COLLOQUIUM MATHEMATICUM

Online First version

WAVELET SYSTEM AND MUCKENHOUPT $\mathcal{A}_2$ CONDITION ON THE HEISENBERG GROUP

BY

S. ARATI and R. RADHA (Chennai)

Abstract. Let $\mathbb{H}^n$ denote the Heisenberg group. It is shown that under certain conditions the wavelet system $\{\psi_{j,k,l,m} : k,l \in \mathbb{Z}^n,\ j,m \in \mathbb{Z}\}$ on $\mathbb{H}^n$ arising from integer translations and nonisotropic dilations forms a Schauder basis for its closed linear span in $L^2(\mathbb{H}^n)$ if and only if the function $\sum_{r\in\mathbb{Z}} \|\widehat{\psi}(\cdot + r)\|^2_{\mathcal{B}_2} |\cdot + r|^n$ satisfies the Muckenhoupt $\mathcal{A}_2$ condition, where $\mathcal{B}_2$ denotes the class of Hilbert–Schmidt operators on $L^2(\mathbb{R}^n)$.

1. Introduction. It is known that for $\varphi \in L^2(\mathbb{R})$, the family $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ is a Schauder basis for its closed linear span (the shift invariant space generated by φ) if and only if w_φ belongs to the Muckenhoupt $\mathcal{A}_2$ class, where $w_\varphi(\xi) = \sum_{k\in\mathbb{Z}} |\widehat{\varphi}(\xi + k)|^2$, $\xi \in \mathbb{R}$ (see [7]). Heil and Powell [4] showed that for $g \in L^2(\mathbb{R})$, the Gabor system $\{M_n T_k g : k, n \in \mathbb{Z}\}$ is a Schauder basis for $L^2(\mathbb{R})$ if and only if $|Z_g|^2 \in \mathcal{A}_2(\mathbb{T}^2)$, where M_n and T_k denote the modulation and translation operators respectively and Z_g denotes the Zak transform of g. Later, Nielsen [6] considered the finitely generated shift invariant space $V(\Phi)$, $\Phi = \{\varphi_1, \ldots, \varphi_N\}$, and studied the problem of characterizing the system of translates generated by Φ as a Schauder basis in terms of the $\mathcal{A}_2$ condition. In fact, he used a product Muckenhoupt $\mathcal{A}_2$ condition for matrix weights. More precisely, he considered the weight $W(\Phi) : \mathbb{T}^n \to \mathbb{C}^{N\times N}$ defined by $W(\Phi) = (\sum_{k\in\mathbb{Z}^n} \widehat{\varphi}_i(\cdot - k)\overline{\widehat{\varphi}_j(\cdot - k)})^N_{i,j=1}$, which is the Gram matrix for Φ. Further, we wish to mention that several references dealing with matrix Muckenhoupt $\mathcal{A}_2$ conditions are provided in [6].

In this paper, we study a similar problem of characterizing the wavelet system $\{\psi_{j,k,l,m} : k,l \in \mathbb{Z}^n,\ j,m \in \mathbb{Z}\}$ in terms of the $\mathcal{A}_2$ condition on the Heisenberg group. This wavelet system arises from the integer left translations $L_{(2k,l,m)}, (k,l,m) \in \mathbb{Z}^n \times \mathbb{Z}^n \times \mathbb{Z}$, and the nonisotropic dilations δ_{2^j}, $j \in \mathbb{Z}$, defined as follows. For $\psi \in L^2(\mathbb{H}^n)$,

2010 *Mathematics Subject Classification*: Primary 42C15; Secondary 43A30.
Key words and phrases: Heisenberg group, Muckenhoupt $\mathcal{A}_2$, Schauder basis, wavelet system.
Received 7 December 2017; revised 28 August 2018.
Published online 17 June 2019.

DOI: 10.4064/cm7467-9-2018

$$\begin{aligned} L_{(u,v,s)}\psi(x,y,t) &= \psi((u,v,s)^{-1}(x,y,t)) \\ &= \psi\big(x-u, y-v, t-s+\tfrac{1}{2}(y\cdot u - x\cdot v)\big), \\ \delta_a\psi(x,y,t) &= |a|^{n+1}\psi(ax, ay, a^2t), \end{aligned}$$

where $(u,v,s)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}$, $a\in\mathbb{R}^*$, $(x,y,t)\in\mathbb{H}^n$. More precisely, letting $\psi_{j,k,l,m}$ denote $\delta_{2^j}L_{(2k,l,m)}\psi$, we prove the following. Under certain conditions the wavelet system $\{\psi_{j,k,l,m} : k,l\in\mathbb{Z}^n,\ j,m\in\mathbb{Z}\}$ forms a Schauder basis for its closed linear span in $L^2(\mathbb{H}^n)$, denoted by $\mathcal{W}(\psi)$, if and only if the function $\sum_{r\in\mathbb{Z}}\|\widehat{\psi}(\cdot+r)\|^2_{\mathcal{B}_2}|\cdot+r|^n$ satisfies the Muckenhoupt $\mathcal{A}_2$ condition, where $\mathcal{B}_2$ denotes the class of Hilbert–Schmidt operators on $L^2(\mathbb{R}^n)$.

Before getting into details, we shall provide the necessary notation and background.

The Heisenberg group $\mathbb{H}^n$ is a nilpotent Lie group whose underlying manifold is $\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}$ with the group law

$$(x,y,t)(u,v,s) = \big(x+u, y+v, t+s+\tfrac{1}{2}(u\cdot y - v\cdot x)\big).$$

It is a nonabelian noncompact locally compact group. The Haar measure on $\mathbb{H}^n$ is the Lebesgue measure $dxdydt$. Let $\mathscr{L}$ denote a lattice in $\mathbb{H}^n$. In other words, $\mathscr{L}$ is a discrete subgroup of $\mathbb{H}^n$ such that $\mathbb{H}^n/\mathscr{L}$ is compact. For $\varphi\in L^2(\mathbb{H}^n)$, the shift invariant space, $V(\varphi)$, is defined to be $\overline{\operatorname{span}}\{L_l\varphi : l\in\mathscr{L}\}$, where $L_l\varphi(X)=\varphi(l^{-1}\cdot X)$ for $X\in\mathbb{H}^n$. However, from the computational point of view, one can work with the standard lattice $\{(2k,l,m) : k,l\in\mathbb{Z}^n,\ m\in\mathbb{Z}\}$ in place of $\mathscr{L}$. For a study of frames and Riesz bases in connection with shift invariant spaces on $\mathbb{H}^n$, we refer to [2], [9].

From the well known Stone–von Neumann theorem it follows that every infinite-dimensional irreducible unitary representation of the Heisenberg group is unitarily equivalent to the representation π_λ, $\lambda\in\mathbb{R}^*$, defined by

$$\pi_\lambda(x,y,t)\varphi(\xi) = e^{2\pi i\lambda t}e^{2\pi i\lambda(x\cdot\xi+\frac{1}{2}x\cdot y)}\varphi(\xi+y), \qquad \varphi\in L^2(\mathbb{R}^n).$$

For $f\in L^1(\mathbb{H}^n)$, the *group Fourier transform* $\hat{f}$ is defined as follows: for $\lambda\in\mathbb{R}^*$,

$$\hat{f}(\lambda) = \int_{\mathbb{C}^n\times\mathbb{R}} f(z,t)\pi_\lambda(z,t)\,dz\,dt.$$

More explicitly, $\hat{f}(\lambda)$ is the bounded operator acting on $L^2(\mathbb{R}^n)$ (i.e., $\hat{f}(\lambda)\in\mathscr{B}(L^2(\mathbb{R}^n))$) given by $\hat{f}(\lambda)\varphi=\int_{\mathbb{C}^n\times\mathbb{R}}f(z,t)\pi_\lambda(z,t)\varphi\,dz\,dt$ for $\varphi\in L^2(\mathbb{R}^n)$, where the integral is a Bochner integral taking values in the Hilbert space $L^2(\mathbb{R}^n)$. Further, $\|\hat{f}(\lambda)\|_{\mathscr{B}}\le\|f\|_{L^1(\mathbb{H}^n)}$. The inverse Fourier transform of

$f \in L^1(\mathbb{H}^n)$ in the t variable, denoted by f^λ, is defined as

$$f^\lambda(z) = \int_{\mathbb{R}} f(z,t) e^{2\pi i \lambda t}\, dt. \tag{1.1}$$

It can be seen that $f^\lambda \in L^1(\mathbb{C}^n)$. For $f \in L^1(\mathbb{C}^n)$, the operator $W_\lambda(f)$ on $L^2(\mathbb{R}^n)$ is defined as

$$W_\lambda(f) = \int_{\mathbb{C}^n} f(z)\pi_\lambda(z,0)\, dz.$$

Clearly, there is a relation between the group Fourier transform and W_λ given by

$$\hat{f}(\lambda) = W_\lambda(f^\lambda). \tag{1.2}$$

Moreover, $W_\lambda(f)$ is an integral operator on $L^2(\mathbb{R}^n)$ with kernel K_f^λ given by

$$K_f^\lambda(\xi,\eta) = \int_{\mathbb{R}^n} f(x, \eta-\xi) e^{\pi i \lambda x\cdot(\xi+\eta)}\, dx.$$

In particular when $\lambda = 1$, $W_\lambda(f)$ is denoted by $W(f)$ and called the *Weyl transform* of f, and the associated kernel is denoted by K_f.

As in the case of the Euclidean Fourier transform, the definitions of W_λ and the group Fourier transform $\hat{f}$ can be extended to functions in $L^2(\mathbb{C}^n)$ and $L^2(\mathbb{H}^n)$ respectively by a density argument. In fact, for $f \in L^2(\mathbb{C}^n)$, $W_\lambda(f)$ is a Hilbert–Schmidt operator on $L^2(\mathbb{R}^n)$ which satisfies

$$\|W_\lambda(f)\|_{\mathcal{B}_2} = \|K_f^\lambda\|_{L^2(\mathbb{C}^n)} = |\lambda|^{-n/2}\|f\|_{L^2(\mathbb{C}^n)},$$

where $\mathcal{B}_2 = \mathcal{B}_2(L^2(\mathbb{R}^n))$ denotes the class of Hilbert–Schmidt operators on $L^2(\mathbb{R}^n)$. In other words, for $f, g \in L^2(\mathbb{C}^n)$,

$$\langle W_\lambda(f), W_\lambda(g)\rangle_{\mathcal{B}_2} = \langle K_f^\lambda, K_g^\lambda\rangle_{L^2(\mathbb{C}^n)} = |\lambda|^{-n}\langle f, g\rangle_{L^2(\mathbb{C}^n)}. \tag{1.3}$$

Furthermore, the group Fourier transform satisfies the Plancherel formula

$$\|\hat{f}\|_{L^2(\mathbb{R}^*,\mathcal{B}_2;d\mu)} = \|f\|_{L^2(\mathbb{H}^n)},$$

where $L^2(\mathbb{R}^*,\mathcal{B}_2;d\mu)$ stands for the space of functions on $\mathbb{R}^*$ taking values in $\mathcal{B}_2$ and square integrable with respect to the measure $d\mu(\lambda) = |\lambda|^n d\lambda$. For a further study of the Heisenberg group, we refer to [3] and [10].

For $\psi \in L^2(\mathbb{H}^n)$ and $k, l \in \mathbb{Z}^n$, the function $G_{k,l}^\psi$ is defined by

$$G_{k,l}^\psi(\lambda) = \sum_{r\in\mathbb{Z}} \langle \widehat{\psi}(\lambda+r), \widehat{L_{(2k,l,0)}\psi}(\lambda+r)\rangle_{\mathcal{B}_2} |\lambda+r|^n, \quad \lambda \in (0,1]. \tag{1.4}$$

It can also be written in terms of the kernel of W_λ as

$$G_{k,l}^\psi(\lambda) = \sum_{r\in\mathbb{Z}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K_{\psi^{\lambda+r}}^{\lambda+r}(\xi,\eta)\overline{K_{\psi^{\lambda+r}}^{\lambda+r}(\xi+l,\eta)} \times e^{-2\pi i(\lambda+r)k\cdot(2\xi+l)}\, d\xi\, d\eta |\lambda+r|^n. \tag{1.5}$$

We refer to [9] for the proof.

Definition 1.1. A nonnegative, integrable function w on $\mathbb{T}$ is said to satisfy the *Muckenhoupt $\mathcal{A}_2$ condition* if there exists a positive constant C satisfying

$$\left(\frac{1}{|I|}\int_I w(\xi)d\xi\right)\left(\frac{1}{|I|}\int_I \frac{1}{w(\xi)}\,d\xi\right) \leq C \tag{1.6}$$

for all intervals $I \subset \mathbb{T}$.

Definition 1.2. For $(k,l) \in \mathbb{Z}^{2n}$ and $j \in \mathbb{Z}$, the *λ-twisted translation* $(T^t_{(k,l)})^\lambda$ and the *dilation* $\mathcal{D}_{2^j}$ are operators on $L^2(\mathbb{C}^n)$ defined by

$$(T^t_{(k,l)})^\lambda \varphi(x,y) = e^{\pi i\lambda(x\cdot l - y\cdot k)}\varphi(x-k, y-l),$$
$$\mathcal{D}_{2^j}\varphi(x,y) = 2^{nj}\varphi(2^j x, 2^j y), \qquad \varphi \in L^2(\mathbb{C}^n).$$

The following results are proved in [1].

Lemma 1.3 ([1]). *For $\psi \in L^2(\mathbb{H}^n)$, $k,l \in \mathbb{Z}^n$, and $j,m \in \mathbb{Z}$, the inverse Fourier transform of $\delta_{2^j}L_{(k,l,m)}\psi$ with respect to the t variable satisfies*

$$(\delta_{2^j}L_{(k,l,m)}\psi)^\lambda = 2^{-j}e^{2\pi i\lambda 2^{-2j}m}\mathcal{D}_{2^j}(T^t_{(k,l)})^{\lambda 2^{-2j}}\psi^{\lambda 2^{-2j}},$$

where $\psi^{\lambda 2^{-2j}}$ is the inverse Fourier transform of ψ in the t variable given by (1.1).

Lemma 1.4 ([1]). *Let $\varphi \in L^2(\mathbb{C}^n)$. Then the kernel, $K^\lambda_{\mathcal{D}_{2^j}(T^t_{(k,l)})^{\lambda 2^{-2j}}\varphi}$ of $W_\lambda(\mathcal{D}_{2^j}(T^t_{(k,l)})^{\lambda 2^{-2j}}\varphi)$ satisfies*

$$K^\lambda_{\mathcal{D}_{2^j}(T^t_{(k,l)})^{\lambda 2^{-2j}}\varphi}(\xi,\eta) = e^{\pi i\lambda 2^{-2j}k\cdot(2^{j+1}\xi+l)}K^{\lambda 2^{-2j}}_\varphi(2^j\xi + l, 2^j\eta).$$

2. The main result

Definition 2.1. For $\psi \in L^2(\mathbb{H}^n)$, $k,l \in \mathbb{Z}^n$ and $j \in \mathbb{Z}$, the function $H^\psi_{j,k,l}$ on $(0,1]$ is defined as

$$H^\psi_{j,k,l}(\lambda) = \sum_{r\in\mathbb{Z}}\langle\widehat{\psi}(2^{2j}(\lambda+r)), (\delta_{2^j}L_{(2k,l,0)}\psi)^\wedge(2^{2j}(\lambda+r))\rangle_{\mathcal{B}_2}|2^{2j}(\lambda+r)|^n$$

for $\lambda \in (0,1]$.

When $j=0$, $H^\psi_{j,k,l}$ coincides with $G^\psi_{k,l}$. Further, $H^\psi_{j,k,l}$ can be expressed in terms of the kernel of W_λ. In fact, using (1.2) and Lemma 1.3, we have

$$\begin{aligned}
&H^\psi_{j,k,l}(\lambda)\\
&= \sum_{r\in\mathbb{Z}}\langle W_{2^{2j}(\lambda+r)}(\psi^{2^{2j}(\lambda+r)}), W_{2^{2j}(\lambda+r)}((\delta_{2^j}L_{(2k,l,0)}\psi)^{2^{2j}(\lambda+r)})\rangle_{\mathcal{B}_2}|2^{2j}(\lambda+r)|^n\\
&= 2^{-j}\sum_{r\in\mathbb{Z}}\langle W_{2^{2j}(\lambda+r)}(\psi^{2^{2j}(\lambda+r)}), W_{2^{2j}(\lambda+r)}(\mathcal{D}_{2^j}(T^t_{(2k,l)})^{\lambda+r}\psi^{\lambda+r})\rangle_{\mathcal{B}_2}\\
&\qquad\qquad \times |2^{2j}(\lambda+r)|^n.
\end{aligned}$$

Now using (1.3) it turns out that

$$
\begin{aligned}
(2.1)\qquad & H^{\psi}_{j,k,l}(\lambda) \\
&= 2^{-j}\sum_{r\in\mathbb{Z}}\langle K^{2^{2j}(\lambda+r)}_{\psi^{2^{2j}(\lambda+r)}}, K^{2^{2j}(\lambda+r)}_{\mathcal{D}_{2^j}(T^t_{(2k,l)})^{\lambda+r}\psi^{\lambda+r}}\rangle_{L^2(\mathbb{C}^n)}|2^{2j}(\lambda+r)|^n \\
&= 2^{(2n-1)j}\sum_{r\in\mathbb{Z}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K^{2^{2j}(\lambda+r)}_{\psi^{2^{2j}(\lambda+r)}}(\xi,\eta)\overline{K^{\lambda+r}_{\psi^{\lambda+r}}(2^j\xi+l,2^j\eta)} \\
&\qquad\qquad \times e^{-2\pi i(\lambda+r)k\cdot(2^{j+1}\xi+l)}\,d\xi\,d\eta\,|\lambda+r|^n,
\end{aligned}
$$

by applying Lemma 1.4.

For $\psi\in L^2(\mathbb{H}^n)$, the wavelet system $\{\psi_{j,k,l,m} : k,l\in\mathbb{Z}^n,\ j,m\in\mathbb{Z}\}$ is explicitly given by

$$
\begin{aligned}
\psi_{j,k,l,m}(x,y,t) &= \delta_{2^j}L_{(2k,l,m)}\psi(x,y,t) \\
&= 2^{(n+1)j}\psi\big(2^jx-2k,2^jy-l,2^{2j}t-m+\tfrac{1}{2}2^j(y\cdot 2k-x\cdot l)\big)
\end{aligned}
$$

for $(x,y,t)\in\mathbb{H}^n$. We are now in a position to state our main result.

THEOREM 2.2. *Let $\psi\in L^2(\mathbb{H}^n)$ satisfy*

(i) $G^{\psi}_{k,l}(\lambda)=0$ *for a.e.* $\lambda\in(0,1]$ *and all* $(k,l)\in\mathbb{Z}^{2n}\setminus\{(0,0)\}$, *and*
(ii) $H^{\psi}_{j,k,l}(\lambda)=0$ *for a.e.* $\lambda\in(0,1]$ *and all* $j>0$ *in* $\mathbb{Z}$ *and* $(k,l)\in\mathbb{Z}^{2n}$,

where $G^{\psi}_{k,l}$ and $H^{\psi}_{j,k,l}$ are as in (1.4) *and Definition* 2.1 *respectively. Then $\{\psi_{j,k,l,m} : k,l\in\mathbb{Z}^n,\ j,m\in\mathbb{Z}\}$ is a Schauder basis for $\mathcal{W}(\psi)$ if and only if $G^{\psi}_{0,0}\in\mathcal{A}_2$.*

Before proving this result, we establish an isometric isomorphism between $\mathcal{W}(\psi)$, the closed linear span of $\{\psi_{j,k,l,m} : k,l\in\mathbb{Z}^n,\ j,m\in\mathbb{Z}\}$, and $L^2(\mathbb{T},l^2(\mathbb{Z}^{2n+1});G^{\psi}_{0,0})$, the space of 1-periodic functions on $\mathbb{T}$ taking values in $l^2(\mathbb{Z}^{2n+1})$ and square integrable with weight $G^{\psi}_{0,0}$. Let $\sigma(\mathbb{T})$ denote the space of trigonometric polynomials on $\mathbb{T}$. Let $c_{00}(\mathbb{Z}^{2n+1},\sigma(\mathbb{T}))$ be the space of sequences indexed by $\mathbb{Z}^{2n+1}$ and consisting of only finitely many nonzero terms, each term being a trigonometric polynomial on $\mathbb{T}$. Finally, denote $\operatorname{span}\{\psi_{j,k,l,m} : k,l\in\mathbb{Z}^n,\ j,m\in\mathbb{Z}\}$ by $\mathcal{A}(\psi)$.

THEOREM 2.3. *Let $\psi\in L^2(\mathbb{H}^n)$ satisfy* (i) *and* (ii) *of Theorem* 2.2. *For $f\in\mathcal{A}(\psi)$ given by $f=\sum c_{j,k,l,m}\psi_{j,k,l,m}$, the sequence R defined by $R(\lambda)=\{R_{j,k,l}(\lambda)\}_{(j,k,l)\in\mathbb{Z}\times\mathbb{Z}^n\times\mathbb{Z}^n}$ with $R_{j,k,l}(\lambda)=\sum_m c_{j,k,l,m}e^{2\pi im\lambda}$ for $\lambda\in\mathbb{T}$ is in $c_{00}(\mathbb{Z}^{2n+1},\sigma(\mathbb{T}))$. Then the map $f\mapsto R$ defined initially between $\mathcal{A}(\psi)$ and $c_{00}(\mathbb{Z}^{2n+1},\sigma(\mathbb{T}))$ can be extended to an isometric isomorphism of $\mathcal{W}(\psi)$ onto $L^2(\mathbb{T},l^2(\mathbb{Z}^{2n+1});G^{\psi}_{0,0})$.*

Proof. Let $f \in \mathcal{A}(\psi)$ be given by

$$f = \sum_{(j',k',l',m') \in \mathcal{F}} c_{j',k',l',m'} \psi_{j',k',l',m'},$$

where $\mathcal{F}$ is a finite set. For $\lambda \in \mathbb{R}^*$,

$$\widehat{\psi_{j',k',l',m'}}(\lambda) = \int_{\mathbb{H}^n} \psi_{j',k',l',m'}(x,y,t)\pi_\lambda(x,y,t)\,dx\,dy\,dt$$
$$= \int_{\mathbb{H}^n} 2^{(n+1)j'} \psi\big(2^{j'}x - 2k', 2^{j'}y - l', 2^{2j'}t - m' + \tfrac{1}{2}2^{j'}(y\cdot 2k' - x\cdot l')\big) \times \pi_\lambda(x,y,t)\,dx\,dy\,dt$$
$$= \int_{\mathbb{H}^n} 2^{(n+1)j'} \psi\big(2^{j'}x - 2k', 2^{j'}y - l', 2^{2j'}t' + \tfrac{1}{2}2^{j'}(y\cdot 2k' - x\cdot l')\big) \times \pi_\lambda(x,y,t' + 2^{-2j'}m')\,dx\,dy\,dt'$$
$$= \int_{\mathbb{H}^n} \psi_{j',k',l',0}(x,y,t') e^{2\pi i m' 2^{-2j'}\lambda} \pi_\lambda(x,y,t')\,dx\,dy\,dt'$$
$$= e^{2\pi i m' 2^{-2j'}\lambda} \widehat{\psi_{j',k',l',0}}(\lambda)$$

and so

$$\hat{f}(\lambda) = \sum_{(j',k',l',m') \in \mathcal{F}} c_{j',k',l',m'} e^{2\pi i m' 2^{-2j'}\lambda} \widehat{\psi_{j',k',l',0}}(\lambda)$$
$$= \sum_{j',k',l'} \Big(\sum_{m'} c_{j',k',l',m'} e^{2\pi i m' 2^{-2j'}\lambda}\Big) \widehat{\psi_{j',k',l',0}}(\lambda)$$
$$= \sum_{j',k',l'} R_{j',k',l'}(2^{-2j'}\lambda) \widehat{\psi_{j',k',l',0}}(\lambda).$$

Then

$$\|f\|^2_{L^2(\mathbb{H}^n)} = \|\hat{f}\|^2_{L^2(\mathbb{R}^*, \mathcal{B}_2; d\mu)} = \int_{\mathbb{R}^*} \|\hat{f}(\lambda)\|^2_{\mathcal{B}_2} |\lambda|^n\,d\lambda$$
$$= \int_{\mathbb{R}} \Big\langle \sum_{j',k',l'} R_{j',k',l'}(2^{-2j'}\lambda)\widehat{\psi_{j',k',l',0}}(\lambda), \sum_{j,k,l} R_{j,k,l}(2^{-2j}\lambda)\widehat{\psi_{j,k,l,0}}(\lambda) \Big\rangle_{\mathcal{B}_2} |\lambda|^n\,d\lambda$$
$$= \int_{\mathbb{R}} \sum_{j',k',l'} \|R_{j',k',l'}(2^{-2j'}\lambda)\widehat{\psi_{j',k',l',0}}(\lambda)\|^2_{\mathcal{B}_2} |\lambda|^n\,d\lambda$$
$$+ \int_{\mathbb{R}} \sum_{\substack{(j',k',l') \\ \neq (j,k,l)}} \langle R_{j',k',l'}(2^{-2j'}\lambda)\widehat{\psi_{j',k',l',0}}(\lambda), R_{j,k,l}(2^{-2j}\lambda)\widehat{\psi_{j,k,l,0}}(\lambda)\rangle_{\mathcal{B}_2} |\lambda|^n\,d\lambda$$
$$= I_1 + I_2.$$

Now,

$$I_1 = \sum_{j',k',l'} \int_{\mathbb{R}} |R_{j',k',l'}(2^{-2j'}\lambda)|^2 \|\widehat{\psi_{j',k',l',0}}(\lambda)\|^2_{\mathcal{B}_2} |\lambda|^n\,d\lambda.$$

But by Lemmas 1.3 and 1.4,

$$\begin{aligned}\|\widehat{\psi_{j,k,l,0}}(\lambda)\|^2_{\mathcal{B}_2} &= \|W_\lambda((\psi_{j,k,l,0})^\lambda)\|^2_{\mathcal{B}_2} = \|K^\lambda_{(\psi_{j,k,l,0})^\lambda}\|^2_{L^2(\mathbb{C}^n)}\\ &= \|K^\lambda_{(\delta_{2^j}L_{(2k,l,0)}\psi)^\lambda}\|^2_{L^2(\mathbb{C}^n)} = \|2^{-j}K^\lambda_{\mathcal{D}_{2^j}(T^t_{(2k,l)})^{\lambda 2^{-2j}}\psi^{\lambda 2^{-2j}}}\|^2_{L^2(\mathbb{C}^n)}\\ &= \int_{\mathbb{C}^n} 2^{-2j}|e^{2\pi i\lambda 2^{-2j}k\cdot(2^{j+1}\xi+l)}K^{\lambda 2^{-2j}}_{\psi^{\lambda 2^{-2j}}}(2^j\xi+l,2^j\eta)|^2\,d\xi\,d\eta\\ &= \int_{\mathbb{C}^n} 2^{-2j}2^{-2nj}|K^{\lambda 2^{-2j}}_{\psi^{\lambda 2^{-2j}}}(u,v)|^2\,du\,dv = 2^{-2(n+1)j}\|K^{\lambda 2^{-2j}}_{\psi^{\lambda 2^{-2j}}}\|^2_{L^2(\mathbb{C}^n)}\\ &= 2^{-2(n+1)j}\|W_{\lambda 2^{-2j}}(\psi^{\lambda 2^{-2j}})\|^2_{\mathcal{B}_2} = 2^{-2(n+1)j}\|\hat{\psi}(\lambda 2^{-2j})\|^2_{\mathcal{B}_2}.\end{aligned}$$

Then we have

$$\begin{aligned}I_1 &= \sum_{j',k',l'}\int_{\mathbb{R}}|R_{j',k',l'}(2^{-2j'}\lambda)|^2 2^{-2(n+1)j'}\|\hat{\psi}(\lambda 2^{-2j'})\|^2_{\mathcal{B}_2}|\lambda|^n\,d\lambda\\ &= \int_{\mathbb{R}}\sum_{j',k',l'}|R_{j',k',l'}(\lambda)|^2\|\hat{\psi}(\lambda)\|^2_{\mathcal{B}_2}|\lambda|^n\,d\lambda = \int_0^1\sum_{j',k',l'}|R_{j',k',l'}(\lambda)|^2 G^\psi_{0,0}(\lambda)\,d\lambda\\ &= \|R\|^2_{L^2(\mathbb{T},l^2(\mathbb{Z}^{2n+1});G^\psi_{0,0})}.\end{aligned}$$

Next, we can write

$$\begin{aligned}I_2 &= \int_{\mathbb{R}}\sum_{\substack{j'=j\\(k',l')\neq(k,l)}}\langle R_{j',k',l'}(2^{-2j'}\lambda)\widehat{\psi_{j',k',l',0}}(\lambda), R_{j,k,l}(2^{-2j}\lambda)\widehat{\psi_{j,k,l,0}}(\lambda)\rangle_{\mathcal{B}_2}|\lambda|^n\,d\lambda\\ &\quad+\int_{\mathbb{R}}\sum_{\substack{j'>j\\k',l',k,l}}\langle R_{j',k',l'}(2^{-2j'}\lambda)\widehat{\psi_{j',k',l',0}}(\lambda), R_{j,k,l}(2^{-2j}\lambda)\widehat{\psi_{j,k,l,0}}(\lambda)\rangle_{\mathcal{B}_2}|\lambda|^n\,d\lambda\\ &\quad+\int_{\mathbb{R}}\sum_{\substack{j>j'\\k',l',k,l}}\langle R_{j',k',l'}(2^{-2j'}\lambda)\widehat{\psi_{j',k',l',0}}(\lambda), R_{j,k,l}(2^{-2j}\lambda)\widehat{\psi_{j,k,l,0}}(\lambda)\rangle_{\mathcal{B}_2}|\lambda|^n\,d\lambda\\ &= J_1+J_2+J_3.\end{aligned}$$

Consider

$$J_1 = \int_{\mathbb{R}}\sum_{\substack{j\\(k',l')\neq(k,l)}} R_{j,k',l'}(2^{-2j}\lambda)\overline{R_{j,k,l}(2^{-2j}\lambda)}\langle\widehat{\psi_{j,k',l',0}}(\lambda),\widehat{\psi_{j,k,l,0}}(\lambda)\rangle_{\mathcal{B}_2}|\lambda|^n\,d\lambda.$$

Again using Lemmas 1.3 and 1.4 we have

$$\begin{aligned}\langle\widehat{\psi_{j,k',l',0}}(\lambda),\widehat{\psi_{j,k,l,0}}(\lambda)\rangle_{\mathcal{B}_2} &= \langle K^\lambda_{(\delta_{2^j}L_{(2k',l',0)}\psi)^\lambda}, K^\lambda_{(\delta_{2^j}L_{(2k,l,0)}\psi)^\lambda}\rangle_{L^2(\mathbb{C}^n)}\\ &= \int_{\mathbb{C}^n}2^{-2j}K^\lambda_{\mathcal{D}_{2^j}(T^t_{(2k',l')})^{\lambda 2^{-2j}}\psi^{\lambda 2^{-2j}}}(\xi,\eta)\overline{K^\lambda_{\mathcal{D}_{2^j}(T^t_{(2k,l)})^{\lambda 2^{-2j}}\psi^{\lambda 2^{-2j}}}(\xi,\eta)}\,d\xi\,d\eta\end{aligned}$$

$$= \int_{\mathbb{C}^n} 2^{-2j} e^{2\pi i\lambda 2^{-2j} k'\cdot(2^{j+1}\xi+l')} K^{\lambda 2^{-2j}}_{\psi^{\lambda 2^{-2j}}}(2^j\xi+l',2^j\eta) \times e^{-2\pi i\lambda 2^{-2j} k\cdot(2^{j+1}\xi+l)} \overline{K^{\lambda 2^{-2j}}_{\psi^{\lambda 2^{-2j}}}(2^j\xi+l,2^j\eta)}\, d\xi\, d\eta$$

$$= 2^{-2j} e^{2\pi i\lambda 2^{-2j}(k'\cdot l'-k\cdot l)} \times \int_{\mathbb{C}^n} e^{2\pi i\lambda 2^{-2j}(k'-k)\cdot 2^{j+1}\xi} K^{\lambda 2^{-2j}}_{\psi^{\lambda 2^{-2j}}}(2^j\xi+l',2^j\eta) \overline{K^{\lambda 2^{-2j}}_{\psi^{\lambda 2^{-2j}}}(2^j\xi+l,2^j\eta)}\, d\xi\, d\eta$$

$$= 2^{-2(n+1)j} e^{2\pi i\lambda 2^{-2j}(k\cdot l'-k'\cdot l)} \times \int_{\mathbb{C}^n} e^{-2\pi i\lambda 2^{-2j}(k-k')\cdot(2x+l-l')} K^{\lambda 2^{-2j}}_{\psi^{\lambda 2^{-2j}}}(x,y) \overline{K^{\lambda 2^{-2j}}_{\psi^{\lambda 2^{-2j}}}(x+l-l',y)}\, dx\, dy.$$

Substituting this in J_1 and changing variables again, we obtain

$$J_1 = \int_{\mathbb{R}} \sum_{\substack{j \\ (k',l')\neq(k,l)}} R_{j,k',l'}(\lambda)\overline{R_{j,k,l}(\lambda)} e^{2\pi i\lambda(k\cdot l'-k'\cdot l)} \times \int_{\mathbb{C}^n} e^{-2\pi i\lambda(k-k')\cdot(2x+l-l')} K^{\lambda}_{\psi^\lambda}(x,y)\overline{K^{\lambda}_{\psi^\lambda}(x+l-l',y)}\, dx\, dy |\lambda|^n\, d\lambda.$$

Further, by discretizing with respect to λ and using (1.5) we get

$$J_1 = \int_0^1 \sum_{\substack{j \\ (k',l')\neq(k,l)}} R_{j,k',l'}(\lambda)\overline{R_{j,k,l}(\lambda)} e^{2\pi i\lambda(k\cdot l'-k'\cdot l)} G^{\psi}_{k-k',l-l'}(\lambda)\, d\lambda = 0,$$

in view of condition (i) in the hypothesis. In order to calculate J_2, we carry out similar steps, appeal to Lemmas 1.3 and 1.4 and arrive at

$$\langle \widehat{\psi_{j',k',l',0}}(\lambda), \widehat{\psi_{j,k,l,0}}(\lambda)\rangle_{\mathcal{B}_2} = \int_{\mathbb{C}^n} 2^{-j'} 2^{-j} 2^{-2nj} e^{2\pi i\lambda 2^{-2j'} k'\cdot(2^{j'-j+1}(x-l)+l')} e^{-2\pi i\lambda 2^{-2j} k\cdot(2x-l)} \times K^{\lambda 2^{-2j'}}_{\psi^{\lambda 2^{-2j'}}}(2^{j'-j}(x-l)+l', 2^{j'-j}y) \overline{K^{\lambda 2^{-2j}}_{\psi^{\lambda 2^{-2j}}}(x,y)}\, dx\, dy$$

after a change of variables. Then we get

$$J_2 = \sum_{\substack{j'>j \\ k',l',k,l}} 2^{j'-j} 2^{2n(j'-j)} \int_{\mathbb{R}} R_{j',k',l'}(\lambda')\overline{R_{j,k,l}(2^{2(j'-j)}\lambda')} \times \int_{\mathbb{C}^n} e^{2\pi i\lambda' k'\cdot(2^{j'-j+1}(x-l)+l')} e^{-2\pi i\lambda' 2^{2(j'-j)} k\cdot(2x-l)} \times K^{\lambda'}_{\psi^{\lambda'}}(2^{j'-j}x+(l'-2^{j'-j}l), 2^{j'-j}y) \overline{K^{\lambda' 2^{2(j'-j)}}_{\psi^{\lambda' 2^{2(j'-j)}}}(x,y)}\, dx\, dy\, |\lambda'|^n\, d\lambda'$$

by changing variables once again. Further,

$$e^{2\pi i\lambda' k'\cdot(2^{j'-j+1}(x-l)+l')}e^{-2\pi i\lambda' 2^{2(j'-j)}k\cdot(2x-l)}$$
$$= e^{2\pi i\lambda' 2^{j'-j+1}x\cdot(k'-2^{j'-j}k)}e^{2\pi i\lambda'(k'-2^{j'-j}k)\cdot(l'-2^{j'-j}l)}e^{2\pi i\lambda' 2^{j'-j}(k\cdot l'-k'\cdot l)}.$$

Thus,

$$J_2 = \sum_{\substack{j'>j\\ k',l',k,l}} 2^{(2n+1)(j'-j)} \int_{\mathbb{R}} R_{j',k',l'}(\lambda)\overline{R_{j,k,l}(2^{2(j'-j)}\lambda)}e^{2\pi i\lambda 2^{j'-j}(k\cdot l'-k'\cdot l)}$$
$$\times \int_{\mathbb{C}^n} e^{2\pi i\lambda(k'-2^{j'-j}k)\cdot(2^{j'-j+1}x+(l'-2^{j'-j}l))}$$
$$\times K^{\lambda}_{\psi^\lambda}(2^{j'-j}x+(l'-2^{j'-j}l), 2^{j'-j}y)\overline{K^{\lambda 2^{2(j'-j)}}_{\psi^{\lambda 2^{2(j'-j)}}}(x,y)}\,dx\,dy\,|\lambda|^n\,d\lambda.$$

Now discretizing with respect to λ, we get

$$J_2 = \sum_{\substack{j'>j\\ k',l',k,l}} 2^{(2n+1)(j'-j)} \int_0^1 R_{j',k',l'}(\lambda)\overline{R_{j,k,l}(2^{2(j'-j)}\lambda)}e^{2\pi i\lambda 2^{j'-j}(k\cdot l'-k'\cdot l)}$$
$$\times \sum_{r\in\mathbb{Z}} \int_{\mathbb{C}^n} e^{2\pi i(\lambda+r)(k'-2^{j'-j}k)\cdot(2^{j'-j+1}x+(l'-2^{j'-j}l))}$$
$$\times K^{\lambda+r}_{\psi^{\lambda+r}}(2^{j'-j}x+(l'-2^{j'-j}l), 2^{j'-j}y)\overline{K^{(\lambda+r)2^{2(j'-j)}}_{\psi^{(\lambda+r)2^{2(j'-j)}}}(x,y)}\,dx\,dy\,|\lambda+r|^n\,d\lambda$$
$$= \sum_{\substack{j'>j\\ k',l',k,l}} 2^{2(j'-j)} \int_0^1 R_{j',k',l'}(\lambda)\overline{R_{j,k,l}(2^{2(j'-j)}\lambda)}e^{2\pi i\lambda 2^{j'-j}(k\cdot l'-k'\cdot l)}$$
$$\times \overline{H^{\psi}_{j'-j,k'-2^{j'-j}k,l'-2^{j'-j}l}(\lambda)}\,d\lambda$$
$$= 0,$$

which follows from (2.1) and condition (ii) of the hypothesis. Similarly, we obtain $J_3=0$, thereby proving $I_2=0$. Hence $\|f\|^2_{L^2(\mathbb{H}^n)} = \|R\|^2_{L^2(\mathbb{T},l^2(\mathbb{Z}^{2n+1});G^{\psi}_{0,0})}$. Thus we derive the isometric isomorphism of $\mathcal{A}(\psi)$ onto $c_{00}(\mathbb{Z}^{2n+1},\sigma(\mathbb{T}))$. Since $\mathcal{A}(\psi)$ is dense in $\mathcal{W}(\psi)$ and $c_{00}(\mathbb{Z}^{2n+1},\sigma(\mathbb{T}))$ is dense in the space $L^2(\mathbb{T},l^2(\mathbb{Z}^{2n+1});G^{\psi}_{0,0})$, the isometric isomorphism can be extended as required. ∎

In the remaining part of this paper, we shall use the following notation. Let α and Λ denote the triplet of indices $(j,k,l)\in\mathbb{Z}\times\mathbb{Z}^n\times\mathbb{Z}^n$ and $(J,K,L)\in\mathbb{N}^3$ respectively. Let Ω denote the rectangle $\{(j,k,l)\in\mathbb{Z}\times\mathbb{Z}^n\times\mathbb{Z}^n : |j|\le J,\ |k|\le K,\ |l|\le L\}$, where $J,K,L\in\mathbb{N}$.

We shall now prove our main result.

Proof of Theorem 2.2. Throughout this proof we shall assume that $\mathbb{Z}$ is ordered as $\{0, 1, -1, 2, -2, \ldots\}$. For $\alpha \in \mathbb{Z}^{2n+1}$, $m \in \mathbb{Z}$, let

$$R^{\alpha,m}(\lambda) = \{(R^{\alpha,m})_{\alpha'}(\lambda)\}_{\alpha' \in \mathbb{Z}^{2n+1}}, \qquad (R^{\alpha,m})_{\alpha'}(\lambda) = \begin{cases} e^{2\pi i m\lambda}, & \alpha' = \alpha, \\ 0, & \alpha' \neq \alpha, \end{cases}$$

for $\lambda \in \mathbb{T}$. Then by making use of the isometric isomorphism in Theorem 2.3, it can be easily shown that $\{\psi_{\alpha,m} : \alpha \in \mathbb{Z}^{2n+1},\, m \in \mathbb{Z}\}$ is a Schauder basis for $\mathcal{W}(\psi)$ if and only if $\{R^{\alpha,m} : \alpha \in \mathbb{Z}^{2n+1},\, m \in \mathbb{Z}\}$ is a Schauder basis for $L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^{\psi}_{0,0})$.

Assume that $\{R^{\alpha,m} : \alpha \in \mathbb{Z}^{2n+1},\, m \in \mathbb{Z}\}$ is a Schauder basis for the space $L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^{\psi}_{0,0})$. Let $x \in L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^{\psi}_{0,0})$. Then there exists a unique $\{c_{\alpha,m}(x)\}_{\alpha \in \mathbb{Z}^{2n+1}, m \in \mathbb{Z}}$ such that $x = \sum_{\alpha,m} c_{\alpha,m}(x) R^{\alpha,m}$. By the Riesz representation theorem, for each $(\alpha, m) \in \mathbb{Z}^{2n+1} \times \mathbb{Z}$, there exists $S^{\alpha,m} \in L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^{\psi}_{0,0})$ such that $c_{\alpha,m}(x) = \langle x, S^{\alpha,m}\rangle$ for every $x \in L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^{\psi}_{0,0})$. Hence $\langle R^{\alpha',m'}, S^{\alpha,m}\rangle_{L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^{\psi}_{0,0})} = \delta_{\alpha\alpha'}\delta_{mm'}$ for $\alpha, \alpha' \in \mathbb{Z}^{2n+1}$, $m, m' \in \mathbb{Z}$, which shows that $\{S^{\alpha,m} : \alpha \in \mathbb{Z}^{2n+1},\, m \in \mathbb{Z}\}$ is a biorthogonal dual system. Further,

$$\begin{aligned}\int_0^1 |\overline{(S^{\alpha,m})_{\alpha'}(\lambda)} G^{\psi}_{0,0}(\lambda)|\, d\lambda &= \int_0^1 |(S^{\alpha,m})_{\alpha'}(\lambda)\sqrt{G^{\psi}_{0,0}(\lambda)}||\sqrt{G^{\psi}_{0,0}(\lambda)}|\, d\lambda \\ &\leq \|S^{\alpha,m}\|_{L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^{\psi}_{0,0})} \|\psi\|_{L^2(\mathbb{H}^n)} < \infty,\end{aligned}$$

by the Cauchy–Schwarz inequality, showing that $\overline{(S^{\alpha,m})_{\alpha'}} G^{\psi}_{0,0} \in L^1(\mathbb{T})$. Now

$$\begin{aligned}\langle R^{\alpha',m'}, S^{\alpha,m}\rangle_{L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^{\psi}_{0,0})} &= \int_0^1 \langle R^{\alpha',m'}(\lambda), S^{\alpha,m}(\lambda)\rangle_{l^2(\mathbb{Z}^{2n+1})} G^{\psi}_{0,0}(\lambda)\, d\lambda \\ &= \int_0^1 e^{2\pi i m'\lambda}\overline{(S^{\alpha,m})_{\alpha'}(\lambda)} G^{\psi}_{0,0}(\lambda)\, d\lambda\end{aligned}$$

is the $(-m')$th Fourier coefficient of $\overline{(S^{\alpha,m})_{\alpha'}} G^{\psi}_{0,0}$. Since the $(-m')$th Fourier coefficient of $\overline{(S^{\alpha,m})_{\alpha'}} G^{\psi}_{0,0}$ is $\delta_{\alpha\alpha'}\delta_{mm'}$, we get $\overline{(S^{\alpha,m})_{\alpha'}(\lambda)} G^{\psi}_{0,0}(\lambda) = 0$ for a.e. $\lambda \in \mathbb{T}$ if $\alpha' \neq \alpha$ and $\overline{(S^{\alpha,m})_{\alpha}(\lambda)} G^{\psi}_{0,0}(\lambda) = e^{-2\pi i m\lambda}$ for a.e. $\lambda \in \mathbb{T}$. The latter equation shows that $G^{\psi}_{0,0}(\lambda) \neq 0$ for a.e. $\lambda \in \mathbb{T}$. So, we have

$$(S^{\alpha,m})_{\alpha'}(\lambda) = \begin{cases} \frac{1}{G^{\psi}_{0,0}(\lambda)} e^{2\pi i m\lambda}, & \alpha' = \alpha, \\ 0, & \alpha' \neq \alpha, \end{cases} \qquad \text{a.e. } \lambda \in \mathbb{T}. \tag{2.2}$$

Moreover, $1/G^{\psi}_{0,0} \in L^1(\mathbb{T})$. In fact,

$$\left\| \frac{1}{G^{\psi}_{0,0}} \right\|_1 = \int_0^1 \|S^{\alpha,m}(\lambda)\|^2_{l^2(\mathbb{Z}^{2n+1})} G^{\psi}_{0,0}(\lambda)\, d\lambda = \|S^{\alpha,m}\|^2_{L^2(\mathbb{T},l^2(\mathbb{Z}^{2n+1});G^{\psi}_{0,0})},$$

which is finite. It is known that a complete sequence $\{x_n : n \in \mathbb{N}\}$ with dual sequence $\{y_n : n \in \mathbb{N}\}$ is a Schauder basis for a Hilbert space $\mathbb{H}$ if and only if the partial sum operators $S_N(x) = \sum_{n=1}^{N} \langle x, y_n\rangle x_n$ are uniformly bounded on $\mathbb{H}$. So, if we define the partial sum operators $\tilde{T}_{\Lambda,M}$ on $L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^{\psi}_{0,0})$ for $\Lambda \in \mathbb{N}^3$, $M \in \mathbb{N}$ by

$$\tilde{T}_{\Lambda,M}(x) = \sum_{\substack{\alpha\in\Omega\\ |m|\le M}} \langle x, S^{\alpha,m}\rangle_{L^2(\mathbb{T},l^2(\mathbb{Z}^{2n+1});G^{\psi}_{0,0})} R^{\alpha,m}$$

for $x \in L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1});\ G^{\psi}_{0,0})$, then $\sup_{(\Lambda,M)\in\mathbb{N}^4} \|\tilde{T}_{\Lambda,M}\| < \infty$. We also consider the symmetric Fourier partial sum operators $T_{\Lambda,M}$ on $L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1});$ $G^{\psi}_{0,0})$ given by

(2.3)
$$T_{\Lambda,M}(x) = \sum_{\substack{\alpha\in\Omega\\ |m|\le M}} \langle x, R^{\alpha,m}\rangle_{L^2(\mathbb{T},l^2(\mathbb{Z}^{2n+1}))} R^{\alpha,m},\ \ x \in L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^{\psi}_{0,0}).$$

Since for $x \in L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^{\psi}_{0,0})$,

$$\begin{aligned}\langle x, S^{\alpha,m}\rangle_{L^2(\mathbb{T},l^2(\mathbb{Z}^{2n+1});G^{\psi}_{0,0})} &= \int_0^1 x_\alpha(\lambda)\overline{\frac{1}{G^{\psi}_{0,0}(\lambda)} e^{2\pi i m\lambda}}\ G^{\psi}_{0,0}(\lambda)\, d\lambda\\ &= \int_0^1 \langle x(\lambda), R^{\alpha,m}(\lambda)\rangle_{l^2(\mathbb{Z}^{2n+1})}\, d\lambda = \langle x, R^{\alpha,m}\rangle_{L^2(\mathbb{T},l^2(\mathbb{Z}^{2n+1}))},\end{aligned}$$

we have $\tilde{T}_{\Lambda,M} = T_{\Lambda,M}$ and so $A := \sup_{(\Lambda,M)\in\mathbb{N}^4} \|T_{\Lambda,M}\| < \infty$. We shall now explicitly compute the terms of the sequence $(T_{\Lambda,M}x)(\lambda)$ indexed by $\mathbb{Z}^{2n+1}$, where $x \in L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^{\psi}_{0,0})$ and $\lambda \in \mathbb{T}$. For $\alpha' \in \Omega$ we have

$$(T_{\Lambda,M}x)_{\alpha'}(\lambda) = \sum_{\alpha\in\Omega} \sum_{|m|\le M} \langle x, R^{\alpha,m}\rangle_{L^2(\mathbb{T},l^2(\mathbb{Z}^{2n+1}))} (R^{\alpha,m})_{\alpha'}(\lambda).$$

But

$$\begin{aligned}\sum_{\alpha\in\Omega} \langle x, R^{\alpha,m}\rangle_{L^2(\mathbb{T},l^2(\mathbb{Z}^{2n+1}))} (R^{\alpha,m})_{\alpha'}(\lambda) &= \langle x, R^{\alpha',m}\rangle_{L^2(\mathbb{T},l^2(\mathbb{Z}^{2n+1}))} e^{2\pi i m\lambda}\\ &= \Big(\int_0^1 x_{\alpha'}(\lambda')\overline{e^{2\pi i m\lambda'}}\, d\lambda'\Big) e^{2\pi i m\lambda}.\end{aligned}$$

Further, by using the definition of $R^{\alpha,m}$, it follows that

$$(T_{\Lambda,M}x)_{\alpha'}(\lambda)=\begin{cases}\int_0^1 x_{\alpha'}(\lambda')D_M(\lambda-\lambda')\,d\lambda', & \alpha'\in\Omega\\ 0, & \text{otherwise},\end{cases}\qquad \lambda\in\mathbb{T},$$

where D_M is the Dirichlet kernel given by $D_M(x)=\sum_{m=-M}^{M}e^{2\pi imx}$.

Now, for any $M\in\mathbb{N}$, $\|D_M\|_\infty=\sup_{\lambda\in\mathbb{T}}|D_M(\lambda)|=2M+1$ and by Bernstein's inequality, $\|(D_M)'\|_\infty\leq(2M+1)^2$. By the mean value theorem, one can show that $D_M(\lambda)\geq(2M+1)-(2M+1)^2|\lambda|$. So, if $|\lambda|\leq\frac{1}{NM}$, $N\in\mathbb{N}$, then $D_M(\lambda)\geq\|D_M\|_\infty\big(1-\frac{2M+1}{NM}\big)$. Now, we can choose $N\in\mathbb{N}$, independent of M, so that $1-\frac{2M+1}{NM}\geq\frac{1}{2}$. Thus the choice of N is such that for any $M\in\mathbb{N}$, $D_M(\lambda)\geq\frac{1}{2}\|D_M\|_\infty$ whenever $|\lambda|\leq\frac{1}{NM}$.

Now, we are in a position to prove that $G^\psi_{0,0}\in\mathcal{A}_2$. Let $I\subseteq\mathbb{T}$ and $|I|>\frac{1}{2N}$. Then

$$\begin{aligned}\Big(\frac{1}{|I|}\int_I G^\psi_{0,0}(\lambda)\,d\lambda\Big)&\Big(\frac{1}{|I|}\int_I\frac{1}{G^\psi_{0,0}(\lambda)}\,d\lambda\Big)\\ &\leq(2N)^2\int_0^1 G^\psi_{0,0}(\lambda)\,d\lambda\int_0^1\frac{1}{G^\psi_{0,0}(\lambda)}\,d\lambda=(2N)^2\|\psi\|^2_{L^2(\mathbb{H}^n)}\Big\|\frac{1}{G^\psi_{0,0}}\Big\|_{L^1(\mathbb{T})}.\end{aligned}$$

Let $C_1:=(2N)^2\|\psi\|^2_{L^2(\mathbb{H}^n)}\|1/G^\psi_{0,0}\|_{L^1(\mathbb{T})}$. Then $0<C_1<\infty$ and (1.6) holds with $C=C_1$. Let $I\subseteq\mathbb{T}$ and $|I|\leq\frac{1}{2N}$. Further, if $\int_I\frac{1}{G^\psi_{0,0}(\lambda)}\,d\lambda=0$, then (1.6) holds for any $C>0$. So, we may assume that $\int_I\frac{1}{G^\psi_{0,0}(\lambda)}\,d\lambda>0$. Since $1/G^\psi_{0,0}\in L^1(\mathbb{T})$, we have $0<\int_I\frac{1}{G^\psi_{0,0}(\lambda)}\,d\lambda<\infty$. Choose $M\in\mathbb{N}$ such that $\frac{1}{4NM}\leq|I|\leq\frac{1}{2NM}$. Define $x\in L^2(\mathbb{T},l^2(\mathbb{Z}^{2n+1});G^\psi_{0,0})$ by

$$x(\lambda)=\{x_{\alpha'}(\lambda)\}_{\alpha'\in\mathbb{Z}^{2n+1}},\qquad x_{\alpha'}(\lambda)=\begin{cases}f(\lambda), & \alpha'=\mathbf{0},\\ 0, & \text{otherwise},\end{cases}\qquad\lambda\in\mathbb{T},$$

where $f\in L^2(\mathbb{T};G^\psi_{0,0})$, $f\geq0$ on I and $f=0$ on $\mathbb{T}\setminus I$. Then for any $\Lambda\in\mathbb{N}^3$,

$$\|T_{\Lambda,M}x\|_{L^2(\mathbb{T},l^2(\mathbb{Z}^{2n+1});G^\psi_{0,0})}\leq A\|x\|_{L^2(\mathbb{T},l^2(\mathbb{Z}^{2n+1});G^\psi_{0,0})}.$$

Now, for $\lambda\in\mathbb{T}$,

$$\begin{aligned}\|T_{\Lambda,M}x(\lambda)\|^2_{l^2(\mathbb{Z}^{2n+1})}&=\sum_{\alpha'\in\Omega}\Big|\int_0^1 x_{\alpha'}(\lambda')D_M(\lambda-\lambda')\,d\lambda'\Big|^2\\ &=\Big|\int_0^1 f(\lambda')D_M(\lambda-\lambda')\,d\lambda'\Big|^2=\Big|\int_I f(\lambda')D_M(\lambda-\lambda')\,d\lambda'\Big|^2.\end{aligned}$$

Consequently,

$$(2.4)\qquad \int_I \Big|\int_I f(\lambda')D_M(\lambda-\lambda')\,d\lambda'\Big|^2 G^{\psi}_{0,0}(\lambda)\,d\lambda \le \int_0^1 \Big|\int_I f(\lambda')D_M(\lambda-\lambda')\,d\lambda'\Big|^2 G^{\psi}_{0,0}(\lambda)\,d\lambda$$
$$= \|T_{\Lambda,M}x\|^2_{L^2(\mathbb{T},l^2(\mathbb{Z}^{2n+1});G^{\psi}_{0,0})} \le A^2\|x\|^2_{L^2(\mathbb{T},l^2(\mathbb{Z}^{2n+1});G^{\psi}_{0,0})}$$
$$= A^2\int_0^1 \|x(\lambda)\|^2_{l^2(\mathbb{Z}^{2n+1})} G^{\psi}_{0,0}(\lambda)\,d\lambda = A^2\int_I |f(\lambda)|^2 G^{\psi}_{0,0}(\lambda)\,d\lambda.$$

For $\lambda,\lambda'\in I$, we have $|\lambda-\lambda'|\le|I|\le\frac{1}{2NM}\le\frac{1}{NM}$ and so $D_M(\lambda-\lambda')\ge\frac{1}{2}\|D_M\|_\infty=\frac{1}{2}(2M+1)\ge M\ge\frac{1}{4N|I|}$. Then $\int_I f(\lambda')D_M(\lambda-\lambda')\,d\lambda'\ge\frac{1}{4N|I|}\int_I f(\lambda')\,d\lambda'\ge 0$ for $\lambda\in I$. Hence

$$(2.5)\qquad \int_I \Big|\int_I f(\lambda')D_M(\lambda-\lambda')\,d\lambda'\Big|^2 G^{\psi}_{0,0}(\lambda)\,d\lambda \ge \frac{1}{(4N|I|)^2}\Big(\int_I f(\lambda')\,d\lambda'\Big)^2\int_I G^{\psi}_{0,0}(\lambda)\,d\lambda.$$

From (2.4) and (2.5), we have

$$\frac{1}{(4N)^2|I|^2}\Big(\int_I f(\lambda')\,d\lambda'\Big)^2\int_I G^{\psi}_{0,0}(\lambda)\,d\lambda \le A^2\int_I |f(\lambda)|^2 G^{\psi}_{0,0}(\lambda)\,d\lambda.$$

Let $f=1/G^{\psi}_{0,0}$ on I and $f=0$ on $\mathbb{T}\setminus I$. Then clearly $f\in L^2(\mathbb{T};G^{\psi}_{0,0})$. Also,

$$\frac{1}{(4N)^2|I|^2}\Big(\int_I \frac{1}{G^{\psi}_{0,0}(\lambda)}\,d\lambda\Big)^2\Big(\int_I G^{\psi}_{0,0}(\lambda)\,d\lambda\Big) \le A^2\Big(\int_I \frac{1}{G^{\psi}_{0,0}(\lambda)}\,d\lambda\Big).$$

In other words,

$$\Big(\frac{1}{|I|}\int_I \frac{1}{G^{\psi}_{0,0}(\lambda)}\,d\lambda\Big)\Big(\frac{1}{|I|}\int_I G^{\psi}_{0,0}(\lambda)\,d\lambda\Big) \le A^2(4N)^2.$$

Thus, (1.6) holds with $C=\max\{C_1, A^2(4N)^2\}>0$ for all $I\subseteq\mathbb{T}$, thereby proving that $G^{\psi}_{0,0}\in\mathcal{A}_2$.

Conversely, suppose $G^{\psi}_{0,0}\in\mathcal{A}_2$. By using the isometric isomorphism between the spaces $\mathcal{W}(\psi)$ and $L^2(\mathbb{T},l^2(\mathbb{Z}^{2n+1});G^{\psi}_{0,0})$, we need only show that $\{R^{\alpha,m}:\alpha\in\mathbb{Z}^{2n+1},\ m\in\mathbb{Z}\}$ is a Schauder basis for $L^2(\mathbb{T},l^2(\mathbb{Z}^{2n+1});G^{\psi}_{0,0})$. Since $G^{\psi}_{0,0}\in\mathcal{A}_2$, (1.6) holds and hence $1/G^{\psi}_{0,0}\in L^1(\mathbb{T})$ as $\int_0^1(1/G^{\psi}_{0,0}(\lambda))\,d\lambda\le C/\|\psi\|^2_{L^2(\mathbb{H}^n)}<\infty$, showing that $G^{\psi}_{0,0}(\lambda)>0$ for a.e. $\lambda\in\mathbb{T}$. For $(\alpha,m)\in\mathbb{Z}^{2n+1}\times\mathbb{Z}$, if we define $S^{\alpha,m}\in L^2(\mathbb{T},l^2(\mathbb{Z}^{2n+1});G^{\psi}_{0,0})$ by $S^{\alpha,m}(\lambda)=$

$\{(S^{\alpha,m})_{\alpha'}(\lambda)\}_{\alpha'\in\mathbb{Z}^{2n+1}}$ for $\lambda \in \mathbb{T}$, where $(S^{\alpha,m})_{\alpha'}(\lambda)$ is as in (2.2), then $\{S^{\alpha,m} : \alpha \in \mathbb{Z}^{2n+1}, m \in \mathbb{Z}\}$ is a biorthogonal dual to $\{R^{\alpha,m} : \alpha \in \mathbb{Z}^{2n+1}, m \in \mathbb{Z}\}$, for

$$\begin{aligned}\langle R^{\alpha',m'}, S^{\alpha,m}\rangle_{L^2(\mathbb{T},l^2(\mathbb{Z}^{2n+1});G^{\psi}_{0,0})} &= \int_0^1 (R^{\alpha',m'})_{\alpha'}(\lambda)\overline{(S^{\alpha,m})_{\alpha'}(\lambda)}G^{\psi}_{0,0}(\lambda)\,d\lambda \\ &= \begin{cases} 0, & \alpha' \neq \alpha, \\ \int_0^1 e^{2\pi i(m'-m)\lambda}\,d\lambda, & \alpha' = \alpha, \end{cases}\end{aligned}$$

which shows $\langle R^{\alpha',m'}, S^{\alpha,m}\rangle_{L^2(\mathbb{T},l^2(\mathbb{Z}^{2n+1});G^{\psi}_{0,0})} = \delta_{\alpha\alpha'}\delta_{mm'}$ for $\alpha, \alpha' \in \mathbb{Z}^{2n+1}$ and $m, m' \in \mathbb{Z}$.

We shall now show that the operators $T_{\Lambda,M}$ on $L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^{\psi}_{0,0})$ defined in (2.3) are uniformly bounded. Let $x \in L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^{\psi}_{0,0})$. For $\alpha \in \mathbb{Z}^{2n+1}$, let

$$\overrightarrow{u^{\alpha}} = \{(u^{\alpha})_{\alpha'}\}_{\alpha'\in\mathbb{Z}^{2n+1}}, \qquad (u^{\alpha})_{\alpha'} = \begin{cases} 1, & \alpha' = \alpha, \\ 0, & \text{otherwise.} \end{cases}$$

Then, for $\lambda \in \mathbb{T}$,

$$\begin{aligned}(T_{\Lambda,M}x)(\lambda) &= \sum_{\alpha\in\Omega}\sum_{|m|\leq M}\langle x, R^{\alpha,m}\rangle_{L^2(\mathbb{T},l^2(\mathbb{Z}^{2n+1}))}R^{\alpha,m}(\lambda) \\ &= \sum_{\alpha\in\Omega}\Big(\sum_{|m|\leq M}\langle x, R^{\alpha,m}\rangle_{L^2(\mathbb{T},l^2(\mathbb{Z}^{2n+1}))}e^{2\pi i m\lambda}\Big)\overrightarrow{u^{\alpha}}.\end{aligned}$$

Splitting the index $|m| \leq M$ as $m = 0$, $1 \leq m \leq M$ and $-M \leq m \leq -1$, and using $e^{2\pi i m\lambda} = \cos 2\pi m\lambda + i \sin 2\pi m\lambda$, we get, upon further simplification,

$$\begin{aligned}(2.6)\qquad &(T_{\Lambda,M}x)(\lambda) \\ &= \sum_{\alpha\in\Omega}\Big[\int_0^1 x_{\alpha}(\lambda')\,d\lambda' + \sum_{m=1}^{M}\Big(2\int_0^1 x_{\alpha}(\lambda')\cos 2\pi m\lambda'\ d\lambda'\Big)\cos 2\pi m\lambda \\ &\qquad - i\sum_{m=1}^{M}\Big(2i\int_0^1 x_{\alpha}(\lambda')\sin 2\pi m\lambda'\ d\lambda'\Big)\sin 2\pi m\lambda\Big]\overrightarrow{u^{\alpha}} \\ &= \sum_{\alpha\in\Omega}\Big(\int_0^1 x_{\alpha}(\lambda')\Big(1 + 2\sum_{m=1}^{M}\cos 2\pi m(\lambda' - \lambda)\Big)\,d\lambda'\Big)\overrightarrow{u^{\alpha}} \\ &= \sum_{\alpha\in\Omega}\Big(\int_0^1 x_{\alpha}(\lambda')D_M(\lambda' - \lambda)\,d\lambda'\Big)\overrightarrow{u^{\alpha}},\end{aligned}$$

where D_M is the Dirichlet kernel. In order to find $\|T_{\Lambda,M}x\|$, we also consider the modified partial sum operator $T^{*}_{\Lambda,M}$ on $L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^{\psi}_{0,0})$ similar to that in [11] defined as follows. For $x \in L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^{\psi}_{0,0})$,

$$T^*_{\Lambda,M}(x) = \sum_{\alpha\in\Omega}\Big(\sum_{|m|\le M-1}\langle x, R^{\alpha,m}\rangle_{L^2(\mathbb{T},l^2(\mathbb{Z}^{2n+1}))}R^{\alpha,m} + \frac{1}{2}\sum_{|m|=M}\langle x, R^{\alpha,m}\rangle_{L^2(\mathbb{T},l^2(\mathbb{Z}^{2n+1}))}R^{\alpha,m}\Big).$$

Computing as earlier we get, for $\lambda \in \mathbb{T}$,

$$\begin{aligned}(T^*_{\Lambda,M}x)(\lambda) &= \sum_{\alpha\in\Omega}\Big(\int_0^1 x_\alpha(\lambda')(D_{M-1}(\lambda'-\lambda)+\cos 2\pi M(\lambda'-\lambda))\,d\lambda'\Big)\overrightarrow{u^\alpha}\\ &= \sum_{\alpha\in\Omega}\Big(\int_0^1 x_\alpha(\lambda')(D_{M}(\lambda'-\lambda)-\cos 2\pi M(\lambda'-\lambda))\,d\lambda'\Big)\overrightarrow{u^\alpha}\\ &= \sum_{\alpha\in\Omega}\Big(\int_0^1 x_\alpha(\lambda')D^*_{M}(\lambda'-\lambda)\,d\lambda'\Big)\overrightarrow{u^\alpha},\end{aligned} \tag{2.7}$$

where D^*_M is the modified Dirichlet kernel [11] given by

$$D^*_M(x) = D_M(x) - \cos 2\pi Mx. \tag{2.8}$$

Now writing

$$D^*_M(x) = \frac{\sin 2\pi(M+\frac{1}{2})x}{\sin\frac{2\pi x}{2}} - \cos 2\pi Mx = \frac{\sin 2\pi Mx}{\tan \pi x},$$

one can show that

$$\begin{aligned}\int_0^1 x_\alpha(\lambda')D^*_M(\lambda'-\lambda)\,d\lambda' &= \int_0^1 x_\alpha(\lambda+t)\frac{\sin 2\pi Mt}{\tan \pi t}\,dt\\ &= \Big(\int_0^1 \frac{x_\alpha(\lambda+t)\sin 2\pi M(t+\lambda)}{\tan \pi t}\,dt\Big)\cos 2\pi M\lambda\\ &\quad - \Big(\int_0^1 \frac{x_\alpha(\lambda+t)\cos 2\pi M(t+\lambda)}{\tan \pi t}\,dt\Big)\sin 2\pi M\lambda.\end{aligned}$$

As in [11], we define for $\lambda \in \mathbb{T}$ and $M \in \mathbb{N}$ the sequences $p_M(\lambda)$, $q_M(\lambda)$, $\tilde{p}_M(\lambda)$ and $\tilde{q}_M(\lambda)$ as follows:

$$\begin{aligned} p_M(\lambda) &= \{(p_M)_\alpha(\lambda)\}_{\alpha\in\mathbb{Z}^{2n+1}}, & (p_M)_\alpha(\lambda) &= x_\alpha(\lambda)\cos 2\pi M\lambda,\\ q_M(\lambda) &= \{(q_M)_\alpha(\lambda)\}_{\alpha\in\mathbb{Z}^{2n+1}}, & (q_M)_\alpha(\lambda) &= x_\alpha(\lambda)\sin 2\pi M\lambda,\\ \tilde{p}_M(\lambda) &= \{(\tilde{p}_M)_\alpha(\lambda)\}_{\alpha\in\mathbb{Z}^{2n+1}}, & \tilde{q}_M(\lambda) &= \{(\tilde{q}_M)_\alpha(\lambda)\}_{\alpha\in\mathbb{Z}^{2n+1}},\end{aligned}$$

where $(\tilde{p}_M)_\alpha$ is the conjugate function of $(p_M)_\alpha$ given by

$$(\tilde{p}_M)_\alpha(\lambda) = -\int_0^1 \frac{(p_M)_\alpha(\lambda+t)}{\tan \pi t}\,dt$$

and $(\tilde{q}_M)_\alpha$ is the conjugate function of $(q_M)_\alpha$ defined similarly. Using these definitions, we have

$$(T^*_{\Lambda,M}x)(\lambda) = \sum_{\alpha\in\Omega} \big((\tilde{p}_M)_\alpha(\lambda)\sin 2\pi M\lambda - (\tilde{q}_M)_\alpha(\lambda)\cos 2\pi M\lambda\big)\overrightarrow{u^\alpha}. \tag{2.9}$$

For $x \in L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^\psi_{0,0})$, using (2.6)–(2.9), we have

$$\begin{aligned}
&\|T_{\Lambda,M}x\|_{L^2(\mathbb{T},l^2(\mathbb{Z}^{2n+1});G^\psi_{0,0})} \\
&\quad\le \Big(\int_0^1 \|T_{\Lambda,M}x(\lambda) - T^*_{\Lambda,M}x(\lambda)\|^2_{l^2(\mathbb{Z}^{2n+1})} G^\psi_{0,0}(\lambda)\,d\lambda\Big)^{1/2} \\
&\qquad + \Big(\int_0^1 \|T^*_{\Lambda,M}x(\lambda)\|^2_{l^2(\mathbb{Z}^{2n+1})} G^\psi_{0,0}(\lambda)\,d\lambda\Big)^{1/2} \\
&\quad\le \Big(\int_0^1 \sum_{\alpha\in\Omega}\Big(\int_0^1 |x_\alpha(\lambda')|\,d\lambda'\Big)^2 G^\psi_{0,0}(\lambda)\,d\lambda\Big)^{1/2} \\
&\qquad + \Big(\int_0^1 \sum_{\alpha\in\Omega}\big(|(\tilde{p}_M)_\alpha(\lambda)| + |(\tilde{q}_M)_\alpha(\lambda)|\big)^2 G^\psi_{0,0}(\lambda)\,d\lambda\Big)^{1/2}.
\end{aligned} \tag{2.10}$$

But

$$\sum_{\alpha\in\Omega}\Big(\int_0^1 |x_\alpha(\lambda')|\,d\lambda'\Big)^2 \le \|x\|^2_{L^2(\mathbb{T},l^2(\mathbb{Z}^{2n+1});G^\psi_{0,0})} \frac{C}{\|\psi\|^2_{L^2(\mathbb{H}^n)}},$$

by the Cauchy–Schwarz inequality. Further, by defining $\tilde{P}_M(\lambda)$ and $\tilde{Q}_M(\lambda)$ to be the sequences whose αth terms are the moduli of the αth terms of $\tilde{p}_M(\lambda)$ and $\tilde{q}_M(\lambda)$ respectively, we obtain from (2.10)

$$\begin{aligned}
&\|T_{\Lambda,M}x\|_{L^2(\mathbb{T},l^2(\mathbb{Z}^{2n+1});G^\psi_{0,0})} \\
&\quad\le \|x\|_{L^2(\mathbb{T},l^2(\mathbb{Z}^{2n+1});G^\psi_{0,0})} \frac{\sqrt{C}}{\|\psi\|_{L^2(\mathbb{H}^n)}} \Big(\int_0^1 G^\psi_{0,0}(\lambda)\,d\lambda\Big)^{1/2} \\
&\qquad + \Big(\int_0^1 \sum_{\alpha\in\Omega} |(\tilde{P}_M + \tilde{Q}_M)_\alpha(\lambda)|^2 G^\psi_{0,0}(\lambda)\,d\lambda\Big)^{1/2} \\
&\quad\le \sqrt{C}\,\|x\|_{L^2(\mathbb{T},l^2(\mathbb{Z}^{2n+1});G^\psi_{0,0})} + \|\tilde{P}_M + \tilde{Q}_M\|_{L^2(\mathbb{T},l^2(\mathbb{Z}^{2n+1});G^\psi_{0,0})} \\
&\quad\le \sqrt{C}\,\|x\| + \|\tilde{P}_M\| + \|\tilde{Q}_M\|.
\end{aligned} \tag{2.11}$$

By [5, Theorem 1], there exists $C' > 0$ independent of M, α and x such that

$$\int_0^1 |(\tilde{p}_M)_\alpha(\lambda)|^2 G^\psi_{0,0}(\lambda)\, d\lambda \leq C' \int_0^1 |(p_M)_\alpha(\lambda)|^2 G^\psi_{0,0}(\lambda)\, d\lambda.$$

Then

$$\begin{aligned} \|\tilde{P}_M\|^2_{L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^\psi_{0,0})} &\leq C' \sum_{\alpha\in\mathbb{Z}^{2n+1}} \int_0^1 |(p_M)_\alpha(\lambda)|^2 G^\psi_{0,0}(\lambda)\, d\lambda \\ &\leq C' \int_0^1 \Big(\sum_{\alpha\in\mathbb{Z}^{2n+1}} |x_\alpha(\lambda)|^2\Big) G^\psi_{0,0}(\lambda)\, d\lambda = C'\|x\|^2_{L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^\psi_{0,0})}, \end{aligned}$$

by using the definition of $(p_M)_\alpha(\lambda)$. Similarly, one can show that

$$\|\tilde{Q}_M\|^2_{L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^\psi_{0,0})} \leq C'\|x\|^2_{L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^\psi_{0,0})}.$$

Then it follows from (2.11) that

$$\|T_{\Lambda,M}x\|_{L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^\psi_{0,0})} \leq (\sqrt{C} + 2\sqrt{C'})\|x\|_{L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^\psi_{0,0})}.$$

Since C and C' are independent of x, Λ and M, we obtain

$$\sup_{(\Lambda,M)\in\mathbb{N}^4} \|T_{\Lambda,M}\| \leq \sqrt{C} + 2\sqrt{C'} < \infty.$$

Since $\tilde{T}_{\Lambda,M} = T_{\Lambda,M}$, we get $\sup_{(\Lambda,M)\in\mathbb{N}^4} \|\tilde{T}_{\Lambda,M}\| < \infty$. Let R belong to the space $L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^\psi_{0,0}) = \overline{\operatorname{span}}\{R^{\alpha',m'} : \alpha' \in \mathbb{Z}^{2n+1},\ m' \in \mathbb{Z}\}$. By the uniform boundedness of the operators $\tilde{T}_{\Lambda,M}$ and the biorthogonality between $\{R^{\alpha',m'}\}$ and $\{S^{\alpha,m}\}$, we get

$$R = \lim_{\Lambda,M\to\infty} \sum_{\substack{\alpha\in\Omega \\ |m|\leq M}} \langle R, S^{\alpha,m}\rangle_{L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^\psi_{0,0})} R^{\alpha,m},$$

where the limit is in $L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^\psi_{0,0})$. The uniqueness of the coefficients $\{\langle R, S^{\alpha,m}\rangle_{L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^\psi_{0,0})}\}$ follows again from the biorthogonality between $\{R^{\alpha',m'}\}$ and $\{S^{\alpha,m}\}$.

Thus, we have shown that $\{R^{\alpha,m} : \alpha \in \mathbb{Z}^{2n+1},\ m \in \mathbb{Z}\}$ is a Schauder basis for $L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^\psi_{0,0})$, thereby proving our assertion. ■

REMARK 2.4. Condition (i) in Theorem 2.2 is used in connection with the orthonormality of the system of left translates on the Heisenberg group. In fact, for $\varphi \in L^2(\mathbb{H}^n)$, the system $\{L_{(2k,l,m)}\varphi : (k,l,m) \in \mathbb{Z}^{2n+1}\}$ is an orthonormal system in $L^2(\mathbb{H}^n)$ if and only if $G^\varphi_{0,0}(\lambda) = 1$ for a.e. $\lambda \in (0,1]$ and φ satisfies (i) in Theorem 2.2 (see [2] and [9]). In other words, condition (i) is analogous to condition C in [8], where the orthogonality of the system of twisted translates on $\mathbb{C}^n$ is discussed. Similarly, condition (ii) is satisfied by a function $\psi \in L^2(\mathbb{H}^n)$ when the wavelet system $\{\psi_{j,k,l,m}\}$ forms an orthonormal system in $L^2(\mathbb{H}^n)$. On the other hand, conditions (i) and (ii) together do not lead to the orthogonality of the wavelet system. This is similar to the situation in the classical case, where one cannot obtain the orthogonality of the wavelet system in the absence of the condition $\sigma_\varphi(\xi) = \sum_{k\in\mathbb{Z}} |\widehat{\varphi}(\xi + k)|^2 = 1$ for a.e. $\xi \in \mathbb{T}$. But, to prove the orthonormality of the system of translates on $\mathbb{R}^n$, no condition analogous to condition (i) in Theorem 2.2 or condition C in [8] is required.

Remark 2.5. Assuming condition (i) of Theorem 2.2 alone, we can show that the system of left translates on the Heisenberg group forms a Schauder basis for its closed linear span if and only if $G^{\psi}_{0,0} \in \mathcal{A}_2$.

Acknowledgements. We thank the referee for meticulously reading our manuscript and providing valuable suggestions and comments.

REFERENCES

[1] S. Arati and R. Radha, *Orthonormality of a wavelet system on the Heisenberg group*, J. Math. Pures Appl. (online, 2019).

[2] D. Barbieri, E. Hernández and A. Mayeli, *Bracket map for the Heisenberg group and the characterization of cyclic subspaces*, Appl. Comput. Harmon. Anal. 37 (2014), 218–234.

[3] G. B. Folland, *Harmonic Analysis in Phase Space*, Princeton Univ. Press, Princeton, NJ, 1989.

[4] C. Heil and A. M. Powell, *Gabor Schauder bases and the Balian–Low theorem*, J. Math. Phys. 47 (2006), no. 11, art. 113506, 21 pp.

[5] R. Hunt, B. Muckenhoupt and R. Wheeden, *Weighted norm inequalities for the conjugate function and Hilbert transform*, Trans. Amer. Math. Soc. 176 (1973), 227–251.

[6] M. Nielsen, *On stability of finitely generated shift-invariant systems*, J. Fourier Anal. Appl. 16 (2010), 901–920.

[7] M. Nielsen and H. Šikić, *Schauder bases of integer translates*, Appl. Comput. Harmon. Anal. 23 (2007), 259–262.

[8] R. Radha and S. Adhikari, *Frames and Riesz bases of twisted shift-invariant spaces in* $L^2(\mathbb{R}^{2n})$, J. Math. Anal. Appl. 434 (2016), 1442–1461.

[9] R. Radha and S. Adhikari, *Shift-invariant spaces with countably many mutually orthogonal generators on the Heisenberg group*, arXiv:1711.06902v2 (2017).

[10] S. Thangavelu, *Harmonic Analysis on the Heisenberg Group*, Birkhäuser, Boston, 1998.

[11] A. Zygmund, *Trigonometric Series*, 3rd ed., Cambridge Univ. Press, Cambridge, 2002.

S. Arati, R. Radha (corresponding author)
Department of Mathematics
Indian Institute of Technology Madras
Chennai 600 036, India
E-mail: aratishashi@gmail.com
radharam@iitm.ac.in