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Realizing spaces as path-component spaces

by

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Abstract. The path-component space $\pi_0(X)$ of a topological space X is the quotient space of X whose points are the path components of X. We show that every Tychonoff space X is the path-component space of a Tychonoff space Y of weight $w(Y) = w(X)$ such that the natural quotient map $Y \to \pi_0(Y) = X$ is a perfect map. Hence, many topological properties of X transfer to Y. We apply this result to construct a compact space $X \subset \mathbb{R}^3$ for which the fundamental group $\pi_1(X, x_0)$ is an uncountable T_4 topological group but the canonical homomorphism $\psi : \pi_1(X, x_0) \to \check{\pi}_1(X, x_0)$ to the first shape homotopy group is trivial.

1. Introduction. The *path-component space* $\pi_0(X)$ of a topological space X is the quotient space of X whose points are the path components of X. We let $q_X : X \to \pi_0(X)$ denote the natural quotient map identifying each path component to a point. It is a beautiful and surprising result of Douglas Harris that every topological space is a path-component space [19]. In particular, for every space X, Harris constructed a paracompact Hausdorff space $H(X)$ and a natural homeomorphism $\pi_0(H(X)) \cong X$. Harris' result plays a key role in the proof that every topological group is isomorphic to the topological fundamental group of some space [6]. Harris' result is used in a similar fashion to prove a topological group analogue of the Nielsen–Schreier Theorem [7].

A remaining problem of relevance to topological fundamental groups is: given a particular class of spaces $\mathcal{C}$, identify a subclass $\mathcal{D}$ such that for every $Y \in \mathcal{C}$, we have $Y \cong \pi_0(X)$ for some $X \in \mathcal{D}$. For instance, Banakh, Vovk, and Wójcik [3] show that every first countable space X is the path-component space of a complete metric space $\circledast X$ called the *cobweb* of X.

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The construction of $\circledast X$ is actually quite similar to that of $H(X)$; however, it is not necessarily compact or separable when X is. In the current paper, we prove the following theorem using techniques quite different from those in [3] and [19].

Theorem 1.1. *Every Tychonoff space X is the path-component space of a Tychonoff space Y of weight $w(Y) = w(X)$ such that $q_Y : Y \to X$ is perfect.*

A topological property $\mathscr{P}$ is an *inverse invariant of perfect maps* if whenever $f : Y \to X$ is a perfect map and $A \subseteq X$ has property $\mathscr{P}$, then so does the preimage $f^{-1}(A)$. Such properties are numerous and include compactness, paracompactness, metrizability, Lindelöf number, Čech completeness, Borel type, etc. We refer to [11, Section 3.7] for more on perfect maps as well as the column of inverse invariants of perfect maps in [11, table, p. 510].

Corollary 1.2. *Every Tychonoff space X is the path-component space of a Tychonoff space Y such that Y shares every topological property of X that is an inverse invariant of perfect maps.*

The path-component space of a based loop space $\Omega(X, x_0)$ is the fundamental group $\pi_1(X, x_0)$ equipped with its natural quotient topology. In general, $\pi_1(X, x_0)$ need not be a topological group [13, 14]; however, it is a quasitopological group in the sense that inversion is continuous and multiplication is continuous in each variable [5]. We refer to [2] for basic theory of quasitopological groups.

Since multiplication in $\pi_1(X, x_0)$ need not be continuous, one cannot take separation axioms for granted. While $T_0 \Leftrightarrow T_1$ holds for all quasitopological groups, it need not be the case that T_1 implies T_2 or that T_2 implies higher separation axioms. The relevance of separation axioms in fundamental groups was noted in [8] where it is shown that if $\pi_1(X, x_0)$ is T_1, then X has the homotopically path Hausdorff property introduced in [15]. The converse holds if X is locally path-connected. Furthermore, if $\pi_1(X, x_0)$ is T_1, then X admits a generalized universal covering in the sense of [17].

To verify that $\pi_1(X, x_0)$ is Hausdorff, it is sufficient to know that X is *π_1-shape injective*, meaning the canonical homomorphism $\psi : \pi_1(X, x_0) \to \check{\pi}_1(X, x_0)$ to the first shape homotopy group is injective. Motivated by a construction in [5], we construct what is apparently the first example of a compact metric space for which $\pi_1(X, x_0)$ has been verified to be Hausdorff but for which ψ is not injective. In doing so, we show π_1-shape injectivity is, in a strong way, not a necessary condition for the T_4 separation axiom in fundamental groups of compact metric spaces.

Theorem 1.3. *There exists a compact metric space $X \subset \mathbb{R}^3$ such that $\pi_1(X, x_0)$ is an uncountable T_4 topological group and $\psi : \pi_1(X, x_0) \to \check{\pi}_1(X, x_0)$ is the trivial homomorphism.*

2. Path-component spaces. If $f : X \to Y$ is a map, then $f(q_X(x)) \subseteq q_Y(f(x))$ for all $x \in X$. Thus f determines a well-defined map $f_0 : \pi_0(X) \to \pi_0(Y)$ given by $f_0(q_X(x)) = q_Y(f(x))$. Moreover, since q_X is quotient, f_0 is continuous. Altogether, we obtain an endofunctor of the usual category **Top** of topological spaces. From the following basic examples, we begin to see that separation axioms are not passed to path-component spaces even if X is a compact metric space.

Example 2.1. If $P_1 = \{0\} \times [-1, 1]$ and $P_2 = \{(x, -\sin(\pi/x)) \mid x \in (0, 1]\}$ so that $T = P_1 \cup P_2$ is the closed topologists sine curve, then $\pi_0(X) = \{P_1, P_2\}$ is the two-point Sierpiński space with topology $\{\emptyset, \{P_2\}, X\}$. Thus $\pi_0(X)$ is T_0 but not T_1.

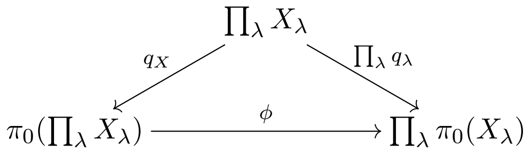
Example 2.2. Let $X \subseteq \mathbb{R}^2$ be the classical Brouwer–Janiszewski–Knaster bucket handle continuum. The *composant* of a point $x \in X$ is the union of all proper subcontinua containing x. The path components of the bucket handle X are precisely the composants of X, of which there are uncountably many [1]. Since every composant of a continuum is dense in that continuum (see [20, Ch. 5, §48.VI, Theorem 2]), it follows that $\pi_0(X)$ is uncountable and indiscrete.

Example 2.3. Let L be a linearly ordered space and $X = L \times [0, 1]$ have the order topology induced by the lexicographical ordering. A straightforward application of the intermediate value theorem shows that the sets $\{\ell\} \times [0, 1]$, $\ell \in L$, are the path components of X. Hence $\pi_0(X) \cong L$ where q_X may be identified with the projection onto the first coordinate. Taking $L = [0, 1]$ provides an example of a non-metrizable X for which $\pi_0(X)$ is Hausdorff and admits non-constant paths.

To prove our main theorem, we consider path-component spaces of products.

Proposition 2.4. *Let $\{X_\lambda\}$ be a family of spaces and $X = \prod_\lambda X_\lambda$. Let $q_\lambda : X_\lambda \to \pi_0(X_\lambda)$ and $q_X : X \to \pi_0(X)$ be the canonical quotient maps. There is a natural continuous bijection $\phi : \pi_0(X) \to \prod_\lambda \pi_0(X_\lambda)$ such that $\phi \circ q_X = \prod_\lambda q_\lambda$, which is a homeomorphism if and only if $\prod_\lambda q_\lambda$ is a quotient map.*

Proof. The fibers of $\prod_\lambda q_\lambda$ are precisely the path components of X. Hence, $\phi(q_X(x)) = (q_\lambda(x))$ is a well-defined bijection.



The continuity of ϕ and the final statement of the lemma follow directly from the fact that q_X is quotient, ϕ is a bijection, and $\phi \circ q_X = \prod_\lambda q_\lambda$. ∎

Corollary 2.5. *If, for all λ, X_λ is compact and $\pi_0(X_\lambda)$ is Hausdorff, then $\phi : \pi_0(\prod_\lambda X_\lambda) \to \prod_\lambda \pi_0(X_\lambda)$ is a homeomorphism.*

Proof. Given the assumptions, $\prod_\lambda X_\lambda$ is compact and $\prod_\lambda \pi_0(X_\lambda)$ is Hausdorff. Thus $\prod_\lambda q_\lambda$ is a quotient map and we may apply the final statement of Proposition 2.4. ∎

Definition 2.6. We say a map $f : X \to Y$ is *perfect* if X is Hausdorff, f is a closed map, and the point-preimage $f^{-1}(y)$ is compact for each $y \in Y$.

Certainly every onto perfect map is quotient. Moreover, a perfect map $f : X \to Y$ has the property that if $S \subseteq Y$, then the restriction $f|_{f^{-1}(S)} : f^{-1}(S) \to S$ is a perfect map [11, Proposition 3.7.6]. If we use this observation, the following lemma becomes evident.

Lemma 2.7. *Suppose A is a compact Hausdorff space such that $\pi_0(A)$ is Hausdorff and $X \subseteq \pi_0(A)$. If $Y = q_A^{-1}(X)$, then $q_Y : Y \to X$ is a perfect map and $\pi_0(Y) \cong X$.*

3. Realizing Tychonoff spaces as path-component spaces. This section is dedicated to proving Theorem 1.1 as well as a functorial version of the construction. We begin by realizing $[0,1]$ as a path-component space.

Let $C = \{\sum_{i=1}^{\infty} x_i/3^i \mid x_i \in \{0,2\}\}$ be the standard middle third Cantor set and $\mathscr{C}$ be the set of connected components of $[0,1]\setminus C$. Let $\{D_1, D_2, \dots\}$ be an enumeration of $\mathscr{C}$ and define

$$K = (C \times [0,1]) \cup \bigcup_{i=1}^{\infty} D_i \times \{n(i)\}$$

where $n(i) = 0$ if i is even and $n(i) = 1$ if i is odd. With a certain enumeration of $\mathscr{C}$, one can picture K as in Figure 1.

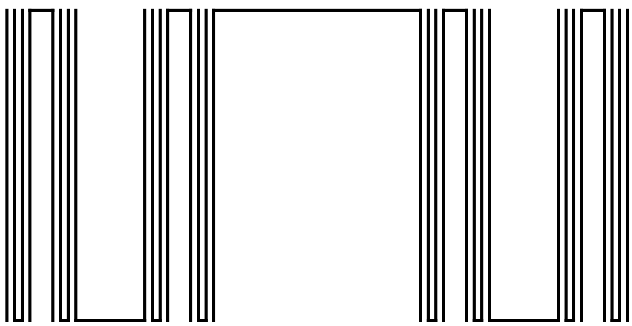


Fig. 1. The space K

Clearly, $K = \bigcap_n K_n$ for $K_n = (I \setminus \bigcup_{i=1}^n D_i) \times I \cup \bigcup_{i=1}^n D_i \times \{n(i)\}$. Since each K_n is a contractible, compact CW-complex, K is a compact and shape equivalent to a point.

Lemma 3.1. *$\pi_0(K) \cong [0,1]$ where $q_K : K \to \pi_0(K)$ is a perfect map.*

Proof. The path components of the space K are sets of the form $D_i \times \{n(i)\} \cup \partial D_i \times [0,1]$ for $i \in \mathbb{N}$ and $\{x\} \times [0,1]$ for $x \in C \setminus \bigcup_{i=1}^{\infty} \overline{D}_i$. Let $p : K \to [0,1]$ be the projection onto the first coordinate and $c : [0,1] \to [0,1]$ be the standard Cantor ternary function which is onto, constant on each interval $\overline{D}_i$, and injective on $C \setminus \bigcup_{i=1}^{\infty} \overline{D}_i$. Since p and c are onto perfect maps, so is $c \circ p$. Moreover, the fibers of $c \circ p$ are precisely the path-components of K. Hence, there is a canonical homeomorphism $g : \pi_0(K) \to [0,1]$ such that $g \circ q_K = c \circ p$. It follows that q_K is perfect. ∎

Example 3.2. The space K may be used to construct reasonably nice spaces whose path-component spaces satisfy only weak separation axioms. The space $K' = (K \times [0,1]) \setminus (\{1\} \times [0,1] \times (0,1)) \subset \mathbb{R}^3$ is not compact but is a separable metric space. The path components of K' are precisely the sets:

(1) $C \times [0,1]$ where $C \subset [0,1) \times [0,1]$ is a path component of K,
(2) $C_0 = \{1\} \times [0,1] \times \{0\}$,
(3) $C_1 = \{1\} \times [0,1] \times \{1\}$.

From Lemma 3.1, it follows that $\pi_0(K') \cong [0,1] \times \{0,1\}/\sim$ where $(t,0) \sim (t,1)$ if $t < 1$. Hence, $\pi_0(K')$ is the T_1 but non-Hausdorff "closed unit interval with two copies of 1."

Recall that the *weight* $w(X)$ of a topological space X is the minimal cardinality of a basis generating the topology of X.

Proof of Theorem 1.1. Suppose X is a Tychonoff space of weight $\mathbf{m} = w(X)$. Recall that for any cardinal $\mathbf{m} \geq \aleph_0$, the direct product $[0,1]^{\mathbf{m}}$ (of weight $\mathbf{m}$) is universal for all Tychonoff spaces X of weight $\mathbf{m}$ [11, 2.3.23], i.e. X homeomorphically embeds as a subspace of $[0,1]^{\mathbf{m}}$. Hence, we may identify X as a subspace of $[0,1]^{\mathbf{m}}$. Recalling the compact space $K \subset [0,1]^2$ constructed above, we may identify $\pi_0(K) = [0,1]$ by Lemma 3.1. By Corollary 2.5, the map $Q = (q_K)^{\mathbf{m}} : K^{\mathbf{m}} \to [0,1]^{\mathbf{m}}$ is a perfect map whose fibers are the path components of $K^{\mathbf{m}}$ and the canonical bijection $\phi : \pi_0(K^{\mathbf{m}}) \to [0,1]^{\mathbf{m}}$ such that $\phi \circ q_{K^{\mathbf{m}}} = Q$ is a homeomorphism.

$$\begin{array}{ccc} & K^{\mathbf{m}} & \\ {\scriptstyle q_{K^{\mathbf{m}}}} \swarrow & & \searrow {\scriptstyle Q} \\ \pi_0(K^{\mathbf{m}}) & \xrightarrow{\quad \phi \quad} & [0,1]^{\mathbf{m}} \end{array}$$

Set $Y = Q^{-1}(X) \subseteq K^{\mathbf{m}}$. Note that since K is second countable, $w(Y) \leq w(K^{\mathbf{m}}) = \mathbf{m} = w(X)$. Since the restriction $Q|_Y : Y \to X$ is an onto perfect map, we have $w(X) \leq w(Y)$ [11, 3.7.19]. Hence $w(Y) = w(X)$. By Lemma 2.7, $q_Y : Y \to \pi_0(Y)$ is a perfect map and $\pi_0(Y) \cong \phi^{-1}(X)$. Hence $\pi_0(Y) \cong X$. ∎

The construction of Y in the proof of Theorem 1.1 is not functorial since a choice of basis for the topology of X is required for the embedding $X \subseteq [0,1]^{\mathbf{m}}$. We now show that if one is willing to give up the equality $w(Y) = w(X)$, the construction can be made functorial.

Theorem 3.3. *Let* **Tych** $\subset$ **Top** *be the full subcategory of Tychonoff spaces. Then there is a functor* $\Phi : \mathbf{Tych} \to \mathbf{Tych}$ *and a natural isomorphism* $\pi_0 \circ \Phi \cong \mathrm{Id}_{\mathbf{Tych}}$ *such that the maps* $q_{\Phi(X)} : \Phi(X) \to \pi_0(\Phi(X))$ *are onto perfect maps.*

Proof. Let X be a Tychonoff space and $C(X, I)$ denote the set of all continuous functions $X \to [0,1]$. Since this set of functions separates points and closed sets, the natural map $i_X : X \to [0,1]^{C(X,I)}$, $i(x)(f) = f(x)$ to the direct product is an embedding. For convenience, we identify $X = i(X)$. We follow the same line of argument as in the proof of Theorem 1.1 except that we replace $[0,1]^{\mathbf{m}}$ and $K^{\mathbf{m}}$ with the compact spaces $[0,1]^{C(X,I)}$ and $K^{C(X,I)}$ respectively. The product map

$$Q = (q_K)^{C(X,I)} : K^{C(X,I)} \to [0,1]^{C(X,I)}$$

is perfect and we define $\Phi(X) = Q^{-1}(X)$. It follows that $\pi_0(\Phi(X)) \cong X$ where $q|_{\Phi(X)} : \Phi(X) \to \pi_0(\Phi(X))$ is a perfect map.

To construct Φ on morphisms, suppose $h : X \to Y$ is a map of Tychonoff spaces. Let $m : K^{C(X,I)} \to K^{C(Y,I)}$ and $M : [0,1]^{C(X,I)} \to [0,1]^{C(Y,I)}$ be the canonical induced maps. Set $A = K^{C(X,I)}$ and $B = K^{C(Y,I)}$. The naturality of the embeddings i_X and quotient maps q_X indicates that the following diagram commutes:

$$\begin{array}{ccc}
K^{C(X,I)} & \xrightarrow{\ m\ } & K^{C(Y,I)} \\
{\scriptstyle q_A}\downarrow & & \downarrow{\scriptstyle q_B} \\
[0,1]^{C(X,I)} & \xrightarrow{\ M\ } & [0,1]^{C(Y,I)} \\
{\scriptstyle i_X}\uparrow & & \uparrow{\scriptstyle i_Y} \\
X & \xrightarrow{\ f\ } & Y
\end{array}$$

Since $M(i_X(X)) = i_Y(f(X)) \subseteq i_Y(Y)$, we have

$$q_B(m(\Phi(X))) = q_B(m(q_A^{-1}(i(X)))) = M(i_X(X)) = i_Y(f(X)) \subseteq i_Y(Y)$$

and thus $m(\Phi(X)) \subseteq q_B^{-1}(i_Y(Y)) = \Phi(Y)$. Hence, we define $\Phi(h) : \Phi(X) \to \Phi(Y)$ to be the restriction of m to $\Phi(X)$. The remaining details of confirming functorality of Φ and naturality of the homeomorphisms $\pi_0(\Phi(X)) \to X$ are straightforward. ∎

4. Application to topologized fundamental groups

DEFINITION 4.1 (Markov [22]). The *free Markov topological group* $F_M(Y)$ on the space Y is the topological group equipped with a map $\sigma_M : Y \to F_M(Y)$ such that every map $f : Y \to G$ to a topological group G induces a unique continuous homomorphism $\widetilde{f} : F_M(Y) \to G$ such that $\widetilde{f} \circ \sigma_M = f$.

DEFINITION 4.2 (Graev [18]). The *free Graev topological group* $F_G(Y, y)$ on the based space (Y, y) is the topological group equipped with a map $\sigma_G : Y \to F_G(Y, y)$ taking y to the identity element e such that every map $f : (Y, y) \to (G, 1_G)$ to a topological group G induces a unique continuous homomorphism $\widetilde{f} : F_G(Y, y) \to G$ such that $\widetilde{f} \circ \sigma_G = f$.

REMARK 4.3. One can show that the free Markov (Graev) topological group exists for all (based) spaces Y using categorical methods [23]. The underlying group of $F_M(Y)$ is the free group on the underlying set of Y and the underlying group of $F_G(Y, y)$ is the free group on $Y \setminus \{y\}$. The groups $F_M(Y)$ and $F_G(Y, y_0)$ are Hausdorff if Y is functionally Hausdorff ([1]). The universal maps σ_M and σ_G, which are the inclusion of generators, are closed embeddings if Y is Tychonoff. The homomorphism $F_M(Y) \to F_G(Y, y)$ which factors the normal subgroup generated by $\{y\}$ is a topological quotient map. We refer to [2, 24, 25, 26] for more on the existence and structure of free topological groups.

PROPOSITION 4.4. *If Y is path-connected, then so is $F_G(Y, y)$.*

Proof. The image of $\sigma_G : Y \to F_G(Y, y)$ is path connected and algebraically generates the entire group. Since the path component of the identity in a topological group is a subgroup, this must be the entire group $F_G(Y, y)$. ∎

Recall that the path-component space $\pi_1(X, x_0) = \pi_0(\Omega(X, x_0))$ is the fundamental group equipped with the natural quotient topology. Since the groups $\pi_1(X, x_0)$ need not be Hausdorff, we do not assume any separation axioms on groups.

For an unbased space X, let $W(X) = X \times [0, 1]/X \times \{0, 1\}$ and if $x_0 \in X$, then $\Sigma X = X \times [0, 1]/(X \times \{0, 1\} \cup \{x_0\} \times [0, 1])$ is the reduced suspension. We take the image of $X \times \{0, 1\}$ to be the basepoint a_0 in both $W(X)$ and ΣX.

A topological space X is said to be a *k_ω-space* if X is the inductive limit of compact subspaces $X_1 \subset X_2 \subset \cdots$, i.e. $X = \bigcup_{n \geq 1} X_n$ and $C \subseteq X$ is

([1]) A space X is *functionally Hausdorff* if $C(Y, I)$ separates the points of Y.

closed if and only if $C \cap X_n$ is closed in X_n for all n. The next theorem follows from [5, Corollaries 1.2 and 4.23].

THEOREM 4.5. *If X is a k_ω-space, then $\pi_1(W(X), a_0)$ is naturally isomorphic to the free Markov topological group $F_M(\pi_0(X))$ on the path-component space $\pi_0(X)$.*

By combining Theorem 4.5 with Theorem 1.1, we obtain the following corollary; compare with [5, Corollary 1.2], which relies on Harris' construction and thus provides no control over compactness or metrizability.

COROLLARY 4.6. *For every Tychonoff space X, there is a Tychonoff space Y such that $w(Y) = w(X)$, $q : Y \to \pi_0(Y) = X$ is a perfect map, and*

$$\pi_1(W(Y), a_0) \cong F_M(X).$$

REMARK 4.7. For the case $X = [0,1]^n$, we need not use the full power of Theorem 1.1. In this case, we may replace Y with K^n since $\pi_0(K^n) \cong [0,1]^n$. Since $K^n \subseteq \mathbb{R}^{2n}$, the space $W(K^n)$ embeds as a subspace of $\mathbb{R}^{2n+1}$.

LEMMA 4.8 ([4, Lemma 4.4]). *If a 2-cell e^2 is attached to a path-connected space Y, then the inclusion $Y \to Y \cup e^2$ induces a homomorphism $\pi_1(Y, y_0) \to \pi_1(Y \cup e^2, y_0)$ which is a topological quotient map.*

THEOREM 4.9. *Suppose X is a compact Hausdorff space which is well-pointed at $x_0 \in X$, i.e. $\{x_0\} \to X$ is a cofibration. Then*

$$\pi_1(\Sigma X, a_0) \cong F_G(\pi_0(X), C_0)$$

where C_0 is the path component of x_0 in X.

Since $\{x_0\} \to X$ is a cofibration, ΣX is homotopy equivalent to $W(X) \cup e^2$ where the 2-cell is attached along the loop $\ell : [0,1] \to W(X)$ where $\ell(t)$ is the image of (x_0, t) in $W(X)$. Recall by Theorem 4.5 that $\pi_1(W(X), a_0) \cong F_M(\pi_0(X))$. According to Remark 4.3, there is an isomorphism of topological groups $F_G(\pi_0(X), C_0) \cong F_M(\pi_0(X))/N$ where N is the normal subgroup generated by $\{C_0\}$. The inclusion $W(X) \to W(X) \cup e^2$ induces a homomorphism $\pi_1(W(X), a_0) \to \pi_1(W(X) \cup e^2, a_0)$, which, according to Lemma 4.8, is a topological quotient map whose kernel is precisely N. Thus

$$\pi_1(W(X) \cup e^2, a_0) \cong F_G(\pi_0(X), C_0). \blacksquare$$

COROLLARY 4.10 ([5, Corollary 1.2]). *For every compact Hausdorff (resp. metrizable) space X and $x_0 \in X$, there is a path-connected compact Hausdorff (resp. metrizable) space Z and $a_0 \in Z$ such that $\pi_1(Z, a_0) \cong F_G(X, x_0)$.*

Proof. Using Theorem 1.1, construct a space Y such that $\pi_0(Y) \cong X$ and $q_Y : Y \to X$ is perfect. Pick a point $y \in Y$ and attach a copy of the unit interval to Y where $1 \sim y$. Call the resulting space Y^+ and let y_0 be the image of 0, which we take to be the basepoint. Now (Y^+, y_0) is well-pointed, and $\pi_0(Y^+) \cong \pi_0(Y) \cong X$. Set $Z = \Sigma Y^+$ and apply Theorem 4.9. ■

Example 4.11. As in Remark 4.7, we consider a more direct construction. Recalling that $\pi_0(K^n) = [0,1]^n$, pick $y_0 \in K^n$ and set $x_0 = q_{K^n}(y_0)$. Let $(K^n)^+$ be the space obtained by attaching a copy of the unit interval to K^n where $1 \sim y_0$. Now we have an isomorphism $\pi_1(\Sigma(K^n)^+, a_0) \cong F_G([0,1]^n, x_0)$. Since $(K^n)^+$ embeds in $\mathbb{R}^{2n}$, $\Sigma(K^n)^+$ embeds as a compact subspace of $\mathbb{R}^{2n+1}$. Moreover, since K^n is the intersection of contractible polyhedra, so are the spaces $(K^n)^+$ and $\Sigma(K^n)^+$. Hence, $\Sigma(K^n)^+$ is shape equivalent to a point.

The first shape homotopy group of a paracompact Hausdorff space is the inverse limit $\check{\pi}_1(X, x_0) = \varprojlim_{\mathscr{U}} \pi_1(|N(\mathscr{U})|, v_0)$ of fundamental groups of geometric realizations of nerves of open covers $\mathscr{U}$ of X. Since we only require general properties of this group, we refer the reader to [21] for details of the construction. The nerve $|N(\mathscr{U})|$ is a simplicial complex and hence has discrete fundamental group [9]. Hence $\check{\pi}_1(X, x_0)$ is naturally topologized as an inverse limit of discrete groups. It is known that there is a canonical homomorphism $\psi : \pi_1(X, x_0) \to \check{\pi}_1(X, x_0)$ which is continuous with respect to the quotient topology on $\pi_1(X, x_0)$ [8]. Hence the injectivity of ψ implies that $\pi_1(X, x_0)$ is functionally Hausdorff. For example, ψ is known to be injective for all one-dimensional spaces [10] and planar spaces [16]. However, ψ is not typically an embedding [12]. Hence, even when ψ is injective, one cannot necessarily conclude that $\pi_1(X, x_0)$ enjoys stronger separation axioms.

Proof of Theorem 1.3. Using the case $n = 1$ from Example 4.11, we obtain a compact subset $X \subset \mathbb{R}^3$ such that $\pi_1(X, x_0) \cong F_G([0,1], 0)$, which is Hausdorff. Additionally, since X is a compact metric space, $\Omega(X, x_0)$ is a separable metric space. It follows that the quotient $\pi_1(X, x_0)$ is Lindelöf. Every Hausdorff topological group is T_3 and every T_3 Lindelöf space is T_4. Finally, since $\check{\pi}_1(X, x_0)$ is an inverse limit of discrete groups, it is totally path-disconnected. But $\pi_1(X, x_0)$ is path-connected by Proposition 4.4. Hence, the continuous homomorphism $\psi : \pi_1(X, x_0) \to \check{\pi}_1(X, x_0)$ is trivial. ■

Remark 4.12. A topological space is *cosmic* if it is the continuous image of a separable metric space. The group $\pi_1(X, x_0) \cong F_G([0,1], 0)$ in Theorem 1.3 is, in fact, a cosmic k_ω-group that is homeomorphic to $\mathbb{R}^\infty$ topologized as the direct limit of finite Euclidean spaces $\mathbb{R}^n$ [27].

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