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SINGULARITY OF THE GENERATOR SUBALGEBRA IN MIXED q-GAUSSIAN ALGEBRAS

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Abstract. We prove that for the mixed q-Gaussian algebra $\Gamma_Q(H_{\mathbb{R}})$ associated to a real Hilbert space $H_{\mathbb{R}}$ and a real symmetric matrix $Q = (q_{ij})$ with $\sup|q_{ij}| < 1$, the generator subalgebra is singular.

In this note we discuss the generator masas in mixed q-Gaussian algebras. In the early 90's, motivated by mathematical physics and quantum probability, Bożejko and Speicher introduced the von Neumann algebra generated by q-deformed Gaussian variables [3]. Since then, this family of von Neumann algebras as well as several generalizations has attracted a lot of attention. Recently, the generator subalgebras in these q-deformed von Neumann algebras have been fruitfully investigated in [7, 10, 9, 2, 1, 5].

In this note we are interested in the case of mixed q-Gaussian algebras introduced in [4], and we prove that the associated generator subalgebras are singular. Our methods are adapted from [9, 10].

Before stating the main results let us fix some notation. Let $N \in \mathbb{N}\cup\{\infty\}$, let $Q = (q_{ij})_{i,j=1}^N$ be a symmetric matrix with $q = \max_{i,j}|q_{ij}| < 1$, and let $H_{\mathbb{R}}$ be a real Hilbert space with orthonormal basis $e_1, \ldots, e_N$. Write $H = H_{\mathbb{R}} + \mathrm{i}H_{\mathbb{R}}$ for the complexification of $H_{\mathbb{R}}$. Let $\mathcal{F}_Q(H)$ be the Fock space associated to the Yang–Baxter operator

$$T : H \otimes H \to H \otimes H, \quad e_i \otimes e_j \mapsto q_{ij} e_j \otimes e_i,$$

constructed in [4]. Let Ω be the vacuum vector. The left and right *creation operators* l_i and r_i are defined by the formulas

$$l_i\xi = e_i \otimes \xi, \quad r_i\xi = \xi \otimes e_i, \quad \xi \in \mathcal{F}_Q(H),$$

and their adjoints l^*, r^* are called the left and right *annihilation operators* respectively. We consider the associated mixed q-Gaussian algebra $\Gamma_Q(H_{\mathbb{R}})$ generated by the self-adjoint variables $s_j = l_j^* + l_j$. Consider the *Wick product*

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map $W : \Gamma_Q(H_{\mathbb{R}})\Omega \to \Gamma_Q(H_{\mathbb{R}})$ such that $W(\xi)\Omega = \xi$, and define the right Wick product map W_r on the commutant similarly. Take $\xi_0 \in H_{\mathbb{R}}$ and let M_{ξ_0} be the von Neumann subalgebra generated by $W(\xi_0)$ in $\Gamma_Q(H_{\mathbb{R}})$. Note that M_{ξ_0} is a diffuse abelian subalgebra. We refer to [4, 9] for any unexplained notation and terminology on the mixed q-Gaussian algebra $\Gamma_Q(H_{\mathbb{R}})$.

Recall that a von Neumann subalgebra $A \subset M$ is called *singular* if the normalizer $\{u \in \mathcal{U}(M) : uAu^* = A\}$ is contained in A. For a finite von Neumann algebra (M, τ), we denote by $L^2(M)$ the completion of M with respect to the norm $\|x\|_2^2 := \tau(x^*x)$ for any $x \in M$. A subalgebra A is called *mixing* in M if for any sequence $\{v_n\}$ of unitaries in A which weakly converges to 0, we have

$$\lim_{n\to\infty} \|\mathbb{E}_A(xv_ny) - \mathbb{E}_A(x)v_n\mathbb{E}_A(y)\|_2 = 0, \quad \forall x, y \in M,$$

where $\mathbb{E}_A$ stands for the conditional expectation onto A. It is easy to see that for diffuse subalgebras, mixing implies singularity. We refer to [8] for more details on the theory of finite von Neumann algebras. Our results will be based on the following well-known property (see for example [8, Theorem 11.4.1] and [10, Proposition 1].

LEMMA 1. *Let M be a finite von Neumann algebra and $A \subset M$ a diffuse subalgebra. Assume that $Y \subset M$ is a subset whose linear span is $\|\cdot\|_2$-dense in $L^2(M)$ and $\{v_n\} \subset A$ is an orthonormal basis for $L^2(A)$. If*

$$\sum_n \|\mathbb{E}_A(xv_ny) - \mathbb{E}_A(x)v_n\mathbb{E}_A(y)\|_2^2 < \infty$$

for all $x, y \in Y$, then A is mixing in M. In particular, A is singular in M.

The following is our main result.

THEOREM 2. *Let $v_n = \|W(\xi_0^{\otimes n})\|_2^{-1} W(\xi_0^{\otimes n})$, $n \in \mathbb{N}$. Then for any words $x = W(\xi_1 \otimes \cdots \otimes \xi_m)$ and $y = W(\eta_1 \otimes \cdots \otimes \eta_k)$ with $\xi_i, \eta_j \in H_{\mathbb{R}}$, we have*

$$\sum_n \|\mathbb{E}_{M_{\xi_0}}(xv_ny) - \mathbb{E}_{M_{\xi_0}}(x)v_n\mathbb{E}_{M_{\xi_0}}(y)\|_2^2 < \infty.$$

Consequently, M_{ξ_0} is mixing and singular in $\Gamma_Q(H_{\mathbb{R}})$.

It is a standard argument to deduce from the above theorem that M_{ξ_0} is maximal abelian in $\Gamma_Q(H_{\mathbb{R}})$ and hence $\Gamma_Q(H_{\mathbb{R}})$ is a II_1 factor if $\dim H_{\mathbb{R}} \geq 2$. As a result we recover the main theorem in [9].

Before the proof of the above theorem, let us recall the following estimate given in [9, proof of Lemma 1].

LEMMA 3. *Let $(H_n)_{n\geq 1}$ be a sequence of Hilbert spaces and write $H = \bigoplus_{n\geq 1} H_n$. Let $r, s \in \mathbb{N}$ and let $(a_i)_{1\leq i\leq r}$, $(b_j)_{1\leq j\leq s}$ be two families of operators on H which send each H_n into H_{n+1} or H_{n-1} such that there exists*

$0<q<1$ with

$$\|[a_i,b_j]|_{H_n}\| \le q^n, \qquad n\in\mathbb{N}.$$

Assume that $K_n\subset H_n$ is a finite-dimensional Hilbert subspace for each $n\ge 1$ such that for $K=\bigoplus_n K_n$ we have

$$a_i(K)\subset K, \quad 1\le i\le r-1, \quad \text{and} \quad a_r|_K=0.$$

Then for any $n\ge 1$, there is a constant $C>0$, independent of n, such that

$$\|(a_r\cdots a_1b_1\cdots b_s)|_{K_n}\|\le Cq^n.$$

Proof. For each i we may write

$$a_ib_1\cdots b_s\xi-b_1\cdots b_sa_i\xi=\sum_{j=1}^{s}b_1\cdots b_{j-1}[a_i,b_j]|_{H_{m(j,n)}}b_{j+1}\cdots b_s\xi, \quad \xi\in K_n,$$

where $m(j,n)$ is an integer greater than $n-s$. Iterating this formula we obtain

$$\begin{aligned}
&a_r\cdots a_1b_1\cdots b_s\xi\\
&\quad=b_1\cdots b_sa_r\cdots a_1\xi\\
&\qquad+\sum_{i=1}^{r}(a_r\cdots a_ib_1\cdots b_sa_{i-1}\cdots a_1\xi-a_r\cdots a_{i+1}b_1\cdots b_sa_i\cdots a_1\xi)\\
&\quad=b_1\cdots b_sa_r\cdots a_1\xi\\
&\qquad+\sum_{i=1}^{r}a_r\cdots a_{i+1}\Big(\sum_{j=1}^{s}b_1\cdots b_{j-1}[a_i,b_j]|_{H_{m'(i,j,n)}}b_{j+1}\cdots b_s\Big)a_{i-1}\cdots a_1\xi,
\end{aligned}$$

where $\xi\in K_n$ and for each i,j,n the integer $m'(i,j,n)$ is greater than $n-s-r$. Note that $a_r\cdots a_1\xi=0$ by the assumption on a_i. Since the sum above is independent of n, the lemma is established. ∎

Now we may prove our main result.

Proof of Theorem 2. Note that if $x\in M_{\xi_0}$ or $y\in M_{\xi_0}$, then the summation is trivially 0. So without loss of generality we assume that $\{x\Omega,y\Omega\}\perp\mathcal{F}_Q(\mathbb{R}\xi_0)$. In this case we have $\mathbb{E}_{M_{\xi_0}}(x)=\mathbb{E}_{M_{\xi_0}}(y)=0$. By Lemma 1, it is enough to show that $\sum_n\|\mathbb{E}_{M_{\xi_0}}(xW(\xi_0^{\otimes n})y)\Omega\|^2_{\mathcal{F}_Q(H_\mathbb{R})}/\|\xi_0^{\otimes n}\|^2_{\mathcal{F}_Q(H_\mathbb{R})}<\infty$. By the second quantization we know that

$$\begin{aligned}
\mathbb{E}_{M_{\xi_0}}(xW(\xi_0^{\otimes n})y)\Omega&=P_{\mathcal{F}_Q(\mathbb{R}\xi_0)}\big(W(\xi_1\otimes\cdots\otimes\xi_m)W(\xi_0^{\otimes n})\eta_1\otimes\cdots\otimes\eta_k\big)\\
&=P_{\mathcal{F}_Q(\mathbb{R}\xi_0)}\big(W(\xi_1\otimes\cdots\otimes\xi_m)W_r(\eta_1\otimes\cdots\otimes\eta_k)\xi_0^{\otimes n}\big),
\end{aligned}$$

where $P_{\mathcal{F}_Q(\mathbb{R}\xi_0)}$ is the orthogonal projection from $\mathcal{F}_Q(H_\mathbb{R})$ to $\mathcal{F}_Q(\mathbb{R}\xi_0)$. So by the Wick formula in [6, Theorem 1], it suffices to prove that if

$$\zeta_n=P_{\mathcal{F}_Q(\mathbb{R}\xi_0)}(l_{i_1}\cdots l_{i_p}l^*_{i_{p+1}}\cdots l^*_{i_s}r_{j_1}\cdots r_{j_l}r^*_{j_{l+1}}\cdots r^*_{j_t}\xi_0^{\otimes n})$$

with at least one pair of vectors $\{e_{i_{s'}}, e_{j_{t'}}\} \perp \xi_0$, then

$$\sum_{n\geq 0} \|\zeta_n\|^2_{\mathcal{F}_Q(H_\mathbb{R})}/\|\xi_0^{\otimes n}\|^2_{\mathcal{F}_Q(H_\mathbb{R})} < \infty.$$

Take s'' to be the largest index in $\{k : e_{i_k} \perp \xi_0\}$. Note that we only need to consider the case $s'' \geq p+1$ since otherwise $\zeta_n = 0$ by orthogonality. Note that it is easy to see that $\|[l_i^*, r_j]|_{H^{\otimes n}}\| \leq q^n$ (see e.g. [9, Lemma 2]). Now applying Lemma 3 to the operator $l^*_{i_{s''}} \cdots l^*_{i_s} r_{j_1} \cdots r_{j_l} r^*_{j_{l+1}} \cdots r^*_{j_t}$, we see that for all n large enough,

$$\|\zeta_n\|_{\mathcal{F}_Q(H_\mathbb{R})} \leq Cq^n \|\xi_0^{\otimes n}\|_{\mathcal{F}_Q(H_\mathbb{R})},$$

where C is a constant independent of n. Thus $\sum_{n\geq 0} \|\zeta_n\|^2_{\mathcal{F}_Q(H_\mathbb{R})}/\|\xi_0^{\otimes n}\|^2_{\mathcal{F}_Q(H_\mathbb{R})} < \infty$, as desired. ∎

Remark 4. The above arguments can be adapted to the setting of q-Araki–Woods algebras $\Gamma_q(H_\mathbb{R}, U_t)$, recently studied in [1, 2]. In particular, this provides a simple proof of the key estimate

$$\sum_{n\geq 0} \|T_{x,y}(H_n^q(s_q(\xi_0^{\otimes n}))\Omega)\|_q^2/\|\xi_0^{\otimes n}\|_q^2 < \infty$$

in [1, Lemma 3.1]. According to the discussion in [1], the result is closely related to the factoriality of $\Gamma_q(H_\mathbb{R}, U_t)$ and implies that M_{ξ_0} is a singular masa if ξ_0 is invariant under U_t.

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