor $M \otimes_R -$ leaves any $\mathcal{S}$-pure short exact sequence exact. Thus M is $\mathcal{S}$-pure flat.

Now, assume that the functor $\operatorname{Tor}_1^{\mathcal{SF}}(M,-)$ vanishes. By Remark 2.5, there exists an $\mathcal{SFR}$-pure (and so $\mathcal{S}$-pure) exact sequence $0 \to \operatorname{Ker}\varphi \xrightarrow{\iota} F \xrightarrow{\varphi} M \to 0$ where F is $\mathcal{S}$-pure flat and φ is the $\mathcal{S}$-pure flat cover of M. Hence, for a given $\mathcal{S}$-pure flat left R-module F', we have the short exact sequence

$$0 \to K \otimes_R F' \xrightarrow{\iota \otimes_R \mathrm{id}_{F'}} F \otimes_R F' \xrightarrow{\varphi \otimes_R \mathrm{id}_{F'}} M \otimes_R F' \to 0,$$

and hence the short exact sequence

$$0 \to (M \otimes_R F')^+ \xrightarrow{(\varphi \otimes_R \mathrm{id}_{F'})^+} (F \otimes_R F')^+ \xrightarrow{(\iota \otimes_R \mathrm{id}_{F'})^+} (K \otimes_R F')^+ \to 0,$$

which yields the short exact sequence

$$0 \to \operatorname{Hom}_R(F', M^+) \xrightarrow{\operatorname{Hom}_R(\mathrm{id}_{F'}, \varphi^+)} \operatorname{Hom}_R(F', F^+) \xrightarrow{\operatorname{Hom}_R(\mathrm{id}_{F'}, \iota^+)} \operatorname{Hom}_R(F', K^+) \to 0.$$

We can therefore conclude that

$$(\natural) \qquad 0 \to M^+ \xrightarrow{\varphi^+} F^+ \xrightarrow{\iota^+} K^+ \to 0$$

is $\mathcal{SFL}$-pure exact. On the other hand, by hypothesis and Theorem 4.5, the functor $M \otimes_R -$ leaves any $\mathcal{SFL}$-pure exact sequence exact. We deduce that M is $\mathcal{SFL}$-pure flat. Since ($\natural$) is $\mathcal{SFL}$-pure exact, we get the short exact sequence

$$0 \to M \otimes_R M^+ \xrightarrow{\mathrm{id}_M \otimes_R \varphi^+} M \otimes_R F^+ \xrightarrow{\mathrm{id}_M \otimes_R \iota^+} M \otimes_R K^+ \to 0,$$

and hence the exact sequence

$$0 \to (M \otimes_R K^+)^+ \xrightarrow{(\mathrm{id}_M \otimes_R \iota^+)^+} (M \otimes_R F^+)^+ \xrightarrow{(\mathrm{id}_M \otimes_R \varphi^+)^+} (M \otimes_R M^+)^+ \to 0.$$

Thus we deduce that the sequence

$$0 \to \mathrm{Hom}_R(K^+, M^+) \xrightarrow{\mathrm{Hom}_R(\iota^+, \mathrm{id}_{M^+})} \mathrm{Hom}_R(F^+, M^+) \xrightarrow{\mathrm{Hom}_R(\varphi^+, \mathrm{id}_{M^+})} \mathrm{Hom}_R(M^+, M^+) \to 0$$

is exact, or equivalently $\mathrm{Hom}_R(\varphi^+, \mathrm{id}_{M^+})$ is epic. This implies that there exists a left R-homomorphism $\psi : F^+ \to M^+$ with $\psi \circ \varphi^+ = \mathrm{id}_{M^+}$. Consequently, ($\natural$) splits, so M^+ is isomorphic to a direct summand of F^+. But, by [25, Lemma 2.2], F^+ is $\mathcal{S}$-pure injective, hence so also is M^+. Now, by [25, Lemma 2.2] again, M is $\mathcal{S}$-pure flat, as desired. ∎

COROLLARY 4.9. *Let R be a virtually Gorenstein ring and M be an R-module. Then the following conditions hold:*

(i) *If M is Gorenstein flat, then $\mathrm{Tor}_n^{\mathcal{GF}}(M, -) = 0$ for all $n \geq 1$.*
(ii) *If M is Gorenstein projective, then $\mathrm{Tor}_n^{\mathcal{GP}}(M, -) = \mathrm{Tor}_n^{\mathcal{GF}}(M, -) = 0$ for all $n \geq 1$.*
(iii) *If either $\mathrm{Tor}_1^{\mathcal{GF}}(M, -)$ or $\mathrm{Tor}_1^{\mathcal{GP}}(M, -)$ vanishes, then M is Gorenstein flat.*

We are now in a position to determine the $\mathcal{S}$-pure flat dimension of a right (respectively left) R-module M to be the least integer n for which $\mathrm{Tor}_{n+1}^{\mathcal{SF}}(M, -)$ (respectively $\mathrm{Tor}_{n+1}^{\mathcal{SF}}(-, M)$) vanishes.

THEOREM 4.10. *Let M be a right* (*respectively left*) *R-module and n be a non-negative integer. The following conditions are equivalent:*

(i) *M is of $\mathcal{S}$-pure flat dimension at most n.*
(ii) $\mathrm{Tor}_m^{\mathcal{SF}}(M, -)$ (*respectively* $\mathrm{Tor}_m^{\mathcal{SF}}(-, M)$) *vanishes for all $m > n$.*
(iii) $\mathrm{Tor}_{n+1}^{\mathcal{SF}}(M, -)$ (*respectively* $\mathrm{Tor}_{n+1}^{\mathcal{SF}}(-, M)$) *vanishes.*
(iv) *The nth $\mathcal{SFR}$-syzygy* (*respectively the nth $\mathcal{SFL}$-syzygy*) *of M is $\mathcal{S}$-pure flat.*

Proof. We may assume that M is a right R-module.

(i)⇒(ii) It follows from the hypothesis that the minimal left $\mathcal{SFR}$-resolution of M is of the form

$$\mathbf{F}_\bullet : 0 \to F_k \to \cdots \to F_0 \to M \to 0,$$

with $k \leq n$. Hence, by definition of $\mathrm{Tor}_m^{\mathcal{SF}}(M,-)$, we get the result.

The implication (ii)⇒(iii) is clear.

(iii)⇒(iv) Let N be a left R-module. Assume that $\varphi : F \to M$ is the $\mathcal{S}$-pure flat cover of M. Then we have the short $\mathcal{SFR}$-pure exact sequence

$$(\flat) \qquad 0 \to K_1 \overset{\iota}{\hookrightarrow} F \overset{\varphi}{\to} M \to 0$$

where $K_1 := \mathrm{Ker}\,\varphi$ is the first $\mathcal{SFR}$-syzygy of M. We argue by induction on n. Firstly, assume that $n = 1$. Applying Theorem 4.6 to $(\flat)$ yields the exact complex

$$\mathrm{Tor}_2^{\mathcal{SF}}(F,N) \to \mathrm{Tor}_2^{\mathcal{SF}}(M,N) \to \mathrm{Tor}_1^{\mathcal{SF}}(K_1,N) \to \mathrm{Tor}_1^{\mathcal{SF}}(F,N).$$

By hypothesis and Theorem 4.8(i), $\mathrm{Tor}_2^{\mathcal{SF}}(M,N)=\mathrm{Tor}_1^{\mathcal{SF}}(F,N)=0$. Therefore $\mathrm{Tor}_1^{\mathcal{SF}}(K_1,N) = 0$. Hence, by Theorem 4.8(iii), the first $\mathcal{SFR}$-syzygy of M is $\mathcal{S}$-pure flat. We suppose, inductively, that $n \geq 2$ and the result has been proved for smaller values of n. Once again, we apply Theorem 4.6 to $(\flat)$ to get the exact complex

$$\mathrm{Tor}_{n+1}^{\mathcal{SF}}(F,N) \to \mathrm{Tor}_{n+1}^{\mathcal{SF}}(M,N) \to \mathrm{Tor}_n^{\mathcal{SF}}(K_1,N) \to \mathrm{Tor}_n^{\mathcal{SF}}(F,N).$$

Since F is $\mathcal{S}$-pure flat, Theorem 4.8(i) yields

$$\mathrm{Tor}_{n+1}^{\mathcal{SF}}(F,N) = \mathrm{Tor}_n^{\mathcal{SF}}(F,N) = 0.$$

Thus

$$\mathrm{Tor}_n^{\mathcal{SF}}(K_1,N) \cong \mathrm{Tor}_{n+1}^{\mathcal{SF}}(M,N) = 0.$$

So, by the inductive assumption, the $(n-1)$th $\mathcal{SFR}$-syzygy of K_1 is $\mathcal{S}$-pure flat. In fact, the $(n-1)$th $\mathcal{SFR}$-syzygy of K_1 is just the nth $\mathcal{SFR}$-syzygy of M. This completes the inductive step, and the proof.

(iv)⇒(i) Consider the following part of the minimal left $\mathcal{SFR}$-resolution of M:

$$F_{n-1} \xrightarrow{d_{n-1}} F_{n-2} \xrightarrow{d_{n-2}} \cdots \xrightarrow{d_1} F_0 \overset{\varphi}{\to} M \to 0.$$

There is no loss of generality in assuming that F_{n-1} is non-zero. Let $K_n := \mathrm{Ker}\,d_{n-1}$ and $K_{n-1} := \mathrm{Ker}\,d_{n-2}$ be the nth $\mathcal{SFR}$-syzygy and the $(n-1)$th $\mathcal{SFR}$-syzygy of M, respectively. There is no loss of generality in assuming that K_n is non-zero. By hypothesis, the short exact sequence

$$0 \to K_n \overset{\iota}{\hookrightarrow} F_{n-1} \xrightarrow{d_{n-1}} K_{n-1} \to 0$$

is $\mathcal{SFR}$-pure exact where K_n is $\mathcal{S}$-pure flat and ι is the inclusion map. Hence the $\mathcal{S}$-pure flat cover of K_n is just the identity map. We therefore obtain the $\mathcal{SFR}$-pure exact complex

$$0 \to K_n \overset{\iota}{\hookrightarrow} F_{n-1} \xrightarrow{d_{n-1}} F_{n-2} \xrightarrow{d_{n-2}} \cdots \xrightarrow{d_1} F_0 \overset{\varphi}{\to} M \to 0,$$

which is the minimal left $\mathcal{SFR}$-resolution of M. Equivalently, the $\mathcal{S}$-pure flat dimension of M is equal to n. ∎

Over a virtually Gorenstein ring R, Theorem 4.10 and Example 3.12 allow us to determine the Gorenstein flat dimension of an R-module M to be the least integer n for which $\mathrm{Tor}_{n+1}^{\mathcal{GF}}(M,-)$ vanishes.

COROLLARY 4.11. *Suppose that R is a virtually Gorenstein ring, M is an R-module and n is a non-negative integer. The following conditions are equivalent:*

(i) *M is of Gorenstein flat dimension at most n.*
(ii) $\mathrm{Tor}_{m}^{\mathcal{GF}}(M,-)=0$ *for all $m>n$.*
(iii) $\mathrm{Tor}_{n+1}^{\mathcal{GF}}(M,-)=0$.
(iv) *The nth $\mathcal{GF}$-syzygy of M is Gorenstein flat.*

THEOREM 4.12. *Let M be a right* (*respectively left*) *R-module and n be a non-negative integer. Consider the following conditions:*

(i) $\mathrm{Tor}_{m}^{\mathcal{SP}}(M,-)$ (*respectively* $\mathrm{Tor}_{m}^{\mathcal{SP}}(-,M)$) *vanishes for all $m>n$.*
(ii) $\mathrm{Tor}_{n+1}^{\mathcal{SP}}(M,-)$ (*respectively* $\mathrm{Tor}_{n+1}^{\mathcal{SP}}(-,M)$) *vanishes.*
(iii) *The nth $\mathcal{SPR}$-syzygy* (*respectively the nth $\mathcal{SPL}$-syzygy*) *of M is $\mathcal{S}$-pure flat.*

Then the implications (i)⇒(ii)⇒(iii) *hold.*

Proof. The implication (i)⇒(ii) is clear.

(ii)⇒(iii) We may assume that M is a right R-module. Let N be a left R-module and $\varphi : P_0 \to M$ be an $\mathcal{S}$-pure projective precover of M. Consider the short $\mathcal{SPR}$-pure (equivalently $\mathcal{S}$-pure) exact sequence

$$(\dagger)\qquad 0 \to K_1 \overset{\iota}{\hookrightarrow} P_0 \overset{\varphi}{\to} M \to 0$$

where $K_1 := \operatorname{Ker}\varphi$ is the first $\mathcal{SPR}$-syzygy of M. We proceed by induction on n. Suppose that $n=1$. By using Theorem 4.6, (†) gives the exact complex

$$\mathrm{Tor}_{2}^{\mathcal{SP}}(P_0,N) \to \mathrm{Tor}_{2}^{\mathcal{SP}}(M,N) \to \mathrm{Tor}_{1}^{\mathcal{SP}}(K_1,N) \to \mathrm{Tor}_{1}^{\mathcal{SP}}(P_0,N).$$

But the hypothesis and Theorem 4.8(ii) yield $\mathrm{Tor}_{2}^{\mathcal{SP}}(M,N)=\mathrm{Tor}_{1}^{\mathcal{SP}}(P_0,N) = 0$. Hence $\mathrm{Tor}_{1}^{\mathcal{SP}}(K_1,N)=0$, and then Theorem 4.8(iii) implies that the first $\mathcal{SPR}$-syzygy of M must be $\mathcal{S}$-pure flat. We assume inductively that $n\geq 2$ and that the result has been proved for smaller values of n. Once again, by Theorem 4.6, (†) yields the exact complex

$$\mathrm{Tor}_{n+1}^{\mathcal{SP}}(P_0,N) \to \mathrm{Tor}_{n+1}^{\mathcal{SP}}(M,N) \to \mathrm{Tor}_{n}^{\mathcal{SP}}(K_1,N) \to \mathrm{Tor}_{n}^{\mathcal{SP}}(P_0,N).$$

Now, Theorem 4.8(ii) implies that

$$\mathrm{Tor}_{n+1}^{\mathcal{SP}}(P_0,N)=\mathrm{Tor}_{n}^{\mathcal{SP}}(P_0,N)=0.$$

Then

$$\mathrm{Tor}_{n}^{\mathcal{SP}}(K_1,N)\cong\mathrm{Tor}_{n+1}^{\mathcal{SP}}(M,N)=0.$$

Thus the $(n-1)$th $\mathcal{SPR}$-syzygy of K_1 is $\mathcal{S}$-pure flat, by the inductive hypothesis. We deduce that the nth $\mathcal{SPR}$-syzygy of M is $\mathcal{S}$-pure flat. This completes the inductive step, and the proof. ∎

REMARK 4.13. It seems that $\mathrm{Tor}_n^{\mathcal{SF}}(-,\sim)$ is more similar to the left derived functor $\mathrm{Tor}_n^{\mathcal{R}}(-,\sim)$ than $\mathrm{Tor}_n^{\mathcal{SP}}(-,\sim)$ is. There are two pertinent questions: whether the conditions of Theorem 4.12 are all equivalent and whether we can link the $\mathcal{S}$-pure flat dimension of a right (respectively left) R-module to $\mathrm{Tor}_n^{\mathcal{SP}}(M,-)$ (respectively $\mathrm{Tor}_n^{\mathcal{SP}}(-,M)$). However, when R is commutative and there exists a class $\mathcal{S}'$ of R-modules such that every module in $\mathcal{S}'$ is $\mathcal{S}'$-pure flat and the class of all $\mathcal{S}$-pure projective R-modules coincides with the class of all $\mathcal{S}'$-pure flat R-modules, we can link the $\mathcal{S}$-pure projective dimension (not the $\mathcal{S}$-pure flat dimension) of an R-module to $\mathrm{Tor}_n^{\mathcal{SP}}(-,\sim)$. Theorem 4.15 below illustrates this point very well.

DEFINITION 4.14. Let R be a commutative Noetherian ring. A finitely generated R-module M is called *totally reflexive* if:

(i) $\mathrm{Ext}_R^i(M,R) = \mathrm{Ext}_R^i(\mathrm{Hom}_R(M,R),R) = 0$ for all $i > 0$.
(ii) The natural R-homomorphism $M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M,R),R)$ is an isomorphism.

It is easy to see that, over a commutative Noetherian ring R, the class of all totally reflexive R-modules coincides with the class of all finitely generated Gorenstein projective R-modules.

In view of Example 3.12, for a given virtually Gorenstein Artinian ring R, the following theorem links the Gorenstein projective dimension of an R-module to $\mathrm{Tor}_n^{\mathcal{GP}}(-,\sim)$.

THEOREM 4.15. *Let R be a virtually Gorenstein Artinian ring, M be an R-module and n be a non-negative integer. The following conditions are equivalent:*

(i) *M is of Gorenstein projective dimension at most n.*
(ii) $\mathrm{Tor}_m^{\mathcal{GP}}(M,-) = 0$ *for all $m > n$.*
(iii) $\mathrm{Tor}_{n+1}^{\mathcal{GP}}(M,-) = 0$.
(iv) *The nth $\mathcal{GP}$-syzygy of M is Gorenstein projective, and hence Gorenstein flat.*

Proof. It follows from [1, Theorem 5(2)] that every Gorenstein projective R-module is the direct limit of a direct system of totally reflexive R-modules. But, by [23, Example 4.7], the class of all $\mathcal{G}$-pure flat R-modules coincides with the class of all direct limits of direct systems of R-modules in $\mathcal{G}$. Hence the class of all Gorenstein projective R-modules coincides with the class of all $\mathcal{G}$-pure flat R-modules. Thus Theorem 4.10 yields the desired equivalences. ∎

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