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FINITE GROUPS WHOSE NON-NORMAL SUBGROUPS ARE OF PRIME ORDER

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Abstract. We determine the structure of finite groups whose non-normal subgroups are of prime order.

1. Introduction. The structure of groups whose subgroups are all normal (called *Dedekind groups*) was completely determined by R. Dedekind and R. Baer (see [R93, Theorem 5.3.7]). Since then, many authors dealt with generalizations of this kind of groups. Here we mention some of these. Pic [P49] considered finite groups in which every subgroup S is quasinormal, that is, $SH = HS$ for all subgroups H of G. Buckley etc. [BLW92] dealt with groups whose subgroups form at most two conjugacy classes. In particular, Huaguo Shi etc. [SHC17] and Brandl [B95] classified groups whose non-normal subgroups are all pairwise conjugate.

In this paper, we consider a generalization of groups considered in [SHC17] and [B95], namely finite groups whose non-normal subgroups are all of prime order. Our main result is a classification of such groups:

Main Theorem 1.1. *Let G be a group. Then all non-normal subgroups of G are of prime order if and only if G is isomorphic to one of the following groups:*

(1) $G = \langle u, v \mid u^p = 1,\ v^{q^m} = 1,\ v^{-1}uv = u^k\rangle$ *for some different primes p, q and an integer k with $p \equiv 1 \pmod q$, $k \not\equiv 1 \pmod p$ and $k^q \equiv 1 \pmod p$;*
(2) $G = \langle u, v \mid u^{p^n} = 1,\ v^p = 1,\ v^{-1}uv = u^{1+p^{n-1}}\rangle$ *for some $n \geq 2$ and some prime $p \geq 3$;*
(3) $G = \langle u, v \mid u^{2^n} = 1,\ v^2 = 1,\ v^{-1}uv = u^{1+2^{n-1}}\rangle$ *for some $n \geq 3$;*
(4) $G = \langle u, v \mid u^p = v^p = 1,\ [u, v] = c,\ c^p = 1,\ [u, c] = [v, c] = 1\rangle$ *for some odd prime p;*

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(5) $G = \langle u, v, w \mid u^p = v^p = w^{p^m} = 1,\ [u, v] = w^{p^{m-1}},\ [u, w] = 1,\ [v, w] = 1$ *or* $w^{p^{m-1}} \rangle$ *for some* $m \geq 2$ *and some odd prime* p*;*
(6) $G = \langle a_1, \ldots, a_n \mid a_k^2 = 1,\ [a_i, a_j] = c,\ c^2 = 1,\ [a_k, c] = 1,\ 1 \leq i, j, k \leq n,\ i \neq j \rangle$ *for some* $2 \leq n \leq 4$*;*
(7) $G = \langle a_1, a_2, b \mid a_1^2 = a_2^2 = b^4 = 1,\ [a_1, a_2] = [a_2, b] = b^2,\ [a_1, b] = 1 \rangle$*;*
(8) $G = \langle a_1, a_2, a_3, b \mid a_i^2 = b^{2^m} = 1,\ [a_i, a_j] = b^{2^{m-1}},\ a_1a_2a_3 = b^{2^{m-2}},\ [a_i, b] \in \{1, b^{2^{m-1}}\},\ 1 \leq i, j \leq 3,\ i \neq j \rangle$ *for some* $m \geq 3$*;*
(9) $G = \langle u, v, w \mid u^p = v^p = w^q = 1,\ v^{-1}uv = u,\ w^{-1}uw = v,\ w^{-1}vw = u^m v^n \rangle$ *for some different primes* p, q *and some integers* n, m *such that* $x^2 - nx - m$ *is irreducible in the polynomial ring* $F_p[x]$*, where* F_p *is the field with* p *elements, and* $x^2 - nx - m$ *is a factor of the polynomial* $x^q - 1$.

Throughout this paper, only finite groups are considered. Our notation and terminology are standard. For example, we denote by $A \rtimes B$ the semidirect product of A and B, and $|g|$ denotes the order of an element g in a group G. All unexplained notation and terminology can be found in [G80] and [R93].

2. Some preliminaries. Here we first note that the classification of finite groups whose non-normal subgroups are all pairwise conjugate was given in [SHC17] and [B95]:

LEMMA 2.1 ([SHC17, Main Theorem]). *Let* G *be a finite group. Then all non-normal subgroups of* G *are pairwise conjugate in* G *if and only if* G *is isomorphic to one of the following groups:*

(1) $G = \langle u, v \mid u^p = 1,\ v^{q^m} = 1,\ v^{-1}uv = u^k \rangle$ *for some primes* p, q *and some integer* k*, where* $p \equiv 1 \pmod{q}$*,* $k \not\equiv 1 \pmod{p}$*, and* $k^q \equiv 1 \pmod{p}$*;*
(2) $G = \langle u, v \mid u^{p^n} = 1,\ v^p = 1,\ v^{-1}uv = u^{1+p^{n-1}} \rangle$ *for some* $n \geq 2$ *and* $p \geq 3$*;*
(3) $G = \langle u, v \mid u^{2^n} = 1,\ v^2 = 1,\ v^{-1}uv = u^{1+2^{n-1}} \rangle$ *for some* $n \geq 3$.

By Lemma 2.1, to classify finite groups whose non-normal subgroups are all of prime order, it is enough to consider groups G such that not all non-normal subgroups of G are conjugate. In the following when writing about groups all of whose non-normal subgroups are of prime order, we always assume that G actually contains a non-normal subgroup, since the structure of Dedekind groups is known.

LEMMA 2.2. *Suppose that all non-normal subgroups of* G *are of prime order. If* G *is nilpotent, then* G *is a* p*-group.*

Proof. Let A be a non-normal subgroup of prime order in G. Then there is a Sylow p-subgroup P in G such that $A < P \trianglelefteq G$ since G is nilpotent.

If $G \neq P$, then there exists a Sylow q-subgroup $Q \trianglelefteq G$. Now $AQ \trianglelefteq G$ by hypothesis, and hence $A = P \cap AQ \trianglelefteq G$, a contradiction. Thus G is a p-group. ■

LEMMA 2.3. *Suppose that each non-normal subgroup in G is of prime order. If G is not nilpotent, then $G = P \rtimes Q$, where P is a Sylow p-subgroup of G, and Q is a Sylow q-subgroup of G which is of order q, where p, q are primes.*

Proof. Let H be the subgroup of G generated by all normal *Sylow* subgroups of G. Then H is a nilpotent normal subgroup of G. Hence all Sylow subgroups of G/H are cyclic since all non-normal subgroups of G are of prime order. Thus G is solvable.

Let $\{P_1, \ldots, P_n\}$ be a Sylow basis of G, where $P_i \in \mathrm{Syl}_{p_i}(G)$, $i = 1, \ldots, n$, and $P_1 \ntrianglelefteq G$ without loss of generality. Hence $P_1 P_i \trianglelefteq G$ for $2 \leq i \leq n$ since all non-normal subgroups of G are of prime order. If $n > 2$, then we get $P_1 P_2 \trianglelefteq G$ and $P_1 P_3 \trianglelefteq G$. Hence $P_1 = P_1 P_2 \cap P_1 P_3 \trianglelefteq G$, a contradiction. Therefore $n = 2$, and $G = P_1 P_2$. Now $P = P_2$ and $Q = P_1$ are the groups we need. ■

LEMMA 2.4 ([G80, Ch. 3, Theorem 3.2], Maschke's Theorem). *Let G be a p'-group of automorphisms of an abelian p-group V and suppose V_1 is a G-invariant direct factor of V. Then $V = V_1 \times V_2$, where V_2 is also G-invariant.*

3. The proof of Main Theorem. In this section, we will prove our main theorem by using several lemmas and theorems.

THEOREM 3.1. *Let G be a p-group, where $p \geq 3$ is a prime. Suppose that all non-normal subgroups of G are of prime order. Then not all non-normal subgroups of G are conjugate in G if and only if G is isomorphic to one of the following groups:*

(i) $G = \langle u, v \mid u^p = v^p = 1, [u, v] = c, c^p = 1, [u, c] = [v, c] = 1\rangle$ *for some odd prime p;*
(ii) $G = \langle u, v, w \mid u^p = v^p = w^{p^m} = 1, [u, v] = w^{p^{m-1}}, [u, w] = 1, [v, w] = 1$ *or* $w^{p^{m-1}}\rangle$ *for some $m \geq 2$ and some odd prime p.*

Proof. We divide the proof into two cases:

(1) Suppose the exponent $\exp(G)$ of G is p. Let A, B be two non-normal subgroups of G of order p which are not conjugate in G, and let $C \leq Z(G)$ be a minimal subgroup of G. Then $AC \trianglelefteq G$ and $BC \trianglelefteq G$ since all non-normal subgroups of G are of prime order. Moreover, AC is a proper subgroup of G since A is non-normal in G. If $AB \leq G$, then $AB \trianglelefteq G$. On the other hand, AB, AC are both subgroups of order p^2, so $A = AB \cap AC$. Hence

$A \trianglelefteq G$, a contradiction since $A \ntrianglelefteq G$. So $AB \nleq G$. Hence $\langle A, B\rangle = ACB$ since $ACB = AC \cdot CB \leq G$. Moreover, $[A, B] = C$.

Now we claim that each subgroup D of G of order p is contained in $\langle A, B\rangle$. Otherwise we would obtain $[A, D] = C$. Choose three proper generators a, b, d in A, B, D respectively such that $a^b = a^d$. Then $bd^{-1} \in C_G(A)$ and $A\langle bd^{-1}\rangle \trianglelefteq G$. Hence $A = A\langle bd^{-1}\rangle \cap AC \trianglelefteq G$, contradicting $A \ntrianglelefteq G$.

It is easy to see that every non-normal subgroup of $\langle A, B\rangle$ is of order p and the non-normal subgroups of $\langle A, B\rangle$ are not all conjugate. Let u, v be generators of A, B respectively. Then $G = \langle u, v\rangle$, that is, G is a group of type (i).

(2) Suppose $\exp(G) \neq p$. Let A, B be two non-normal subgroups of G of order p which are not conjugate in G. By the same argument as in (1), we deduce that $\langle A, B\rangle \leq G$, and every subgroup of order p in G is contained in $\langle A, B\rangle$. Hence $\Omega_1(G) = \langle A, B\rangle = ACB$, where C is the unique subgroup in $Z(G)$ of order p. Since all non-normal subgroups of G are of prime order, we have $\langle g, \Omega_1(G)\rangle \trianglelefteq G$ for any $g \in G$. Hence $G/\Omega_1(G)$ is a Dedekind group. Thus $G/\Omega_1(G)$ is abelian.

Now we claim that $G/\Omega_1(G)$ is cyclic. Indeed, otherwise there are two elements $g\Omega_1(G), h\Omega_1(G)$ in $G/\Omega_1(G)$ such that $\langle g\Omega_1(G), h\Omega_1(G)\rangle = \langle g\Omega_1(G)\rangle \times \langle h\Omega_1(G)\rangle$ and $g^{p^{m_1}} = h^{p^{m_2}} = 1$. Clearly $m_1, m_2 \geq 2$. Without loss of generality we may assume that $m_1 \geq m_2$. Since all non-normal subgroups of G are of prime order, we have $\Omega_1(\langle g\rangle) = C = \Omega_1(\langle h\rangle)$. Thus we can choose two proper elements g, h such that $g^{p^{m_1-1}} = h^{p^{m_2-1}}$. Hence $(g^{p^{m_1-m_2}}h)^{p^{m_2-1}} = 1$, and thus the order of $\langle g^{p^{m_1-m_2}}h\Omega_1(G)\rangle$ is less than the order of $\langle h\Omega_1(G)\rangle$. But $\langle g\Omega_1(G), h\Omega_1(G)\rangle = \langle g\Omega_1(G), g^{p^{m_1-m_2}}h\Omega_1(G)\rangle$, which contradicts $\langle g\Omega_1(G), h\Omega_1(G)\rangle = \langle g\Omega_1(G)\rangle \times \langle h\Omega_1(G)\rangle$. Hence $G/\Omega_1(G)$ is cyclic.

Now there exists a cyclic subgroup H in G such that $G = \langle A, B, H\rangle$, where H is of order $> p$. If AH or BH is abelian (say AH is abelian), then we can choose three proper generators u, v, w in A, B, H respectively such that $G = \langle u, v, w\rangle$. That is, G is a group of type (ii).

If AH and BH are non-abelian, then we can choose three proper generators a, b, w in A, B, H respectively such that $w^{p^m} = 1$ and $[a, w] = [b, w] = w^{p^{m-1}}$. Let $u = ab^{-1}$ and $v = b$. Then $G = \langle u, v, w\rangle$ is a group of type (ii) too.

It is easy to check that all non-normal subgroups of type (ii) are of order p and not all of them are conjugate. ■

Theorem 3.2. *Let G be a 2-group. Suppose that all non-normal subgroups of G are of prime order. Then not all non-normal subgroups of G are conjugate in G if and only if G is isomorphic to one of the following groups:*

(i) $G = \langle a_1, \dots, a_n \mid a_k^2 = 1,\ [a_i, a_j] = c,\ c^2 = 1,\ [a_k, c] = 1 \rangle$ *for some* $2 \le n \le 4$*;*
(ii) $G = \langle a_1, a_2, b \mid a_1^2 = a_2^2 = b^4 = 1,\ [a_1, a_2] = [a_2, b] = b^2,\ [a_1, b] = 1 \rangle$*;*
(iii) $G = \langle a_1, a_2, a_3, b \mid a_i^2 = b^{2^m} = 1,\ [a_i, a_j] = b^{2^{m-1}},\ a_1a_2a_3 = b^{2^{m-2}},\ [a_i, b] \in \{1, b^{2^{m-1}}\},\ 1 \le i, j \le 3,\ i \ne j \rangle$ *for some* $m \ge 3$*.*

Proof. Let A, B be two non-normal subgroups of G which are not conjugate in G, and let $C \le Z(G)$ be a minimal subgroup in G. Then $AC \trianglelefteq G$, $BC \trianglelefteq G$ and $AB \not\trianglelefteq G$. Now $ACB \le G$, thus $[A, B] = C$.

Let $\{a_1, \dots, a_n\}$ be a minimal generating system of $\Omega_1(G)$, and $c \in Z(G)$ an element of order 2. Then by the above paragraph, we have the following conclusions:

(a) $Z(G)$ contains a unique element of order 2.
(b) The conjugacy class of each a_i has two elements.
(c) $\langle a_i \rangle$ is not normal in G for $i = 1, \dots, n$.

By (a)–(c) we can deduce that $[a_i, a_j] = c$ when $i \ne j$, that is, $a_ia_ja_ia_j = c$ and $a_ia_j = a_ja_ic$. Thus $a_ia_ja_k = a_ja_ica_k = a_j(a_ia_k)c = a_ja_ka_icc = a_ja_ka_i$ for i, j, k pairwise distinct. In addition, if $n > 4$, we have $(a_2a_3a_4a_5)^2 = (a_2a_3)^2(a_4a_5)^2 = cc = 1$, and $[a_1, a_2a_3a_4a_5] = 1$. Hence $\langle a_1 \rangle \langle a_2a_3a_4a_5 \rangle \trianglelefteq G$. Also $\langle a_1 \rangle \langle c \rangle \trianglelefteq G$. Thus $\langle a_1 \rangle = \langle a_1 \rangle \langle a_2a_3a_4a_5 \rangle \cap \langle a_1 \rangle \langle c \rangle \trianglelefteq G$, a contradiction. Therefore $n \le 4$.

If $G = \Omega_1(G)$, then $G = \langle a_1, \dots, a_n \mid a_k^2 = 1,\ [a_i, a_j] = c,\ c^2 = 1,\ [a_k, c] = 1 \rangle$ for some $2 \le n \le 4$. It is easy to check that all non-normal subgroups of G are of order 2. That is, G is a group of type (i).

In the following we assume that $G \ne \Omega_1(G)$. In this case, $\Omega_1(G)$ is a group of type (i).

Now we claim that if $\exp(G) > 4$, then $n = 3$. Indeed, otherwise there is an element b of order 8 in G. Then $b^2 \in C_G(\Omega_1(G))$ since the size of the conjugacy class of each non-normal subgroup is 2 and $b^4 = (a_1a_2)^2$. Hence $b^{-2}a_1a_2 \in \Omega_1(G)$. Therefore $b^2 \in Z(\Omega_1(G))$. By calculation, if $n = 2$ or 4, then $|Z(\Omega_1(G))| = 2$. Thus we have $n = 3$, proving the claim. By the previous discussion, we obtain $\Omega_2(\langle g \rangle) = \langle a_1a_2a_3 \rangle$ for any element g of order more than 4.

(1) If $n = 2$, then $\exp(G) = 4$ by the above paragraph. Choose $b \in G \backslash \Omega_1(G)$. Then $b^2 = (a_1a_2)^2$. If $[b, a_1a_2] = 1$, then $(b^{-1}a_1a_2)^2 = 1$, hence $b^{-1}a_1a_2 \in \Omega_1(G)$. This implies that $b^{-1} \in \Omega_1(G)$, a contradiction. Thus $[b, a_1a_2] \ne 1$ and hence only one element in $\{a_1, a_2\}$ (say a_1) commutes with b. If $G \ne \langle \Omega_1(G), b \rangle$, let $d \notin \langle \Omega_1(G), b \rangle$; then $d^2 = (a_1a_2)^2$ and $[d, a_1a_2] \ne 1$. Therefore $[bd, a_1a_2] = 1$. If $(bd)^2 = 1$, then $bd \in \Omega_1(G)$. Thus $d \in \langle \Omega_1(G), b \rangle$, a contradiction. If $(bd)^2 \ne 1$, then $(bda_1a_2)^2 = 1$. Hence we also get $d \in \langle \Omega_1(G), b \rangle$, a contradiction. Therefore $G = \langle \Omega_1(G), b \rangle$, and

hence $G = \langle a_1, a_2, b \mid a_1^2 = a_2^2 = b^4 = 1,\ [a_1, a_2] = [a_2, b] = b^2,\ [a_1, b] = 1\rangle$. It is easy to check that all non-normal subgroups of G are of order 2. That is, G is a group of type (ii).

(2) If $n = 3$, we assume that $\exp(G) = 2^m$ with $m \geq 3$. Choose an element b in G with order 2^m. We claim that all elements of order 4 in G are in $\Omega_1(G)$. Indeed, otherwise choose $d \notin \Omega_1(G)$ of order 4. Then $\langle d\rangle \trianglelefteq G$, and $d^{a_i} = d$ or d^3, where $i = 1, 2, 3$. Hence at least one element in $\{a_1a_2, a_1a_3, a_2a_3\}$ commutes with d, say a_1a_2. Then $(a_1a_2d)^2 = (a_1a_2)^2d^2 = c \cdot c = 1$, and hence $d \in \Omega_1(G)$, a contradiction. Thus our claim is proved.

If $G \neq \langle \Omega_1(G), b\rangle$, let $g \notin \langle \Omega_1(G), b\rangle$. Clearly $|g| \geq 8$. If $g^2 \in \langle b\rangle$, then $\langle b, g\rangle$ is a generalized quaternion group of order more than 16, or G is a split extension of $\langle b\rangle$ by an element of order 2 in G by [KS04, 5.3.2]. If $\langle b, g\rangle$ is a generalized quaternion group of order more than 16, then there exists a non-normal subgroup of order 4, contradicting the hypothesis. If $\langle b, g\rangle$ is a split extension of $\langle b\rangle$ by an element of order 2, then $g \in \langle \Omega_1(G), b\rangle$, a contradiction too. Thus $g^2 \notin \langle b\rangle$. By the same argument, we have $b^2 \notin \langle g\rangle$. Since b^2, g^2 are both in $C_G(\Omega_1(G))$, we find that $\langle b^2, g^2\rangle$ is a quaternion group of order 8. Let $K = \langle b, g\rangle$, $b^8 = 1$, $g^8 = 1$, $\langle b\rangle \trianglelefteq K$, $\langle g\rangle \trianglelefteq G$. Then $\langle b^2, g^2\rangle$ is an abelian group by calculation, a contradiction too.

Thus we get $G = \langle \Omega_1(G), b\rangle$. Now $G = \langle a_1, a_2, a_3, b \mid a_i^2 = b^{2^m} = 1,$ $[a_i, a_j] = b^{2^{m-1}}, a_1a_2a_3 = b^{2^{m-2}},\ [a_i, b] \in \{1, b^{2^{m-1}}\},\ 1 \leq i, j \leq 3,\ i \neq j\rangle$ for some $m \geq 3$. It is easy to check that all non-normal subgroups of G are of order 2. That is, G is a group of type (iii).

(3) If $n = 4$, then $\exp(G) = 4$. By the same argument as in case (2), we find that all elements of order 4 in G are in $\Omega_1(G)$, which implies that $G = \Omega_1(G)$, a contradiction. ∎

In the following we will classify non-nilpotent groups whose non-normal subgroups are of prime order.

THEOREM 3.3. *Let G be a non-nilpotent group. Suppose that all non-normal subgroups of G are of prime order. Then not all non-normal subgroups of G are conjugate in G if and only if $G = \langle u, v, w \mid u^p = v^p = w^q = 1,\ u^v = u,\ u^w = v,\ v^w = u^m v^n\}$, where p and q are different primes, $x^2 - nx - m$ is irreducible in $F_p[x]$, and $x^2 - nx - m$ is a factor of $x^q - 1$.*

Proof. By Lemma 2.3, we have $G = P \rtimes Q$. It is easy to see that either P is a Dedekind group or all non-normal subgroups of P are of prime order. By Lemmas 2.1 and 2.2 and Theorem 3.1, either $\Omega_1(P) = \langle u, v \mid u^p = v^p = 1,$ $[u, v] = c,\ c^p = 1,\ [u, c] = [v, c] = 1\rangle$ for some odd prime p, or $\Omega_1(P)$ is an abelian group. Our proof will be divided into the following six steps:

(1) *$\Omega_1(P)$ is an abelian group.*

Suppose that $\Omega_1(P) = \langle u, v \mid u^p = v^p = 1,\ [u,v] = c,\ c^p = 1,\ [u,c] = [v,c] = 1\rangle$ where p is a prime. It is easy to see that $\langle c\rangle = \Omega_1(P)'$. Hence $\langle c\rangle \trianglelefteq G$, since $P \trianglelefteq G$, and thus $Q\langle c\rangle \trianglelefteq G$. By Lemma 2.3, the number of subgroups conjugate to Q in G is p, and so these subgroups are contained in $Q\langle c\rangle$. This implies that $H = N_{\Omega_1(P)}(Q)$ is of order p^2. Now by hypothesis, Q is normal in HQ, and both H and HQ are normal in G. Hence $Q \trianglelefteq G$, a contradiction.

(2) *$\Omega_1(P)$ is not a cyclic subgroup.*

If $\Omega_1(P)$ is cyclic, then all subgroups of order p are normal in G. Hence, all non-normal subgroups are conjugate to Q since they are all of prime order; this is a contradiction since not all non-normal subgroups of G are conjugate in G.

(3) *$\Omega_1(P)$ is an abelian group of type (p, p).*

Suppose that there exist three subgroups T_1, T_2, T_3 of order p in $\Omega_1(P)$ such that $T_1 \times T_2 \times T_3 \leq \Omega_1(P)$. Then $T_1T_2Q \trianglelefteq G$ and $T_1T_3Q \trianglelefteq G$ by hypothesis. This implies that $T_1Q \trianglelefteq G$. By the same argument as in (1) we get $Q \trianglelefteq G$, a contradiction.

(4) *P is an abelian group of type (p, p).*

Suppose that P has a cyclic subgroup P_1 of order p^2 satisfying $P_1 \trianglelefteq G$. Then there is a subgroup P_2 of order p which is also normal in G, hence $P_2Q \trianglelefteq G$ by hypothesis. Again by the same argument as in (1) we get $Q \trianglelefteq G$, a contradiction.

(5) *Q acts irreducibly on P.*

Suppose that Q acts reducibly on P. Then by Lemma 2.4 there exist two Q-invariant proper subgroups A and B of P such that $P = A \times B$. By hypothesis, $AQ \trianglelefteq G$ and $BQ \trianglelefteq G$. Therefore $Q = AQ \cap BQ \trianglelefteq G$, a contradiction.

(6) *Conclusion.*

By (5), G itself is not an abelian group, but every proper subgroup of G is abelian, that is, G is a minimal non-abelian group. By the structure of minimal non-abelian groups, $G = \langle u, v, w \mid u^p = v^p = w^q = 1,\ u^v = u,\ u^w = v,\ v^w = u^m v^n\}$, where p and q are different primes, $x^2 - nx - m$ is irreducible in $F_p[x]$, and $x^2 - nx - m$ is a factor of $x^q - 1$.

It is easy to check that all non-normal subgroups of G are of order p and not all of them are conjugate. ∎

Proof of Main Theorem 1.1. This follows from Lemma 2.1 and Theorems 3.1, 3.2 and 3.3. ∎

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