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A REFINEMENT OF A HARDY TYPE INEQUALITY FOR NEGATIVE EXPONENTS, AND SHARP APPLICATIONS TO MUCKENHOUPT WEIGHTS ON $\mathbb{R}$

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Abstract. We prove a sharp integral inequality that generalizes the well known Hardy type integral inequality for negative exponents. We also give sharp applications in two directions for Muckenhoupt weights on $\mathbb{R}$. This work refines the results of Nikolidakis (2014).

1. Introduction. In 1920, Hardy [2, 3] proved the following inequality, now known as Hardy's inequality.

Theorem A. *If $p > 1$, $a_n \geq 0$ and $A_n = a_1 + \cdots + a_n$ for $n = 1, 2, \ldots$ then*

$$\sum_{n=1}^{\infty}\left(\frac{A_n}{n}\right)^p \leq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n^p. \tag{1.1}$$

Moreover, inequality (1.1) *is best possible, that is, the constant on the right side cannot be decreased.*

In 1926, Copson [1] generalized Theorem A by replacing the arithmetic mean of a sequence by a weighted arithmetic mean. More precisely, he proved the following

Theorem B. *Let $p > 1$ and $a_n, \lambda_n > 0$ for $n = 1, 2, \ldots$. Set $\Lambda_n = \sum_{i=1}^{n} \lambda_i$ and $A_n = \sum_{i=1}^{n} \lambda_i a_i$. Then*

$$\sum_{n=1}^{\infty} \lambda_n \left(\frac{A_n}{\Lambda_n}\right)^p \leq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} \lambda_n a_n^p, \tag{1.2}$$

where the constant is best possible.

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Generalizations of (1.1) have been given in [6], [7] and elsewhere. For example, in [8] Hardy's and Copson's inequalities are generalized by replacing means by more general linear transforms.

Theorem A has a continuous analogue:

THEOREM C. *If* $p > 1$ *and* $f : [0, +\infty) \to \mathbb{R}^+$ *is* L^p*-integrable, then*

$$\int_0^\infty \left(\frac{1}{t} \int_0^t f(u)\, \mathrm{d}u \right)^p \mathrm{d}t \le \left(\frac{p}{p-1} \right)^p \int_0^\infty f(t)^p \, \mathrm{d}t. \tag{1.3}$$

The constant on the right side of (1.3) *is best possible.*

It is easy to see that Theorems A and C are equivalent, by standard approximation arguments which involve step functions. In [4], there is a continuous analogue of (1.3) for negative exponents, presented without proof:

THEOREM D. *Let* $f : [a, b] \to \mathbb{R}^+$*. Then for every* $p > 0$*,*

$$\int_a^b \left(\frac{1}{t-a} \int_a^t f(u)\, \mathrm{d}u \right)^{-p} \mathrm{d}t \le \left(\frac{p+1}{p} \right)^p \int_a^b f(t)^{-p} \, \mathrm{d}t. \tag{1.4}$$

Moreover, (1.4) *is best possible.*

In [9], a generalization of (1.4) was given:

THEOREM E. *Let* $p \ge q > 0$ *and* $f : [a, b] \to \mathbb{R}^+$*. The following sharp inequality is true:*

$$\begin{aligned} \int_a^b \left(\frac{1}{t-a} \int_a^t f(u)\, \mathrm{d}u \right)^{-p} \mathrm{d}t \\ \le \left(\frac{p+1}{p} \right)^q \int_a^b \left(\frac{1}{t-a} \int_a^t f(u)\, \mathrm{d}u \right)^{-p+q} f(t)^{-q} \, \mathrm{d}t. \end{aligned} \tag{1.5}$$

In fact, in [9] a more general weighted discrete analogue of (1.5) is proved:

THEOREM F. *Let* $p \ge q > 0$ *and* $a_n, \lambda_n > 0$ *for* $n = 1, 2, \ldots$*. Define* A_n *and* Λ_n *as in Theorem B. Then*

$$\sum_{n=1}^\infty \lambda_n \left(\frac{A_n}{\Lambda_n} \right)^{-p} \le \left(\frac{p+1}{p} \right)^q \sum_{n=1}^\infty \lambda_n \left(\frac{A_n}{\Lambda_n} \right)^{-p+q} a_n^{-q}. \tag{1.6}$$

Also, an application of Theorems E and F is given in [9], concerning Muckenhoupt weights.

In this paper we generalize and refine inequality (1.5) by specifying the integral of f over $[a, b]$. We also assume, for simplicity, that f is Riemann integrable on $[a, b]$. More precisely, we will prove

THEOREM 1. *Let* $p \geq q > 0$ *and* $f : [a, b] \to \mathbb{R}^+$ *with* $\frac{1}{b-a}\int_a^b f = \ell$. *Then*

$$\int_a^b \left(\frac{1}{t-a} \int_a^t f(u)\,\mathrm{d}u \right)^{-p} \mathrm{d}t \leq \left(\frac{p+1}{p} \right)^q \int_a^b \left(\frac{1}{t-a} \int_a^t f(u)\,\mathrm{d}u \right)^{-p+q} f(t)^{-q}\,\mathrm{d}t - \frac{q}{p+1}(b-a)\ell^{-p}. \tag{1.7}$$

Moreover, inequality (1.7) *is sharp if one considers all weights* f *that have mean integral average over* $[a, b]$ *equal to* ℓ.

The sharpness means that the constant in front of the integral on the right side cannot be decreased, while the one in front of ℓ^{-p} cannot be increased. In fact, more is true:

THEOREM 2. *Let* $p \geq q > 0$ *and* $a_n, \lambda_n > 0$ *for every* $n = 1, 2, \ldots$. *Define* A_n *and* Λ_n *as in Theorem B. Then for all* $N = 1, 2, \ldots$,

$$\sum_{n=1}^{N} \lambda_n \left(\frac{A_n}{\Lambda_n} \right)^{-p} \leq \left(\frac{p+1}{p} \right)^q \sum_{n=1}^{N} \lambda_n \left(\frac{A_n}{\Lambda_n} \right)^{-p+q} a_n^{-q} - \frac{q}{p+1} \Lambda_N \left(\frac{A_N}{\Lambda_N} \right)^{-p}. \tag{1.8}$$

In Section 2 we prove Theorem 2 and also the validity and sharpness of (1.7). For the whole subject of generalizations of inequalities (1.1) or (1.2), see [5] and [10].

In the last section we prove an application of Theorem 1:

THEOREM 3. *Let* $\varphi : [0, 1) \to \mathbb{R}^+$ *be a non-decreasing function satisfying the Muckenhoupt-type inequality*

$$\left(\frac{1}{t} \int_0^t \varphi(y)\,\mathrm{d}y \right) \left(\frac{1}{t} \int_0^t \varphi(y)^{-1/(q-1)}\,\mathrm{d}y \right)^{q-1} \leq M \tag{1.9}$$

for every $t \in (0, 1]$, *where* $q > 1$ *and* $M > 0$ *are given. Let* $p_0 \in (1, q)$ *be the solution of the equation*

$$\frac{q - p_0}{q - 1} (M p_0)^{1/(q-1)} = 1. \tag{1.10}$$

Then for every $p \in (p_0, q]$ *the inequality*

$$\frac{1}{t} \int_0^t \left(\frac{1}{s} \int_0^s \varphi \right)^{-1/(p-1)} \mathrm{d}s \leq \left(\frac{1}{t} \int_0^t \varphi \right)^{-1/(p-1)} \frac{1}{K'}\, c\, \frac{q}{p} \left(\frac{p-1}{q-1} \right)^2 \tag{1.11}$$

holds for all $t \in (0,1]$, *where*

$$c = M^{1/(q-1)} \quad \textit{and} \quad K' = K'(p,q,c) = \frac{1}{p^{1/(q-1)}} - c\,\frac{q-p}{q-1}.$$

This inequality is also sharp for $t = 1$.

The above theorem immediately implies

Corollary. *Let* φ *be as in Theorem* 3. *Then for every* $t \in (0,1]$ *and every* $p \in (p_0, q]$,

$$\left(\frac{1}{t}\int_0^t \varphi^{-1/(p-1)}\right)^{p-1}\left(\frac{1}{t}\int_0^t \varphi\right) \le \left[\frac{1}{K'}\,c\,\frac{q}{p}\left(\frac{p-1}{q-1}\right)^2\right]^{p-1}.$$

This gives the best possible range of p*'s for which the Muckenhoupt condition* (1.9) *with* q *replaced by* p *still holds, under the hypothesis of* (1.9).

The above corollary is in [9] but with another constant. Thus by proving Theorem 3 we refine the results in [9] by improving the constants that appear there and by giving certain sharp inequalities that involve Muckenhoupt weights on $\mathbb{R}$.

2. The Hardy inequality

Proof of Theorem 2. Following [9], we prove two lemmas.

Lemma 1. *For all* $n \ge 1$,

$$\left(\frac{p+1}{p}\right)^q a_n^{-q}\left(\frac{A_n}{\Lambda_n}\right)^{-p+q} + p\left(\frac{p}{p+1}\right)^{q/p} a_n^{q/p}\left(\frac{A_n}{\Lambda_n}\right)^{-p-q/p} \ge (p+1)\left(\frac{A_n}{\Lambda_n}\right)^{-p}. \tag{2.1}$$

Proof. It is well known that

$$y_1^{-p} + p y_1 y_2^{-p-1} - (p+1) y_2^{-p} \ge 0 \tag{2.2}$$

for all $y_1, y_2 > 0$: this is an immediate consequence (set $y = y_1/y_2$) of the inequality

$$y^{-p} + py \ge (p+1) \quad \text{for all } y, p \ge 0;$$

the latter follows from Young's inequality

$$\frac{1}{q}t^q + \frac{1}{q'}s^{q'} \ge ts,$$

valid for $t, s \ge 0$, $q > 1$ and $1/q + 1/q' = 1$, by choosing $q = p+1$ and $t = 1/y$.

Applying (2.2) when

$$y_1 = \left(\frac{p}{p+1}\right)^{1+q/p} a_n^{q/p} \left(\frac{A_n}{\Lambda_n}\right)^{1-q/p} \quad \text{and} \quad y_2 = \left(\frac{p}{p+1}\right)\frac{A_n}{\Lambda_n},$$

we get (2.1). ■

As a consequence of Lemma 1 we clearly have

$$\left(\frac{p+1}{p}\right)^q \sum_{n=1}^N \lambda_n a_n^{-q} \left(\frac{A_n}{\Lambda_n}\right)^{-p+q} + p\left(\frac{p}{p+1}\right)^{q/p} \sum_{n=1}^N \lambda_n a_n^{q/p} \left(\frac{A_n}{\Lambda_n}\right)^{-p-q/p} \geq (p+1)\sum_{n=1}^N \left(\frac{A_n}{\Lambda_n}\right)^{-p} \lambda_n \tag{2.3}$$

for all $N \geq 1$.

We proceed to the proof of

Lemma 2. *For all* $N \geq 1$,

$$\sum_{n=1}^N \lambda_n \left(\frac{A_n}{\Lambda_n}\right)^{-p} - \left(\frac{p}{p+1}\right) \sum_{n=1}^N \lambda_n a_n \left(\frac{A_n}{\Lambda_n}\right)^{-p-1} \geq \frac{\Lambda_n}{p+1}\left(\frac{A_n}{\Lambda_n}\right)^{-p}. \tag{2.4}$$

Proof. We follow [9]. For $N = 1$, inequality (2.4) is in fact an equality. Suppose (2.6) is true with $N-1$ in place of N. Define

$$\begin{aligned} S_N &= \sum_{n=1}^N \left[\lambda_n \left(\frac{A_n}{\Lambda_n}\right)^{-p} - \left(\frac{p}{p+1}\right)\lambda_n a_n \left(\frac{A_n}{\Lambda_n}\right)^{-p-1}\right] \\ &= \sum_{n=1}^{N-1} \left[\lambda_n \left(\frac{A_n}{\Lambda_n}\right)^{-p} - \left(\frac{p}{p+1}\right)\lambda_n a_n \left(\frac{A_n}{\Lambda_n}\right)^{-p-1}\right] \\ &\quad + \lambda_N \left(\frac{A_N}{\Lambda_N}\right)^{-p} - \left(\frac{p}{p+1}\right)(A_N - A_{N-1})\left(\frac{A_N}{\Lambda_N}\right)^{-p-1}. \end{aligned}$$

By the inductive assumption, we see that

$$\begin{aligned} S_N &\geq \frac{\Lambda_{N-1}}{p+1}\left(\frac{A_{N-1}}{\Lambda_{N-1}}\right)^{-p} + \lambda_N \left(\frac{A_N}{\Lambda_N}\right)^{-p} \\ &\quad - \left(\frac{p}{p+1}\right)(A_N - A_{N-1})\left(\frac{A_N}{\Lambda_N}\right)^{-p-1}. \\ &= \frac{\Lambda_{N-1}}{p+1}\left(\frac{A_{N-1}}{\Lambda_{N-1}}\right)^{-p} + \lambda_N \left(\frac{A_N}{\Lambda_N}\right)^{-p} - \frac{p}{p+1}\Lambda_N \left(\frac{A_N}{\Lambda_N}\right)^{-p} \\ &\quad + \frac{\Lambda_{N-1}}{p+1}\left[p\,\frac{A_{N-1}}{\Lambda_{N-1}}\left(\frac{A_N}{\Lambda_N}\right)^{-p-1}\right]. \end{aligned} \tag{2.5}$$

By (2.2),

$$p\left(\frac{A_{N-1}}{\Lambda_{N-1}}\right)\left(\frac{A_N}{\Lambda_N}\right)^{-p-1} \geq -\left(\frac{A_{N-1}}{\Lambda_{N-1}}\right)^{-p} + (p+1)\left(\frac{A_N}{\Lambda_N}\right)^{-p}.$$

Inserting this in (2.5), we obtain

$$\begin{aligned} S_N &\geq \frac{\Lambda_{N-1}}{p+1}\left(\frac{A_{N-1}}{\Lambda_{N-1}}\right)^{-p} + \lambda_N\left(\frac{A_N}{\Lambda_N}\right)^{-p} - \left(\frac{p}{p+1}\right)\Lambda_N\left(\frac{A_N}{\Lambda_N}\right)^{-p} \\ &\quad + \frac{\Lambda_{N-1}}{p+1}\left[(p+1)\left(\frac{A_N}{\Lambda_N}\right)^{-p} - \left(\frac{A_{N-1}}{\Lambda_{N-1}}\right)^{-p}\right] \\ &= \left(\frac{A_N}{\Lambda_N}\right)^{-p}\left(\lambda_N - \frac{p}{p+1}\Lambda_N + \Lambda_{N-1}\right) = \frac{\Lambda_N}{p+1}\left(\frac{A_N}{\Lambda_N}\right)^{-p} \end{aligned}$$

as desired. ∎

We now consider the quantity

$$\begin{aligned} y &= \sum_{n=1}^{N} \lambda_n a_n^{q/p}\left(\frac{A_n}{\Lambda_n}\right)^{-p-q/p} \\ y &= \sum_{n=1}^{N} \lambda_n \left[a_n^{q/p}\left(\frac{A_n}{\Lambda_n}\right)^{-q-q/p}\right]\left[\frac{A_n}{\Lambda_n}\right]^{-p+q}. \end{aligned}$$

Suppose that $p > q$; the case $p = q$ will be discussed at the end of the proof. Applying Hölder's inequality with exponents $r = \frac{p}{q}$ and $r' = \frac{p}{p-q}$, we get

$$\begin{aligned} y &\leq \left\{\sum_{n=1}^{N} \lambda_n a_n\left(\frac{A_n}{\Lambda_n}\right)^{-p-1}\right\}^{q/p}\left\{\sum_{n=1}^{N} \lambda_n\left(\frac{A_n}{\Lambda_n}\right)^{-p}\right\}^{1-q/p} \\ &\leq \left\{\frac{p+1}{p}\sum_{n=1}^{N} \lambda_n\left(\frac{A_n}{\Lambda_n}\right)^{-p} - \frac{1}{p}\Lambda_N\left(\frac{A_N}{\Lambda_N}\right)^{-p}\right\}^{q/p} \\ &\quad \cdot\left\{\sum_{n=1}^{N} \lambda_n\left(\frac{A_n}{\Lambda_n}\right)^{-p}\right\}^{1-q/p}, \end{aligned} \tag{2.6}$$

in view of Lemma 2.

We set now $z = \sum_{n=1}^{N} \lambda_n a_n^{-q}(A_n/\Lambda_n)^{-p+q}$ and $x = \sum_{n=1}^{N} \lambda_n (A_n/\Lambda_n)^{-p}$. Because of (2.6) we have

$$y \leq \left\{\frac{p+1}{p}x - \frac{1}{p}\Lambda_N\left(\frac{A_N}{\Lambda_N}\right)^{-p}\right\}^{q/p} \cdot x^{1-q/p}.$$

Hence, by setting $c = \Lambda_N\left(\frac{A_N}{\Lambda_N}\right)^{-p}$, we have

$$y \leq \left(\frac{p+1}{p}x - \frac{c}{p}\right)^{q/p} \cdot x^{1-q/p} = \left(\frac{p+1}{p}\right)^{q/p}\left[x - \frac{c}{p+1}\right]^{q/p} x^{1-q/p}. \tag{2.7}$$

It follows that $x - \frac{c}{p+1} > 0$, so $x > \frac{c}{p+1}$. Now because of Lemma 1, it is immediate that

$$\left(\frac{p+1}{p}\right)^q z + p\left(\frac{p}{p+1}\right)^{q/p} y \geq (p+1)x,$$

so by (2.7),

$$\left(\frac{p+1}{p}\right)^q z + p\left(\frac{p}{p+1}\right)^{q/p}\left(\frac{p+1}{p}\right)^{q/p}\left[x - \frac{c}{p+1}\right]^{q/p} x^{1-q/p} \geq (p+1)x,$$

and hence

$$\begin{aligned}(2.8)\qquad \left(\frac{p+1}{p}\right)^q z &\geq (p+1)x - p\left[x - \frac{c}{p+1}\right]^{q/p} x^{1-q/p} \\ &= x + \left\{p\,x - p\left[x - \frac{c}{p+1}\right]^{q/p} x^{1-q/p}\right\} = x + pG(x),\end{aligned}$$

where $G(x) = x - \left[x - \frac{c}{p+1}\right]^{q/p} x^{1-q/p}$ for $x > \frac{c}{p+1}$. Thus

$$(2.9)\qquad \left(\frac{p+1}{p}\right)^q z - x \geq p\,G(x) \geq p\inf\left\{G(x) : x > \frac{c}{p+1}\right\}.$$

We will now find the above infimum. Note that

(2.10)

$$\begin{aligned}G'(x) &= 1 - \left(1 - \frac{q}{p}\right)x^{-q/p}\left(x - \frac{c}{p+1}\right)^{q/p} - x^{1-q/p}\left(\frac{q}{p}\right)\left(x - \frac{c}{p+1}\right)^{q/p-1} \\ &= 1 - \left(1 - \frac{q}{p}\right)\left(1 - \frac{c}{(p+1)x}\right)^{q/p} - \frac{q}{p}\left(1 - \frac{c}{(p+1)\,x}\right)^{q/p-1}.\end{aligned}$$

Consider the function

$$H(t) = 1 - \left(1 - \frac{q}{p}\right)t^{q/p} - \frac{q}{p}\,t^{q/p-1}, \quad t \in (0,1).$$

Then $H'(t) = -t^{q/p-2}\left(1 - \frac{q}{p}\right)\frac{q}{p}(t-1) > 0$ for every $t \in (0,1)$. Thus $H(t)$ is strictly increasing, so $H(t) \leq H(1) = 0$ for all $t \in (0,1)$. By setting $t = 1 - \frac{c}{(p+1)x}$, we conclude that the right hand side of (2.10) is negative for $x > \frac{c}{p+1}$, so G is decreasing in $\left(\frac{c}{p+1}, +\infty\right)$. Thus

$$\begin{aligned}G(x) &\geq \lim_{x\to+\infty} G(x) = \lim_{x\to+\infty}\left[x - x^{1-q/p}\left(x - \frac{c}{p+1}\right)^{q/p}\right] \\ &= \lim_{x\to+\infty}\frac{1 - \left(1 - \frac{c}{(p+1)x}\right)^{q/p}}{\frac{1}{x}} = \lim_{y\to0^+}\frac{1 - \left(1 - \frac{c}{p+1}y\right)^{q/p}}{y} \\ &= -\frac{q}{p}\left(-\frac{c}{p+1}\right) = \frac{qc}{p(p+1)},\end{aligned}$$

by de L'Hospital's rule. Hence (2.9) implies

$$\left(\frac{p+1}{p}\right)^q z - x \geq p\frac{q\,c}{p\,(p+1)} = \frac{q\,c}{p+1},$$

which gives inequality (1.8), by the definitions of x, z and c.

The case $p = q$ is obtained by letting $p \to q^+$ in (1.8). ∎

Proof of Theorem 1. We first prove (1.7). For simplicity we assume $a = 0$ and $b = 1$. We also assume that $f : [0,1] \to \mathbb{R}^+$ is continuous. The general case of Riemann integrable functions can be handled by approximation. We additionally assume that there exists a positive constant δ which is a lower bound for f on $[0,1]$. Otherwise we consider the family of functions $(f_\delta)_{\delta>0}$ on $[0,1]$ defined by $f_\delta(t) = \max(f(t),\delta)$. If (1.7) is true for every f_δ, it is also true for f, by letting $\delta \to 0^+$.

We now suppose that $\int_0^1 f = \ell$ and define $F : (0,1] \to \mathbb{R}^+$ by $F(t) = \frac{1}{t}\int_0^t f(u)\,\mathrm{d}u$. Then

$$\int_0^1 \left(\frac{1}{t}\int_0^t f(u)\,\mathrm{d}u\right)^{-p} \mathrm{d}t = \int_0^1 F(t)^{-p}\,\mathrm{d}t.$$

The last integral can be approximated by Riemann sums of the type

$$\sum_{n=1}^{2^k} \frac{1}{2^k} F(n/2^k)^{-p} = \frac{1}{2^k}\sum_{n=1}^{2^k}\left(\frac{\sum_{i=1}^n a_i^{(k)}}{n}\right)^{-p}$$

where

$$a_i^{(k)} = 2^k \int_{i-1/2^k}^{i/2^k} f \qquad \text{for } i = 1,\ldots,2^k.$$

By (1.8) the Riemann sum above is less than or equal to

$$\left(\frac{p+1}{p}\right)^q \frac{1}{2^k}\sum_{n=1}^{2^k}\left(\frac{\sum_{i=1}^n a_i^{(k)}}{n}\right)^{-p+q}(a_n^{(k)})^{-q} - \frac{q}{p+1}\left(\frac{\sum_{n=1}^{2^k} a_n^{(k)}}{2^k}\right)^{-p}.$$

Now obviously $\sum_{n=1}^{2^k} a_n^{(k)}/2^k = \ell$, while since f is continuous, for every $n = 1,\ldots,2^k$ there exists $b_n^{(k)} \in \left[\frac{n-1}{2^k},\frac{n}{2^k}\right]$ such that $a_n^{(k)} = f(b_n^{(k)})$. Thus the quantity above equals

$$\left(\frac{p+1}{p}\right)^q \frac{1}{2^k}\sum_{n=1}^{2^k} F(n/2^k)^{-p+q} f(b_n^{(k)})^{-q} - \frac{q}{p+1}\ell^{-p}.$$

This quantity is equivalent to

$$R := \left(\frac{p+1}{p}\right)^q \frac{1}{2^k}\sum_{n=1}^{2^k} F(b_n^{(k)})^{-p+q} f(b_n^{(k)})^{-q} - \frac{q}{p+1}\ell^{-p}$$

as $k \to \infty$. Indeed, since $f(t) \geq \delta$ for every $t \in [0,1]$, $p > q$ and $F(t) \to f(0)$ as $t \to 0^+$, the function F^{-p+q} can be extended to a uniformly continuous function on $[0,1]$, thus by the choice of $b_n^{(k)}$, for every $\epsilon > 0$ there exists a positive integer k_0 such that

$$\left| \frac{1}{2^k} \sum_{n=1}^{2^k} [F(n/2^k)^{-p+q} - F(b_n^{(k)})^{-p+q}] f(b_n^{(k)})^{-q} \right| < \delta^{-q}\epsilon$$

for every $k \geq k_0$.

It is now clear that R approximates the right side of (1.7) as $k \to \infty$.

To prove the sharpness of (1.7), fix $\ell > 0$ and $p \geq q > 0$. For any $a \in (-1/p, 0)$ let $g_a(t) = \ell\,(1-a)\,t^{-a}$, $t \in [0,1]$. It is easy to see that $\int_0^1 g_a = \ell$, $\frac{1}{t}\int_0^t g_a = \frac{1}{1-a} g_a(t)$ for every $t \in (0,1]$ and $\int_0^1 g_a^{-p} = \frac{\ell^{-p}(1-a)^{-p}}{1+ap}$. We now consider the difference

$$L_a = \int_0^1 \left(\frac{1}{t} \int_0^t g_a \right)^{-p} \mathrm{d}t - \left(\frac{p+1}{p} \right)^q \int_0^1 \left(\frac{1}{t} \int_0^t g_a \right)^{-p+q} g_a(t)^{-q} \,\mathrm{d}t.$$

By the properties of g_a,

$$L_a = \ell^{-p} \frac{1 - (1-a)^{-q} \left(\frac{p+1}{p}\right)^q}{1+ap}.$$

Hence

$$\lim_{a \to -(1/p)^+} L_a = \ell^{-p} q (1-a)^{-q-1}|_{a=-1/p} (-1) \left(\frac{p+1}{p} \right)^q = -\frac{q}{p+1} \ell^{-p},$$

which proves the sharpness or (1.7). ∎

3. Proof of Theorem 3. Let $\varphi : [0,1) \to \mathbb{R}^+$ be non-decreasing with

$$\left(\frac{1}{t} \int_0^t \varphi \right) \left(\frac{1}{t} \int_0^t \varphi^{-1/(q-1)} \right)^{q-1} \leq M \tag{3.1}$$

for every $t \in (0,1]$, where q is fixed such that $q > 1$ and $M > 0$. We also assume that there exists an $\varepsilon > 0$ such that $\varphi(t) \geq \varepsilon > 0$ for all $t \in [0,1)$. The general case can be handled by adding a small constant $\varepsilon > 0$ to φ.

We need the following from [9].

Lemma A. *Let $\psi : (0,1) \to [0,+\infty)$ with $\lim_{t \to 0} t\psi(t)^a = 0$, where $a > 1$ and $\psi(t)$ is continuous and monotone on $(0,1)$. Then for any $a \in (0,1)$,*

$$a \int_0^u \psi(t)^{a-1} [t\,\psi(t)]' \,\mathrm{d}t = u\psi(u)^a + (a-1) \int_0^u \psi(t)^a \,\mathrm{d}t.$$

We define $h : [0,1) \to \mathbb{R}^+$ by $h(t) = \varphi(t)^{-1/(q-1)}$. Then obviously $h(t) \leq \varepsilon^{-1/(q-1)}$ for all $t \in [0,1)$. Let also $p_0 \in [1,q]$ be defined via

$$\frac{q-p_0}{q-1}(Mp_0)^{1/(q-1)} = 1,$$

and let $p \in (p_0, q]$. Define $\psi(t) = \frac{1}{t}\int_0^t \varphi^{-1/(q-1)}$, $t \in (0,1)$. By Lemma A, for $a = \frac{q-1}{p-1} > 1$,

$$\frac{q-1}{p-1}\int_0^t \varphi(s)^{-1/(q-1)}\left(\frac{1}{s}\int_0^s \varphi^{-1/(q-1)}\right)^{\frac{q-p}{p-1}} \mathrm{d}s - \frac{q-p}{p-1}\int_0^t\left(\frac{1}{s}\int_0^s \varphi^{-1/(q-1)}\right)^{\frac{q-1}{p-1}} \mathrm{d}s = t\left(\frac{1}{t}\int_0^t \varphi^{-1/(q-1)}\right)^{\frac{q-1}{p-1}}. \tag{3.2}$$

For every $y > 0$ define

$$g_y(x) = \frac{q-1}{q-p} y x^{(q-p)/(p-1)} - x^{(q-1)/(p-1)}, \qquad x \in [y, +\infty).$$

Then $g_y'(x) = \frac{q-1}{p-1} x^{(q-1)/(p-1)-2}(y-x) \leq 0$ for all $x \geq y$, so g_y is strictly decreasing on $[y, +\infty)$.

Therefore if $y \leq x \leq w$ then $g_y(x) \geq g_y(w)$. For every $s \in (0,t]$ set now

$$x = \frac{1}{s}\int_0^s \varphi^{-1/(q-1)}, \quad y = \varphi(s)^{-1/(q-1)}, \quad c = M^{1/(q-1)}, \quad z = \left(\frac{1}{s}\int_0^s \varphi\right)^{-\frac{1}{q-1}}.$$

By (3.1) we have $y \leq x \leq cz =: w$. Hence $g_y(x) \geq g_y(w)$, so

$$\frac{q-1}{q-p}\varphi(s)^{-1/(q-1)}\left(\frac{1}{s}\int_0^s \varphi^{-1/(q-1)}\right)^{\frac{q-p}{p-1}} - \left(\frac{1}{s}\int_0^s \varphi^{-1/(q-1)}\right)^{\frac{q-1}{p-1}} \geq \frac{q-1}{q-p}\varphi(s)^{-1/(q-1)}\left(\frac{1}{s}\int_0^s \varphi\right)^{\frac{1}{q-1}-\frac{1}{p-1}} c^{\frac{q-p}{p-1}} - c^{\frac{q-1}{p-1}}\left(\frac{1}{s}\int_0^s \varphi\right)^{-\frac{1}{p-1}}.$$

Integrating this on $s \in (0,t]$ we get

$$\frac{q-1}{q-p}\int_0^t \varphi(s)^{-1/(q-1)}\left(\frac{1}{s}\int_0^s \varphi\right)^{-\frac{1}{p-1}+\frac{1}{q-1}} \mathrm{d}s \cdot c^{\frac{q-p}{p-1}} \leq \frac{q-1}{q-p}\int_0^t \varphi(s)^{-1/(q-1)}\left(\frac{1}{s}\int_0^s \varphi^{-1/(q-1)}\right)^{\frac{q-p}{p-1}} \mathrm{d}s - \int_0^t\left(\frac{1}{s}\int_0^s \varphi^{-1/(q-1)}\right)^{\frac{q-1}{p-1}} \mathrm{d}s + c^{\frac{q-1}{p-1}}\int_0^t\left(\frac{1}{s}\int_0^s \varphi\right)^{-1/(p-1)} \mathrm{d}s. \tag{3.3}$$

Now because of (3.2) we get

$$\frac{q-1}{q-p}\int_0^t \varphi(s)^{-1/(q-1)}\left(\frac{1}{s}\int_0^s \varphi^{-1/(q-1)}\right)^{\frac{q-p}{p-1}} \mathrm{d}s - \int_0^t \left(\frac{1}{s}\int_0^s \varphi^{-1/(q-1)}\right)^{\frac{q-1}{p-1}} \mathrm{d}s$$
$$= \frac{p-1}{q-p}\,\frac{1}{t^{(q-p)/(p-1)}}\left(\int_0^t \varphi^{-1/(q-1)}\right)^{\frac{q-1}{p-1}}.$$

Thus (3.3) gives

$$c^{\frac{q-p}{p-1}}\,\frac{q-1}{q-p}\int_0^t \varphi(s)^{-1/(q-1)}\left(\frac{1}{s}\int_0^s \varphi\right)^{-\frac{1}{p-1}+\frac{1}{q-1}} \mathrm{d}s$$
$$\le c^{\frac{q-1}{p-1}}\int_0^t \left(\frac{1}{s}\int_0^s \varphi\right)^{-\frac{1}{p-1}} \mathrm{d}s + \frac{p-1}{q-p} t\left(\frac{1}{t}\int_0^t \varphi^{-1/(q-1)}\right)^{\frac{q-1}{p-1}}. \tag{3.4}$$

But

$$\left[\frac{1}{t}\int_0^t \varphi^{-1/(q-1)}\right]^{\frac{q-1}{p-1}} \le M^{1/(p-1)}\left(\frac{1}{t}\int_0^t \varphi\right)^{-\frac{1}{p-1}},$$

so by (3.4),

$$c^{\frac{q-p}{p-1}}\,\frac{q-1}{q-p}\int_0^t \varphi(s)^{-1/(q-1)}\left(\frac{1}{s}\int_0^s \varphi\right)^{-\frac{1}{p-1}+\frac{1}{q-1}} \mathrm{d}s$$
$$\le c^{\frac{q-1}{p-1}}\int_0^t \left(\frac{1}{s}\int_0^s \varphi\right)^{-\frac{1}{p-1}} \mathrm{d}s + \frac{p-1}{q-p}\, t\, M^{1/(p-1)}\left(\frac{1}{t}\int_0^t \varphi\right)^{-\frac{1}{p-1}}.$$

Consequently,

$$A_1 := \frac{q-1}{q-p}\int_0^t \varphi(s)^{-1/(q-1)}\left(\frac{1}{s}\int_0^s \varphi\right)^{-\frac{1}{p-1}+\frac{1}{q-1}} \mathrm{d}s$$
$$\le c\int_0^t \left(\frac{1}{s}\int_0^s \varphi\right)^{-\frac{1}{p-1}} \mathrm{d}s + \frac{p-1}{q-p}\,\frac{M^{1/(p-1)}}{c^{(q-p)/(p-1)}}\, t\left(\frac{1}{t}\int_0^t \varphi\right)^{-\frac{1}{p-1}}. \tag{3.5}$$

Now by using Theorem 1 we get

$$\int_0^t \left(\frac{1}{s}\int_0^s \varphi\right)^{-\frac{1}{p-1}} \mathrm{d}s$$
$$\le \left(\frac{1+\frac{1}{p-1}}{\frac{1}{p-1}}\right)^{\frac{1}{q-1}}\int_0^t \left(\frac{1}{s}\int_0^s \varphi\right)^{-\frac{1}{p-1}+\frac{1}{q-1}} \varphi(s)^{-\frac{1}{q-1}}\,\mathrm{d}s - \frac{\frac{1}{q-1}}{1+\frac{1}{p-1}}\, t\left(\frac{1}{t}\int_0^t \varphi\right)^{-\frac{1}{p-1}}$$
$$= p^{1/(q-1)} A_1\,\frac{q-p}{q-1} - \frac{p-1}{(q-1)p}\, t\left(\frac{1}{t}\int_0^t \varphi\right)^{-\frac{1}{p-1}}.$$

Thus (3.5) becomes

$$A_1 \le cp^{1/(q-1)} A_1 \, \frac{q-p}{q-1} - c\,\frac{p-1}{(q-1)p}\, t\left(\frac{1}{t}\int_0^t \varphi\right)^{-\frac{1}{p-1}} + \frac{p-1}{q-p}\,\frac{M^{1/(p-1)}}{c^{(q-p)/(p-1)}}\, t\left(\frac{1}{t}\int_0^1 \varphi\right)^{-\frac{1}{p-1}},$$

hence

$$\left[1 - c\,p^{1/(q-1)}\frac{q-p}{q-1}\right] A_1 \le \left[\frac{M^{1/(p-1)}}{c^{(q-p)/(p-1)}}\,\frac{p-1}{q-p} - c\,\frac{p-1}{(q-1)\,p}\right]\cdot t\left(\frac{1}{t}\int_0^t \varphi\right)^{-\frac{1}{p-1}},$$

and consequently

$$K(p,q,c)\left[\frac{1}{t}\int_0^t \varphi(s)^{-1/(q-1)}\left(\frac{1}{s}\int_0^s \varphi\right)^{-\frac{1}{p-1}+\frac{1}{q-1}} \mathrm{d}s\right] \le \left[\frac{p-1}{q-1}\,\frac{M^{1/(p-1)}}{c^{(q-p)/(p-1)}} - c\,\frac{(p-1)(q-p)}{p\,(q-1)^2}\right]\left(\frac{1}{t}\int_0^t \varphi\right)^{-\frac{1}{p-1}}$$

where $K = K(p,q,c) = 1 - c\,p^{1/(q-1)}\frac{q-p}{q-1} > 0$ for all $p \in (p_0, q]$. As a consequence,

$$K\left[\frac{1}{t}\int_0^t \varphi(s)^{-1/(p-1)}\left(\frac{1}{s}\int_0^s \varphi\right)^{-\frac{1}{p-1}+\frac{1}{q-1}} \mathrm{d}s\right] \le \left(\frac{1}{t}\int_0^t \varphi\right)^{-\frac{1}{p-1}}\left(\frac{p-1}{q-1}\right)^2 c\,\frac{q}{p}. \tag{3.6}$$

Now we use the inequality

$$\frac{1}{t}\int_0^t \varphi(s)^{-1/(q-1)}\left(\frac{1}{s}\int_0^s \varphi\right)^{-\frac{1}{p-1}+\frac{1}{q-1}} \mathrm{d}s \ge \left[\frac{1/(p-1)}{1+1/(p-1)}\right]^{\frac{1}{q-1}}\cdot \frac{1}{t}\int_0^t\left(\frac{1}{s}\int_0^s \varphi\right)^{-\frac{1}{p-1}} \mathrm{d}s,$$

following from Theorem E. Thus (3.6) gives

$$\frac{K'}{t}\int_0^t\left(\frac{1}{s}\int_0^s \varphi\right)^{-\frac{1}{p-1}} \mathrm{d}s \le \left(\frac{1}{t}\int_0^t \varphi\right)^{-\frac{1}{p-1}}\left(\frac{p-1}{q-1}\right)^2 c\frac{q}{p} \tag{3.7}$$

where $K' = K/p^{1/(q-1)}$, $K = 1 - c\,p^{1/(q-1)}\frac{q-p}{q-1}$.

Thus the inequality stated in Theorem 3 is proved.

We need to prove the sharpness of (3.7). We consider a such that $0 < a < q-1$ and the function $\varphi_a : (0,1] \to \mathbb{R}^+$ defined by $\varphi_a(t) = t^a$, $t \in (0,1]$. Then

φ_a is strictly increasing and $\frac{1}{t}\int_0^t \varphi_a = \frac{1}{t}\frac{t^{a+1}}{a+1} = \frac{1}{a+1}\varphi_a(t)$ for all $t \in (0,1]$, while $\int_0^t \varphi_a^{-1/(q-1)} = \frac{1}{1-a/(q-1)} t^{1-a/(q-1)}$. Thus

$$\left(\frac{1}{t}\int_0^t \varphi_a\right)\left[\frac{1}{t}\int_0^t \varphi_a^{-1/(q-1)}\right]^{q-1} = \left[\frac{q-1}{q-1-a}\right]^{q-1}[t^{-a/(q-1)}]^{q-1}\cdot\left(\frac{1}{t}\int_0^t \varphi_a\right)$$
$$= \frac{1}{a+1}\left(\frac{q-1}{q-1-a}\right)^{q-1} =: M(q,a)$$

and

$$c_a = c(q,a) = [M(q,a)]^{1/(q-1)} = \left[\frac{q-1}{(q-1)-a}\right]\frac{1}{(1+a)^{1/(q-1)}}.$$

Let now $p \in (p_0, q]$ and suppose additionally that $a < p-1$ so that $\int_0^1 \varphi_a^{-1/(p-1)} = (p-1)/(p-1-a)$. We prove the sharpness of (1.11) for $t = 1$, that is, we prove that the inequality

$$\frac{K'}{t}\int_0^t \left(\frac{1}{s}\int_0^s \varphi\right)^{-\frac{1}{p-1}} \mathrm{d}s \le \left(\frac{1}{t}\int_0^t \varphi\right)^{-\frac{1}{p-1}} c\,\frac{q}{p}\left(\frac{p-1}{q-1}\right)^2$$

becomes sharp for $t = 1$. Obviously if

$$\int_0^1 \left(\frac{1}{s}\int_0^s \varphi_a\right)^{-\frac{1}{p-1}} \mathrm{d}s = \frac{1}{(1+a)^{-1/(p-1)}}\int_0^1 \varphi_a^{-1/(p-1)} = (1+a)^{1/(p-1)}\frac{1}{1-a/(p-1)}$$

while

$$\left(\int_0^1 \varphi_a\right)^{-\frac{1}{p-1}} = \left(\frac{1}{a+1}\right)^{-\frac{1}{p-1}}.$$

Thus in order to prove the desired sharpness we just need to prove that

$$\left[\frac{1}{p^{1/(q-1)}} - \frac{q-p}{q-1}c_a\right]\frac{p-1}{(p-1)-a} \cong c_a\,\frac{q}{p}\left[\frac{p-1}{q-1}\right]^2 \qquad \text{as } a \to (p-1)^-,$$

or equivalently

$$\left[\frac{1}{p^{1/(q-1)}} - \frac{q-p}{q-1}\frac{1}{(1+a)^{1/(q-1)}}\frac{q-1}{(q-1)-a}\right]\frac{1}{(p-1)-a}$$
$$\cong \frac{q}{p}\frac{p-1}{(q-1)^2}\frac{1}{(1+a)^{1/(p-1)}}\frac{q-1}{(q-1)-a} \qquad \text{as } a \to (p-1)^-.$$

Note that $a \to (p-1)^-$ is equivalent to $x := (a+1) \to p^-$. Thus we just need to observe that

$$\frac{p^{-\frac{1}{1/(q-1)}} - \frac{q-p}{q-x}\frac{1}{x^{1/(q-1)}}}{p-x} \cong \frac{q}{p}\frac{p-1}{q-1}\frac{1}{p^{1/(q-1)}}\frac{1}{q-p} \qquad \text{as } x \to p^-,$$

which is a simple application of de L'Hospital's rule.

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