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HOM-GROUPS, REPRESENTATIONS AND HOMOLOGICAL ALGEBRA

BY

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Abstract. A Hom-group G is a nonassociative generalization of a group where associativity, invertibility, and unitality are twisted by a map $\alpha : G \to G$. Introducing the Hom-group algebra $\mathbb{K}G$, we observe that Hom-groups provide examples of Hom-algebras, Hom-Lie algebras and Hom-Hopf algebras. We introduce two types of modules over a Hom-group G. To find out more about those modules, we introduce Hom-group (co)homology with coefficients in those modules. Our (co)homology theories generalize group (co)homology for groups. In contrast to the associative case, the coefficients of Hom-group homology are different from the ones for Hom-group cohomology. We show that the inverse elements provide a relation between Hom-group (co)homology with coefficients in right and left G-modules. It is shown that our (co)homology theories for Hom-groups with coefficients could be reduced to Hochschild (co)homology of Hom-group algebras. For certain coefficients the functoriality of Hom-group (co)homology is shown.

1. Introduction. The notion of Hom-Lie algebra is a generalization of Lie algebras which appeared first in q-deformations of Witt and Virasoro algebras where the Jacobi identity is deformed by a linear map [AS], [CKL], [CZ]. There are several interesting examples of Hom-Lie algebras. For example, it is shown in [GR] that any algebra of dimension 3 is a Hom-Lie algebra. The related algebra structure is called Hom-algebra and introduced in [MS1]. Later, other objects such as Hom-bialgebras and Hom-Hopf algebras were studied in [MS2], [MS3], [Ya2], [Ya3], [Ya4]. For more work on Hom-Lie algebras we refer the reader to [CS], [HLS], [LS], [BM], for Hom-algebras to [GMMP], [FG], [HMS], and for representations of Hom-objects to [CQ], [GW], [PSS]. Hopf algebras are known to have close relations to groups and Lie algebras. The sets of group-like elements and of primitive elements of a Hopf algebra form a group and a Lie algebra respectively. Conversely, any group gives a Hopf algebra which is called the group algebra. For any Lie algebra we have the universal enveloping algebra. There

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have been many works relating Hom-Lie algebras and Hom-Hopf algebras. However some relations were missing in the context of Hom-type objects due to the lack of Hom-type notions for groups and group algebras. Here we briefly explain how Hom-groups have appeared in the context of Hom-type objects. The universal enveloping algebra of a Hom-Lie algebra has a Hom-bialgebra structure [Ya4]. However, it does not have a Hom-Hopf algebra structure in the sense of [MS2]. This is due to the fact that the antipode is not an inverse of the identity map with respect to the convolution product. This motivated the authors of [LMT] to modify the notion of invertibility in Hom-algebras and introduce a new definition for the antipode of Hom-Hopf algebras. Solving this problem, they came to axioms of Hom-groups which naturally appear in the structure of the group-like elements of Hom-Hopf algebras. They were also motivated by constructing a Hom-Lie group integrating a Hom-Lie algebra. Simultaneously with this paper, the present author [H1] introduced and studied several fundamental notions for Hom-groups for which the twisting map α is invertible. It was shown that Hom-groups are examples of quasigroups. Furthermore, the Lagrange theorem for finite Hom-groups was shown. The Hom-Hopf algebra structure of the Hom-group of the Hopf algebra $\mathbb{K}G$ was introduced in [H2].

In this paper we investigate various aspects of Hom-groups such as modules and homological algebra. In Section 2, we give the basics of Hom-groups. We introduce the Hom-algebra associated to a Hom-group G, called the Hom-group algebra and denoted by $\mathbb{K}G$. It has been shown in [MS1] that the commutator of a Hom-associative algebra A is a Hom-Lie algebra $\mathfrak{g}_A$. In [Ya4], [LMT] it is shown that the universal enveloping algebra of a Hom-Lie algebra is endowed with a Hom-Hopf algebra structure. Therefore Hom-groups are sources of examples for Hom-algebras, Hom-Lie algebras, and Hom-Hopf algebras as follows:

$$G \hookrightarrow \mathbb{K}G \hookrightarrow \mathfrak{g}_{\mathbb{K}G} \hookrightarrow U(\mathfrak{g}_{\mathbb{K}G}).$$

For more examples and fundamental notions on Hom-groups we refer the reader to [H1].

In Section 3, we introduce two types of modules over Hom-groups. The first type is called dual Hom-modules. Using inverse elements in a Hom-group G, we show that a left dual G-module can be turned into a right dual G-module and vice versa. Then we introduce G-modules and we show that for any left G-module M the algebraic dual $\mathrm{Hom}(M, \mathbb{K})$ is a dual right G-module where $\mathbb{K}$ is a field.

It is known that group (co)homology provides an important set of tools for studying modules over a group. This motivates us to introduce (co)homology theories for Hom-groups to find out more about representations of Hom-groups.

Generally introducing homological algebra for nonassociative objects is a difficult task. The first attempts to introduce homological tools for Hom-algebras and Hom-Lie algebras appeared in [AEM], [MS3], [MS4], [Ya1]. In [HSS] Hochschild and cyclic (co)homology for Hom-algebras was defined. In Section 4, we introduce Hom-group cohomology with coefficients in dual left (right) G-modules. The conditions on M in our work also appeared in other contexts such as [CG] where the category of Hom-modules over Hom-algebras was used to obtain a monoidal category for modules over Hom-bialgebras. A noticeable difference between homology theories of Hom-algebras introduced in [HSS] and the ones for Hom-groups in this paper is that the former needs bimodules over Hom-algebras and the latter requires one-sided modules (left or right).

We show that Hom-group cohomology with coefficients in a dual right G-module is isomorphic to Hom-group cohomology with coefficients in the dual left G-module where the left action is given by the inverse elements. We compute 0- and 1-cocycles and we show the functoriality of Hom-group cohomology for certain coefficients.

Since any Hom-group gives the Hom-group algebra, a natural question is the relation of these two different objects to cohomology theories. We show that the Hom-group cohomology of a Hom-group G with coefficients in a dual left module is isomorphic to the Hom-Hochschild cohomology of the Hom-group algebra $\mathbb{K}G$ with coefficients in the dual $\mathbb{K}G$-bimodule whose right $\mathbb{K}G$-action is trivial. Then we introduce Hom-group homology with coefficients in left (right) G-modules. In contrast to the associative case, the Hom-associativity condition leads us to use different type of representations for cohomology and homology theories for Hom-groups. We look into similar results in the homology case.

The (co)homology theories for Hom-algebras in [GZW], [HSS], [AEM], [MS3], [ZL], [ZZ], and for Hom-groups in this paper, gives us some hope of solving the open problems of introducing homological tools for other non-associative objects such as Jordan algebras and alternative algebras.

2. Hom-groups. Here we recall the definition of a Hom-group from [LMT].

DEFINITION 2.1. A *Hom-group* consists of a set G together with a distinguished element 1 of G, a set map $\alpha : G \to G$, an operation $\mu : G \times G \to G$, and an operation $^{-1} : G \to G$, subject to the following axioms:

(i) The product map $\mu : G \times G \to G$ has the Hom-associativity property

$$\mu(\alpha(g), \mu(h, k)) = \mu(\mu(g, h), \alpha(k)).$$

For simplicity, when there is no confusion, we omit the sign μ.

(ii) The map α is multiplicative, i.e., $\alpha(gk) = \alpha(g)\alpha(k)$.
(iii) The element 1 is called the unit and it satisfies the Hom-unitality condition

$$g1 = 1g = \alpha(g), \quad \alpha(1) = 1.$$

(iv) The map $g \mapsto g^{-1}$ has the anti-morphism property $(gh)^{-1} = h^{-1}g^{-1}$.
(v) For any $g \in G$ there exists a natural number n satisfying the Hom-invertibility condition

$$\alpha^n(gg^{-1}) = \alpha^n(g^{-1}g) = 1.$$

The smallest such n is called the *invertibility index* of g.

Since we have the anti-morphism $g \mapsto g^{-1}$, the inverse of any element $g \in G$ is unique although different elements may have different invertibility indices. The inverse of the unit element 1 of a Hom-group (G, α) is 1 itself because $\alpha(\mu(1, 1)) = \alpha(1) = 1$. For any Hom-group (G, α) we have $\alpha(g)^{-1} = \alpha(g^{-1})$, because if the invertibility index od g is k then

$$\alpha^{k-1}(\alpha(g)\alpha(g^{-1})) = \alpha^k(g)\alpha^k(g^{-1}) = 1.$$

So the invertibility index of $\alpha(g)$ is $k - 1$. If $k = 1$ then the invertibility index of each element of G is one. Nonassociativity of the product makes it difficult to define the notion of order for an element g. Therefore many basic results of group theory will be affected by missing the associativity condition.

Example 2.2. Let $(G, \mu, 1)$ be any group and $\alpha : G \to G$ be a group homomorphism. We define a new product $\mu_\alpha : G \times G \to G$ by

$$\mu_\alpha(g, h) = \alpha(\mu(g, h)) = \mu(\alpha(g), \alpha(h)).$$

Then (G, μ_α) is a Hom-group, denoted by G_α. We note that the inverse of any element $g \in G_\alpha$ is also g^{-1} because

$$\alpha(\mu_\alpha(g, g^{-1})) = \alpha(g)\alpha(g^{-1}) = \alpha(1) = 1.$$

The invertibility index of every element of G_α is 1.

Remark 2.3. In this paper we use the general definition of Hom-groups as given in Definition 2.1. However, in [H1] the author considered the special case when α is invertible. Therefore the invertibility axiom will change to: for any $g \in G$, there exists a $g^{-1} \in G$ where

$$gg^{-1} = g^{-1}g = 1.$$

It was shown that the inverse element g^{-1} is unique and also $(gh)^{-1} = h^{-1}g^{-1}$. Therefore some of the axioms in Definition 2.1 will be obtained by Hom-associativity. See [H1].

Definition 2.4. Let $\mathbb{K}$ be a field. For any Hom-group (G, α) we can define a free k-Hom algebra $\mathbb{K}G$ which is called the *Hom-group algebra*. More precisely, $\mathbb{K}G$ denotes the set of all formal expressions of the form $\sum cg$ where $c \in k$ and $g \in G$. Multiplication of $\mathbb{K}G$ is defined by $(cg)(c'g') = (cc')(gg')$ for all $c, c' \in k$ and $g, g' \in G$. For the Hom-algebra structure we extend $\alpha : G \to G$ to a $\mathbb{K}$-linear map $\mathbb{K}G \to \mathbb{K}G$ in the obvious way.

Remark 2.5. It is shown in [MS1] that the commutator of a Hom-associative algebra A given by $[a, b] = ab - ba$ makes it a Hom-Lie algebra $\mathfrak{g}_A$. Furthermore, in [Ya4], [LMT] it is shown that the universal enveloping algebra of a Hom-Lie algebra is endowed with a Hom-Hopf algebra structure. The Hopf algebra structure in [Ya4] is different from the one in [LMT]. Therefore Hom-groups are a source of examples of Hom-algebras, Hom-Lie algebras, and Hom-Hopf algebras as follows:

$$G \hookrightarrow \mathbb{K}G \hookrightarrow \mathfrak{g}_{\mathbb{K}G} \hookrightarrow U(\mathfrak{g}_{\mathbb{K}G}).$$

By [LMT], an element x in a unital Hom-algebra $(A, \alpha, 1)$ is invertible if there is an element $x^{-1} \in A$ and a nonnegative integer k such that

$$\alpha^k(xx^{-1}) = \alpha^k(x^{-1}x) = 1.$$

The element x^{-1} is called a *Hom-inverse* of x. The Hom-inverse of an element in a Hom-algebra may not be unique. This differs from Hom-groups where the inverse of an element is unique, and prevents the Hom-invertible elements in a Hom-algebra from being a Hom-group in general. In [LMT] it is shown that for any unital Hom-algebra, the unit 1 is Hom-invertible, the product of any two Hom-invertible elements is Hom-invertible, and every inverse of a Hom-invertible element is Hom-invertible. Furthermore, the set of group-like elements in a Hom-Hopf algebra is a Hom-group. The inverse of an element cg in the Hom-group algebra $\mathbb{K}G$ is $c^{-1}g^{-1}$, where $c \in \mathbb{K}$ and $g \in G$.

Definition 2.6. A subset H of a Hom-group (G, α) is called a *Hom-subgroup* of G if (H, α) itself is a Hom-group.

Note that if H is a Hom-subgroup of G then $\alpha(h) = 1h \in H$ for all $h \in H$. Therefore $\alpha(H) \subseteq H$.

Example 2.7. Let G be a group and $\alpha : G \to G$ be a group homomorphism. If H is a subgroup of G with $\alpha(H) \subseteq H$ then (H_α, α) is a Hom-subgroup of G_α.

Definition 2.8. Let (G, α) and (H, β) be Hom-groups. A morphism $f : G \to H$ is called a *morphism of Hom-groups* if $\beta(f(g)) = f(\alpha(g))$ and $f(gk) = f(g)f(k)$ for all $g, k \in G$. The Hom-groups G and H are called *isomorphic* if there exist a bijective morphism $f : G \to H$ of Hom-groups.

Proposition 2.9. *Let (G, α) and (H, β) be Hom-groups and $f : G \to H$ be a morphism of Hom-groups. If the invertibility index of the element $f(1_G) \in H$ is n then $\beta^{n+2}(f(1_G)) = 1_H$.*

Proof. Since f is multiplicative, we have

$$f(1_G)f(1_G) = f(1_G 1_G) = f(1_G).$$

Also

$$1_H f(1_G) = \beta(f(1_G)) = f(\alpha(1_G)) = f(1_G).$$

Therefore

$$f(1_G)f(1_G) = 1_H f(1_G).$$

Then $\beta^n(f(1_G))\beta^n(f(1_G))$=$\beta^n(1)\beta^n(f(1_G))$. So we have $\beta^n(f(1_G))\beta^n(f(1_G))$ $= 1_H\beta^n(f(1_G))$. Thus

$$[\beta^n(f(1_G))\beta^n(f(1_G))]\beta^{n+1}(f(1_G)^{-1}) = [1_H\beta^n(f(1_G))]\beta^{n+1}(f(1_G)^{-1}).$$

So

$$\beta^{n+1}(f(1_G))[\beta^n(f(1_G)f(1_G)^{-1})] = \beta(1_H)[\beta^n(f(1_G)f(1_G)^{-1})].$$

Consequently,

$$\beta^{n+1}(f(1_G))1_H = b(1_H)1_H.$$

Therefore $\beta^{n+2}(f(1_G)) = \beta^2(1_H) = 1_H$. ∎

The following lemma shows that in general for a Hom-group homomorphism $f : G \to H$ the unitality condition $f(1) = 1$ does not hold.

LEMMA 2.10. *Let (G, α) and (H, β) be Hom-groups and $f : G \to H$ be a morphism of Hom-groups. If $f(1_G) = 1_H$ then $f(g^{-1}) = f(g)^{-1}$.*

Proof. Let the invertibility index of g be n. Then

$$\beta^n(f(g)f(g^{-1})) = f(\alpha^n(gg^{-1})) = f(1_G) = 1_H.$$

So $f(g^{-1}) = f(g)^{-1}$. ∎

LEMMA 2.11. *Let (G, α) and (H, β) be Hom-groups and $f : G \to H$ be a morphism of Hom-groups. If $f(1_G) = 1_H$ then* $\ker f = \{g \in G : f(g) = 1_H\}$ *is a Hom-subgroup of G.*

Proof. Since f is multiplicative, $\ker f$ is closed under multiplication. Also if $x \in \ker f$ then $x^{-1} \in \ker f$ because by the previous lemma

$$f(x^{-1}) = f(x)^{-1} = 1_H^{-1} = 1_H. \blacksquare$$

3. G-modules. In this section we introduce two different types of modules over a Hom-group G. The first type is called dual G-modules and we use them later to introduce a cohomology theory for Hom-groups. The other type is called G-modules and they will be used to define a homology theory of Hom-groups.

DEFINITION 3.1. Let (G, α) be a Hom-group. An abelian group M is called a *dual left G-module* if there are linear maps $\cdot : G \times M \to M$ and

$\beta : M \to M$ such that

$$g \cdot (\alpha(h) \cdot m) = \beta((gh) \cdot m), \quad g, h \in G, \tag{3.1}$$

and

$$1 \cdot m = \beta(m).$$

Similarly, M is called a *dual right G-module* if

$$(m \cdot \alpha(h)) \cdot g = \beta(m \cdot (hg)), \quad m \cdot 1 = \beta(m). \tag{3.2}$$

Finally, we call M a *dual G-bimodule* if it is both a dual left and a dual right G-module with the following bimodule property:

$$\alpha(a) \cdot (v \cdot b) = (a \cdot v) \cdot \alpha(b).$$

LEMMA 3.2. *If (G, α) is a Hom-group and M a dual left G-module, then*

$$g \cdot \beta(m) = \beta(\alpha(g) \cdot m), \quad g \in G, \, m \in M. \tag{3.3}$$

Similarly for a dual right G-module we have

$$\beta(m \cdot \alpha(g)) = \beta(m) \cdot g. \tag{3.4}$$

Proof. This follows by substituting $h = 1$ in (3.1) and (3.2) and using $1 \cdot m = \beta(m) = m \cdot 1$. ∎

It is known that for every group G, a right G-module M can be turned into a left G-module $\widetilde{M} = M$ where the left action is given by $g \cdot m := mg^{-1}$. This can also be done for Hom-groups:

LEMMA 3.3. *Let (G, α) be a Hom-group. A dual right G-module M can be turned into a dual left G-module $\widetilde{M} = M$ by defining the left action by*

$$g \cdot m := m \cdot g^{-1}. \tag{3.5}$$

Proof. This follows from

$$\begin{aligned} g \cdot (\alpha(k) \cdot m) &= g \cdot (m \cdot \alpha(k)^{-1}) \\ &= (m \cdot \alpha(k)^{-1}) \cdot g^{-1} = (m \cdot \alpha(k^{-1})) \cdot g^{-1} \\ &= \beta(m \cdot (k^{-1} g^{-1})) = \beta(m \cdot (gk)^{-1}) = \beta((gk) \cdot m), \end{aligned}$$

and also

$$1 \cdot m = m \cdot 1^{-1} = m \cdot 1 = \beta(m). \ ∎$$

The following notion of modules over Hom-groups will be used to introduce Hom-group homology.

DEFINITION 3.4. Let (G, α) be a Hom-group. An abelian group M equipped with $\cdot : G \times M \to M$, $g \times m \mapsto g \cdot m$, and $\beta : M \to M$ is called a *left G-module* if

$$(gk) \cdot \beta(m) = \alpha(g) \cdot (k \cdot m), \quad 1 \cdot m = \beta(m) \tag{3.6}$$

for all $g, k \in G$ and $m \in M$. Similarly, (M, β) is called a *right G-module* if

$$\beta(m) \cdot (gk) = (m \cdot g) \cdot \alpha(k), \qquad m \cdot 1 = \beta(m).$$

Furthermore, M is called an *G-bimodule* if

$$\alpha(g) \cdot (m \cdot k) = (g \cdot m) \cdot \alpha(k) \tag{3.7}$$

for all $g, k \in G$ and $m \in M$.

Example 3.5. For a Hom-group G, the Hom-group algebra $\mathbb{K}G$ is a bimodule over G with the left and right actions defined by its multiplication and $\beta = \alpha$. More precisely the left action is defined by $g \cdot (ch) = c(gh)$ where $g, h \in G$ and $c \in k$.

Lemma 3.6. *Let (G, α) be a Hom-group. A right G-module M can be turned into a left G-module $\widetilde{M} = M$ by defining the left action by*

$$g \cdot m := m \cdot g^{-1}. \tag{3.8}$$

Proof. This is because

$$\begin{aligned}(gk) \cdot \beta(m) &= \beta(m) \cdot (k^{-1} g^{-1}) = (m \cdot k^{-1}) \cdot \alpha(g^{-1}) \\ &= (m \cdot k^{-1}) \cdot \alpha(g)^{-1} = \alpha(g) \cdot (m \cdot k^{-1}) = \alpha(g) \cdot (k \cdot m). \quad \blacksquare\end{aligned}$$

Example 3.7. Let G be a Hom-group and M be a right G-module. If $\mathbb{K}$ is a field, then the algebraic dual $M^* = \mathrm{Hom}(M, \mathbb{K})$ can be turned into a left dual G-module by defining the left dual action by

$$(g \cdot f)(m) = f(m \cdot g). \tag{3.9}$$

4. Hom-group cohomology. In this section we introduce Hom-group cohomology for Hom-groups. To do this we need to use the dual modules for appropriate coefficients.

Theorem 4.1. *Let (G, α) be a Hom-group and (M, β) be a dual left G-module. Let $C^n_{\mathrm{Hom}}(G, M)$ be the space of all maps $\varphi : G^{\times n} \to M$. Then*

$$C^*_{\mathrm{Hom}}(G, M) = \bigoplus_{n \geq 0} C^n_{\mathrm{Hom}}(G, M)$$

with the coface maps

$$\begin{aligned}&\delta_0 \varphi(g_1, \dots, g_{n+1}) = g_1 \cdot \varphi(\alpha(g_2), \dots, \alpha(g_{n+1})), \\ &\delta_i \varphi(g_1, \dots, g_{n+1}) = \beta\big(\varphi(\alpha(g_1), \dots, g_i g_{i+1}, \dots, \alpha(g_{n+1}))\big), \quad 1 \leq i \leq n, \\ &\delta_{n+1} \varphi(g_1, \dots, g_{n+1}) = \beta\big(\varphi(\alpha(g_1), \dots, \alpha(g_n))\big),\end{aligned} \tag{4.1}$$

is a cosimplicial module.

Proof. We need to show that $\delta_i\delta_j = \delta_j\delta_{i-1}$ for $0 \le j < i \le n-1$. Let us first show that $\delta_1\delta_0 = \delta_0\delta_0$:

$$\begin{aligned}\delta_0(\delta_0\varphi)(g_1,\dots,g_{n+2}) &= g_1 \cdot \delta_0\varphi(\alpha(g_2),\dots,\alpha(g_{n+2}))\\ &= g_1 \cdot \big(\alpha(g_2)\cdot\varphi(\alpha^2(g_3),\dots,\alpha^2(g_{n+2}))\big)\\ &= \beta\big((g_1g_2)\cdot\varphi(\alpha^2(g_3),\dots,\alpha^2(g_{n+2}))\big)\\ &= \beta\big(\delta_0\varphi(g_1g_2\,\alpha(g_3),\dots,\alpha(g_{n+2}))\big) = \delta_1\delta_0\varphi(g_1,\dots,g_{n+2}).\end{aligned}$$

We have used the left dual module property in the third equality. Now we show that $\delta_{n+1}\delta_n = \delta_n\delta_n$:

$$\begin{aligned}\delta_{n+1}\delta_n\varphi(g_1,\dots,g_{n+1}) &= \beta\big(\delta_n\varphi(\alpha(g_1),\dots,\alpha(g_n))\big)\\ &= \beta^2\big(\varphi(\alpha(g_1),\dots,\alpha(g_{n-1}))\big) = \beta\big(\delta_n\varphi(\alpha(g_1),\dots,\alpha(g_{n-1}),\alpha(g_ng_{n+1}))\big)\\ &= \beta\big(\delta_n\varphi(\alpha(g_1),\dots,\alpha(g_{n-1}),\alpha(g_n)\alpha(g_{n+1}))\big) = \delta_n\delta_n\varphi(g_1,\dots,g_{n+1}).\end{aligned}$$

We have used the multiplicativity of α in the fourth equality. The following demonstrates that $\delta_{n+1}\delta_0 = \delta_0\delta_n$:

$$\begin{aligned}\delta_{n+1}\delta_0\varphi(g_1,\dots,g_{n+1}) &= \beta\big(\delta_0\varphi(\alpha(g_1),\dots,\alpha(g_n))\big)\\ &= \beta\big(\alpha(g_1)\cdot\varphi(\alpha^2(g_2),\dots,\alpha^2(g_n))\big) = g_1\cdot\beta\big(\varphi(\alpha^2(g_2),\dots,\alpha^2(g_n))\big)\\ &= g_1\cdot\delta_n\varphi(\alpha(g_2),\dots,\alpha(g_{n+1})) = \delta_0\delta_n\varphi(g_1,\dots,g_{n+1}).\end{aligned}$$

We have used Lemma 3.2 in the third equality. The relation $\delta_{j+1}\delta_j = \delta_j\delta_j$ follows from the Hom-associativity of G. ∎

Similarly we have the following result.

PROPOSITION 4.2. *Let (G,α) be a Hom-group and (M,β) be a dual right G-module. Let $C^n_{\mathrm{Hom}}(G,M)$ be the space of all maps $\varphi : G^{\times n} \to M$. Then $C^*_{\mathrm{Hom}}(G,M) = \bigoplus_{n\ge 0} C^n_{\mathrm{Hom}}(G,M)$ with the coface maps*

$$\begin{aligned}&\delta_0\varphi(g_1,\dots,g_{n+1}) = \varphi(\alpha(g_1),\dots,\alpha(g_n))\cdot g_{n+1},\\ &\delta_i\varphi(g_1,\dots,g_{n+1}) = \beta\big(\varphi(\alpha(g_1),\dots,g_ig_{i+1},\dots,\alpha(g_{n+1}))\big), \quad 1\le i\le n,\\ &\delta_{n+1}\varphi(g_1,\dots,g_{n+1}) = \beta\big(\varphi(\alpha(g_2),\dots,\alpha(g_{n+1}))\big),\end{aligned} \tag{4.2}$$

is a cosimplicial module.

Proof. We only show that $\delta_{n+1}\delta_0 = \delta_0\delta_n$:

$$\begin{aligned}\delta_{n+1}\delta_0\varphi(g_1,\dots,g_{n+1}) &= \beta\big(\delta_0\varphi(\alpha(g_2),\dots,\alpha(g_{n+1}))\big)\\ &= \beta\big(\varphi(\alpha^2(g_2),\dots,\alpha^2(g_n))\cdot\alpha(g_{n+1})\big) = \beta\big(\varphi(\alpha^2(g_2),\dots,\alpha^2(g_n))\big)\cdot g_{n+1}\\ &= \delta_n\varphi(\alpha(g_1),\dots,\alpha(g_n))\cdot g_{n+1} = \delta_0\delta_n\varphi(g_1,\dots,g_{n+1}).\end{aligned}$$

We have used Lemma 3.2 in the third equality. The other relations can be proved as in the proof of Theorem 4.1. ∎

Now we define the coboundary $b = \sum_{i=0}^{n} d_i$. The previous theorem and proposition imply $b^2 = 0$. The cohomology of the cochain complex

$$0 \xrightarrow{b} C^0_{\mathrm{Hom}}(G, M) \xrightarrow{b} C^1_{\mathrm{Hom}}(G, M) \xrightarrow{b} C^2_{\mathrm{Hom}}(G, M) \xrightarrow{b} C^3_{\mathrm{Hom}}(G, M) \to \cdots$$

is called the *Hom-group cohomology* of G with coefficients in M. Here $M = C^0_{\mathrm{Hom}}(G, M)$. The previous theorem and proposition imply a relation between Hom-group cohomology with coefficients in dual left and dual right modules.

PROPOSITION 4.3. *Let (G, α) be a Hom-group and M be a dual right G-module. Then $\widetilde{M} = M$ with the left action $g \cdot m = m \cdot g^{-1}$ is a dual left G-module. Furthermore,*

$$H^*(G, M) \cong H^*(G, \widetilde{M}).$$

Proof. The space $\widetilde{M}$ is a dual left G-module by Lemma 3.3. We define

$$F : C^n(G, \widetilde{M}) \to C^n(G, M)$$

by

$$F(\varphi)(g_1, \ldots, g_n) = \varphi(g_n^{-1}, \ldots, g_1^{-1}).$$

We show $F\delta_0^{\widetilde{M}} = \delta_0^M F$ where $\delta^{\widetilde{M}}$ and δ^M stand for the coface maps when the coefficients are $\widetilde{M}$ and M, respectively:

$$\begin{aligned} F\delta_0^{\widetilde{M}}\varphi(g_1, \ldots, g_{n+1}) &= \delta_0^{\widetilde{M}}\varphi(g_{n+1}^{-1}, \ldots, g_1^{-1}) = g_{n+1}^{-1} \cdot \varphi(\alpha(g_n^{-1}), \ldots, \alpha(g_1^{-1})) \\ &= \varphi(\alpha(g_n^{-1}), \ldots, \alpha(g_1^{-1})) \cdot g_{n+1} = \varphi(\alpha(g_n)^{-1}, \ldots, \alpha(g_1)^{-1}) \cdot g_{n+1} \\ &= F\varphi(\alpha(g_1), \ldots, \alpha(g_n)) \cdot g_{n+1} = \delta_0^M F(g_1, \ldots, g_{n+1}). \end{aligned}$$

Similarly, F commutes with all δ_i's and therefore with the coboundary maps $b = \sum_i \delta_i$. Thus F is a map of cochain complexes and induces a map on cohomology. Furthermore, F is a bijection at the level of cochain complexes because inverse elements are unique in Hom-groups. ∎

The following two examples show that cohomology classes can contain important information about a Hom-group.

EXAMPLE 4.4 (H^0 and twisted invariant elements). Let (G, α) be a Hom-group and M be a dual right G-module. Then

$$H^0(G, M) = \{m \in M : mg = \beta(m), \forall g \in G\}.$$

So the zero cohomology group is the subspace of M which contains those elements that are invariant under the G-action with respect to β.

EXAMPLE 4.5 (H^1 and twisted crossed homomorphisms). Let (G, α) be a Hom-group and M be a dual right G-module. To compute $H^1(G, M)$ we need to compute $\ker b$ which contains the 1-cochains $f : G \to M$ with

$df(g,h)=0$. This means

$$f(\alpha(g))\cdot h-\beta(f(gh))+\beta(f(\alpha(h)))=0,$$

or

$$\beta(f(gh))=f(\alpha(g))\cdot h+\beta(f(\alpha(h))).$$

These maps are called *twisted crossed homomorphisms* of G. Also $\operatorname{Im} b$ contains all $\varphi: G\to M$ for which there exists $m\in M$ such that $\varphi(g)=mg-\beta(m)$. These maps are called *twisted principal crossed homomorphisms* of G. Therefore the first cohomology is the quotient of the twisted crossed homomorphisms by the twisted principal crossed homomorphisms.

EXAMPLE 4.6. We recall that for a Hom-group G the Hom-group algebra $V=\mathbb{K}G$ is a G-bimodule under multiplication of G. Therefore by the examples of the previous section, $(\mathbb{K}G)^*$ is a G-dual bimodule. Now we consider the Hom-group cohomology of G with coefficients in the dual G-bimodule $(\mathbb{K}G)^*$. We show that the coboundary map can be written differently in this case. One identifies $\varphi\in C^n(G,G^*)$ with

$$\phi: G^{\times\, n+1}\to k,\qquad \phi(g_0,g_1,\dots g_n):=\varphi(g_1\otimes\cdots\otimes g_n)(g_0).$$

As a result the coboundary map will be $b: C^n(G,G^*)\to C^{n+1}(G,G^*)$, where

$$\begin{aligned}b\phi(g_0,\dots,g_{n+1})&=\phi(g_0g_1,\alpha(g_2),\dots,\alpha(g_{n+1}))\\&\quad+\sum_{j=1}^{n}(-1)^j\phi(\alpha(g_0),\dots,g_jg_{j+1},\dots,\alpha(g_{n+1}))\\&\quad+(-1)^{n+1}\phi(g_{n+1}g_0\otimes\alpha(g_1),\dots,\alpha(g_n)).\end{aligned}$$

Also the cosimplicial structure is translated into

$$\begin{aligned}&\delta_0\phi(g_0,\dots,g_n)=\phi(g_0g_1,\alpha(g_2),\dots,\alpha(g_n)),\\&\delta_i\phi(g_0,\dots,g_n)=\phi(\alpha(g_0),\dots,g_ig_{i+1},\dots,\alpha(g_n)),\quad 1\le i\le n-1,\\&\delta_n\phi(g_0,\dots,g_n)=\phi(g_ng_0,\alpha(g_1),\dots,\alpha(g_n)).\end{aligned}\tag{4.3}$$

The following proposition shows the functoriality of Hom-group cohomology with certain coefficients.

PROPOSITION 4.7. *Let (G,α_G) and $(G',\alpha_{G'})$ be Hom-groups. Then any morphism $f: G\to G'$ of Hom-groups induces a map*

$$\widehat{f}: H^n_{\mathrm{Hom}}(G',(\mathbb{K}G')^*)\to H^n_{\mathrm{Hom}}(G,(\mathbb{K}G)^*).$$

Proof. We define $F: C^n(G',\mathbb{K}G'^*)\to C^n(G,\mathbb{K}G^*)$ by

$$F\varphi(g_0,\dots,g_n)=\varphi(f(g_0),\dots,f(g_n)).$$

The map F commutes with all differentials δ_i in (4.3) because $f(\alpha_G(g))=\alpha_{G'}(f(g))$ and $f(gk)=f(g)f(k)$. Here we only show that F commutes with

δ_0 and we leave the other commutativity relations to the reader:

$$\begin{aligned}\delta_0^G F\varphi(g_0,\ldots,g_n) &= F\varphi(g_0g_1,\alpha_G(g_2),\ldots,\alpha_G(g_n))\\ &= \varphi(f(g_0g_1),f(\alpha_G(g_2)),\ldots,f(\alpha_G(g_n)))\\ &= \varphi(f(g_0)f(g_1),\alpha_{G'}(f(g_2)),\ldots,\alpha_{G'}(f(g_n)))\\ &= \delta_0^{G'}(f(g_0),\ldots,f(g_n)) = F\delta_0^{G'}\varphi(g_0,\ldots,g_n). \blacksquare\end{aligned}$$

Even in the case of associative groups, $H^*(G,\mathbb{K}G)$ does not have a similar functoriality property. This reminds us that the coefficients $(\mathbb{K}G)^*$ as dual G-module have an important role in Hom-group cohomology.

EXAMPLE 4.8 (Trace 0-cocycles). Let (G,α) be a Hom-group. Using the differentials in (4.3) we have

$$H^0_{\text{Hom}}(G,\mathbb{K}G^*) = \{\varphi : G \to k : \varphi(gh) = \varphi(hg)\}.$$

More precisely, 0-cocycles of G are trace maps on G.

Here we aim to find the relation between the Hom-group cohomology of a Hom-group G and the Hochschild cohomology of the Hom-group algebra $\mathbb{K}G$. For this we recall the Hochschild cohomology of Hom-algebras introduced in [HSS]. First we need to recall from [HSS] the definition of dual modules for Hom-algebras. Let $(\mathcal{A},\alpha)$ be a Hom-algebra. A vector space V is called a *dual left $\mathcal{A}$-module* if there are linear maps $\cdot : \mathcal{A}\otimes V \to V$ and $\beta : V \to V$ with

$$a\cdot(\alpha(b)\cdot v) = \beta((ab)\cdot v). \tag{4.4}$$

Similarly, V is called a *dual right $\mathcal{A}$-module* if $v\cdot(\alpha(a))\cdot b = \beta(v\cdot(ab))$. Finally, we call V a *dual $\mathcal{A}$-bimodule* if $\alpha(a)\cdot(v\cdot b) = (a\cdot v)\cdot\alpha(b)$. Let $(\mathcal{A},\alpha)$ be a Hom-algebra, (M,β) be a dual $\mathcal{A}$-bimodule and $C^n(\mathcal{A},M)$ be the space of all k-linear maps $\varphi : \mathcal{A}^{\otimes n} \to M$. Then the authors in [HSS] showed that $C^*(\mathcal{A},M) = \bigoplus_{n\geq 0} C^n(\mathcal{A},M)$ with the coface maps

$$\begin{aligned} &d_0\varphi(a_1\otimes\cdots\otimes a_{n+1}) = a_1\cdot\varphi(\alpha(a_2)\otimes\cdots\otimes\alpha(a_{n+1})),\\ &d_i\varphi(a_1\otimes\cdots\otimes a_{n+1}) = \beta(\varphi(\alpha(a_1)\otimes\cdots\otimes a_ia_{i+1}\otimes\cdots\otimes\alpha(a_{n+1}))),\\ &\hspace{22em} 1\leq i\leq n,\\ &d_{n+1}\varphi(a_1\otimes\cdots\otimes a_{n+1}) = \varphi(\alpha(a_1)\otimes\cdots\otimes\alpha(a_n))\cdot a_{n+1}, \end{aligned} \tag{4.5}$$

is a cosimplicial module. The cohomology of the complex $(C^*(\mathcal{A},M),b)$, where $b = \sum_{i=0}^{n+1} d_i$, is the Hochschild cohomology of the Hom-algebra $\mathcal{A}$ with coefficients in M, and is denoted by $H^*(\mathcal{A},M)$.

The following theorem shows that if the dual left G-module M satisfies an extra condition $\alpha(a)\cdot\beta(m) = \beta(a\cdot m)$ for $a\in\mathbb{K}G$ and $m\in M$, then the group cohomology of a Hom-group G with coefficients in M will reduce to the Hochschild cohomology of the Hom-group algebra $\mathbb{K}G$ with coefficients in the dual $\mathbb{K}G$-bimodule $\widetilde{M} = M$ where the left action comes from the left action of G and the right action is trivial. If $\alpha = \beta = \mathrm{Id}$ then we obtain the corresponding well-known result in the associative case.

THEOREM 4.9. *Let (G, α) be a Hom-group and let the vector space M be a dual left G-module. If $\alpha(a) \cdot \beta(m) = \beta(a \cdot m)$, then $\widetilde{M} = M$ is a dual $\mathbb{K}G$-bimodule where the left action comes from the original left action of G and the right action is the trivial action $m \cdot g := m \cdot 1 = \beta(m)$. Furthermore*

$$H^*(G, M) \cong H^*(\mathbb{K}G, \widetilde{M}).$$

Proof. The condition $\alpha(a) \cdot \beta(m) = \beta(a \cdot m)$ ensures that $\widetilde{M}$ with the given dual left action and the trivial right action $m \cdot g = m \cdot 1 = \beta(m)$ is a dual G-bimodule, and therefore a dual $\mathbb{K}G$-bimodule, because

$$\alpha(g) \cdot (m \cdot k) = \alpha(g) \cdot \beta(m) = \beta(g \cdot m) = (g \cdot m) \cdot \alpha(k).$$

Now all differentials d_i of the Hochschild cohomology of $\mathbb{K}G$ will be the same as the ones, δ_i, for the group cohomology of G. Therefore the identity map $\mathrm{Id} : C^n(G, M) \to C^n(\mathbb{K}G, \widetilde{M})$ induces an isomorphism at the level of complexes. ∎

If G is an associative group and $\mathbb{K}G$ is the group algebra, then any $\mathbb{K}G$-bimodule M can be turned into a right (or left) G-module by the adjoint action. How to obtain a similar result for Hom-groups is not clear because we do not know how to define the adjoint action for Hom-groups.

5. Hom-group homology. In this section we introduce homology theory for Hom-groups. To do this we need to use modules instead of dual modules for the coefficients.

THEOREM 5.1. *Let (G, α) be a Hom-group and (M, β) be a right G-module satisfying*

$$\beta(m \cdot g) = \beta(m) \cdot \alpha(g).$$

Let $C_n^{\mathrm{Hom}}(G, M) = M \times G^{\times n}$. Then $C_*^{\mathrm{Hom}}(G, M) = \bigoplus_{n \geq 0} C_n^{\mathrm{Hom}}(G, M)$ with the face maps

$$\begin{aligned} &d_0(m, g_1, \ldots, g_n) = (m \cdot g_1, \alpha(g_2), \ldots, \alpha(g_n)), \\ &d_i(m, g_1, \ldots, g_{n+1}) = (\beta(m), \alpha(g_1), \ldots, g_i g_{i+1}, \ldots, \alpha(g_n)), \\ &\qquad\qquad\qquad\qquad 1 \leq i \leq n-1, \\ &d_n(m, g_1, \ldots, g_n) = (\beta(m), \alpha(g_1), \ldots, \alpha(g_{n-1})), \end{aligned} \tag{5.1}$$

is a simplicial module.

Proof. First we show $d_0d_0 = d_0d_1$:

$$\begin{aligned}
&d_0d_0(m, g_1, \dots, g_n) = d_0(m \cdot g_1, \alpha(g_2), \dots, \alpha(g_n))\\
&\quad = ((m \cdot g_1) \cdot \alpha(g_2), \alpha^2(g_3), \dots, \alpha^2(g_n))\\
&\quad = ((m \cdot g_1) \cdot \alpha(g_2), \alpha^2(g_3), \dots, \alpha^2(g_n)) = (\beta(m) \cdot (g_1g_2), \alpha^2(g_3), \dots, \alpha^2(g_n))\\
&\quad = d_0(\beta(m), g_1g_2, \alpha(g_3), \dots, \alpha(g_n)) = d_0d_1(m, g_1, \dots, g_n).
\end{aligned}$$

We have used the dual right property in the fourth equality. Now we show $d_0d_j = d_{j-1}d_0$ for $j > 1$:

$$\begin{aligned}
d_0d_j(m, g_1, \dots, g_n) &= d_0(\beta(v) \otimes \alpha(a_1) \otimes \cdots \otimes a_ja_{j+1} \otimes \cdots \otimes \alpha(a_n))\\
&= \beta(v) \cdot \alpha(a_1) \otimes \alpha^2(a_2) \otimes \cdots \otimes \alpha(a_ja_{j+1}) \otimes \cdots \otimes \alpha^2(a_n)\\
&= \beta(v \cdot a_1) \otimes \alpha^2(a_2) \otimes \cdots \otimes \alpha(a_j)\alpha(a_{j+1}) \otimes \cdots \otimes \alpha^2(a_n)\\
&= d_{j-1}(v \cdot a_1 \otimes \alpha(a_2) \otimes \cdots \otimes \alpha(a_n)) = d_{j-1}\delta_0(m \otimes a_1 \otimes \cdots \otimes a_n).
\end{aligned}$$

We have used the condition $\beta(m \cdot g) = \beta(m)\alpha(g)$ in the third equality. Now we show $d_0d_n = d_{n-1}d_0$:

$$\begin{aligned}
d_0d_n(m, g_1, \dots, g_n) &= d_0(\beta(m), \alpha(g_1) \cdots, \alpha(g_{n-1}))\\
&= (\beta(m) \cdot \alpha(g_1), \alpha^2(g_2), \dots \alpha^2(g_{n-1})) = \beta(m \cdot g_1), \alpha^2(g_2) \cdots \alpha^2(g_{n-1})\\
&= d_{n-1}(m \cdot g_1, \alpha(g_2) \cdots, \alpha(g_n)) = d_{n-1}\delta_0(m, g_1 \cdots g_n).
\end{aligned}$$

Now we show $d_id_n = d_{n-1}d_i$:

$$\begin{aligned}
d_id_n(m, g_1, \dots, g_n) &= d_i(\beta(m), \alpha(g_1), \dots, \alpha(g_{n-1}))\\
&= (\beta^2(m), \alpha^2(g_1), \dots, \alpha(g_i)\alpha(g_{i+1}), \dots, \alpha^2(g_{n-1}))\\
&= d_{n-1}(\beta(m), \alpha(g_1), \dots, g_ig_{i+1}, \dots, \alpha(g_n)) = d_{n-1}d_i(m, g_1, \dots, g_n).
\end{aligned}$$

The other commutativity relations can be verified similarly. ∎

We define the boundary map $b = \sum_{i=0}^{n} d_i$. By the previous theorem we have $b^2 = 0$. The homology of the chain complex

$$\begin{aligned}
0 \overset{b}{\leftarrow} M = C_0^{\text{Hom}}(G, M) \overset{b}{\leftarrow} C_1^{\text{Hom}}(G, M)&\\
\overset{b}{\leftarrow} C_2^{\text{Hom}}(G, M) \overset{b}{\leftarrow} C_3^{\text{Hom}}(G, M) \leftarrow \cdots&
\end{aligned}$$

is called the *Hom-group homology* of G with coefficients in M. Similarly, one has Hom-group homology with coefficients in left modules:

Proposition 5.2. *Let (G, α) be a Hom-group and (M, β) be a left G-module satisfying*

$$\beta(g \cdot m) = \alpha(g) \cdot \beta(m).$$

Let $C_n^{\text{Hom}}(G, M) = G^{\times n}$. Then $C_*^{\text{Hom}}(G, M) = \bigoplus_{n \geq 0} C_n^{\text{Hom}}(G, M)$ with the *face maps*

$$(5.2)\quad \begin{aligned} d_0(g_1,\dots,g_n,m) &= (\alpha(g_1),\alpha(g_2),\dots,\alpha(g_{n-1}),g_n\cdot m),\\ d_i(g_1,\dots,g_{n+1},m) &= (\alpha(g_1),\dots,g_ig_{i+1},\dots,\alpha(g_n),\beta(m)),\\ &\qquad\qquad 1\le i\le n-1,\\ d_n(g_1,\dots,g_n,m) &= (\alpha(g_2),\dots,\alpha(g_n),\beta(m)),\end{aligned}$$

is a simplicial module.

Proof. Similar to that of the previous theorem. ∎

Now we state a relation between Hom-group homology with coefficients in left and right modules.

PROPOSITION 5.3. *Let (G,α) be a Hom-group and M be a right G-module. Then $\widetilde{M}=M$ with the left action $g\cdot m=m\cdot g^{-1}$ is a left G-module and*

$$H_*(G,M)\cong H_*(G,\widetilde{M}).$$

Proof. The space $\widetilde{M}$ is a left module by Lemma 3.6. We define

$$F:C_n(G,\widetilde{M})\to C_n(G,M)\quad\text{by}\quad F(g_1,\dots,g_n,m)=(m,g_n^{-1},\dots,g_1^{-1}).$$

We show $d_0^M F=Fd_0^{\widetilde{M}}$:

$$\begin{aligned} d_0^M F(g_1,\dots,g_n,m) &= d_0^M(m,g_n^{-1},\dots,g_1^{-1})\\ &= (m\cdot g_n^{-1},\alpha(g_{n-1}^{-1}),\dots,\alpha(g_1^{-1})) = (m\cdot g_n^{-1},\alpha(g_{n-1})^{-1},\dots,\alpha(g_1)^{-1})\\ &= F(\alpha(g_1),\dots,\alpha(g_{n-1}),m\cdot g_n^{-1})) = F(\alpha(g_1),\dots,\alpha(g_{n-1}),g_n\cdot m))\\ &= Fd_0^{\widetilde{M}}(g_1,\dots,g_n,m).\end{aligned}$$

Now we prove $d_n^M F=Fd_n^{\widetilde{M}}$:

$$\begin{aligned} d_n^M F(g_1,\dots,g_n,m) &= d_n^M(m,g_n^{-1},\dots,g_1^{-1})\\ &= (\beta(m),\alpha(g_n^{-1}),\dots,\alpha(g_2^{-1})) = (\beta(m),\alpha(g_n)^{-1},\dots,\alpha(g_2)^{-1})\\ &= F(\alpha(g_2),\dots,\alpha(g_n),\beta(m)) = Fd_n^{\widetilde{M}}(g_1,\dots,g_n,m). \ ∎\end{aligned}$$

Next we show that the Hom-group homology of a Hom-group with coefficients in a right (or left) module reduces to the Hochschild homology of the Hom-group algebra with coefficients in a certain bimodule. To do this we recall that the authors of [HSS] introduce Hochschild homology of a Hom-algebra A as follows. Let (A,μ,α) be a Hom-algebra, and (V,β) be an A-bimodule such that

$$\beta(v\cdot a)=\beta(v)\cdot\alpha(a)\quad\text{and}\quad\beta(a\cdot v)=\alpha(a)\cdot\beta(v).$$

Then

$$C_*^{\mathrm{Hom}}(A,V)=\bigoplus_{n\ge 0}C_n^{\mathrm{Hom}}(A,V),\qquad C_n^{\mathrm{Hom}}(A,V):=V\otimes A^{\otimes n},$$

with the face maps

$$\delta_0(v \otimes a_1 \otimes \cdots \otimes a_n) = v \cdot a_1 \otimes \alpha(a_2) \otimes \cdots \otimes \alpha(a_n),$$
$$\delta_i(v \otimes a_1 \otimes \cdots \otimes a_n) = \beta(v) \otimes \alpha(a_1) \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes \alpha(a_n), \quad 1 \leq i \leq n-1,$$
$$\delta_n(v \otimes a_1 \otimes \cdots \otimes a_n) = a_n \cdot v \otimes \alpha(a_1) \otimes \cdots \otimes \alpha(a_{n-1}),$$

is a simplicial module. Similar to the cohomology case we have the following result.

THEOREM 5.4. *Let (G, α) be a Hom-group and let the vector space M be a right G-module satisfying $\beta(m \cdot g) = \beta(m) \cdot \alpha(g)$. Let $\widetilde{M} = M$ be a left module with the trivial left action $g \cdot m := m \cdot 1 = \beta(m)$. Then $\widetilde{M}$ is a $\mathbb{K}G$-bimodule and*

$$H_*(G, M) \cong H_*(\mathbb{K}G, \widetilde{M}).$$

Proof. The condition $\beta(m \cdot g) = \beta(m) \cdot \alpha(g)$ ensures that $\widetilde{M}$ with the given right action and the trivial left action is a G-bimodule, and therefore a $\mathbb{K}G$-bimodule, because

$$\alpha(g) \cdot (m \cdot k) = \beta(m \cdot k) = \beta(m) \cdot \alpha(k) = (g \cdot m) \cdot \alpha(k).$$

Therefore all the differentials δ_i of the Hochschild homology of $\mathbb{K}G$ are the same as the d_i for group homology of G. Therefore the identity map $\mathrm{Id} : C_n(G, M) \to C_n(\mathbb{K}G, \widetilde{M})$ induces an isomorphism at the level of complexes. ∎

The conditions of M in the above theorem also appeared in [CG] where the authors used the category of Hom-modules over Hom-algebras to obtain a monoidal category for modules over Hom-bialgebras.

The functoriality of Hom-group homology is shown as follows.

PROPOSITION 5.5. *Let (G, α_G) and $(G', \alpha'_{G'})$ be Hom-groups. The morphism $f : G \to G'$ of Hom-groups induces a map*

$$\widehat{f} : H_n^{\mathrm{Hom}}(G, \mathbb{K}G) \to H_n^{\mathrm{Hom}}(G', \mathbb{K}G')$$

given by

$$g_0, \ldots, g_n \mapsto f(g_0), \ldots, f(g_n).$$

Proof. The map $\widehat{f}$ commutes with all faces δ_i because $f(\alpha(g)) = \alpha'(f(g))$ and $f(gk) = f(g)f(k)$. ∎

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