

On the reflection of the countable chain condition

by

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Abstract. We study the question of when an uncountable ccc topological space X contains a ccc subspace of size \aleph_1 . We show that it does if X is compact Hausdorff and more generally if X is Hausdorff with $\text{pct}(X) \leq \aleph_1$. For each regular cardinal κ , an example is constructed of a ccc Tikhonov space of size κ and countable pseudocharacter but with no uncountable ccc subspace of size less than κ . We also give a ccc compact T_1 space of size κ with no uncountable ccc subspace of size less than κ .

A topological space (X, τ) satisfies the *countable chain condition* (X is *ccc*) if for any uncountable $\mathcal{F} \subseteq \tau$ there are distinct $U, V \in \mathcal{F}$ such that $U \cap V \neq \emptyset$. We are interested in the following

QUESTION 1. *Does any uncountable ccc topological space contain an uncountable ccc subspace of size \aleph_1 ?*

The corresponding questions for *second-countability* (in place of ccc) and for *separability* have trivial affirmative answers. The question for *Lindelöfness* is highly non-trivial and has received substantial attention (see, for instance, [BT], [HJ], [KT] and [S]). It was claimed in the first paragraph of [BT] that the answer to Question 1 was affirmative “by a standard easy Löwenheim–Skolem argument”. It turns out that this is not the case, as the following simple example shows.

EXAMPLE 2. *For each infinite cardinal κ , there is a Hausdorff ccc space with no uncountable ccc subspace of size less than 2^κ .*

Proof. Let $X = 2^\kappa$ with the topology τ generated by sets of the form $P \cap C$ where P is basic clopen in the product topology and $C \subseteq X$ with

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$|X \setminus C| < 2^\kappa$. Note that the generating set is closed under finite intersections so it is in fact a base for τ .

Clearly τ contains the product topology on X so (X, τ) is a Hausdorff space. Moreover since any clopen in the product topology has size 2^κ , for any two basic open sets $P_1 \cap C_1$ and $P_2 \cap C_2$ we have

$$(P_1 \cap C_1) \cap (P_2 \cap C_2) \neq \emptyset \iff P_1 \cap P_2 \neq \emptyset.$$

It follows that (X, τ) is ccc (since the product topology is ccc).

Finally, if Y is a subspace of X with $|Y| < 2^\kappa$ then Y is discrete. To see this, just note that given $y \in Y$ we can take $C = (X \setminus Y) \cup \{y\}$, and since $|X \setminus C| = |Y \setminus \{y\}| < 2^\kappa$, we see that $\{y\} = Y \cap (X \cap C)$ is open in Y . Hence Y is not ccc unless Y is countable. ■

A standard way to approach Question 1 would be to take an elementary submodel M of (a large enough fragment of) the universe with $\omega_1 \cup \{X, \tau\} \subseteq M$ and $|M| = \aleph_1$, and then try to prove that the subspace $X \cap M$ is ccc (see [D] for more information on elementary submodels). By elementarity one can easily see that $X \cap M$ with the topology τ_M generated by $\{U \cap M : U \in \tau \cap M\}$ is ccc. However, τ_M is usually strictly coarser than the subspace topology on $X \cap M$ (see [JT] for many examples of this phenomenon), so this gives us a positive answer only if we can guarantee that the two topologies coincide. It is well known and easy to see by elementarity that this is the case whenever $\chi(X) \subseteq M$, so we have

THEOREM 3. *Any uncountable ccc space of character at most \aleph_1 contains a ccc subspace of size \aleph_1 .*

Since ccc linearly ordered spaces are first countable we get

COROLLARY 4. *Every uncountable ccc linearly ordered topological space contains a ccc subspace of size \aleph_1 .*

Our goal now is to prove a stronger version of Theorem 3 for the class of Hausdorff spaces (see Theorem 7 below). For this, recall that the *pointwise compactness type* of a space X , denoted by $\text{pct}(X)$, is the smallest infinite cardinal κ for which X can be covered by compact subsets K with $\chi(K, X) \leq \kappa$. When $\text{pct}(X) = \aleph_0$ we say that X has *pointwise countable type*.

Fix a Hausdorff topological space (X, τ) and an elementary submodel M of $H(\theta)$ (for a large enough regular cardinal θ) such that $(X, \tau) \in M$. We let

$$\mathcal{K} = \{K \subseteq X : K \text{ is compact and } \chi(K, X) \leq \text{pct}(X)\}.$$

By definition, \mathcal{K} is a cover of X and it is not hard to see, using the fact that in a compact Hausdorff space any open set is a union of compact G_δ subsets, that \mathcal{K} is a network of X (i.e. any open subset of X is a union of

elements of \mathcal{K}). Note also that \mathcal{K} is closed under finite intersections. Now for each $p \in X \cap M$ we let

$$\Delta_p = \bigcap \{U \in \tau \cap M : p \in U\}.$$

Note that by elementarity, since X is Hausdorff, if $p, q \in X \cap M$ are distinct then $\Delta_p \cap \Delta_q = \emptyset$. Also, if A is any intersection of open subsets of X which are in M and $p \in A \cap M$ then clearly $\Delta_p \subseteq A$. In particular, this is true of any $A \in \mathcal{P}(X) \cap M$ for which $\chi(A, X) \subseteq M$.

LEMMA 5. *If $p \in X \cap M$ and $\text{pct}(X) \subseteq M$ then*

$$\Delta_p = \bigcap \{K \in \mathcal{K} \cap M : p \in K\}.$$

Proof. Note that $\psi\chi(K, X) \subseteq M$ for all $K \in \mathcal{K}$, so $\Delta_p \subseteq \bigcap \{K \in \mathcal{K} \cap M : p \in K\}$. On the other hand, by elementary and the fact that \mathcal{K} is a network of X , for each $U \in \tau \cap M$ with $p \in U$ we can find $K_U \in \mathcal{K} \cap M$ such that $p \in K_U \subseteq U$. But then $\bigcap \{K \in \mathcal{K} \cap M : p \in K\} \subseteq \bigcap \{K_U : p \in U \in \tau \cap M\} \subseteq \bigcap \{U : p \in U \in \tau \cap M\} = \Delta_p$, which finishes the proof. ■

As we mentioned before, the topology in $X \cap M$ generated by $\tau \cap M$ is often strictly coarser than the subspace topology. So given $U \in \tau$ and $p \in U \cap M$ there is no guarantee that there is a $V \in \tau \cap M$ with $p \in V \subseteq U$. However, we have the following

LEMMA 6. *Suppose that $\text{pct}(X) \subseteq M$. For any $U \in \tau$ and $p \in U \cap M$, if $\Delta_p \subseteq U$ then there is a $V \in \tau \cap M$ such that $p \in V \subseteq U$.*

Proof. Let $\mathcal{C} = \{K \cap (X \setminus U) : K \in \mathcal{K} \cap M, p \in K\}$. By Lemma 5 we have $\bigcap \mathcal{C} = \Delta_p \cap (X \setminus U) = \emptyset$. But \mathcal{C} is a collection of compact subsets of X closed under finite intersections and therefore $\emptyset \in \mathcal{C}$, so there is a $K \in \mathcal{K} \cap M$ with $p \in K \subseteq U$. Since $\chi(K, X) \subseteq M$, by elementarity K has an outer base entirely contained in M and hence there is a $V \in \tau \cap M$ such that $p \in K \subseteq V \subseteq U$. ■

We are ready to prove our main result.

THEOREM 7. *Let X be a ccc Hausdorff space and κ a cardinal such that $\text{pct}(X) \leq \kappa \leq |X|$. Then there is a ccc subspace $Y \subseteq X$ with $|Y| = \kappa$.*

Proof. Take M such that $\kappa \subseteq M$ and $|M| = \kappa$. Consider the set

$$\mathcal{C} = \{C \subseteq X : C \text{ is closed and } C \cap \Delta_p \neq \emptyset \text{ for all } p \in X \cap M\}$$

ordered by inclusion. By Lemma 5 each Δ_p is compact, which allows us to use Zorn's lemma to get a minimal element D of \mathcal{C} . We prove now that D is ccc.

Fix a collection $\{U_\alpha : \alpha \in \omega_1\} \subseteq \tau$ such that $U_\alpha \cap D \neq \emptyset$ for all $\alpha \in \omega_1$. For a given $\alpha \in \omega_1$ the set $D \cap (X \setminus U_\alpha)$ is closed and properly contained in D . Hence by minimality of D there is a $p_\alpha \in X \cap M$ such that $D \cap (X \setminus U_\alpha) \cap \Delta_{p_\alpha} = \emptyset$ and therefore $\Delta_{p_\alpha} \subseteq (X \setminus D) \cup U_\alpha$. Using Lemma 6

we get a $V_\alpha \in \tau \cap M$ such that $p_\alpha \in V_\alpha \subseteq (X \setminus D) \cup U_\alpha$. Since X is ccc there are $\alpha, \beta \in \omega_1$ such that $V_\alpha \cap V_\beta \neq \emptyset$, so by elementarity there is a $q \in V_\alpha \cap V_\beta \cap M$. But then $U_\alpha \cap U_\beta \cap D \supseteq V_\alpha \cap V_\beta \cap D \supseteq \Delta_q \cap D \neq \emptyset$, so $\{U_\alpha \cap D : \alpha \in \omega_1\}$ is not a cellular family in D .

Now for each $p \in X \cap M$ choose a point $y_p \in D \cap \Delta_p$ and let $Y = \{y_p : p \in X \cap M\} \subseteq X$. Since $\kappa \subseteq M$ and $|M| = \kappa$ we know that $|X \cap M| = \kappa$. Moreover, all the Δ_p 's are disjoint so all the y_p 's are different and hence $|Y| = \kappa$. Finally, note that $\overline{Y} \in \mathcal{C}$, so by minimality of D we have $\overline{Y} = D$ and therefore Y is ccc. ■

Since compact (and even locally compact) spaces have pointwise countable type, we get

COROLLARY 8. *Every uncountable ccc (locally) compact Hausdorff space contains a ccc subspace of size \aleph_1 .*

Note that the Hausdorff condition was not needed in Theorem 3. However, the following example shows that Theorem 7 fails for T_1 spaces even in the compact case.

EXAMPLE 9. *For each regular cardinal κ , there is a ccc compact T_1 space of size κ with no uncountable ccc subspace of size less than κ .*

Proof. Let τ_o be the usual order topology on the ordinal $\kappa+1$. Let $X = \kappa$ with the topology $\tau = \{U \cap \kappa : \kappa \in U \in \tau_o\}$. The space X is clearly T_1 and it is ccc because any two nonempty open subsets of X intersect. Note that for any $\alpha \in \kappa$, the subspace $\alpha \subseteq X$ has the usual order topology, so any open subset of X covers all but a compact subspace of X . Hence X is compact. Since, by regularity of κ , any subspace Y of X of size less than κ is contained in some $\alpha \in \kappa$, it follows that Y has a topology finer than the one induced by its order (Y, \in) and therefore Y is not ccc unless Y is countable. ■

The space given in Example 2 has pseudocharacter κ , so taking $\kappa = \aleph_1$ we see that $\chi(X) \leq \aleph_1$ in Theorem 3 cannot be weakened to $\psi\chi(X) \leq \aleph_1$ even for Hausdorff spaces. Although the space in Example 2 is not regular, the following example shows that nonregularity is not the problem.

EXAMPLE 10. *For each regular cardinal κ , there is a ccc Tikhonov space X of size κ and countable pseudocharacter with no uncountable ccc subspace of size less than κ .*

Proof. Fix a function $\varphi : \kappa \rightarrow [\kappa]^{<\omega}$ such that $|\varphi^{-1}(\{s\})| = \kappa$ for every $s \in [\kappa]^{<\omega}$. Given $x \in 2^\kappa$ and $\alpha \in \kappa$ we define $s(\alpha, x) = \{\xi \leq \alpha : x(\xi) = 1\}$. For $x \in 2^\kappa$ we let $G(x) = \{\alpha \in \kappa : x(\alpha) = 1 \text{ and } s(\alpha, x) \text{ is finite}\}$. Now we let

$$X = \{x \in 2^\kappa : \exists \alpha \in G(x), \forall \beta > \alpha [x(\beta) = 1 \leftrightarrow \varphi(\beta) = s(\alpha, x)]\}$$

with the subspace topology inherited from 2^κ (so X is Tikhonov).

First we show that X is dense in 2^κ and therefore X is ccc, being a dense subspace of a ccc space. For this, fix a finite partial function $\sigma : \kappa \rightarrow 2$ and denote by $[\sigma]$ the basic clopen subset of 2^κ determined by σ . Pick $\alpha \in \kappa$ with $\text{dom}(\sigma) < \alpha$ and define $x \in 2^\kappa$ by $\sigma \subseteq x$ and for $\beta \notin \text{dom}(\sigma)$:

$$x(\beta) = 1 \iff \beta = \alpha \text{ or } \beta > \alpha \wedge \varphi(\beta) = \{\alpha\} \cup \sigma^{-1}(\{1\}).$$

Then $x \in [\sigma]$ and $s(\alpha, x) = \{\alpha\} \cup \sigma^{-1}(\{1\})$, hence $\alpha \in G(x)$ and $x \in [\sigma] \cap X$. Thus X is dense in 2^κ .

Note that for each $x \in X$ we can choose $\alpha_x \in \kappa$ witnessing the fact that $x \in X$ and the map $x \mapsto (\alpha_x, s(\alpha_x, x))$ is an injection from X into $\kappa \times [\kappa]^{<\omega}$, so $|X| \leq \kappa$. Moreover, for each $\alpha \in \kappa$ we can define $x_\alpha \in X$ by $x_\alpha(\beta) = 0$ if $\beta < \alpha$, $x_\alpha(\alpha) = 1$ and for $\beta > \alpha$, $x_\alpha(\beta) = 1 \leftrightarrow \varphi(\beta) = \{\alpha\}$. It is clear that all the x_α 's are distinct, so $|X| \geq \kappa$ and hence $|X| = \kappa$.

To see that $\psi\chi(X) = \aleph_0$, fix $p \in X$ and let $A \subseteq \kappa$ consist of the first ω -many elements of κ for which p takes the value 1 (note that any element of X takes the value 1 infinitely many times). Using the fact that the map $x \mapsto (\alpha_x, s(\alpha_x, x))$ defined in the previous paragraph is injective, it is easy to see that p is the only element of X that takes value 1 in all the ordinals in A . But this condition defines a G_δ in X , so we are done.

Finally, let Y be any subspace of X with $|Y| < \kappa$. We will show that Y is discrete and hence not ccc unless countable. For each $y \in Y$ let $\alpha_y \in \kappa$ be a witness of the fact that $y \in X$ and take $\delta = \sup\{\alpha_y : y \in Y\}$. By regularity of κ , $\delta \in \kappa$. Now fix $y \in Y$ and choose $\beta \in \kappa$ such that $\beta > \delta$ and $\varphi(\beta) = s(\alpha_y, y)$. This can be done since $s(\alpha_y, y)$ is finite and therefore $\varphi^{-1}(\{s(\alpha_y, y)\})$ is cofinal in κ . Let $\sigma = \{(\beta, 1)\}$ and note that since $\beta > \delta \geq \alpha_y$ and $\varphi(\beta) = s(\alpha_y, y)$, we have $y(\beta) = 1$ and hence $y \in [\sigma] \cap Y$. On the other hand, if $z \in [\sigma] \cap Y$ then $z(\beta) = 1$ and therefore, since $\beta > \delta \geq \alpha_z$, $s(\alpha_z, z) = \varphi(\beta) = s(\alpha_y, y)$. In particular $\alpha_z = \max s(\alpha_z, z) = \max s(\alpha_y, y) = \alpha_y$. This implies that $z \upharpoonright (\alpha_z + 1) = y \upharpoonright (\alpha_y + 1)$. Since for all $\xi > \alpha_z = \alpha_y$ we have $z(\xi) = 1 \leftrightarrow \varphi(\xi) = s(\alpha_z, z) \leftrightarrow \varphi(\xi) = s(\alpha_y, y) \leftrightarrow y(\xi) = 1$, it follows that $z = y$. Thus $[\sigma] \cap Y = \{y\}$, and Y is discrete. ■

Using this example we can easily set up a situation where we have a ccc Tikhonov space X which contains a ccc subspace of size \aleph_1 but X is such that for any elementary submodel M with $\omega_1 \cup \{X\} \subseteq M$ and $|M| = \aleph_1$, the subspace $X \cap M$ is not ccc. For instance, just take the disjoint union $X = Y \cup Z$ where Y is any subspace of \mathbb{R} of size \aleph_1 and Z is the space in the previous example with $\kappa = \aleph_2$. Then $X \cap M = Y \cup (Z \cap M)$, which is not a ccc space since $Z \cap M$ is an open subspace which is not ccc. However, we do not know the answer to the following

QUESTION 11. *Suppose X is a compact Hausdorff space and M is an elementary submodel with $\omega_1 \cup \{X\} \subseteq M$ and $|M| = \aleph_1$. Is it true that the subspace $X \cap M$ is ccc?*

References

- [BT] J. E. Baumgartner and F. D. Tall, *Reflecting Lindelöfness*, Topology Appl. 122 (2002), 35–49.
- [D] A. Dow, *An introduction to applications of elementary submodels to topology*, Topology Proc. 13 (1988), 17–72.
- [HJ] A. Hajnal and I. Juhász, *Remarks on the cardinality of compact spaces and their Lindelöf subspaces*, Proc. Amer. Math. Soc. 59 (1976), 146–148.
- [JT] L. R. Junqueira and F. D. Tall, *The topology of elementary submodels*, Topology Appl. 82 (1998), 239–266.
- [KT] P. Koszmider and F. D. Tall, *A Lindelöf space with no Lindelöf subspace of size \aleph_1* , Proc. Amer. Math. Soc. 130 (2002), 2777–2787.
- [S] M. Scheepers, *Measurable cardinals and the cardinality of Lindelöf spaces*, Topology Appl. 157 (2010), 1651–1657.

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