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A generalization of the converse of Brolin's theorem

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To the memory of Józef Siciak

Abstract. We prove a generalization of Lopes's theorem, that is, of the converse of Brolin's theorem.

1. Introduction. Let f be a rational function on $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ of degree $d > 1$. The *Julia set* $J(f)$ of f is defined as the set of all non-normality points in $\mathbb{P}^1$ of the family $(f^n)_{n\in\mathbb{N}}$, and the *Fatou set* $F(f)$ of f as $\mathbb{P}^1 \setminus J(f)$. A component of $F(f)$ is called a *Fatou component* of f. Both $F(f)$ and $J(f)$ are totally invariant under f. A Fatou component of f is properly mapped by f to a Fatou component of f, and the preimage of a Fatou component of f under f consists of (at most d) Fatou components of f. Let ω be the Fubini–Study area element on $\mathbb{P}^1$ normalized so that $\omega(\mathbb{P}^1) = 1$. Then the weak limit

$$\mu_f := \lim_{n\to\infty} \frac{(f^n)^*\omega}{d^n}$$

exists on $\mathbb{P}^1$, has no atoms in $\mathbb{P}^1$, and charges no polar subsets in $\mathbb{P}^1$. This probability measure μ_f is called *the equilibrium* or *the* (*non-exceptional*) *balanced* measure of f on $\mathbb{P}^1$, and is in fact the unique probability measure ν on $\mathbb{P}^1$ such that $f^*\nu = d \cdot \nu$ on $\mathbb{P}^1$ and $\nu(E(f)) = 0$, where the exceptional set $E(f) := \{a \in \mathbb{P}^1 : f^{-2}(a) = \{a\}\}$ of f consists of at most two points in $\mathbb{P}^1$ (for any $a \in E(f)$, the probability measure $\nu_a := (\delta_a + \delta_{f(a)})/2$ on $\mathbb{P}^1$ also satisfies $f^*\nu_a = d \cdot \nu_a$ on $\mathbb{P}^1$, but $\nu_a(\{a\}) > 0$). Using Montel's theorem, we have $J(f) = \operatorname{supp} \mu_f$, so in particular $J(f)$ is non-polar.

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When $\infty \in F(f)$ or equivalently when $J(f)$ is a compact subset in $\mathbb{C}$, let us denote by $D_\infty = D_\infty(f)$ the Fatou component of f containing ∞, and by $\nu_\infty = \nu_{D_\infty,\infty}$ the *harmonic measure* of D_∞ with pole ∞, which is a probability measure on ∂D_∞. The measure ν_∞ exists since $J(f)$ is non-polar as mentioned above.

Our aim in this short article is to prove the following.

THEOREM 1. *Let f be a rational function on $\mathbb{P}^1$ of degree $d > 1$, and suppose that $\infty \in F(f)$. Then the following conditions are equivalent:*

(i) *f^2 is a polynomial.*
(ii) *$\mu_f = \nu_\infty$ on $\mathbb{P}^1$.*

The implication (i)$\Rightarrow$(ii) follows from the work of Brolin [1], so we will show the converse (ii)$\Rightarrow$(i).

The condition (i) holds if and only if

(i)$'$ *either f is a polynomial, or $f^{-2}(\infty) = \{\infty\} \not\subset f^{-1}(\infty)$;*

the latter possibility never occurs if $f(D_\infty) = D_\infty$, and in fact occurs if and only if f *has the form $a(z-b)^{-d} + b$ for some $a \in \mathbb{C}^*$ and some $b \in \mathbb{C}$*. In particular, the iteration order 2 of f in (i) is best possible.

The implication (ii)$\Rightarrow$(i) in Theorem 1 was first claimed by Oba and Pitcher [7], also assuming $f(\infty) = \infty$ (so $f(D_\infty) = D_\infty$) and $f'(\infty) = 0$. It was established by Lopes [4] (see also Lalley [3] and Mañé–da Rocha [5]) under a relaxed additional assumption $f(\infty) = \infty$, and by the present authors [8, Theorem 1] under $f(D_\infty) = D_\infty$.

In the proof of Theorem 1, we will first improve some part (see Claim 1 below) in [8, proof of (ii)$\Rightarrow$(i) in Theorem 1] and then give a simple proof of the remaining part, using the following pleasant theorem due to Orevkov.

THEOREM 1.1 (Orevkov [9, a consequence of Corollary 1]). *For every $P \in \mathbb{C}[z]$ of degree > 0, the lemniscate $\{z \in \mathbb{C} : |P(z)| = 1\}$ is an irreducible real-algebraic curve in $\mathbb{C} \cong \mathbb{R}^2$ (identifying $z \in \mathbb{C}$ with $(\Re z, \Im z) \in \mathbb{R}^2$), that is, the lemniscate coincides with the zero set $\{(x,y) \in \mathbb{R}^2 : \phi(x,y) = 0\}$ of some $\phi(x,y) \in \mathbb{R}[x,y]$ irreducible as an element of $\mathbb{C}[x,y]$.*

We conclude this section with some background material. For more details, see e.g. the books [10, 6].

Potential theory. For every probability measure ν on $\mathbb{C}$ having compact support, let p_ν be the (*logarithmic*) *potential* of ν on $\mathbb{C}$ with pole ∞ so that $p_\nu(z) = \int_{\mathbb{C}} \log|z-w|\nu(w)$ on $\mathbb{C}$. Then $dd^c p_\nu = \nu - \delta_\infty$ on $\mathbb{P}^1$ (regarding p_ν as a δ-subharmonic function on $\mathbb{P}^1$) and $p_\nu(z) = \log|z| + O(|z|^{-1})$ as $z \to \infty$. We also set

$$I_\nu := \int_{\mathbb{C}} p_\nu \nu \in \mathbb{R} \cup \{-\infty\},$$

which is called the *energy* of ν with pole ∞.

Let D be a domain in $\mathbb{P}^1$ containing ∞ such that $\mathbb{C}\setminus D$ is non-polar (i.e., $I_\nu > -\infty$ for some probability measure ν supported by $\mathbb{C}\setminus D$). The *equilibrium mass distribution* on $\mathbb{C}\setminus D$ with pole ∞ is the unique probability measure ν on $\mathbb{C}\setminus D$ such that $p_\nu \ge I_\nu$ on $\mathbb{C}$, $p_\nu > I_\nu$ on D, and $p_\nu \equiv I_\nu$ on $\mathbb{C}\setminus D$ except for some (possibly empty) F^σ-polar subset in ∂D. This probability measure is supported by ∂D and coincides with the *harmonic measure* $\nu_{D,\infty}$ of D with pole ∞.

Complex dynamics. Let f be a rational function on $\mathbb{P}^1$ of degree $d > 1$. An ordered pair $F(z_0,z_1) = (F_0(z_0,z_1), F_1(z_0,z_1)) \in (\mathbb{C}[z_0,z_1]_d)^2$ of homogeneous polynomials F_0, F_1 of degree d is called a (non-degenerate homogeneous) *lift* of f if $f(z) = F_1(1,z)/F_0(1,z)$. Such an F is unique up to multiplication in $\mathbb{C}^*$. We note that for every $n \in \mathbb{N}$, $\deg(f^n) = d^n$, and the nth iterate F^n of F, which is written as

$$F^n(z_0,z_1) = \big(F_0^{(n)}(z_0,z_1), F_1^{(n)}(z_0,z_1)\big) \in (\mathbb{C}[z_0,z_1]_{d^n})^2,$$

is a lift of f^n. Let $\|\cdot\|$ be the Euclidean norm on $\mathbb{C}^2$. The limit

$$G^F := \lim_{n\to\infty} \frac{\log\|F^n\|}{d^n} \quad \text{on } \mathbb{C}^2\setminus\{(0,0)\}$$

exists and the convergence is uniform on $\mathbb{C}^2\setminus\{(0,0)\}$ by the homogeneity of F. The function G^F is continuous and plurisubharmonic on $\mathbb{C}^2\setminus\{(0,0)\}$, and is called the *escape rate function* of F on $\mathbb{C}^2\setminus\{(0,0)\}$. We have the equality

$$G^F(cZ) = G^F(Z) + \log|c| \tag{1.1}$$

for every $c \in \mathbb{C}^*$ and every $Z \in \mathbb{C}^2\setminus\{(0,0)\}$, the equality

$$G^F \circ F = d\cdot G^F$$

on $\mathbb{C}^2\setminus\{(0,0)\}$, the identity

$$G^{cF} = G^F + \frac{\log|c|}{d-1} \quad \text{on } \mathbb{C}^2\setminus\{(0,0)\}$$

for every $c\in\mathbb{C}^*$, and the equality $dd^c G^F(1,\cdot) = \mu_f - \delta_\infty$ on $\mathbb{P}^1$ (regarding $G^F(1,\cdot)$ as a δ-subharmonic function on $\mathbb{P}^1$). We also note that for every $n\in\mathbb{N}$, $G^{F^n} = G^F$ on $\mathbb{C}^2\setminus\{(0,0)\}$ and $\mu_{f^n} = \mu_f$ on $\mathbb{P}^1$.

2. Proof of Theorem 1. Let f be a rational function on $\mathbb{P}^1$ of degree $d > 1$. As already mentioned in Section 1, we only need to show the implication (ii)⇒(i).

Fix a lift F of f, and suppose that $\infty \in F(f)$.

Preliminary lemmas. Let us recall the following from [8, §3].

LEMMA 2.1. *The potential p_{μ_f} is continuous on $\mathbb{C}$. More precisely,*

$$G^F(1,\cdot) = p_{\mu_f} + G^F(0,1) \quad \textit{on } \mathbb{C}. \tag{2.1}$$

Proof. Regarding $G^F(1,\cdot)$ and p_{μ_f} as δ-subharmonic functions on $\mathbb{P}^1$, we have $dd^cG^F(1,\cdot) = dd^c p_{\mu_f}(= \mu_f - \delta_\infty)$ on $\mathbb{P}^1$, so that $G^F(1,\cdot) - p_{\mu_f} \equiv C$ on $\mathbb{P}^1$ for some $C \in \mathbb{R}$. Moreover, $C = \lim_{z\to\infty}(G^F(1/z,1) - (p_{\mu_f} - \log|z|)) = G^F(0,1)$, where we have used (1.1). ∎

Lemma 2.2 (the pullback of p_{μ_f} under f). *For every $n \in \mathbb{N}$,*

$$\text{(2.2)}\quad p_{\mu_f} \circ f^n + \log|F_0^{(n)}(1,\cdot)| = d^n \cdot p_{\mu_f} + (d^n - 1)G^F(0,1) \quad \text{on } \mathbb{C} \setminus f^{-n}(\infty).$$

Proof. Without loss of generality, we can assume that $n = 1$. Using (2.1), we compute $p_{\mu_f} \circ f$ as

$$\begin{aligned} p_{\mu_f} \circ f &= G^F(1, f(\cdot)) - G^F(0,1) \\ &= G^F \circ F - \log|F_0(1,\cdot)| - G^F(0,1) \\ &= d \cdot G^F - \log|F_0(1,\cdot)| - G^F(0,1) \\ &= d(p_{\mu_f} + G^F(0,1)) - \log|F_0(1,\cdot)| - G^F(0,1) \\ &= d \cdot p_{\mu_f} - \log|F_0(1,\cdot)| + (d-1)G^F(0,1) \end{aligned}$$

on $\mathbb{C} \setminus f^{-1}(\infty)$. ∎

Proof of (ii)⇒(i)**.** Suppose now that $\mu_f = \nu_\infty$ on $\mathbb{P}^1$ (then $p_{\mu_f} \equiv I_{\mu_f}$ on $\mathbb{C} \setminus D_\infty$). If f is a polynomial, then so is f^2, and we are done. Hence suppose that f is not a polynomial, that is, $F_0(1,\cdot)$ is non-constant on $\mathbb{C}$.

Claim 1. *Either $D_\infty = F(f)$, or f^2 is a polynomial.*

Proof. By (2.2) for $n = 1$ and the identity $p_{\mu_f} \equiv I_{\mu_f}$ on $\mathbb{C} \setminus D_\infty$, we have

$$\text{(2.3)}\qquad |F_0(1,\cdot)| \equiv e^{(d-1)(I_{\mu_f} + G^F(0,1))} \quad \text{on } \mathbb{C} \setminus (D_\infty \cup f^{-1}(D_\infty)).$$

Since $F(f)$ is totally invariant under f, we have $F(f) \supset D_\infty \cup f^{-1}(D_\infty)$. We claim that $F(f) = D_\infty \cup f^{-1}(D_\infty)$; for, otherwise, there is a Fatou component of f in $F(f) \setminus (D_\infty \cup f^{-1}(D_\infty))$. This is impossible by (2.3), since $F_0(1,\cdot)$ is non-constant on $\mathbb{C}$.

Hence $F(f) = D_\infty \cup f^{-1}(D_\infty)$. Then by (2.3) (and since $F_0(1,\cdot)$ is non-constant on $\mathbb{C}$), the Fatou component $f(D_\infty)$ is either D_∞ or a component of $f^{-1}(D_\infty)$. In the latter case, we have $f^2(D_\infty) = D_\infty$.

Hence in any case, $f^2(D_\infty) = D_\infty$, or equivalently $D_\infty \subset f^{-2}(D_\infty)$. From (2.2) for $n = 2$ and $p_{\mu_f} \equiv I_{\mu_f}$ on $\mathbb{C} \setminus D_\infty$, we also have

$$\text{(2.3}'\text{)}\qquad |F_0^{(2)}(1,\cdot)| \equiv e^{(d^2-1)(I_{\mu_f} + G^F(0,1))} \quad \text{on } \mathbb{C} \setminus f^{-2}(D_\infty).$$

Now, if $f^{-2}(D_\infty) \neq D_\infty$, then there is a Fatou component, say U, of f in $f^{-2}(D_\infty) \setminus D_\infty$, so $F(f) \setminus f^{-2}(D_\infty) \supset f^{-2}(U) \neq \emptyset$. Then by (2.3′), $F_0^{(2)}(1,\cdot)$ is constant on $\mathbb{C}$, that is, f^2 is a polynomial. Alternatively, suppose

$f^{-2}(D_\infty) = D_\infty$. Then by (2.3′), we have $|F_0^{(2)}(1,\cdot)| \equiv e^{(d^2-1)(I_{\mu_f}+G^F(0,1))}$ on $\mathbb{C} \setminus D_\infty$, so that $F_0^{(2)}(1,\cdot)$ is constant on $\mathbb{C}$ unless $F(f) = D_\infty$. ∎

Choose $c \in \mathbb{C}^*$ satisfying $|c| = e^{-(d-1)(I_{\mu_f}+G^F(0,1))}$, so that

$$G^{cF}(0,1) = -I_{\mu_f}. \tag{2.4}$$

Then for every $n \in \mathbb{N}$, by (2.2) applied to the lift cF of f and the identity $p_{\mu_f} \equiv I_{\mu_f}$ on $\mathbb{C} \setminus D_\infty$, the lemniscate

$$L_{(cF)^n} := \{z \in \mathbb{C} : |(cF)_0^{(n)}(1,z)| = 1\}$$

contains $J(f)$ $(\subset \mathbb{C} \setminus F(f) \subset \mathbb{C} \setminus (D_\infty \cup f^{-n}(D_\infty)) \subset \mathbb{C} \setminus f^{-n}(\infty))$.

CLAIM 2. *For every $n \in \mathbb{N}$, $L_{(cF)^n} = L_{cF}$. Moreover, $f(L_{cF}) \subset L_{cF}$.*

Proof. For every $n \in \mathbb{N}$, the non-polar (so infinite) set $J(f)$ is contained in both L_{cF} and $L_{(cF)^n}$. Hence, identifying $z \in \mathbb{C}$ with $(\Re z, \Im z) \in \mathbb{R}^2$, we have

$$L_{cF} = \{(x,y) \in \mathbb{R}^2 : \phi(x,y) = 0\} = L_{(cF)^n}$$

for some $\phi(x,y) \in \mathbb{R}[x,y]$ irreducible as an element of $\mathbb{C}[x,y]$, by Orevkov's theorem (Theorem 1.1) and the Bézout theorem (see, e.g., [2, §I.7]; for our purpose, a more elementary [11, p. 4, Lemma] is enough). Hence the former assertion holds. Moreover, for every $z \in \mathbb{C}$, we have $(cF)_0^{(2)}(1,z) = (cF_0)((cF)_0(1,z),(cF)_1(1,z)) = (cF)_0(1,f(z)) \cdot ((cF)_0(1,z))^d$. Hence for every $z \in L_{cF} (= L_{(cF)^2})$, we have $1 = |(cF)_0(1,f(z))| \cdot 1^d$, that is, $f(L_{cF}) \subset L_{cF}$. Hence the latter assertion holds. ∎

Suppose that $D_\infty = F(f)$. Then

CLAIM 3. *$L_{cF} \cap F(f) \neq \emptyset$.*

Proof. Otherwise, we must have $L_{cF} = J(f)$, so that $F(f)$ has also a bounded component (intersecting $f^{-1}(\infty)$). This contradicts the assumption $D_\infty = F(f)$ just above. ∎

Pick $z_0 \in L_{cF} \cap F(f)$, which is possible by Claim 3. Then for every $n \in \mathbb{N}$, $z_0 \in L_{(cF)^n}$ $(\subset \{z \in \mathbb{C} : |(cF)_0^{(n)}(1,z)| \neq 0\} = \mathbb{C} \setminus f^{-n}(\infty))$ by the former half of Claim 2, and $p_{\mu_f}(z_0) > I_{\mu_f}$ by $z_0 \in F(f)$. Hence also by (2.2) (for each $n \in \mathbb{N}$) applied to the lift cF of f and (2.4), we must have

$$p_{\mu_f}(f^n(z_0)) - I_{\mu_f} = d^n(p_{\mu_f}(z_0) - I_{\mu_f}) \to \infty \quad \text{as } n \to \infty.$$

On the other hand, by the latter half of Claim 2, the compactness of L_{cF} in $\mathbb{C}$, and the upper semicontinuity of p_{μ_f} on $\mathbb{C}$, we also have

$$\limsup_{n\to\infty}(p_{\mu_f}(f^n(z_0)) - I_{\mu_f}) \le \sup_{L_{cF}}(p_{\mu_f} - I_{\mu_f}) < \infty.$$

This is a contradiction.

Hence the latter possibility $D_\infty = F(f)$ in Claim 1 never occurs, so f^2 is a polynomial. Now the proof is complete. ■

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