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Accessible points of planar embeddings of tent inverse limit spaces

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Contents

Abstract

In this paper we study a class of embeddings of tent inverse limit spaces. We introduce techniques relying on the Milnor–Thurston kneading theory and use them to study the sets of accessible points and prime ends of given embeddings. We completely characterize the accessible points and prime ends of standard embeddings arising from the Barge–Martin construction of global attractors. In the other embeddings under study we find phenomena which do not occur in the standard embeddings. Furthermore, for the non-standard embeddings we prove that the shift homeomorphism cannot be extended to a planar homeomorphism.

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1. Introduction

The problem of classifying continua that can be embedded in the plane is of substantial interest in continuum theory, mainly because it is intrinsically related to the fixed point property for planar non-separating continua. When a continuum is *chainable*, i.e., it admits an ε-mapping on $[0,1]$ for every $\varepsilon > 0$, it follows from an old result of Bing [6] that the continuum can be embedded in the plane. Therefore, it is natural to ask how many non-equivalent embeddings (see Definition 6.9) of a specific chainable continuum there exist and what these embeddings look like. A straightforward way to describe embeddings is through their sets of accessible points or through their prime end structure.

Inverse limit spaces on intervals are chainable. In [1] Bruin and the present authors showed that there exist uncountably many non-equivalent embeddings of tent map inverse limit spaces for all tent maps with slopes greater than $\sqrt{2}$, but they gave no insight into what the embeddings look like. In this paper we study the class of planar embeddings from [1] in detail, focusing primarily on accessible sets and the prime end structure in the finite critical orbit case.

The study of sets of accessible points of planar embeddings of the Knaster continuum was given by Mayer [21], and a characterization of possible sets of accessible points of embeddings of the Knaster continuum was found by Dębski and Tymchatyn [15]. The study of embeddings of unimodal inverse limit spaces appears in the literature in two forms: corresponding to attractors of orientation preserving (Brucks and Diamond [11]) and orientation reversing (Bruin [13]) planar homeomorphisms. We refer to those embeddings as *standard embeddings*. Barge and Martin [4] showed that every inverse limit space with a single interval bonding map can be realized as an attractor of an orientation preserving planar homeomorphism which acts on the attractor in the same way as the natural shift homeomorphism acts on the inverse limit. Using the construction from [4], Boyland, de Carvalho and Hall [8] recently gave a complete classification of the prime end structure and accessible sets of the Brucks–Diamond embedding of unimodal inverse limit spaces (satisfying certain regularity conditions valid for e.g. tent map inverse limits). For a class of non-standard embeddings of tent inverse limit spaces constructed in [1], the natural shift homeomorphism cannot be extended to the plane, as we show in Chapter 8. With that result we partially answer the question, posed by Boyland, de Carvalho and Hall [8], for which embeddings of tent inverse limit spaces the natural shift homeomorphism can be extended to a planar homeomorphism. Note that recently the present authors together with Bruin [2] constructed a bigger class of embeddings of tent inverse limit spaces, for which it is yet unknown whether they extend to planar homeomorphisms.

Because the embeddings from [1] cannot be extended to planar homomorphisms, we lack dynamical techniques as used in [8]. Therefore, for the construction and study of embeddings we choose a symbolic approach emerging from the Milnor–Thurston kneading theory [22] which was already used in constructions of embeddings in [11], [13] and [1]. It turns out that such a construction gives straightforward calculation techniques on the itineraries which we exploit throughout the paper, even in the case when the dynamical techniques are present, i.e., for the standard embeddings.

We denote $\mathbb{N} = \{1, 2, \dots\}$ and let $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. The *Hilbert cube* is the space $[0,1]^{-\mathbb{N}_0}$ equipped with the product metric

$$d(x, y) := \sum_{i \le 0} 2^i |\pi_i(x) - \pi_i(y)|,$$

where $\pi_i \colon [0,1]^{-\mathbb{N}_0} \to [0,1]$ denotes the coordinate projection for $i \le 0$.

The *tent map family* $T_s \colon [0,1] \to [0,1]$ is defined by $T_s(t) := \min\{st, s(1-t)\}$ where $t \in [0,1]$ and $s \in (0,2]$. Let $c = 1/2$ denote the *critical point* of T_s. In the rest of the paper we work with tent maps for slopes $s \in (\sqrt{2}, 2]$, and when there is no need to specify the slope we set for brevity $T := T_s$. The *inverse limit space with bonding map* T is a subspace of the Hilbert cube defined by

$$X := \varprojlim([0,1], T) = \{x \in [0,1]^{-\mathbb{N}_0} : T(\pi_i(x)) = \pi_{i+1}(x),\ i \le 0\}.$$

The space X is a *continuum*, i.e., a compact and connected metric space. Define the *shift homeomorphism* as $\sigma \colon X \to X$, $\pi_i(\sigma(x)) := T(\pi_i(x))$ for every $i \le 0$.

The space obtained by restricting the bonding map T to its dynamical core is called the *core* of X and will be denoted by X':

$$X' := \varprojlim([T^2(c), T(c)], T|_{[T^2(c), T(c)]}).$$

A continuum is *indecomposable* if it cannot be expressed as a union of two proper subcontinua. When $s \in (\sqrt{2}, 2]$, the core X' is indecomposable and by Bennett's theorem [5], $X = X' \cup \mathcal{C}$, where $\mathcal{C}$ is a ray which contains the fixed point $(\dots, 0, 0)$ and it compactifies on X' (for details see e.g. [19]). The ray $\mathcal{C}$ shields off some points of the continuum X and thus affects the set of accessible points in embeddings of X. However, the interesting phenomena regarding the structure of sets of accessible points occur in X' and thus we will mostly ignore $\mathcal{C}$ in the remainder of the paper. The structure of an embedded X (including $\mathcal{C}$) will be briefly discussed in Chapter 6.

The *composant* $\mathcal{V}_x$ of a point $x \in K$ is the union of all proper subcontinua in K that contain x. If a continuum is indecomposable, it consists of uncountably many pairwise disjoint composants and every composant is dense in the continuum (see [23]). The *arc-component* $\mathcal{U}_x$ of a point $x \in K$ is the union of all arcs from K that contain x.

A point a of a continuum $K \subset \mathbb{R}^2$ is *accessible* (i.e., accessible from the complement of K) if there exists an arc $A \subset \mathbb{R}^2$ such that $A \cap K = \{a\}$. We say that an arc-component $\mathcal{U}_x$ is *fully accessible* if every point of $\mathcal{U}_x$ is accessible. We will be mainly interested in embeddings of inverse limits of indecomposable cores of tent maps with finite critical orbits. In these cases every arc-component of a point coincides with the composant of that point (see [10, Proposition 3]).

We denote by $\mathcal{E}$ the class of embeddings of tent inverse limit spaces X and their cores X' constructed in [1] and refer to them as *$\mathcal{E}$-embeddings*. In [1], every $\mathcal{E}$-embedding of X is represented as a union of uncountably many horizontal segments (called basic arcs) which are aligned along a vertically embedded Cantor set with prescribed identifications between some endpoints of basic arcs (see Chapter 3 of this paper and [1] for details). An $\mathcal{E}$-embedding of X is then uniquely determined by the left-infinite itinerary $L = \dots l_2 l_1$, which is a symbolic description of the largest basic arcs among all basic arcs.

In Chapter 2 and at the beginning of Chapter 3, we recall basic notions of symbolic representation of unimodal inverse limits, recap the construction of embeddings of tent inverse limit spaces from [1], and give a symbolic characterization of arc-components in X, generalizing the result of Brucks and Diamond [11]. In Chapter 4, we characterize the possible sets of accessible points in an arc-component of any indecomposable plane non-separating continuum K. In Chapter 5 we briefly recall Carathéodory's prime end theory and discuss the existence of fourth kind prime ends in special cases which occur for tent map inverse limits. In Chapter 6, we begin our study of the embeddings $\mathcal{E}$. We introduce the notion of cylinders of basic arcs and techniques to explicitly calculate their extrema. We show that two $\mathcal{E}$-embeddings of the same space X are equivalent when they are determined by eventually the same left-infinite tail L. Given an $\mathcal{E}$-embedding of X, we prove that the arc-component of the top basic arc with symbolic description L (this arc-component is denoted by $\mathcal{U}_L$) is fully accessible if the top basic arc is not a spiral point (see Definition 3.3 and Figure 3.4). However, we also show that $\mathcal{U}_L$ is not necessarily the unique fully accessible arc-component. In the same chapter we briefly discuss $\mathcal{E}$-embeddings of a decomposable continuum X and characterize the set of accessible points up to two points on the corresponding circle of prime ends. From Section 6.1 on we study $\mathcal{E}$-embeddings of an indecomposable continuum X'. In Section 6.1 we give sufficient conditions on itineraries of L and kneading sequences ν associated with X' so that the embeddings of X' allow more than one fully accessible arc-component, and we give some interesting examples of such embeddings.

We say that $x \in X$ is a *folding point* if for every $\varepsilon > 0$ there exists a neighbourhood U_ε of x which is not homeomorphic to $C \times (0,1)$, where C is the Cantor set. A point $x \in X$ is called an *endpoint* if for any two subcontinua $X_1, X_2 \subset X$ such that $x \in X_1 \cap X_2$, either $X_1 \subset X_2$ or $X_2 \subset X_1$. Note that endpoints are also folding points. In Chapter 7 we characterize accessible folding points of $\mathcal{E}$-embeddings when the critical orbit of the tent map is finite. Surprisingly, no endpoints will be accessible in any $\mathcal{E}$-embedding of X' with the exception of the Brucks–Diamond embedding. Another surprising phenomenon is the occurrence of *Type 3* folding points (see Definition 7.22 and Figure 7.6) when the orbit of the third iterate of the critical point is periodic but the critical point itself is not periodic. Such a phenomenon does not occur in the standard embeddings of any tent map inverse limit space.

In Chapter 8 we prove that for every embedding constructed in [1] except for the standard embeddings (constructed by Brucks and Diamond [11] and Bruin [13] respectively), the natural shift homeomorphism cannot be extended from an $\mathcal{E}$-embedding of X' to the whole plane. Showing that, we partially answer the question posed by Boyland, de Carvalho and Hall [8] whether only for the standard embeddings of tent inverse limit spaces can the shift homeomorphism be extended to a planar homeomorphism. In Chapter 9 we study special examples of embeddings of X'. We explicitly show that every X' can be embedded with at least two non-degenerate fully accessible arc-components. In the finite orbit case when we have exactly two fully accessible arc-components, we show that there exists an embedding of X' with exactly two simple dense canals.

We conclude the paper with a complete characterization of sets of accessible points (and thus also the prime end structure of the corresponding circle of prime ends) of the two standard embeddings: Bruin's embedding of X' (Chapter 10) and the Brucks–Diamond embedding of X' (Chapter 11) using symbolic dynamics. In Chapter 10 we show that for Bruin's embedding of X' there is exactly one fully accessible non-degenerate arc-component and no other point from the embedding of X' is accessible if X' is different from the Knaster continuum. We show that if X' is not the Knaster continuum, then Bruin's embedding of X' has exactly one simple dense canal. If X' is the Knaster continuum then there is exactly one fully accessible non-degenerate arc-component and the endpoint of $\mathcal{C}$ is also accessible. In particular, there are no simple dense canals in this embedding of the Knaster continuum. In Chapter 11 we explicitly calculate the extrema of cylinders and neighbourhoods of folding points in the second standard embedding and obtain equivalent results to those obtained recently by Boyland, de Carvalho and Hall [8]. Moreover, since the symbolic description makes it possible to distinguish endpoints within the set of folding points, our results extend the classification given in [8].

2. Preliminaries on symbolic dynamics

In [1], uncountably many non-equivalent planar embeddings of indecomposable X' were constructed with the use of symbolic dynamics by making any given $x \in X'$ accessible. We give a short overview of symbolic dynamics but we refer to [1] and [11] for a more complete picture.

The *kneading sequence* of a map T is a right-infinite sequence $\nu = c_1c_2\ldots \in \{0,1\}^\infty$, where

$$c_i = \begin{cases} 0, & T^i(c) \in [0,c], \\ 1, & T^i(c) \in [c,1], \end{cases}$$

for all $i \in \mathbb{N}$. If $T^n(c) = c$ for some $n \in \mathbb{N}$, the critical point c is periodic and the ambiguity in the definition of ν is resolved by defining ν to be the smaller of $(c_1\ldots c_{n-1}0)^\infty$ and $(c_1\ldots c_{n-1}1)^\infty$ in the *parity-lexicographical ordering* on $\{0,1\}^\infty$ defined below.

We denote by $\#_1(a_1\ldots a_n)$ the number of ones in the finite word $a_1\ldots a_n \in \{0,1\}^n$; it can be either even or odd. Choose $t = t_1t_2\ldots \in \{0,1\}^\infty$ and $s = s_1s_2\ldots \in \{0,1\}^\infty$, where we also permit s and t to be finite sequences of the same length. In particular, throughout the paper we will not compare finite words with infinite itineraries. Set $0 < 1$. Take the smallest $k \in \mathbb{N}$ such that $s_k \neq t_k$, if any; otherwise $s = t$. Then the *parity-lexicographical* *ordering* is defined as

$$s \prec t \Leftrightarrow \begin{cases} s_k < t_k \text{ and } \#_1(s_1 \dots s_{k-1}) \text{ is even, or} \\ s_k > t_k \text{ and } \#_1(s_1 \dots s_{k-1}) \text{ is odd.} \end{cases}$$

Fix the kneading sequence $\nu = c_1c_2\dots$. The finite word $a_1 \dots a_n \in \{0,1\}^n$ is called *admissible* if $c_2c_3\dots c_{2+n-i} \preceq a_i \dots a_n \preceq c_1c_2\dots c_{1+n-i}$ for every $i \in \{1,\dots,n\}$. A two-sided infinite sequence $\dots s_{-2}s_{-1}.s_0s_1\dots \in \{0,1\}^{\mathbb{Z}}$ is called *admissible* if every finite subword is admissible. Analogously we define an admissible left- or right-infinite sequence. Additionally, a two-sided sequence $0^\infty s_ks_{k+1}\dots$ will also be called admissible if $s_k = 1$ and every finite subword of the right-infinite sequence $s_ks_{k+1}\dots$ is admissible. Denote the set of all admissible two-sided infinite sequences by Σ_{adm}.

The set $\Sigma_{\mathrm{adm}} \subset \Sigma = \{0,1\}^{\mathbb{Z}}$ inherits the topology of Σ given by the metric

$$d((s_i)_{i\in\mathbb{Z}}, (t_i)_{i\in\mathbb{Z}}) := \sum_{i\in\mathbb{Z}} |s_i - t_i|/2^{|i|}.$$

Define the *shift homeomorphism* $\sigma_\Sigma\colon \Sigma \to \Sigma$ on symbolic sequences as

$$\sigma_\Sigma(\dots s_{-2}s_{-1}.s_0s_1\dots) := \dots s_{-1}s_0.s_1s_2\dots.$$

The continuum X is homeomorphic to the space $\Sigma_{\mathrm{adm}}/\sim$ (see [1, Proposition 2]), where $\sim$ is the equivalence relation on Σ_{adm} given by

$$s \sim t \Leftrightarrow \begin{cases} \text{either } s_i = t_i \text{ for every } i \in \mathbb{Z}, \\ \text{or there exists } k \in \mathbb{Z} \text{ such that } s_i = t_i \text{ for all } i \neq k \text{ but } s_k \neq t_k \\ \qquad \text{and } s_{k+1}s_{k+2}\dots = t_{k+1}t_{k+2}\dots = \nu. \end{cases}$$

Sequences of the form $0^\infty s_ks_{k+1}\dots$, treated differently in the definitions above, correspond to the points of the ray $\mathcal{C}$. By removing these sequences from the definition of Σ_{adm}, we get a space homeomorphic to the core X'. The shifts σ and σ_Σ are conjugate (see [11, Theorem 2.5]). Thus from here on we will abuse notation and denote both σ and σ_Σ by σ.

The homeomorphism between X and $\Sigma_{\mathrm{adm}}/\sim$ is given as follows. A point $x \in X$ is identified with the equivalence class $\bar{x} = \overleftarrow{x}.\overrightarrow{x} = (x_i)_{i\in\mathbb{Z}} \in \Sigma_{\mathrm{adm}}/\sim$ according to the following rule:

$$x_i = \begin{cases} 0, & \pi_i(x) \in [0,c], \\ 1, & \pi_i(x) \in [c,1], \end{cases}$$

for $i \le 0$, and

$$x_i = \begin{cases} 0, & T^i(\pi_0(x)) \in [0,c], \\ 1, & T^i(\pi_0(x)) \in [c,1], \end{cases}$$

for $i \in \mathbb{N}$. If the ambiguity in the definition of x_i happens more than once, then c is periodic and we study the itinerary of the modified kneading sequence instead. That way for every $x \in X$ there are at most two corresponding identified itineraries. If $x \in X$ has two different backward itineraries $\overleftarrow{x}_1$ and $\overleftarrow{x}_2$, then we denote $\overleftarrow{x} := \{\overleftarrow{x}_1, \overleftarrow{x}_2\}$.

An *arc* is a homeomorphic image of the interval $[0,1] \subset \mathbb{R}$. A key fact for constructing embeddings in [1] is that $X \simeq \Sigma_{\mathrm{adm}}/\sim$ can be represented as the union of *basic arcs* defined below. A basic arc will be an arc or a point in X consisting of all points that have the same backward itinerary $\overleftarrow{s}$, with the exception of two possible endpoints with possibly two backward itineraries, one of which is $\overleftarrow{s}$.

From now on, when we speak about left-infinite sequences we omit minuses in indices and write $\overleftarrow{s} = \dots s_2 s_1$ for brevity.

Definition 2.1. Let $\overleftarrow{s} = \dots s_2 s_1 \in \{0,1\}^{\infty}$ be an admissible left-infinite sequence. The set

$$A(\overleftarrow{s}) := \{x \in X : \overleftarrow{s} \in \overleftarrow{x}\} \subset X$$

is called a *basic arc*.

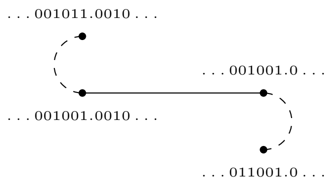


Fig. 2.1. An example of a basic arc. Here it consists of an open arc in X consisting of all points with the same backward itinerary and two pairs of identified points. Identifications are represented by dashed semicircles. Itineraries of joined points are sketched.

It is not difficult to show that $A(\overleftarrow{s})$ is indeed an arc in X (which may be degenerate). See e.g. [13, Lemma 1].

Remark 2.2. Later, when it will be clear from the context that we refer to the basic arc $A(\overleftarrow{s})$, we will often abbreviate this notation to $\overleftarrow{s}$.

To explain two quantities for basic arcs which we will often work with, we need the following definition.

Definition 2.3. For $\nu = c_1 c_2 \dots$ we define

$$\kappa := \min\{i - 2 : i \geq 3,\ c_i = 1\}.$$

Remark 2.4. Definition 2.3 says that the beginning of the kneading sequence is $\nu = 10^{\kappa}1\dots$. If $\kappa = 1$, since we restrict to non-renormalizable tent maps, we can conclude even more, namely that $\nu = 10(11)^n 0\dots$ for some $n \in \mathbb{N}$.

For every admissible left-infinite sequence $\overleftarrow{s} \neq \overleftarrow{0}$ we define two quantities:

$$\tau_L(\overleftarrow{s}) := \sup\{n > 1 : s_{n-1} \dots s_1 = c_1 c_2 \dots c_{n-1},\ \#_1(c_1 \dots c_{n-1}) \text{ odd}\},$$
$$\tau_R(\overleftarrow{s}) := \sup\{n \geq 1 : s_{n-1} \dots s_1 = c_1 c_2 \dots c_{n-1},\ \#_1(c_1 \dots c_{n-1}) \text{ even}\}.$$

In the definition of τ_R we allow $n = 1$, so it follows immediately that the supremum is taken over a non-empty set. On the other hand, such supremum is well defined for τ_L as well. Namely, if $s_1 = 1$, then $s_1 = c_1$ and $\#_1(c_1)$ is odd. In case $s_1 = 0$ we find the smallest $n > 2$ such that $s_{n-1} = 1$, which indeed exists since $\overleftarrow{s} \neq \overleftarrow{0}$. If $n > \kappa + 2$, the word $s_{n-2} \dots s_1 = 0^{n-2}$ is not admissible. Thus $n \leq \kappa + 2$ and $s_{n-1} \dots s_1 = c_1 \dots c_{n-1} = 10\dots0$. Now we restate some lemmas from [13] in our setting.

Lemma 2.5 ([13, Lemmas 2 and 3]). *Let $\overleftarrow{s} \in \{0,1\}^{\infty}$ be an admissible left-infinite sequence such that $\overleftarrow{s} \neq \overleftarrow{0}$. Then*

$$\max \pi_0(A(\overleftarrow{s})) = \inf\{T^n(c) : s_{n-1} \dots s_1 = c_1 \dots c_{n-1},\ \#_1(c_1 \dots c_{n-1}) \text{ even}\},$$
$$\min \pi_0(A(\overleftarrow{s})) = \sup\{T^n(c) : s_{n-1} \dots s_1 = c_1 \dots c_{n-1},\ \#_1(c_1 \dots c_{n-1}) \text{ odd}\}.$$

In particular, if $\tau_L(\overleftarrow{s}), \tau_R(\overleftarrow{s}) < \infty$, then

$$\pi_0(A(\overleftarrow{s})) = [T^{\tau_L(\overleftarrow{s})}(c), T^{\tau_R(\overleftarrow{s})}(c)].$$

If $\overleftarrow{t} \in \{0,1\}^{\infty}$ is another admissible left-infinite sequence such that $s_i = t_i$ for all $i > 0$ except for $i = \tau_R(\overleftarrow{s}) = \tau_R(\overleftarrow{t})$ (or $i = \tau_L(\overleftarrow{s}) = \tau_L(\overleftarrow{t})$), then $A(\overleftarrow{s})$ and $A(\overleftarrow{t})$ have a common endpoint projecting to the right (resp. left) endpoint of $\pi_0(A(\overleftarrow{s}))$.

Example 2.6. Assume $\nu = 10010\dots$. Then $\overleftarrow{s} = \dots 001001$ is an admissible left-infinite sequence. Note that $\tau_L(\overleftarrow{s}) = 2$ and $\tau_R(\overleftarrow{s}) = 5$. So $\pi_0(A(\overleftarrow{s})) = [T^2(c), T^5(c)]$ (see Figure 2.1).

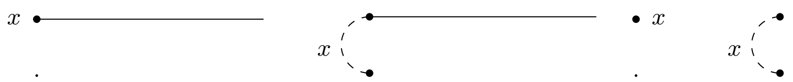


Fig. 2.2. Endpoints of X, all possibilities. Dashed semicircles connect points identified under $\sim$.

Remark 2.7. Note that the basic arc $\overleftarrow{s} = \overleftarrow{0}$ needs to be addressed seperately because $\tau_L(\overleftarrow{0})$ is not well defined. However, it is not difficult to see that $\pi_0(A(\overleftarrow{0})) = [0, T(c)]$.

We recall the symbolic characterization of endpoints in X. It states that endpoints of X are endpoints of basic arcs which are contained in exactly one basic arc (see Figure 2.2). Note that the characterization distinguishes two cases, when $x \in X$ is such that $\pi_i(x) \neq c$ for every $i < 0$ and when there is $i < 0$ such that $\pi_i(x) = c$. In the second case we use the fact that $x \in X$ is an endpoint of X if and only if $\sigma^i(x)$ is an endpoint of X for every $i \in \mathbb{N}$.

Proposition 2.8 ([13, Proposition 2]). *Let $x \in X$ be such that $\pi_i(x) \neq c$ for all $i < 0$. Then x has a unique backward itinerary $\overleftarrow{x}$. The point x is an endpoint of X if and only if $\tau_L(\overleftarrow{x}) = \infty$ and $\pi_0(x) = \min \pi_0(A(\overleftarrow{x}))$ (or $\tau_R(\overleftarrow{x}) = \infty$ and $\pi_0(x) = \max \pi_0(A(\overleftarrow{x}))$). If $x \in X$ is such that $\pi_i(x) = c$ for some $i < 0$, then there exists $j \in \mathbb{N}$ such that the backward itinerary of $\sigma^j(x)$ is unique and we apply the first claim to $\sigma^j(x)$ in this case.*

3. Construction of planar embeddings $\mathcal{E}$

In this chapter we recall the construction of planar embeddings of X in [1].

Roughly speaking (we refer the reader to Figure 3.3), we first embed the Cantor set in the vertical segment and code points of the Cantor set as left-infinite sequences of 0s and 1s respecting some given order. For every admissible left-infinite sequence $\overleftarrow{s}$ (corresponding to a point in an embedded Cantor set) we draw a horizontal arc whose projection on a horizontal planar axis is exactly $\pi_0(A(\overleftarrow{s}))$ and the projection on the y-axis is exactly the point coded by $\overleftarrow{s}$. Then we join with semicircles certain pairs of identified endpoints of basic arcs. If the order on the Cantor set is properly defined, the semicircles will not intersect, and the figure will be homeomorphic to X. The embeddings will be precisely defined in the rest of this chapter.

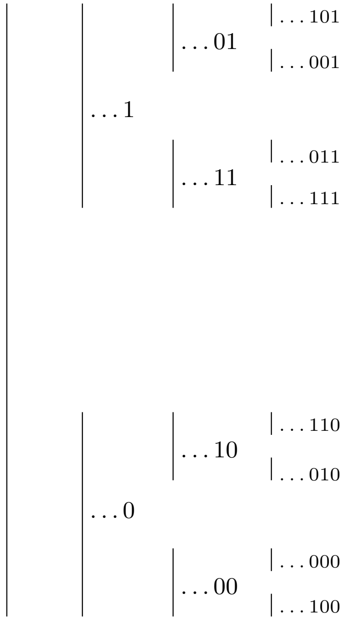


Fig. 3.1. Coding the Cantor set with respect to $L = \dots 010101$.

Fix an admissible left-infinite sequence $L = \dots l_2 l_1 \in \{0,1\}^\infty$. Basic arcs are embedded in the plane with respect to any chosen L as a subset of $[0,1] \times C$, where $C \subset [0,1]$ denotes the Cantor set

$$C := [0,1] \setminus \bigcup_{m=1}^{\infty} \bigcup_{k=0}^{3^{m-1}-1} \left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m} \right).$$

Recall from Lemma 2.5 that we know how to compute $\pi_0(A(\overleftarrow{s}))$ for every admissible $\overleftarrow{s}$. Every basic arc $A(\overleftarrow{s})$ is embedded as the horizontal arc $\pi_0(A(\overleftarrow{s})) \times \{\psi_L(\overleftarrow{s})\}$, where

$$\psi_L(\overleftarrow{s}) := \sum_{i=1}^{\infty} (-1)^{\#_1(l_i \dots l_1) - \#_1(s_i \dots s_1)} 3^{-i} + \frac{1}{2}.$$

This implies a linear order on the basic arcs in which $A(L)$ is the largest. The precise definition is given by:

Definition 3.1. Let $L = \dots l_2 l_1 \in \{0,1\}^\infty$ be fixed. Let $\overleftarrow{s}, \overleftarrow{t} \in \{0,1\}^\infty$ and let $k \in \mathbb{N}$ be the smallest natural number such that $s_k \neq t_k$. Then

$$\overleftarrow{s} \prec_L \overleftarrow{t} \Leftrightarrow \begin{cases} t_k = l_k \text{ and } \#_1(s_{k-1} \dots s_1) - \#_1(l_{k-1} \dots l_1) \text{ even, or} \\ s_k = l_k \text{ and } \#_1(s_{k-1} \dots s_1) - \#_1(l_{k-1} \dots l_1) \text{ odd.} \end{cases} \tag{3.1}$$

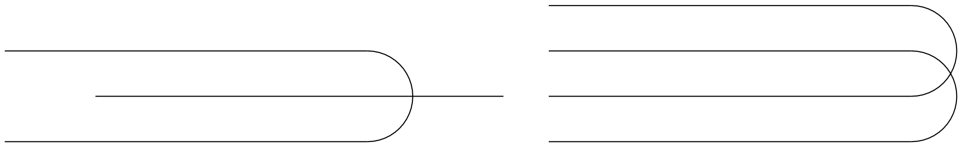


Fig. 3.2. "Self-intersections" in representations of X that are excluded by [1, Proposition 3].

Example 3.2. Assume that the ordering on left-infinite sequences $\{0,1\}^\infty$ (i.e., on the Cantor set) is determined by $L = \dots 010101$ (see Figure 3.1). Then L is the largest sequence and $\dots 010100$ is the smallest. However, if X is a tent inverse limit with bonding map which is not T_2, then there will exist sequences which are not admissible. For example let $\nu = (101)^\infty$. Then $\dots 010100$ (which would be, if admissible, the itinerary of the smallest basic arc in the ordering along the vertically embedded Cantor set) is not admissible and finding the smallest admissible left-infinite sequence requires more work. In the example of $\nu = (101)^\infty$ a word is admissible if and only if it does not contain two consecutive zeros. The smallest admissible left-infinite sequence will be $S = \dots 101010$ (see Example 6.1 for further explanation).

We have embedded basic arcs as horizontal segments ordered along the Cantor set. Now we show how to embed X in the plane. We want to find points in embedded horizontal arcs which are identified under $\sim$ (see Lemma 2.5). If $(s_i)_{i\in\mathbb{Z}}, (t_i)_{i\in\mathbb{Z}} \in \Sigma_{\text{adm}}$ are such that $s_i = t_i$ for all $i \in \mathbb{Z}$ except for exactly one $k < 0$, and $s_{-k+1}s_{-k+2}\dots = t_{-k+1}t_{-k+2}\dots = \nu$, then $A(\overleftarrow{s})$ and $A(\overleftarrow{t})$ contain endpoints which are identified under $\sim$. If $\#_1(s_{-k+1}\dots s_{-1})$ is odd [even], then $\tau_L(\overleftarrow{s}) = \tau_L(\overleftarrow{t})$ [$\tau_R(\overleftarrow{s}) = \tau_R(\overleftarrow{t})$] and thus $\pi_0(A(\overleftarrow{s}))$ and $\pi_0(A(\overleftarrow{t}))$ have a common endpoint on the left [right]. We join embedded $A(\overleftarrow{s})$ and $A(\overleftarrow{t})$ by a semicircle on the left [right] (see Figure 3.3).

Proposition 3 from [1] shows that a representation of X with basic arcs connected by semicircles drawn as above will not intersect other horizontal arcs or semicircles (see Figure 3.2). Moreover, Lemmas 5 and 6 in [1] show that the space consisting of horizontal arcs and semicircles is homeomorphic to $\Sigma_{\text{adm}}/\sim$ and thus gives an embedding φ_L of X in the plane.

From now on, L will denote the left-infinite sequence of the largest basic arc which determines the planar embedding φ_L of X by the rules in (3.1). Let us fix the inverse limit space X. Denote by $\mathcal{E}$ the family of all embeddings of X constructed in [1], i.e., with respect to all admissible tails L, and refer to them as *$\mathcal{E}$-embeddings*. From now on we think of X as a planar continuum obtained by an $\mathcal{E}$-embedding of $\Sigma_{\text{adm}}/\sim$ described above.

3.1. Arc-components. We want to describe the sets of accessible points of an embedded X, focusing primarily on the (fully) accessible arc-components. Since the approach in this study is mostly symbolic, we need to obtain a symbolic description of an arc-component in X. Recall that $\mathcal{U}_x$ denotes the arc-component of $x \in X$.

Definition 3.3. We say that $x \in X$ is a *spiral point* if there exists a ray $R \subset X$ such that x is an endpoint of R and an arc $[x,y] \subset R$ contains infinitely many basic arcs for every $x \neq y \in R$.

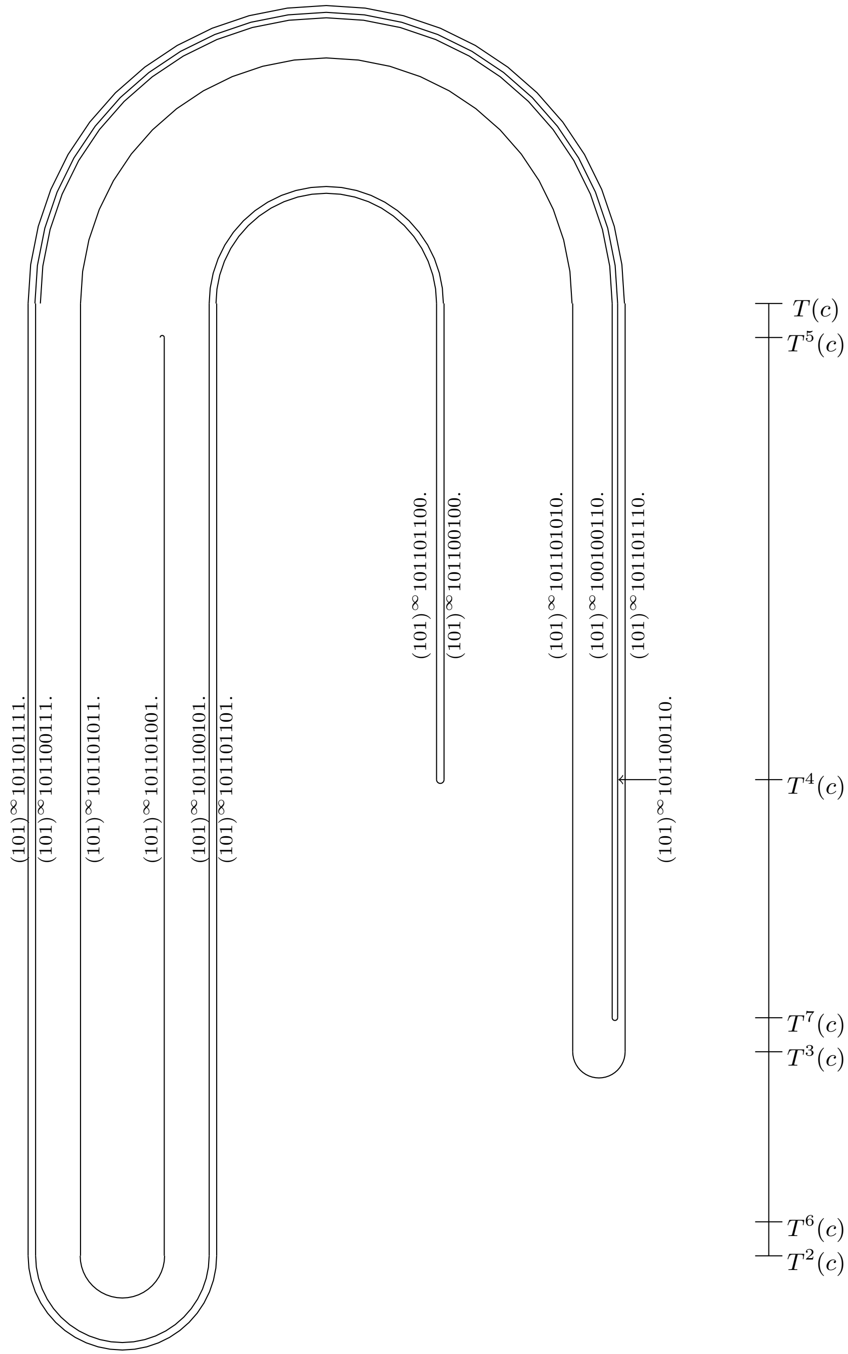


Fig. 3.3. The planar representation of an arc in $\mathcal{U}_{(101)^\infty} \subset X$ with the corresponding kneading sequence $\nu = 100110010\ldots$. The ordering on basic arcs is such that the basic arc coded by $L = 1^\infty$. is the largest. The figure is taken from [1].

Proposition 3.4. *If $x \in X$ is a spiral point, then $A(\overleftarrow{x})$ is degenerate and x is an endpoint of X.*

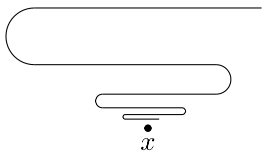


Fig. 3.4. A spiral point $x \in X$.

Proof. Assume that x has a unique left-infinite itinerary $\overleftarrow{x}$, i.e., $\pi_i(x) \neq c$ for every $i < 0$. Assume that $A(\overleftarrow{x})$ is not degenerate. Note that x is not in the interior of $A(\overleftarrow{x})$ since then $R \cup A(\overleftarrow{x})$ is a triod, which contradicts X being chainable. Without loss of generality assume that x is the right endpoint of $A(\overleftarrow{x})$. Then $\tau_R(A(\overleftarrow{x})) = \infty$, otherwise x has two backward itineraries. Therefore, by Proposition 2.8, x is an endpoint of X. Since $x \in A(\overleftarrow{x}) \cup R$, we conclude that $A(\overleftarrow{x})$ is degenerate.

Assume that x does not have a unique left-infinite itinerary, i.e., x corresponds to two endpoints of basic arcs glued together under $\sim$. Denote two left-infinite itineraries of x by $\overleftarrow{x}_1$ and $\overleftarrow{x}_2$. Then there is $k \in \mathbb{N}$ such that $\sigma^{-k}(A(\overleftarrow{x}_1) \cup A(\overleftarrow{x}_2))$ is contained in a single basic arc A and $\sigma^{-k}(x) \in A$ has a unique left-infinite tail. If A is not degenerate, we get a contradiction as in the previous paragraph. Thus $A(\overleftarrow{x}_1) = A(\overleftarrow{x}_2) = A(\overleftarrow{x})$ is degenerate. Again Proposition 2.8 implies that x is an endpoint of X. ■

Remark 3.5. Note that a planar representation of a degenerate basic arc can be either a point or two points joined by a semicircle; see the last two pictures in Figure 2.2.

The following is a direct consequence of Proposition 3.4 since a spiral point cannot be contained in the interior of an arc.

Corollary 3.6. *Non-degenerate arc-components in X are:*

- *lines (i.e., continuous images of $\mathbb{R}$) with no spiral points,*
- *rays (continuous images of $\mathbb{R}^+$), where only the endpoint can be a spiral point,*
- *arcs, where only endpoints can be spiral points.*

Remark 3.7. Let $\overleftarrow{y} \neq \overleftarrow{w}$ be admissible left-infinite sequences. By Lemma 2.5, $A(\overleftarrow{y})$ and $A(\overleftarrow{w})$ are connected by finitely many basic arcs if and only if there exists $k \in \mathbb{N}$ such that $\dots y_{k+1}y_k = \dots w_{k+1}w_k$. We say that $\overleftarrow{y}$ and $\overleftarrow{w}$ have the same *tail*. Thus every arc-component in X is determined by its tail with the exception of (one or two) spiral points with different tails. This generalizes the symbolic representation of arc-components for a finite critical orbit c given in [11] to an arbitrary tent inverse limit space X.

4. General results about accessibility

Definition 4.1. We say that a continuum $K \subset \mathbb{R}^2$ *does not separate the plane* if $\mathbb{R}^2 \setminus K$ is connected.

For $K \subset \mathbb{R}^2$ we denote by $\mathrm{Cl}(K)$ the closure of K in $\mathbb{R}^2$. The following proposition is a special case of [9, Theorem 3.1].

PROPOSITION 4.2. *Let $K \subset \mathbb{R}^2$ be a non-degenerate indecomposable continuum that does not separate the plane and let $Q = [x, y] \subset K$ be an arc. If x and y are accessible, then Q is fully accessible.*

Proof. Assume for contradiction that Q is not fully accessible. Because $x, y \in K$ are both accessible, there exists a point $w \in \mathbb{R}^2 \setminus K$ and arcs $Q_x := [x, w], Q_y := [y, w] \subset \mathbb{R}^2$ such that $(x, w], (y, w] \subset \mathbb{R}^2 \setminus K$ and $Q \cup Q_x \cup Q_y =: S$ is a simple closed curve in $\mathbb{R}^2$ (see Figure 4.1). Thus $\mathbb{R}^2 \setminus S = S_1 \cup S_2$ where S_1 and S_2 are open sets in $\mathbb{R}^2$ such that $\partial S_1 = \partial S_2 = S$. Specifically, S_1 contains no accumulation points of S_2 and vice versa. Denote $K_1 := K \cap \mathrm{Cl}(S_1)$, $K_2 := K \cap \mathrm{Cl}(S_2)$. Note that K_1, K_2 are subcontinua of K and $K_1, K_2 \neq \emptyset$. Because Q is not fully accessible, it follows that $K_1, K_2 \neq K$. Furthermore $K_1 \cup K_2 = K$, which contradicts K being indecomposable. ∎

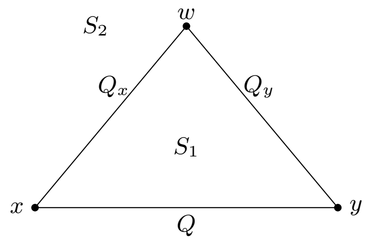


Fig. 4.1. A simple closed curve from the proof of Proposition 4.2.

COROLLARY 4.3. *Let K be an indecomposable planar continuum that does not separate the plane and let $\mathcal{U}$ be an arc-component of K. There are four possibilities regarding the accessibility of $\mathcal{U}$:*

- *$\mathcal{U}$ is fully accessible.*
- *There exists an accessible point $u \in \mathcal{U}$ such that one component of $\mathcal{U} \setminus \{u\}$ is not accessible, and the other one is fully accessible.*
- *There exist two (not necessarily different) accessible points $u, v \in \mathcal{U}$ such that $\mathcal{U} \setminus [u, v]$ is not accessible and $[u, v] \subset \mathcal{U}$ is fully accessible.*
- *$\mathcal{U}$ is not accessible.*

Proof. By Proposition 4.2, the set of accessible points in $\mathcal{U}$ is connected. To see it is closed, take a monotone sequence $(x_i)_{i \in \mathbb{N}}$ (note that we can parametrize every arc-component $\mathcal{U}$ and thus it makes sense to speak about monotone sequences) of accessible points in $\mathcal{U}$ such that $\lim_{i \to \infty} x_i =: x \in \mathcal{U}$. Let $w \in \mathbb{R}^2 \setminus K$ and let $Q_i \subset \mathbb{R}^2$ be disjoint arcs with endpoints x_i and w and such that $Q_i \cap K = x_i$ for every $i \in \mathbb{N}$. Using [20, Remark (i) after the proof of Theorem 6, §61, IV] we can find a planar homeomorphism such that $[x_1, x]$ is a straight arc and then pick a point w and arcs Q_i so that they are straight arcs, and thus they limit on a straight arc as well. Denote by S_i the bounded open set in $\mathbb{R}^2$ with boundary $Q_1 \cup Q_i \cup [x_1, x_i]$, where $[x_1, x_i] \subset \mathcal{U}$. Note that $K \cap S_i = \emptyset$ for every $i \in \mathbb{N}$, since otherwise K is decomposable by analogous arguments to those in the proof of Proposition 4.2. Then also $K \cap \bigcup_{i\in\mathbb{N}} S_i = \emptyset$. Since x is contained in the boundary of $\bigcup_{i\in\mathbb{N}} S_i$, which is arc-connected, we conclude that x can be accessed with a ray from $\bigcup_{i\in\mathbb{N}} S_i \subset \mathbb{R}^2 \setminus K$. ∎

Remark 4.4. Note that it follows from the third item of Corollary 4.3 that there may exist an endpoint $u = v \in \mathcal{U}$ which is accessible while every $x \in \mathcal{U} \setminus \{u\}$ is not accessible. For instance such embeddings of the Knaster continuum are described in [25] and the endpoint is the only accessible point in the arc-component $\mathcal{C}$. In the course of this paper we show that all cases from Corollary 4.3 indeed occur in some embeddings of tent inverse limit spaces.

5. Basic notions from prime end theory

In this chapter we briefly recall Carathéodory's prime end theory. Although the focus of this paper is not on the characterization of prime ends of embeddings of continua, we will include the study of prime ends of some interesting examples throughout the paper. A general study of prime ends of standard planar embeddings appears at the end of the paper.

Definition 5.1. Let $K \subset \mathbb{R}^2$ be a plane non-separating continuum. A *crosscut* of $\mathbb{R}^2 \setminus K$ is an arc $Q \subset \mathbb{R}^2$ which intersects K only in its endpoints. Note that $K \cup Q$ separates the plane into two components, one bounded and the other unbounded. Denote the bounded component by B_Q. A sequence $\{Q_i\}_{i\in\mathbb{N}}$ of crosscuts is called a *chain* if the crosscuts are pairwise disjoint, $\operatorname{diam} Q_i \to 0$ as $i \to \infty$ and $B_{Q_{i+1}} \subset B_{Q_i}$ for every $i \in \mathbb{N}$. We say that two chains $\{Q_i\}_{i\in\mathbb{N}}$ and $\{R_i\}_{i\in\mathbb{N}}$ are *equivalent* if for every $i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that $B_{R_j} \subset B_{Q_i}$ and for every $j \in \mathbb{N}$ there exists $i' \in \mathbb{N}$ such that $B_{Q_{i'}} \subset B_{R_j}$. An equivalence class $[\{Q_i\}_{i\in\mathbb{N}}]$ is called a *prime end*. A basis for the natural topology on the set of all prime ends consists of the sets $\{[\{R_i\}_{i\in\mathbb{N}}] : B_{R_i} \subset B_Q \text{ for all } i\}$ for all crosscuts Q. The set of prime ends equipped with the natural topology is a topological circle, called the *circle of prime ends* (see e.g. [9, Section 2]).

Definition 5.2. Let $P = [\{R_i\}_{i\in\mathbb{N}}]$ be a prime end. The *principal set* of P is $\Pi(P) = \{\lim Q_i : \{Q_i\}_{i\in\mathbb{N}} \in P \text{ is convergent}\}$ and the *impression* of P is $I(P) = \bigcap_i \operatorname{Cl}(B_{R_i})$. Note that both $\Pi(P)$ and $I(P)$ are subcontinua in X' and $\Pi(P) \subseteq I(P)$. We say that P is of the

- *first kind* if $\Pi(P) = I(P)$ is a point,
- *second kind* if $\Pi(P)$ is a point and $I(P)$ is non-degenerate,
- *third kind* if $\Pi(P) = I(P)$ is non-degenerate,
- *fourth kind* if $\Pi(P) \subsetneq I(P)$ are non-degenerate.

Theorem 5.3 (Iliadis [18]). *Let K be a plane non-separating indecomposable continuum. The circle of prime ends corresponding to K can be decomposed into open intervals and their boundary points such that every open interval J uniquely corresponds to a composant* *of K which is accessible at more than one point and $I(e) \subsetneq K$ for every $e \in J$. For the boundary points e we have $I(e) = K$.*

Proposition 5.4. *Let K be a plane non-separating indecomposable continuum such that every proper subcontinuum of K is an arc and every composant contains at most one folding point. Then $\Pi(P)$ is degenerate or equal to K for every prime end P. In particular, there exist no prime ends of the fourth kind.*

Proof. Assume there exists a prime end P such that $\Pi(P)$ is non-degenerate and not equal to K. Then $\Pi(P) = [a, b]$ is an arc in K. We claim that both a and b are folding points. Assume that there exists $\varepsilon > 0$ such that $B(a, \varepsilon) \cap K = C \times (0, 1)$, where C is the Cantor set and $B(a, \varepsilon)$ denotes the open planar ball of radius ε around the point a. Since $a \in \Pi(P)$, there exist a chain $\{Q_i\}_{i\in\mathbb{N}} \in P$ of crosscuts such that $Q_i \to a$ as $i \to \infty$. Note that $Q_i \in B(a, \varepsilon)$ for large enough i, so the endpoints of Q_i are contained in $C \times (0, 1)$, and the interior of Q_i does not intersect K. If there exists $N \in \mathbb{N}$ such that the arc Q_N has both endpoints in the same component of $C \times (0, 1)$, then $\Pi(P)$ is degenerate, a contradiction. Thus the endpoints of Q_i do not lie in the same component of $C \times (0, 1)$, and since $\operatorname{diam} Q_i \to 0$, we can find a subsequence $\{Q_{i_k}\}$ such that all endpoints of Q_{i_k} are contained in different components of $C \times (0, 1)$ (see Figure 5.1). Therefore, it is possible to translate every Q_{i_k} along $(0, 1)$ and find a point $z \notin [a, b]$ in the arc-component of $[a, b]$ for which there exists a chain $\{R_k\}_{k\in\mathbb{N}}$ of crosscuts equivalent to $\{Q_{i_k}\}_{k\in\mathbb{N}}$ such that $R_k \to z$ as $k \to \infty$ (again see Figure 5.1). This contradicts the assumption, i.e., a is a folding point. The proof for b is analogous. We conclude that there exists a composant with at least two folding points, which is a contradiction. ■

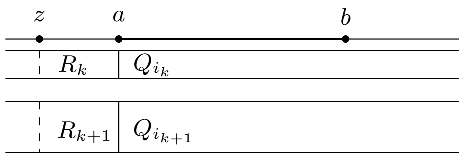


Fig. 5.1. Translating a chain of crosscuts along $(0, 1)$ in Proposition 5.4.

Definition 5.5. Let K be a plane non-separating continuum. A prime end P such that $\Pi(P)$ is non-degenerate but different from K is called an *infinite canal*. A third kind prime end P such that $\Pi(P) = I(P) = K$ is called a *simple dense canal*.

We obtain the following corollary, which we use later to discuss the prime end structure of $\mathcal{E}$-embeddings of X' when the critical orbit is finite.

Corollary 5.6. *Let K be an indecomposable plane non-separating continuum such that every subcontinuum is an arc and every composant contains at most one folding point. Then the circle of prime ends corresponding to K can be partitioned into open intervals and their endpoints. Open intervals correspond to accessible open lines in K. The endpoints of open intervals are the second or third kind prime ends for which the impression is K. The second kind prime end corresponds to an accessible folding point in K and the third kind prime end corresponds to a simple dense canal in K.*

Question. If X' is the core of a tent map inverse limit, is there a planar embedding $\varphi\colon X' \to \mathbb{R}^2$ such that $\varphi(X')$ has a prime end of the fourth kind?

6. An introduction to the study of accessible points of $\mathcal{E}$-embeddings

In this chapter we reduce generality and focus again on our original study of inverse limit spaces of tent maps.

By Corollary 4.3, if $x \in \mathcal{U}_x \subset X$ is accessible it does not a priori follow that every point from $\mathcal{U}_x$ is accessible (see e.g. Figure 6.1). Recall that $X = \mathcal{C} \cup X'$. In this paper we study the sets of accessible points of embeddings of either X or X' and the two cases substantially differ as we shall see in this chapter. In the rest of the paper we are concerned only with embeddings of the cores X'.

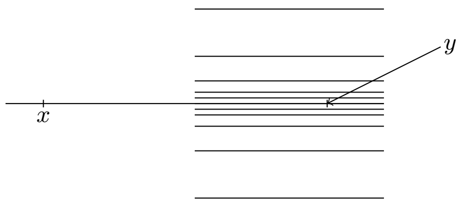


Fig. 6.1. The point x is accessible from the complement, while the point y, which has a neighbourhood of Cantor set of arcs, is not.

We will denote by S the smallest admissible left-infinite tail in X' with respect to $\prec_L$. The arc-component of points from L $[S]$ will be denoted by $\mathcal{U}_L$ $[\mathcal{U}_S]$. The following examples show that $\mathcal{U}_L$ and $\mathcal{U}_S$ do not necessarily coincide. Later in this chapter we will be especially concerned with the accessibility of $\mathcal{U}_L$ and $\mathcal{U}_S$.

Recall that we consider X as a continuum $\Sigma_{\mathrm{adm}}/{\sim}$ embedded in the plane with respect to some admissible L. Recall also that a basic arc consists of all points with the same backward itinerary and that each basic arc is embedded as a horizontal arc in the plane. We abuse notation and identify basic arcs with their left-infinite sequences $\overleftarrow{s}$.

Example 6.1. Assume that the kneading sequence is given by $\nu = (101)^\infty$. Embed X' in the plane according to the ordering in which $L = (01)^\infty$ is the largest. As we commented in Example 3.2, the smallest sequence is then $S = (10)^\infty$. Note that the backward itineraries of L and S differ on infinitely many places. So, the results of Section 3.1 imply that $S \not\subset \mathcal{U}_L$.

Example 6.2. Take the kneading sequence $\nu = 1001(101)^\infty$. Embed X' in the plane according to the ordering in which $L = ((001)(001101))^\infty$ is the largest. The smallest is then $S = ((100)(101100))^\infty \not\subset \mathcal{U}_L$. Note that in comparison with the previous example, this time $S \neq \sigma^k(L)$ for every $k \in \mathbb{N}$.

Definition 6.3. Let ν be a kneading sequence. For any admissible finite word $a_n \dots a_1 \in \{0,1\}^n$ define the *cylinder* $[a_n \dots a_1]$ as

$$[a_n \dots a_1] := \{\overleftarrow{s} = \dots s_{n+2}s_{n+1}a_n \dots a_1 : \overleftarrow{s} \text{ is an admissible left-infinite sequence}\}.$$

Lemma 6.4. *If $a_n \dots a_1$ is admissible, then $[a_n \dots a_1]$ is not an empty set.*

Proof. Say that $1a_n \dots a_1$ is not admissible. In that case $1a_n \dots a_1 \succ c_1 \dots c_{n+1}$, so $a_n \dots a_1 \prec c_2 \dots c_{n+1}$, contrary to $a_n \dots a_1$ being admissible. Analogously, the left-infinite tail $1^\infty a_n \dots a_1$ is admissible. ∎

Definition 6.5. Assume X is embedded in the plane with respect to $L = \dots l_2 l_1$ and take an admissible finite word $a_n \dots a_1$. The *top* of the cylinder $[a_n \dots a_1]$ is the left-infinite sequence, denoted by $L_{a_n \dots a_1} \in [a_n \dots a_1]$, such that $L_{a_n \dots a_1} \succeq_L \overleftarrow{s}$ for all $\overleftarrow{s} \in [a_n \dots a_1]$. Analogously we define the *bottom* of $[a_n \dots a_1]$, denoted by $S_{a_n \dots a_1}$, as the smallest left-infinite sequence in $[a_n \dots a_1]$ with respect to the order $\preceq_L$.

Remark 6.6. Note that each cylinder is a compact set (as a subset of the plane). Thus for admissible finite words $a_n \dots a_1$ there always exist $L_{a_n \dots a_1}$ and $S_{a_n \dots a_1}$ (they can be equal).

Lemma 6.7. *Assume X is embedded in the plane with respect to L. For every admissible finite word $a_n \dots a_1$ the arcs $A(L_{a_n \dots a_1})$ and $A(S_{a_n \dots a_1})$ are fully accessible.*

Proof. Take a point $x \in A(L_{a_n \dots a_1})$ and denote by $p_x = \psi_L(L_{a_n \dots a_1})$ (for the definition of ψ_L see the beginning of Chapter 3) the point in the Cantor set C corresponding to the y-coordinate of x. Then the arc

$$Q = \left\{ \left(\pi_0(x), p_x + \frac{t}{2 \cdot 3^{n+1}} \right) : t \in [0,1] \right\}$$

has the property that $Q \cap X = \{x\}$ (see Figure 6.2). When $x \in A(S_{a_n \dots a_1})$, we can similarly construct the arc Q', accessing x from below, such that $Q' \cap X = \{x\}$ and conclude that x is accessible. ∎

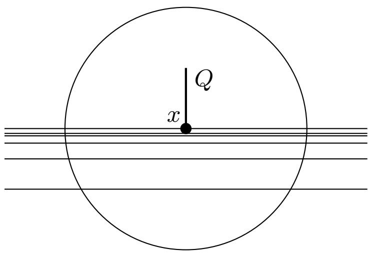


Fig. 6.2. A point at the top of the cylinder $[a_n \dots a_1]$ is accessible by an arc Q.

From Lemma 6.7 it follows in particular that $A(L)$ and $A(S)$ in Examples 6.1 and 6.2 are fully accessible as they are the largest and the smallest arcs respectively among all the arcs in the embedding of X' determined by L.

The following proposition is the first step in determining the set of accessible points of an $\mathcal{E}$-embedding.

PROPOSITION 6.8. *Take $L = \dots l_2 l_1$ and construct the embedding of X with respect to L. Then every point in X with the same symbolic tail as L is accessible. If $A(L)$ is not a spiral point, then $\mathcal{U}_L$ is fully accessible.*

Proof. Take a point $x \in X$, where $\overleftarrow{x} = \dots x_2 x_1$ and there exists $n > 0$ such that $\dots x_{n+2} x_{n+1} = \dots l_{n+2} l_{n+1}$. If $\#_1(x_n \dots x_1)$ and $\#_1(l_n \dots l_1)$ have the same parity, then $\dots l_{n+2} l_{n+1} x_n \dots x_1 = L_{x_n \dots x_1}$, and it is equal to $S_{x_n \dots x_1}$ otherwise. Lemma 6.7, Corollary 4.3 and Remark 3.7 conclude the proof. ∎

DEFINITION 6.9. Let $\varphi, \psi \colon K \to \mathbb{R}^2$ be two embeddings of a continuum K in the plane. We say that the embeddings are *equivalent* if the homeomorphism $\psi \circ \varphi^{-1} \colon \varphi(K) \to \psi(K)$ can be extended to a homeomorphism of the plane.

We denote by φ_L the $\mathcal{E}$-embedding of X such that the arc $A(L)$ is the largest among all basic arcs. In the following proposition we observe that given two left-infinite sequences L^1, L^2 with eventually the same tail, we get equivalent embeddings.

PROPOSITION 6.10. *Let $L^1 = \dots l_2^1 l_1^1$ and $L^2 = \dots l_2^2 l_1^2$ be such that there exists $n \in \mathbb{N}$ such that $l_k^1 = l_k^2$ for every $k > n$. Then the embeddings φ_{L^1} and φ_{L^2} of X are equivalent.*

Proof. If $\#_1(l_n^1 \dots l_1^1)$ and $\#_1(l_n^2 \dots l_1^2)$ are of the same [different] parity, then for every admissible $\overleftarrow{x} = \dots x_2 x_1$ and $\overleftarrow{y} = \dots y_2 y_1$ such that $x_n \dots x_1 = y_n \dots y_1$ it follows that $\overleftarrow{x} \prec_{L^1} \overleftarrow{y}$ if and only if $\overleftarrow{x} \prec_{L^2} \overleftarrow{y}$ [$\overleftarrow{x} \succ_{L^2} \overleftarrow{y}$].

We conclude that $\varphi_{L^2} \circ \varphi_{L^1}^{-1} \colon \varphi_{L^1}(X) \to \varphi_{L^2}(X)$ preserves [reverses] the order in every n-cylinder $[a_n \dots a_1]$. There exists a planar homeomorphism h such that $h|_{\varphi_{L^1}(X)} = \varphi_{L^2}(X)$ and h permutes n-cylinders from the order determined by L^1 to the order determined by L^2. ∎

Now we briefly comment on $\mathcal{E}$-embeddings of X (including the ray $\mathcal{C}$). For the rest of the chapter assume that X is not the Knaster continuum (since then $X = X'$, i.e., $\mathcal{C}$ is contained in the core X'). Let X be embedded in the plane with respect to $L = \dots l_2 l_1 \neq 0^\infty l_n \dots l_1$ for every $n \in \mathbb{N}$. The case when the $\mathcal{E}$-embedding is equivalent to $L = 0^\infty 1$ (the Brucks–Diamond embedding from [11]) will be studied in Chapter 11.

REMARK 6.11. When we study X (i.e., including the ray $\mathcal{C}$), there exist cylinders $[a_n \dots a_1]$ where $a_n \dots a_1 \prec_L c_2 \dots c_{1+n}$, but there is $k \in \{1, \dots, n-1\}$ such that $a_k \dots a_1$ is admissible, $a_k = 1$ and $a_n \dots a_{k+1} = 0^{n-k}$. In that case, $[a_n \dots a_1]$ contains only one basic arc, that is, $[a_n \dots a_1] = \{0^\infty a_n \dots a_1\}$ and $L_{a_n \dots a_1} = S_{a_n \dots a_1} = 0^\infty a_n \dots a_1$.

REMARK 6.12. The ray $\mathcal{C}$ is isolated (when X is not the Knaster continuum), and thus it is fully accessible in any $\mathcal{E}$-embedding of X. On the circle of prime ends, $\mathcal{C}$ corresponds to an open interval with $\bar{0}$ at the centre.

PROPOSITION 6.13. *Take an admissible left-infinite sequence $\overleftarrow{a} = \dots a_2 a_1$ such that $A(\overleftarrow{a}) \not\subset \mathcal{C}$ and $a_n \neq l_n$ for infinitely many $n \in \mathbb{N}$. Then there exist sequences $(\overleftarrow{s}_i)_{i \in \mathbb{N}}$ and $(\overleftarrow{t}_i)_{i \in \mathbb{N}}$ such that $A(\overleftarrow{s}_i), A(\overleftarrow{t}_i) \subset \mathcal{C}$, $\overleftarrow{s}_i, \overleftarrow{t}_i \to \overleftarrow{a}$ as $i \to \infty$ and $\overleftarrow{s}_i \prec_L \overleftarrow{a} \prec_L \overleftarrow{t}_i$.*

Proof. First note that the assumption $A(\overleftarrow{a}) \not\subset \mathcal{C}$ is indeed needed since by Remark 6.12, $\mathcal{C}$ is isolated and thus the statement of the proposition does not hold for basic arcs from $\mathcal{C}$.

Let $(N_i)_{i\in\mathbb{N}}$ be the sequence of natural numbers such that $a_n \neq l_n$ for $n \in \{N_i : i \in \mathbb{N}\}$. Denote

$$\overleftarrow{t_i} := 0^\infty a^*_{N_{2i-1}} a_{N_{2i-1}-1} \dots a_1, \qquad \overleftarrow{s_i} := 0^\infty a^*_{N_{2i}} a_{N_{2i}-1} \dots a_1$$

for every $i\in\mathbb{N}$. For contradiction, if a sequence $\overleftarrow{t_i}$ is not admissible, then $\dots 1 a_{N_{2i-1}-1} \dots a_1 \succ_L \nu$. Thus, $a_{N_{2i-1}-1} \dots a_1 \dots \prec \overleftarrow{c_2}$, which contradicts $a_{N_{2i-1}-1} \dots a_1$ being an admissible word. Thus $\overleftarrow{t_i}$ is an admissible sequence, and the proof goes analogously for $\overleftarrow{s_i}$. Note that $A(\overleftarrow{t_i}), A(\overleftarrow{s_i}) \subset \mathcal{C}$ for every $i \in \mathbb{N}$.

Since $\#_1(a_{N_{2i-1}-1} \dots a_1)$ and $\#_1(l_{N_{2i-1}-1} \dots l_1)$ are of the same parity (the sequences differ on an even number of entries) and $\#_1(a_{N_{2i}-1} \dots a_1)$ and $\#_1(l_{N_{2i}-1} \dots l_1)$ are of different parity (the sequences differ on an odd number of entries), it follows that $\overleftarrow{s_i} \prec_L \overleftarrow{a} \prec_L \overleftarrow{t_i}$ for every $i \in \mathbb{N}$. ∎

Combining Proposition 6.8 with Proposition 6.13 we find that only basic arcs from $\mathcal{U}_L$ or $\mathcal{C}$ can be tops or bottoms of cylinders of $\mathcal{E}$-embeddings of X. Using Proposition 5.4 we obtain the following corollary.

Corollary 6.14. *If $A(L)$ is not a spiral point and X is not Knaster, i.e., $X \neq X'$, then the embedding $\varphi_L(X)$ has exactly two non-degenerate fully accessible arc-components, namely $\mathcal{U}_L$ and $\mathcal{C}$ (however $\mathcal{C} = \mathcal{U}_L$ in the Brucks–Diamond embedding). If $A(L)$ is non-degenerate, there are two remaining points on the circle of prime ends and they correspond either to an infinite canal in X or to an accessible folding point. If $A(L)$ is degenerate then there are no infinite canals in X.* ∎

Remark 6.15. It is easy to check that for the Knaster continuum $X = X'$, only basic arcs from $\mathcal{U}_L$ can be extrema of cylinders in the $\mathcal{E}$-embedding φ_L, i.e., $\mathcal{C}$ is not accessible, except for possibly its endpoint $\bar{0} = (\dots, 0, 0, 0)$. Actually, in $\mathcal{E}$-embeddings the endpoint $\bar{0}$ will always be accessible (see Remark 7.4). So, there is a point on the circle of prime ends corresponding to an accessible $\bar{0}$ and an interval corresponding to a fully accessible line $\mathcal{U}_L$. In particular, there are no simple dense canals.

We return to the embeddings of X' and until the end of this chapter give background for further study. The following statements are going to be often used to determine when an arc-component is fully accessible.

Definition 6.16. Let $\overleftarrow{s} = \dots s_2 s_1$ be an admissible left-infinite sequence. If $\tau_R(\overleftarrow{s}) < \infty$, the tail $\overleftarrow{r}(s) = \dots s_{\tau_R(\overleftarrow{s})+1} s^*_{\tau_R(\overleftarrow{s})} s_{\tau_R(\overleftarrow{s})-1} \dots s_1$ is called the *right neighbour* of $\overleftarrow{s}$, and if $\tau_L(\overleftarrow{s}) < \infty$, the tail $\overleftarrow{l}(s) = \dots s_{\tau_L(\overleftarrow{s})+1} s^*_{\tau_L(\overleftarrow{s})} s_{\tau_L(\overleftarrow{s})-1} \dots s_1$ is called the *left neighbour* of $\overleftarrow{s}$.

Proposition 6.17. *Embed X' in the plane with respect to L. Assume $\overleftarrow{s}$ is at the bottom [top] of some cylinder. If $\overleftarrow{r(s)}$ is not at the top [bottom] of any cylinder, then $A(\overleftarrow{r(s)})$ contains an accessible folding point (see Figure 6.3). An analogous statement holds for $\overleftarrow{l(s)}$.*

Proof. If $\overleftarrow{r(s)}$ is not the top [bottom] of any cylinder, then there exist left-infinite admissible sequences $\overleftarrow{x_i} \succ_L \overleftarrow{r(s)}$ [$\overleftarrow{x_i} \prec_L \overleftarrow{r(s)}$] such that $\overleftarrow{x_i} \to \overleftarrow{r(s)}$ as $i \to \infty$. If $\tau_R(\overleftarrow{x_i}) = \infty$ for infinitely many $i \in \mathbb{N}$, we have found a folding point in $A(\overleftarrow{r(s)})$. So assume without loss of generality that $\tau_R(\overleftarrow{x_i}) < \infty$ for all $i \in \mathbb{N}$. If $\overleftarrow{s} \succ_L \overleftarrow{r(x_i)}$ [$\overleftarrow{s} \prec_L \overleftarrow{r(x_i)}$] for infinitely many $i \in \mathbb{N}$, we get a contradiction to $\overleftarrow{s}$ being the top [bottom] of some cylinder. But then $\overleftarrow{r(x_i)} \prec_L \overleftarrow{r(s)}$ $[\overleftarrow{r(x_i)} \succ_L \overleftarrow{r(s)}]$ for all but finitely many $i \in \mathbb{N}$, which gives a folding point in $A(\overleftarrow{r(s)})$ again. ■

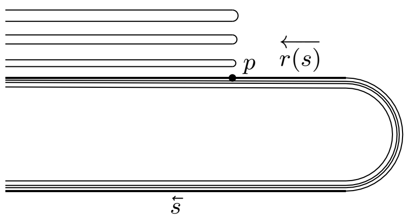


Fig. 6.3. Setup of Proposition 6.17, where p is a folding point.

The corollary below follows directly from Proposition 6.17.

Corollary 6.18. *Let $\mathcal{U} \subset X'$ be an arc-component which contains no folding points and let X' be $\mathcal{E}$-embedded. If there exists a basic arc from $\mathcal{U}$ that is fully accessible, then $\mathcal{U}$ is fully accessible.*

Remark 6.19. When we embed only the core X', there can exist accessible points in $X' \setminus \mathcal{U}_L$; see e.g. Examples 6.1 and 6.2. In these two examples $\mathcal{U}_S \neq \mathcal{U}_L$ and points from $A(S)$ are accessible. In some cases $\mathcal{U}_S$ is fully accessible (see Lemma 9.1), but that is not always the case. In Section 7.2 we explicitly construct examples in which the arc-component $\mathcal{U}_S$ is only partially accessible.

From Lemma 6.7 it follows that the points at the top or bottom of cylinders are accessible. If a point which is not at the top or bottom of any cylinder has a neighbourhood homeomorphic to a Cantor set of arcs, we can conclude that it is not accessible. However, the accessibility of folding points needs to be studied separately, since it is not straightforward to determine if they are accessible or not in a given embedding (see for example Figure 6.4). Thus we need to make a detailed study of conditions for a folding point to be accessible. For instance, in special embeddings of the Knaster continuum in [25] the endpoint is always accessible.

Remark 6.20. When the orbit of c is finite, with (pre)period $n \in \mathbb{N}$, there exist exactly n folding points (see [12]). They are contained in different arc-components which are permuted by the shift homeomorphism. If the orbit of c is periodic, the folding points are endpoints (see [3]). If c is strictly preperiodic, the folding points are not endpoints.

6.1. Tops/bottoms of finite cylinders. In this section we study the symbolics of tops/bottoms of cylinders depending on an $\mathcal{E}$-embedding of X' and we restrict to cases where $L \neq 0^\infty l_n \dots l_1$ for all $n \in \mathbb{N}$.

For $t \in \{0,1\}$, we denote $t^* = 1-t$. For $A = a_1 \dots a_n \in \{0,1\}^n$ denote ${}^*A = a_1^* a_2 \dots a_n$, $A^* = a_1 \dots a_{n-1} a_n^*$ and ${}^*A^* = a_1^* a_2 \dots a_{n-1} a_n^*$.

Definition 6.21. Let ν be a kneading sequence. We say that a finite word $a_1 \dots a_n \in \{0,1\}^n$ is *irreducibly non-admissible* if it is not admissible and $a_2 \dots a_n$ is admissible.

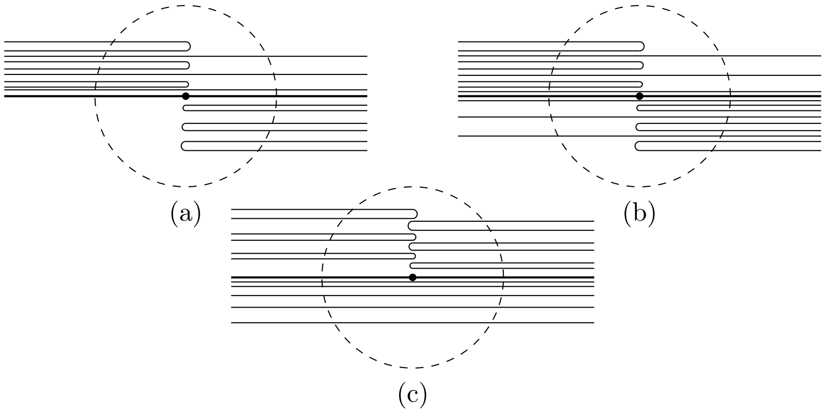


Fig. 6.4. Neighbourhoods of folding points. In cases (a) and (c) the folding point is accessible, while in case (b) it is not.

Definition 6.22. Fix a kneading sequence ν. We say that a finite cylinder $B = [b_n \dots b_1]$ of length $n \in \mathbb{N}$ *alters* $L = \dots l_2 l_1$ if there exist words $(A_i)_{i \in \mathbb{N}}$ such that $\dots A_3 A_2 A_1 = \dots l_{n+2} l_{n+1}$ and the words $A_1 B$ and ${A^*}_i {}^*{A^*}_{i-1} \dots {}^*{A^*}_2 {}^*A_1 B$ are irreducibly non-admissible for every $i \geq 2$.

Proposition 6.23. *If a finite cylinder B alters the admissible sequence L, then L_B or S_B has a different tail than L.*

Proof. Assume B alters L with words A_i as in the definition. If $\#_1(B) - \#_1(l_n \dots l_1)$ is even, then $L_B = \dots {}^*{A^*}_i {}^*{A^*}_{i-1} \dots {}^*{A^*}_2 {}^*A_1 B$. The sequence L_B differs from L on infinitely many places. If $\#_1(B) - \#_1(l_n \dots l_1)$ is odd, then $S_B = \dots {}^*{A^*}_i {}^*{A^*}_{i-1} \dots {}^*{A^*}_2 {}^*A_1 B$. ■

The following example shows that there exist $\mathcal{E}$-embeddings of X' such that none of the extrema of certain cylinders are contained in $\mathcal{U}_L$.

Example 6.24. Let $\nu = (100111011)^\infty$ and $L = (001)^\infty 11$. Note that $S_{10} = (100)^\infty(101) 10 \subset \mathcal{U}_{L_{10}}$ and $L_{10} = (100)^\infty 10 \subset \mathcal{U}_{L_{10}}$. Therefore, $L_{10}, S_{10} \not\subset \mathcal{U}_L$.

In Example 6.24 both extrema belong to the same arc-component. This is not necessarily always the case: see Example 6.26 below.

Proposition 6.25. *If a finite cylinder B is such that $L_B \not\subset \mathcal{U}_L$ or $S_B \not\subset \mathcal{U}_L$, then there exists a finite cylinder B' such that B' alters L.*

Proof. Assume $\#_1(B) - \#_1(l_n \dots l_1)$ is even and $L_B \not\subset \mathcal{U}_L$. Then obviously $B' = B$ alters L. Similarly if $\#_1(B) - \#_1(l_n \dots l_1)$ is odd and $S_B \not\subset \mathcal{U}_L$. So assume $\#_1(B) - \#_1(l_n \dots l_1)$ is even and $S_B \not\subset \mathcal{U}_L$. Then $l^*_{n+1} B$ alters L if $l^*_{n+1} B$ is admissible. If $l^*_{n+1} B$ is not admissible, there exists $i \in \mathbb{N}$ such that $l^*_{n+i} \dots l_n B$ is admissible, since otherwise $S_B = L_B$, which is a contradiction. The word $l^*_{n+i} \dots l_n B$ alters L. Analogously if $\#_1(B) - \#_1(l_n \dots l_1)$ is odd and $L_B \not\subset \mathcal{U}_L$. ■

Example 6.26. Let $\nu = 1001(101)^\infty$ and $L = ((001)(001101))^\infty$. Then $S = S_0 = ((100)(101100))^\infty \not\subset \mathcal{U}_L$. So $B = 0$ alters L and words A_i are divided by brackets.

Now we show there exist $\mathcal{E}$-embeddings with more than two accessible arc-components.

PROPOSITION 6.27. *Assume that ν starts with some finite words $\nu = 1B\ldots = 1ABA\ldots$, where B^* and ABA^* are irreducibly non-admissible. The embedding of X' with respect to $L = (BA)^\infty$ contains at least three tails which are extrema of cylinders.*

Proof. Note that $S = ({}^*B^{**}ABA^*)^\infty$. Take any admissible word D such that $|D| = |A|$ and that $\#_1(D) - \#_1(A)$ is even. Then $S_D = ({}^*ABA^{**}B^*)^\infty D$ and therefore we found three different tails which are extrema of cylinders. ■

The following example shows that it is indeed possible to satisfy the conditions of Proposition 6.27.

EXAMPLE 6.28. Take $\nu = 1001100100111\ldots$, $B = 001$, $A = 0011$, and $L = (BA)^\infty$, which is easily checked to be admissible. For D take e.g. $D = 1111$. Note that $S = ({}^*B^{**}ABA^*)^\infty$ and $S_D = ({}^*ABA^{**}B^*)^\infty D$ and thus we obtain three accessible basic arcs with different tails. If we take e.g. $\nu = (10011001001111)^\infty$, since by Remark 6.20 the only folding points are endpoints and there are no spiral points in X', it follows by Proposition 6.17 that there are three fully accessible non-degenerate dense arc-components. Moreover, none of those arc-components contains an endpoint, so they are all lines. We will return to this particular example in Chapter 7 (Example 7.17).

7. Accessible folding points

In this chapter we study accessibility of folding points which are not at the top or the bottom of any cylinder.

7.1. Accessible endpoints. Let us fix X' and the $\mathcal{E}$-embedding depending on L. Recall that we denote by $\mathcal{U}_L$ the arc-component of $x \in A(L) \subset X'$. By Proposition 6.8, every point with the same symbolic tail as L is accessible.

The following remark is a direct consequence of Proposition 2.8.

REMARK 7.1. If $e \in X'$ is an endpoint with a single itinerary $\overleftarrow{e}$, then there exists a strictly increasing sequence $(n_i)_{i\in\mathbb{N}}$ such that $\bar{e} = \ldots e_{n_i+1}c_1 \ldots c_{n_i}.c_{n_i+1}\ldots = \ldots e_{n_i+1}\nu$ for every $i \in \mathbb{N}$. If an endpoint $e \in X$ has two itineraries, then one of them will have the properties above. That itinerary of e will always be denoted by $\bar{e}$. So in particular by $\overleftarrow{e}$ we always mean the left-itinerary of $\bar{e}$. Note that since the two itineraries are identified by $\sim$ when constructing embeddings of X, this will not affect accessibility of e and thus we may work with either of them.

In this section we work with the concept of an endpoint being capped which is defined below (see Figure 7.1).

DEFINITION 7.2. Let $e \in X'$ be an endpoint with $\tau_L(\overleftarrow{e}) = \infty$ $[\tau_R(\overleftarrow{e}) = \infty]$. We say that a point e is *capped from the left* [*right*] if there exist sequences of admissible itineraries $(\overleftarrow{y}^i)_{i\in\mathbb{N}}, (\overleftarrow{w}^i)_{i\in\mathbb{N}} \subset \{0,1\}^\infty$ such that $\overleftarrow{y}^i, \overleftarrow{w}^i \to \overleftarrow{e}$ as $i \to \infty$, $\overleftarrow{y}^i \prec_L \overleftarrow{e} \prec_L \overleftarrow{w}^i$ for every $i \in \mathbb{N}$ and the arcs $A(\overleftarrow{y}^i)$ and $A(\overleftarrow{w}^i)$ are joined on the left [right].

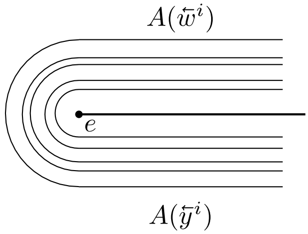


Fig. 7.1. The endpoint e is capped from the left.

Remark 7.3. If $e \in X'$ is a right [left] endpoint which is not capped from the right [left], then e is accessible by a horizontal segment in the plane. Note that if $\overleftarrow{e}$ lies at an extremum of a cylinder (which holds if e.g. e has the same symbolic tail as L), then e is not capped.

Remark 7.4. Let $\nu = 10^\infty$, i.e., $X = X'$ is the Knaster continuum, and let L be arbitrary. Note that any two points $x, y \in X'$ that are $\varepsilon > 0$ close to the point $\bar{0}$ and are identified have the form $x_k x_{k-1} \dots x_1 = y_k y_{k-1} \dots y_1 = 10^{k-1}$ for some $k \in \mathbb{N}$. It follows that either $\overleftarrow{x}, \overleftarrow{y} \prec_L \overleftarrow{0}$ or $\overleftarrow{x}, \overleftarrow{y} \succ_L \overleftarrow{0}$, depending on the parity of $\#_1(l_{k-1} \dots l_1)$. Therefore, the endpoint $\bar{0} \in X'$ is not capped and thus is always accessible in $\mathcal{E}$-embeddings of the Knaster continuum (see Figure 7.2).

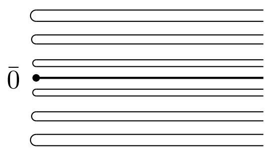


Fig. 7.2. Neighbourhood of the endpoint $\bar{0}$ of the Knaster continuum ($\nu = 10^\infty$) in an $\mathcal{E}$-embedding.

From now on we assume in this section that X' is not the Knaster continuum and thus $\nu \neq 10^\infty$.

It is well known (see e.g. [3]) that X' contains endpoints if and only if the critical point c of a map T is recurrent (i.e., $T^n(c)$ gets arbitrarily close to c as $n \to \infty$).

Definition 7.5. Fix a kneading sequence ν and let $e \in X'$ be an endpoint and thus $\tau_L(\overleftarrow{e}) = \infty$ [$\tau_R(\overleftarrow{e}) = \infty$]. A sequence $(m_i)_{i\in\mathbb{N}} \subset \mathbb{N}$ is called the *complete sequence for* e if for every $n \in \mathbb{N}$ such that $e_n \dots e_1 = c_1 c_2 \dots c_n$ and $\#_1(c_1 c_2 \dots c_n)$ is odd [even] there exists $i \in \mathbb{N}$ such that $m_i = n$.

From $\tau_L(\overleftarrow{e}) = \infty$ (or $\tau_R(\overleftarrow{e}) = \infty$) it follows that the sequence $(m_i)_{i\in\mathbb{N}}$ indeed exists. The main result in this section is that every endpoint of X' (where X' is not the Knaster continuum) which is not contained in $\mathcal{U}_L$ is capped in an $\mathcal{E}$-embedding of X' which is not equivalent to the Brucks–Diamond embedding from [11]. In the proof of Theorem 7.10 we construct an increasing subsequence $(n_i)_{i\in\mathbb{N}} \subset (m_i)_{i\in\mathbb{N}}$ and basic arcs $A(\overleftarrow{x}^{O(i)}), A(\overleftarrow{x}^{I(i)}) \subset \mathcal{R} \subset X'$ such that

$$\overleftarrow{x}^{O(i)} = 1^\infty a^i_k \dots a^i_1 0 c_1 c_2 \dots c_{n_i}, \qquad \overleftarrow{x}^{I(i)} = 1^\infty a^i_k \dots a^i_1 1 c_1 c_2 \dots c_{n_i}, \tag{7.1}$$

and $\overleftarrow{x}^{O(i)} \prec_L \overleftarrow{e} \prec_L \overleftarrow{x}^{I(i)}$ or $\overleftarrow{x}^{I(i)} \prec_L \overleftarrow{e} \prec_L \overleftarrow{x}^{O(i)}$ for some admissible word $a^i_k \dots a^i_1 \in \{0,1\}^k$. Note that the arcs $A(\overleftarrow{x}^{O(i)})$ and $A(\overleftarrow{x}^{I(i)})$ are joined by a left [right] semicircle. Here $\mathcal{R}$ denotes the arc-component of the point $\bar{1}$, which is a dense line in X' independently of the choice of ν (see [10, Proposition 1]).

Remark 7.6. Let $e \in X'$ be an endpoint and thus $\tau_L(\overleftarrow{e}) = \infty$ [$\tau_R(\overleftarrow{e}) = \infty$]. Then $\#_1(c_1 \dots c_{m_i})$ is odd [even] and $\#_1(c_1 \dots c_{m_{i+1}-m_i})$ is even [even] for every $i \in \mathbb{N}$. This follows from the fact that subtracting two odd numbers and two even numbers results in an even number.

Remark 7.7. Fix a kneading sequence ν. Assume that $a_{n-1} \dots a_1 \in \{0,1\}^{n-1}$ is admissible but $a_n \dots a_1 \in \{0,1\}^n$ is not. Then $a_n \dots a_1 \prec c_2 \dots c_{n+1}$.

Lemma 7.8. *Let ν be an admissible kneading sequence. A word $c_2 \dots c_n^*$ is not admissible if and only if either $\#_1(c_2 \dots c_n)$ is odd or there exists $k \in \{3, \dots, n\}$ such that $c_k \dots c_n = c_2 \dots c_{n-k+2}$ and $\#_1(c_k \dots c_n)$ is odd.*

Proof. Assume that $c_2 \dots c_n^*$ is not admissible, so there exists an $i \in \{2, \dots, n\}$ such that $c_i \dots c_n^*$ is not admissible. Take the largest such i and note that $c_i \dots c_n = c_2 \dots c_{n-i+2}$ and $c_2 \dots c_{n-i+2}^* \prec c_2 \dots c_{n-i+2}$. Assume for contradiction that $\#_1(c_2 \dots c_{n-i+2})$ is even. If $c_{n-i+2} = 0$ [$c_{n-i+2} = 1$], it follows that $\#_1(c_2 \dots c_{n-i+1})$ is even [odd] and in both cases $c_2 \dots c_{n-i+2}^* \succ c_2 \dots c_{n-i+2}$ and thus $c_2 \dots c_{n-i+2}^*$ is admissible, a contradiction. ■

Lemma 7.9. *Let ν be an admissible kneading sequence and let $(m_i)_{i\in\mathbb{N}}$ be the complete sequence for an endpoint $e \in X'$. Then for every natural number $k \geq 3$ and $j \in \{0, \dots, m_i\}$, the word $c_k \dots c_{m_{i+1}-m_i}^* c_1 c_2 \dots c_j$ is admissible for every $i \in \mathbb{N}$. Specifically, if $j = 0$, we set $c_1 \dots c_j = \emptyset$.*

Proof. Assume for contradiction that there exist $k \geq 3$ and $j \in \mathbb{N}_0$ such that the word $c_k \dots c_{m_{i+1}-m_i}^* c_1 c_2 \dots c_j$ is not admissible and assume that k is the largest and j is the smallest such index. By the choice of k and j we see that every proper subword of $c_k \dots c_{m_{i+1}-m_i}^* c_1 c_2 \dots c_j$ is admissible. So

$$c_k \dots c_{m_{i+1}-m_i}^* c_1 c_2 \dots c_j = c_2 c_3 \dots c_{m_{i+1}-m_i-k} c_{m_{i+1}-m_i-k+1} \dots c_{m_{i+1}-m_i-k+j+1}^*$$

and $\#_1(c_2 c_3 \dots c_{m_{i+1}-m_i-k+j+1}^*)$ is even by Lemma 7.8. Furthermore, Lemma 7.8 implies that $\#_1(c_k \dots c_{m_{i+1}-m_i}^* = c_2 \dots c_{m_{i+1}-m_i-k-1})$ is even.

If $j = 1$, then both $\#_1(c_k \dots c_{m_{i+1}-m_i}^*)$ and $\#_1(c_k \dots c_{m_{i+1}-m_i}^* c_1)$ are even, which is impossible.

If $j \geq 2$, we deduce from Lemma 7.8 that $\#_1(c_2 \dots c_j)$ is odd. Therefore, $c_2 \dots c_j^* = c_{m_{i+1}-m_i-k+1} \dots c_{m_{i+1}-m_i-k+j+1}$ is not admissible, a contradiction, since $c_2 c_3 \dots c_j^*$ is a subword of ν.

Let $c_1 \dots c_j$ be an empty word, so $c_k \dots c_{m_{i+1}-m_i} = c_2 c_3 \dots c_{m_{i+1}-m_i-k} c_{m_{i+1}-m_i-k+1}$ and $\#_1(c_2 c_3 \dots c_{m_{i+1}-m_i-k} c_{m_{i+1}-m_i-k+1})$ is odd. Let l be the maximal natural number such that $c_{m_{i+1}-m_i+1} \dots c_{m_{i+1}-m_i+l} = c_{m_{i+1}-m_i-k+2} \dots c_{m_{i+1}-m_i-k+l+1}$, i.e.,

$$c_k \dots c_{m_{i+1}-m_i+l} = c_2 c_3 \dots c_{m_{i+1}-m_i-k+l+1}$$

and $c_{m_{i+1}-m_i+l+1} \neq c_{m_{i+1}-m_i-k+l+2}$. Such an l indeed exists since (m_i) is complete. Note that $c_{m_{i+1}-m_i-k+2} \dots c_{m_{i+1}-m_i-k+l+1} = c_1 \dots c_l$ and $\#_1(c_1 \dots c_{l+1})$ is odd because of

Lemma 7.8. Thus $\#_1(c_1 \dots c_l c^*_{l+1})$ is even, so we conclude that $\#_1(c_2 \dots c_{m_{i+1}-m_i-k+l+2})$ is odd. Since $c_2 \dots c^*_{m_{i+1}-m_i-k+l+2} = c_k \dots c_{m_{i+1}-m_i+l+1}$ is admissible, we get a contradiction. ∎

The main idea of the proof of the following theorem is illustrated in Example 7.11.

THEOREM 7.10. *Let $e \in X'$ be an endpoint such that $\tau_R(\overleftarrow{e}) = \infty$ $[\tau_L(\overleftarrow{e}) = \infty]$ and let $L = \dots l_2 l_1 \neq 0^\infty l_n \dots l_1$ be admissible and $\nu \neq 10^\infty$. If L and $\overleftarrow{e}$ have different tails, then e is capped from the right [left].*

Proof. Let $(m_i)_{i\in\mathbb{N}} \subset \mathbb{N}$ be the complete sequence for an endpoint e where $\tau_R(\overleftarrow{e}) = \infty$. The proof works analogously if $\tau_L(\overleftarrow{e}) = \infty$. We will find infinitely many $i \in \mathbb{N}$ such that $\overleftarrow{x}^{O(i)} \succ_L \overleftarrow{e} \succ_L \overleftarrow{x}^{I(i)}$ (or with reversed inequalities) and the arcs $\overleftarrow{x}^{O(i)}$ and $\overleftarrow{x}^{I(i)}$ are joined by a semicircle on the right.

Fix some $i \in \mathbb{N}$ and let $M_1(i) > m_i + 1$ be the smallest natural number such that $l_{M_1(i)} = e^*_{M_1(i)}$. Note that such an $M_1(i)$ exists, otherwise $\overleftarrow{e}$ and L have the same tail.

Assume that $M_1(i) = m_{i+1}$. Note that then we have $e_{M_1(i)-1} \dots e_{m_i+1} = c_2 \dots c_k = 0^\kappa 1 c_{\kappa+2} \dots c_k$ ($c_{\kappa+2} \dots c_k$ may be empty) and $e_{M_1(i)} = 1$. Then we get $l_{M_1(i)} \dots l_{m_i+1} = 0^{\kappa+1} 1 c_{\kappa+3} \dots c_k$, which is not admissible.

Assume that $M_1(i) = m_{i+1} + 1$. By the paragraph above, $M_1(i+1) \neq m_{i+2}$. If $M_1(i+1) = m_{i+2} + 1$, then $l_{m_{i+2}} \dots l_{m_{i+1}+2} l_{m_{i+1}+1} = c_1 \dots c_{m_{i+2}-m_{i+1}+1} c^*_{m_{i+2}-m_{i+1}}$ which is not admissible since $c_1 \dots c_{m_{i+2}-m_{i+1}}$ is even by Remark 7.6. So either $M_1(i) \in \{m_i + 2, \dots, m_{i+1} - 1\}$ or there is $k \in \mathbb{N}$ such that $M_1(i+k) = M_1(i)$. Note that there are infinitely many $i \in \mathbb{N}$ such that $M_1(i) \in \{m_i + 2, \dots, m_{i+1} - 1\}$, and from now on we work with such an $i \in \mathbb{N}$.

If both of the following sequences are admissible, we set

$$\overleftarrow{x}^{O(i)} = 1^\infty e^*_{M_1(i)} e_{M_1(i)-1} \dots e_{m_i+2} 0 e_{m_i} \dots e_1,$$
$$\overleftarrow{x}^{I(i)} = 1^\infty e^*_{M_1(i)} e_{M_1(i)-1} \dots e_{m_i+2} 1 e_{m_i} \dots e_1.$$

(a) Assume that $\#_1(e_{m_i} \dots e_1)$ and $\#_1(l_{m_i} \dots l_1)$ have the same parity and $e_{m_i+1} = l_{m_i+1} = 0$. Then $\overleftarrow{e} \succ_L \overleftarrow{x}^{I(i)}$. Because $l_{M_1(i)-1} \dots l_{m_i+2} = e_{M_1(i)-1} \dots e_{m_i+2}$, the parities of $\#_1(e_{M_1(i)-1} \dots e_1)$ and $\#_1(l_{M_1(i)-1} \dots l_1)$ are the same, and because $l_{M_1(i)} = e^*_{M_1(i)}$, it follows that $\overleftarrow{x}^{O(i)} \succ_L \overleftarrow{e}$.

(b) Assume that $\#_1(e_{m_i} \dots e_1)$ and $\#_1(l_{m_i} \dots l_1)$ have the same parity and $e_{m_i+1} = l_{m_i+1} = 1$. Then $\overleftarrow{e} \succ_L \overleftarrow{x}^{O(i)}$. Because $l_{M_1(i)-1} \dots l_{m_i+2} = e_{M_1(i)-1} \dots e_{m_i+2}$, the parities of $\#_1(e_{M_1(i)-1} \dots e_1)$ and $\#_1(l_{M_1(i)-1} \dots l_1)$ are the same, and because $l_{M_1(i)} = e^*_{M_1(i)}$, it follows that $\overleftarrow{x}^{I(i)} \succ_L \overleftarrow{e}$.

(c) Assume that $\#_1(e_{m_i} \dots e_1)$ and $\#_1(l_{m_i} \dots l_1)$ have the same parity and $e_{m_i+1} = 1 \neq 0 = l_{m_i+1}$. Then we have $\overleftarrow{x}^{O(i)} \succ_L \overleftarrow{e}$. Since the parities of $\#_1(e_{M_1(i)-1} \dots e_1)$ and $\#_1(l_{M_1(i)-1} \dots l_1)$ are different and $l_{M_1(i)} = e^*_{M_1(i)}$, it follows that $\overleftarrow{e} \succ_L \overleftarrow{x}^{I(i)}$.

(d) Assume that $\#_1(e_{m_i} \dots e_1)$ and $\#_1(l_{m_i} \dots l_1)$ have the same parity and $e_{m_i+1} = 0 \neq 1 = l_{m_i+1}$. Then we have $\overleftarrow{x}^{I(i)} \succ_L \overleftarrow{e}$. Since the parities of $\#_1(e_{M_1(i)-1} \dots e_1)$ and $\#_1(l_{M_1(i)-1} \dots l_1)$ are different and $l_{M_1(i)} = e^*_{M_1(i)}$, it follows that $\overleftarrow{e} \succ_L \overleftarrow{x}^{O(i)}$.

Note that if $\#_1(e_{m_i} \dots e_1)$ and $\#_1(l_{m_i} \dots l_1)$ are of different parities, then all the inequalities in cases (a)–(d) are reversed and we use analogous arguments to conclude that either $\overleftarrow{x}^{O(i)} \prec_L \overleftarrow{e} \prec_L \overleftarrow{x}^{I(i)}$ or $\overleftarrow{x}^{I(i)} \prec_L \overleftarrow{e} \prec_L \overleftarrow{x}^{O(i)}$.

Now assume that one of $e^*_{M_1(i)}e_{M_1(i)-1}\dots e_{m_i+2}e^{(*)}_{m_i+1}\dots e_1$ is not admissible (where $s^{(*)}$ means s^* or s). Then we set $x^{O(i)}_{M_1(i)} = x^{I(i)}_{M_1(i)} = e_{M_1(i)}$. If $e_{M_1(i)+1} = l_{M_1(i)+1}$, then we set $x^{O(i)}_{M_1(i)+1} = x^{I(i)}_{M_1(i)+1} = e^*_{M_1(i)+1}$ and we argue that $e^*_{M_1(i)+1}e_{M_1(i)}\dots e_{m_i+2}e^{(*)}_{m_i+1}\dots e_1 = e^*_{M_1(i)+1}10^{\kappa-1}1\dots e_1$ are admissible words. Indeed, $e_{M_1(i)}\dots e_{m_i+2}e^{(*)}_{m_i+1}\dots e_1$ is admissible by Lemma 7.9. If $e^*_{M_1(i)+1}10^{\kappa-1}1\dots$ were not admissible, then $T^3(c) > T^4(c)$, which contradicts T being non-renormalizable. So the following sequences are admissible:

$$\overleftarrow{x}^{O(i)} = 1^\infty e^*_{M_1(i)+1}e_{M_1(i)}\dots e_{m_i+2}0e_{m_i}\dots e_1,$$
$$\overleftarrow{x}^{I(i)} = 1^\infty e^*_{M_1(i)+1}e_{M_1(i)}\dots e_{m_i+2}1e_{m_i}\dots e_1,$$

and $\overleftarrow{x}^{O(i)} \succ_L \overleftarrow{e} \succ_L \overleftarrow{x}^{I(i)}$ or $\overleftarrow{x}^{I(i)} \succ_L \overleftarrow{e} \succ_L \overleftarrow{x}^{O(i)}$.

Assume that $e^*_{M_1(i)+1} = l_{M_1(i)+1}$. Set $x^{O(i)}_{M_1(i)+1} = x^{I(i)}_{M_1(i)+1} = e_{M_1(i)+1}$. Then the words $e_{M_1(i)+1}e_{M_1(i)}\dots e^{(*)}_{m_i+1}e_{m_i}\dots e_1$ are admissible by Lemma 7.9 if $M_1(i)+1 \neq m_{i+1}-1$. Now let $M_1(i) = m_{i+1}-2$. By the assumption at the beginning of this paragraph, at least one of the words $e^*_{m_{i+1}-2}e_{m_{i+1}-3}\dots e_{m_i+1}e^{(*)}_{m_i+1}\dots e_1$ is not admissible.

- Say $\nu = 10^\kappa 1\dots$, where $\kappa > 1$. By Lemma 7.9, $e^*_{m_{i+1}-2}e_{m_{i+1}-3}\dots = c_3^*c_4c_5\dots = 10^{\kappa-2}1\dots$ is always admissible, a contradiction.
- Say $\nu = 10(11)^n0\dots$. Then again $e^*_{m_{i+1}-2}e_{m_{i+1}-3}\dots = 0(11)^{n-1}10\dots$ is always admissible, because $\#_1(0(11)^{n-1}1)$ is odd, a contradiction.

Thus we conclude that caps have been constructed except in the following cases: (one of) $e^*_{M_1(i)}e_{M_1(i)-1}\dots e_{m_i+2}e^{(*)}_{m_i+1}\dots e_1$ is not admissible and $e^*_{M_1(i)+1} = l_{M_1(i)+1}$.

For $j > 1$ denote by $M_j(i)$ the smallest $k \in \mathbb{N}$ such that $k > M_{j-1}(i)$ and $e^*_k = l_k$. By the previous paragraph, it follows that $M_2(i) < m_{i+1}-1$. Take the largest $N \in \mathbb{N}$ such that $M_N(i) < m_{i+1}-1$. Note that for odd $j \in \{1,\dots N\}$ and

$$\overleftarrow{x}^{O(i)} = 1^\infty e^*_{M_j(i)}e_{M_j(i)-1}\dots e_{m_i+2}0e_{m_i}\dots e_1,$$
$$\overleftarrow{x}^{I(i)} = 1^\infty e^*_{M_j(i)}e_{M_j(i)-1}\dots e_{m_i+2}1e_{m_i}\dots e_1,$$

we have $\overleftarrow{x}^{O(i)} \succ_L \overleftarrow{e} \succ_L \overleftarrow{x}^{I(i)}$ or $\overleftarrow{x}^{I(i)} \succ_L \overleftarrow{e} \succ_L \overleftarrow{x}^{O(i)}$. The conclusion follows from the fact that $\#_1(l_{M_j(i)-1}\dots l_{m_i+2})$ and $\#_1(e_{M_j(i)-1}\dots e_{m_i+2})$ are of the same parity since j is odd.

Assume that for every odd $j \in \{1,\dots,N\}$, $1^\infty e^*_{M_j(i)}e_{M_j(i)-1}\dots e_{m_i+2}e^{(*)}_{m_i+1}e_{m_i}\dots e_1$ is not admissible. If $M_{j+1}(i) > M_j(i)+1$, we set

$$\overleftarrow{x}^{O(i)} = 1^\infty e^*_{M_j(i)+1}e_{M_j(i)}\dots e_{m_i+2}0e_{m_i}\dots e_1,$$
$$\overleftarrow{x}^{I(i)} = 1^\infty e^*_{M_j(i)+1}e_{M_j(i)}\dots e_{m_i+2}1e_{m_i}\dots e_1,$$

and argue that both are admissible as in preceding paragraphs. Calculations as above give $\overleftarrow{x}^{O(i)} \succ_L \overleftarrow{e} \succ_L \overleftarrow{x}^{I(i)}$ or $\overleftarrow{x}^{I(i)} \succ_L \overleftarrow{e} \succ_L \overleftarrow{x}^{O(i)}$.

The situation left to consider is when $1^\infty e^*_{M_j(i)}e_{M_j(i)-1}\dots e_{m_i+2}e^{(*)}_{m_i+1}e_{m_i}\dots e_1$ are not admissible and $M_{j+1}(i) = M_j(i)+1$ for every odd $j \in \{1,\dots,N\}$. Note that N must be even. Otherwise $1^\infty e^*_{m_{i+1}-2}e_{m_{i+1}-3}\dots e_{m_i+2}e^{(*)}_{m_i+1}e_{m_i}\dots e_1$ are not admissible and we have already argued that this is not possible.

Thus we conclude that L is of the form

$$\dots e_{M_N(i)+1}e^*_{M_N(i)}e^*_{M_N(i)-1}e_{M_N(i)-2}\dots e_{M_1(i)+2}e^*_{M_1(i)+1}e^*_{M_1(i)}e_{M_1(i)-1}\dots e_{m_i+2}l_{m_i+1}\dots l_1.$$

Note that $\#_1(e^*_{M_N(i)}e^*_{M_N(i)-1}e_{M_N(i)-2}\dots e_{M_1(i)+2}e^*_{M_1(i)+1}e^*_{M_1(i)}e_{M_1(i)-1}\dots e_{m_i+2})$ has the same parity as $\#_1(e_{M_N(i)}e_{M_N(i)-1}e_{M_N(i)-2}\dots e_{M_1(i)+2}e_{M_1(i)+1}e_{M_1(i)}e_{M_1(i)-1}\dots e_{m_i+2})$, because changes in L compared with $\overleftarrow{e}$ always appear in pairs (of two consecutive letters). We set

$$\overleftarrow{x}^{O(i)} = 1^\infty e^*_{m_{i+1}-1}e_{m_{i+1}-2}\dots e_{m_i+2}0e_{m_i}\dots e_1,$$
$$\overleftarrow{x}^{I(i)} = 1^\infty e^*_{m_{i+1}-1}e_{m_{i+1}-2}\dots e_{m_i+2}1e_{m_i}\dots e_1,$$

and note that $\overleftarrow{x}^{O(i)} \succ_L \overleftarrow{e} \succ_L \overleftarrow{x}^{I(i)}$ (or with reversed inequalities). Also note that $\overleftarrow{x}^{I(i)}$ and $\overleftarrow{x}^{O(i)}$ set in such a way are always admissible by Lemma 7.9 and since $e_{m_{i+1}-1} = c_2 = 0$.

We have constructed a sequence corresponding to the basic arc with the following properties: $\overleftarrow{x}^{O(i)} \prec_L \overleftarrow{e} \prec_L \overleftarrow{x}^{I(i)}$ or $\overleftarrow{x}^{I(i)} \prec_L \overleftarrow{e} \prec_L \overleftarrow{x}^{O(i)}$, $\overleftarrow{x}^{O(i)}$ and $\overleftarrow{x}^{I(i)}$ are joined on the right and $\overleftarrow{x}^{O(i)}, \overleftarrow{x}^{I(i)} \to \overleftarrow{e}$ as $i \to \infty$. Since that can be done for infinitely many $i \in \mathbb{N}$, this concludes the proof. ∎

Example 7.11. Let X' be an inverse limit space with the corresponding kneading sequence $\nu = (100111101011010111)^\infty$. Let us study the cappedness of the endpoint $e \in X'$ with the itinerary $\bar{e} = (100111101011010111)^\infty.(100111101011010111)^\infty$ in the embedding determined by $L = (010111110011100111)^\infty$. It follows that $\overleftarrow{x}^{O(i)} \prec_L \overleftarrow{e}$, because $\#_1(100111101011010111)$ and $\#_1(010111110011100111)$ are both even. Note that $M_1(i) := m_i+5$ is the smallest index strictly greater than m_i+1 such that $e^*_{M_1(i)} = l_{M_1(i)}$. We obtain the following situation:

$$\dots(1\mathbf{00}1111\mathbf{01}011\mathbf{01}0111)(100111101011010111)^i = \overleftarrow{e},$$
$$\dots(0\mathbf{10}1111\mathbf{10}011\mathbf{10}0111)(010111110011100111)^i = L,$$
$$1^\infty(1\mathbf{10}1111\mathbf{01}011\mathbf{01}0110)(100111101011010111)^i = \overleftarrow{x}^{O(i)},$$
$$1^\infty(1\mathbf{10}1111\mathbf{01}011\mathbf{01}0111)(100111101011010111)^i = \overleftarrow{x}^{I(i)},$$

where the bold type indicates the letters of $\overleftarrow{e}$ and L which differ for indices larger than m_i. Note that $M_3(i) = m_i + 10$ but the word $00110 = e^*_{m_i+10}e_{m_i+9}\dots e_{m_i+6}$ is not admissible and thus we need to set $x^{O(i)}_{M_3(i)} = x^{I(i)}_{M_3(i)} = e_{M_3(i)} = 1$. Note that $M_5(i) = m_i+17 = m_{i+1}-1$. Thus we set $x^{O(i)}_{M_5(i)} = x^{I(i)}_{M_5(i)} = e^*_{M_5(i)}$. Because $\#_1(e_{m_i+16}\dots e_1)$ and $\#_1(l_{m_i+16}\dots l_1)$ are of the same parity we obtain $\overleftarrow{e} \prec_L \overleftarrow{x}^{I(i)}$. Lemma 7.9 again ensures that every subword of $\overleftarrow{x}^{O(i)}$ is admissible. Therefore the points $x^{O(i)}, x^{I(i)} \in X'$ cap the point e from the right.

If an endpoint e is capped, we still cannot conclude that it is not accessible: see e.g. Figure 7.3. However, if we know that the length of basic arcs arbitrary close to $\overleftarrow{e}$ has a lower bound, the conclusion follows. Therefore we introduce the notion of long-branchedness.

Definition 7.12. Let $T\colon [0,1] \to [0,1]$ be a continuous map. A *lap* of T is a maximal interval of monotonicity of T, and a *branch* of T is the image of a lap. We say that T is *long-branched* if there exists $\delta > 0$ such that the length of all branches of T^n is larger than δ for all $n \in \mathbb{N}$.

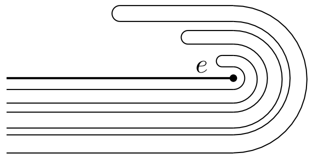


Fig. 7.3. Neighbourhood of an endpoint e. Note that e is capped but also accessible.

REMARK 7.13. Note that if the critical point of T is periodic, then T is long-branched.

COROLLARY 7.14. *Assume $T \neq T_2$ is long-branched and let $e \in X'$ be an endpoint of X'. Assume X' is embedded in the plane with respect to L where $A(L) \not\subset \mathcal{C}$. If $\overleftarrow{e}$ and L have different tails, then e is not accessible.*

Proof. By Theorem 7.10, $e \in X'$ is capped. Since T is long-branched, there exists $\delta > 0$ such that the lengths of $A(\overleftarrow{y}^i)$ and $A(\overleftarrow{w}^i)$ (see Definition 7.2) are greater than δ. It follows that e is not accessible. ■

We merge information from this and the preceding section and give some interesting examples of embeddings of some X'.

EXAMPLE 7.15. Let $\nu = (101)^\infty$ and let $L = (01^k)^\infty$ for any $k \geq 2$. Take an admissible $B = a_n \dots a_1 \in \{0,1\}^n$ for some $n \in \mathbb{N}$. If $l_{n+1}B$ is not admissible, then $\dots l_{n+3}l^*_{n+2}l^*_{n+1}B$ is admissible by the choice of k and since every non-admissible word for $\nu = (101)^\infty$ contains 00. The tail L is thus altered by no finite admissible word (recall Definition 6.22). Therefore, $L_{a_n\dots a_1} = S_{a_n\dots a_1} \subset \mathcal{U}_L$. Since B is an arbitrary finite admissible word, we conclude that $\mathcal{U}_L$ is fully accessible and it is the only non-degenerate accessible set. By Corollary 7.14, the endpoints of X' are not accessible. The remaining point on the circle of prime ends corresponds to a simple dense canal.

EXAMPLE 7.16. Let $\nu = (101)^\infty$ and let $L = (01)^\infty$. Note that $S = (10)^\infty \not\subset \mathcal{U}_L$ and $S = S_0$. Thus, $B = 0$ alters L (recall Definition 6.22; here $A_1 = 0$, $A_i = 01$ for all $i \geq 2$). Since ν is periodic, it follows from Corollary 6.18 that both $\mathcal{U}_L$ and $\mathcal{U}_S$ are fully accessible. As in the example above we can show that no other point from X' is accessible. We conclude that there are two simple dense canals with shores $\mathcal{U}_L$ and $\mathcal{U}_S$.

EXAMPLE 7.17. Take $\nu = (10011001001111)^\infty$, $B = 001$, $A = 0011$, $D = 1111$ and $L = (BA)^\infty$ as in Example 6.28. Recall that at least three arc-components (which are dense lines) are fully accessible. Further calculations show that no other tail can be the top or the bottom of a cylinder. By Corollary 7.14 the endpoints of X' are not accessible. Therefore, the remaining three points on the circle of prime ends correspond to three simple dense canals with shores from pairwise different fully accessible arc-components which are lines. In comparison, the Brucks–Diamond embedding of X' contains seven fully accessible arc-components which are shores of seven simple dense canals (see Chapter 11 or [8]).

7.2. Accessible folding points when ν is preperiodic. First we state a characterization of folding points which we will use implicitly in this section very often. Let $\omega(c)$ denote the set of all accumulation points of the forward orbit of the critical point c by the map T.

PROPOSITION 7.18 ([24, Theorem 2.2]). *A point $x \in X$ is a folding point if and only if $\pi_n(x) \in \omega(c)$ for every $n \in \mathbb{N}$.*

In this section we assume that $\nu = c_1 \dots c_k (c_{k+1} \dots c_{k+n})^\infty$ and that $c_k \neq c_{k+n}$, since otherwise also $\nu = c_1 \dots c_{k-1}(c_k \dots c_{k+n-1})^\infty$. By Remark 6.20 the space X' contains n folding points which are not endpoints with symbolic descriptions

$$\sigma^i((c_{k+1} \dots c_{k+n})^\infty.(c_{k+1} \dots c_{k+n})^\infty)$$

for $i \in \{1, \dots, n\}$. In this section we study the accessibility of folding points that are not contained in extrema of cylinders in $\mathcal{E}$-embeddings of X' when ν is preperiodic.

Let $Q \subset \mathbb{R}^2$ be an arc. From now on let $\operatorname{Int}(Q)$ denote the points from Q which are not endpoints of Q.

REMARK 7.19. Let $\nu = c_1 \dots c_k (c_{k+1} \dots c_{k+n})^\infty$ and let $p \in X'$ be a folding point. Then an arc-component of p in X' can contain at most one folding point. Also, since $c_k \neq c_{n+k}$, we have $p \in \operatorname{Int}(A(\overleftarrow{p}))$. In particular, every folding point $p \in X'$ has a unique two-sided infinite itinerary of zeros and ones assigned to it.

The following result restricts the search for accessible folding points which are not tops/bottoms of cylinders to the case where $\nu = 10(c_3 \dots c_{n+2})^\infty$, i.e., $k = 2$.

PROPOSITION 7.20. *Assume c is preperiodic and such that $T^3(c)$ is not periodic. Embed X' in the plane with respect to $L \neq 0^\infty l_n \dots l_1$. A folding point $p \in X'$ is accessible if and only if the basic arc $A(\overleftarrow{p})$ is the top or the bottom of a finite cylinder.*

Proof. Note that $\nu = c_1 \dots c_k (c_{k+1} \dots c_{k+n})^\infty$ where $k > 2$. Take a folding point $p \in X'$ with the symbolic description

$$\bar{p} = (c_{k+1} \dots c_{k+n})^\infty c_{k+1} \dots c_{k+i} . c_{k+i+1} \dots c_{k+n} (c_{k+1} \dots c_{k+n})^\infty$$

and assume it is not on the top or bottom of any cylinder in X'. Denote $\pi_0(A(\overleftarrow{p})) =: [T^l(c), T^r(c)]$. By Remark 7.19 we have $\pi_0(p) \in (T^l(c), T^r(c))$.

For $M \geq 0$ denote by $p^M \in X'$ any point with the symbolic description

$$\bar{p}^M := \dots c_1 \dots c_k (c_{k+1} \dots c_{k+n})^M c_{k+1} \dots c_{k+i} . c_{k+i+1} \dots c_{k+n} (c_{k+1} \dots c_{k+n})^\infty .$$

Note that the points p^M converge to p as $M \to \infty$ and the corresponding basic arcs $A(\overleftarrow{p}^M)$ project to $[T^l(c), T^{k+i+1}(c)]$ (we refer to these as *left*) or $[T^{k+i+1}(c), T^r(c)]$ (referred to as *right*) depending on the parity of M. We will find *long* basic arcs (i.e., arcs projecting with π_0 also to $[T^l(c), T^r(c)]$) converging to $A(\overleftarrow{p})$ from both sides. Since c is preperiodic, there exists a neighbourhood U of $A(\overleftarrow{p})$ which contains only basic arcs which project to $[T^l(c), T^r(c)]$, $[T^l(c), T^{k+i+1}(c)]$ or $[T^{k+i+1}(c), T^r(c)]$ (i.e., only long or left/right arcs).

Assume that all but finitely many long arcs in U are greater than $A(\overleftarrow{p})$. Since $k > 2$, note that for every $M > 0$ the basic arcs $1^\infty c_k (c_{k+1} \dots c_{k+n})^M c_{k+1} \dots c_{k+i}$ are long. Since $c_k \neq c_{k+n}$, both $1^\infty c_k (c_{k+1} \dots c_{k+n})^M c_{k+1} \dots c_{k+i} \succ_L \overleftarrow{p}$ and $\overleftarrow{p}^M \succ_L \overleftarrow{p}$. Thus, $A(\overleftarrow{p})$ is at the bottom of some cylinder, a contradiction. The proof goes analogously if all but finitely many long arcs are smaller than $A(\overleftarrow{p})$. ∎

Therefore, by Proposition 7.20, if we want to find accessible folding points which are not at the top/bottom of any cylinder, it is enough to study the cases $\nu = 10(c_3 \dots c_{n+2})^\infty$ where $c_{n+2} = 1$.

REMARK 7.21. Assume c is preperiodic and p is an accessible folding point of an embedding of X'. By Corollary 4.3 and since every arc-component contains at most one folding point, only the following three cases can occur:

(1) $\bar{p}$ is the top or the bottom of some cylinder; then $\mathcal{U}_p$ is fully accessible.
(2) $\bar{p}$ is not the top or the bottom of any cylinder, but $\overleftarrow{r(p)}$ or $\overleftarrow{l(p)}$ is; then one component of $\mathcal{U}_p \setminus \{p\}$ is fully accessible, and the other is not accessible (see Figure 6.3).
(3) $\bar{p}$, $\overleftarrow{r(p)}$ and $\overleftarrow{l(p)}$ are not extrema of any cylinder; then c is order reversing and p is the only accessible point of $\mathcal{U}_p$ (see Figure 6.4(c)).

DEFINITION 7.22. We say that an accessible folding point p is accessible of *Type i* if it satisfies the condition (i) from Remark 7.21 for $i \in \{1, 2, 3\}$.

As it turns out, all types of accessible folding points can occur in $\mathcal{E}$-embeddings. In the following chapters we describe how they can be constructed in the preperiodic orbit case (when $T^3(c)$ is periodic) and give examples of such constructions. We will see that the standard Brucks–Diamond embedding does not allow Type 3 folding points for any X' (see Chapter 11).

7.2.1. Type 2. First we give examples of X' which cannot be $\mathcal{E}$-embedded with Type 2 folding points. Then we show in general how to construct a Type 2 accessible folding point and give an example of such construction in both the order preserving and the order reversing case.

LEMMA 7.23. *Let $\nu = 10(c_3 \dots c_{n+2})^\infty$ and assume that $c_i^* c_{i+1} \dots c_{n+2}(c_3 \dots c_{n+2})^M$ is admissible for all $i \in \{3, \dots, n+1\}$ and for all but finitely many $M \in \mathbb{N}$. Then no folding point is Type 2 in any $\mathcal{E}$-embedding of X' which is non-equivalent to the Brucks–Diamond ($L = 0^\infty 1$) embedding.*

Proof. Take a folding point $p \in X'$ with the symbolic description

$$\bar{p} = (c_3 \dots c_{n+2})^\infty . (c_3 \dots c_{n+2})^\infty.$$

We will try to reconstruct L which embeds p as Type 2 and see that this is not possible.

Assume first that $\#_1(c_3 \dots c_{n+2})$ is odd and for some natural number M we have (the following, possibly with reversed inequalities, needs to be satisfied in order for p to be a Type 2 folding point; see Figure 7.5 below):

$$\begin{aligned}
\dots 0(c_3 \dots c_{n+2})^M &\succ_L \dots 1(c_3 \dots c_{n+2})^M, \\
\dots c_i^* c_{i+1} \dots c_{n+2}(c_3 \dots c_{n+2})^M &\prec_L \dots c_i c_{i+1} \dots c_{n+2}(c_3 \dots c_{n+2})^M, \\
\dots 0(c_3 \dots c_{n+2})^{M+k} &\prec_L \dots 1(c_3 \dots c_{n+2})^{M+k}, \\
\dots c_i^* c_{i+1} \dots c_{n+2}(c_3 \dots c_{n+2})^{M+k} &\prec_L \dots c_i c_{i+1} \dots c_{n+2}(c_3 \dots c_{n+2})^{M+k}, \\
\dots 0(c_3 \dots c_{n+2})^{M+N} &\succ_L \dots 1(c_3 \dots c_{n+2})^{M+N}, \\
\dots c_{n+1}^* c_{n+2}(c_3 \dots c_{n+2})^{M+N} &\prec_L \dots c_{n+1} c_{n+2}(c_3 \dots c_{n+2})^{M+N},
\end{aligned}$$

for all $i \in \{3, \dots, n+1\}$ and all $k \in \{1, \dots, N-1\}$, where the natural number $N > 1$ is even. If $\#_1((c_3 \dots c_{n+2})^M)$ is of the same parity as $\#_1(l_{Mn} \dots l_1)$, then $l_{Mn+1} = 0$, and if $\#_1((c_3 \dots c_{n+2})^M)$ is of different parity than $\#_1(l_{Mn} \dots l_1)$, then $l_{Mn+1} = 1$. In any case, $\#_1(c_{n+2}(c_3 \dots c_{n+2})^M)$ is of different parity than $\#_1(l_{Mn+1} \dots l_1)$, so $l_{Mn+2} = c^*_{n+1}$. So $\#_1(c_{n+1}c_{n+2}(c_3 \dots c_{n+2})^M)$ is of the same parity as $\#_1(l_{Mn+2}l_{Mn+1} \dots l_1)$ and thus $l_{Mn+3} = c_n$. Continuing, we get

$$l_{(M+N)n+2} \dots l_{Mn+2} = c^*_{n+1}c^*_{n+2}(c_3 \dots c_{n+2})^{N-1}c_3 \dots c_n c^*_{n+1}.$$

Since $L \subset X'$, it follows that $c^*_{n+1} = 1$, $\#_1(c_3 \dots c_n)$ is even and the word on the right side of the last displayed equation is equal to $10(c_3 \dots c_{n+2})^{N-1}c_3 \dots c_n c^*_{n+1}$. Note that $\#_1(10(c_3 \dots c_{n+2})^{N-1}c_3 \dots c_n c_{n+1})$ is even and thus $10(c_3 \dots c_{n+2})^{N-1}c_3 \dots c_n c^*_{n+1}$ is not admissible by Lemma 7.8, a contradiction.

Assume that $\#_1(c_3 \dots c_{n+2})$ is even. Note that in this case N is not necessarily even, but now the conclusion $c_{n+1} = 0$ implies that $\#_1(c_3 \dots c_n)$ is odd. We continue with arguments as in the paragraphs above. Since $\#_1(c_3 \dots c_{n+2})$ is even, the word $\#_1(10(c_3 \dots c_{n+2})^{N-1}c_3 \dots c_n c_{n+1})$ is even and thus $10(c_3 \dots c_{n+2})^{N-1}c_3 \dots c_n c^*_{n+1}$ is not admissible by Lemma 7.8, a contradiction.

Note that the proof works analogously for other folding points from the space X'. ■

Next we give examples of preperiodic ν where no folding point can be $\mathcal{E}$-embedded as Type 2, except possibly using the Brucks–Diamond embedding (see Chapter 11), in particular the rational endpoint case.

Example 7.24. The assumptions of Lemma 7.23 are satisfied e.g. for $\nu = 10(0^\alpha 1^\beta)$ for all $\alpha, \beta \in \mathbb{N}$.

Lemma 7.25 (Order preserving case). *Let $\nu = 10(c_3 \dots c_{n+2})^\infty$, $c_{n+2} = 1$, and assume $\#_1(c_3 \dots c_{n+2})$ even. Let $\bar{p} = (c_3 \dots c_{n+2})^\infty c_3 \dots c_i . c_{i+1} \dots c_{n+2}(c_3 \dots c_{n+2})^\infty$ be a symbolic description of a folding point $p \in X'$. Then p is a Type 2 folding point if and only if there exists a natural number M such that*

$$\dots c^*_j c_{j+1} \dots c_{n+2}(c_3 \dots c_{n+2})^{M+N}c_3 \dots c_i \succ_L \dots c_j c_{j+1} \dots c_{n+2}(c_3 \dots c_{n+2})^{M+N}c_3 \dots c_i$$

for all $N \in \mathbb{N}$ and all $j \in \{3, \dots, 1+n\}$ for which $c^*_j c_{j+1} \dots c_{n+2}(c_3 \dots c_{n+2})^{M+N}c_3 \dots c_i$ is admissible, and

$$\dots 0(c_3 \dots c_{n+2})^{M+N'}c_3 \dots c_i \prec_L \dots 1(c_3 \dots c_{n+2})^{M+N'}c_3 \dots c_i$$

for infinitely many $N' \in \mathbb{N}$, or the whole statement with reversed inequalities.

Proof. Assume that $\#_1(c_3 \dots c_i)$ is odd [even]. Note that

$$\dots c^*_j c_{j+1} \dots c_{n+2}(c_3 \dots c_{n+2})^{M+N}c_3 \dots c_i$$

are exactly the itineraries of all “long" basic arcs in a sufficiently small cylinder around $A(\bar{p})$, and

$$\dots 0(c_3 \dots c_{n+2})^{M+N'}c_3 \dots c_i$$

are the itineraries of all “left” [“right”] basic arcs in the same cylinder (see Figure 6.3). Here we use terms “long” and “left” [“right”] as in the proof of Proposition 7.20, i.e., if we denote $\pi_0(A(\bar{p})) = [T^l(c), T^r(c)]$, then long basic arcs are the ones which project to $[T^l(c), T^r(c)]$ and left [right] basic arcs are the ones which project to $[T^l(c), \pi_0(p)]$ $[[\pi_0(p), T^r(c)]]$. Note that there exists a neighbourhood of $A(\overleftarrow{p})$ which contains only long and left [right] basic arcs. The condition in the statement forces that all long basic arcs are above $A(\overleftarrow{p})$ and infinitely many left [right] are below $A(\overleftarrow{p})$. The statement also holds true if all long basic arcs were forced to be under $A(\overleftarrow{p})$ and infinitely many left [right] are above $A(\overleftarrow{p})$ (which refers to "the whole statement with reversed inequalities"). ∎

We give an example that satisfies the assumptions of Lemma 7.25.

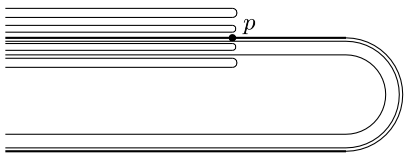
EXAMPLE 7.26 (Type 2, order preserving case). Take $\nu = 10(01101001)^\infty$ and $L = (1010010111001001)^\infty$ and

$$\bar{p} = (01101001)^\infty 01.101001(01101001)^\infty.$$

Then $\overleftarrow{r(p)}$ is the smallest left-infinite tail so it is the smallest in the cylinder [0]. As the calculations below show, all long basic arcs in small neighbourhood of $A(\overleftarrow{p})$ are below $A(\overleftarrow{p})$ and left arcs are both above and below $A(\overleftarrow{p})$, depending on the parity of period which corresponds to $\overleftarrow{p}$ in the left-infinite description of basic arcs (see Figure 7.4).

$$\begin{aligned}
\dots 0(01101001)^{2N}01 &\succ_L \overleftarrow{p}, & \dots 11001(01101001)^N 01 &\prec_L \overleftarrow{p},\\
\dots 0(01101001)^{2N+1}01 &\prec_L \overleftarrow{p}, & \dots 001001(01101001)^N 01 &\prec_L \overleftarrow{p},\\
\dots 11(01101001)^N 01 &\prec_L \overleftarrow{p}, & \dots 0101001(01101001)^N 01 &\prec_L \overleftarrow{p},\\
\dots 101(01101001)^N 01 &\prec_L \overleftarrow{p}, & \dots 11101001(01101001)^N 01 &\prec_L \overleftarrow{p},
\end{aligned}$$

for all $N \in \mathbb{N}$. Further calculations show that only tails of L and S can appear as the extrema of cylinders. By Proposition 6.8, the arc-component $\mathcal{U}_L$ is fully accessible and since $\mathcal{U}_L$ contains no folding points, it corresponds to an open interval on the circle of prime ends. The accessible part of $\mathcal{U}_S$ corresponds to a half-open interval on the circle of prime ends, where the endpoint of the half-open interval corresponds to the accessible folding point p. By further calculations we conclude that the other folding points are not accessible, so the remaining point on the circle of prime ends corresponds to a simple dense canal with shores $\mathcal{U}_L$ and $\mathcal{U}_S$.



$S = \overleftarrow{r(p)}$

Fig. 7.4. Type 2 folding point from Example 7.26.

LEMMA 7.27 (Order reversing case). *Let* $\nu = 10(c_3 \dots c_{n+2})^\infty$, $c_{n+2} = 1$, *and assume* $\#_1(c_3 \dots c_{n+2})$ *is odd. Let* $\bar{p} = (c_3 \dots c_{n+2})^\infty c_3 \dots c_i . c_{i+1} \dots c_{n+2}(c_3 \dots c_{n+2})^\infty$ *be a symbolic description of a folding point* $p \in X'$. *Then* p *is a Type* 2 *folding point if and only if there exists a natural number* M *such that*

$$\dots c_j^* c_{j+1} \dots c_{n+2}(c_3 \dots c_{n+2})^{M+N} c_3 \dots c_i \prec_L \dots c_j c_{j+1} \dots c_{n+2}(c_3 \dots c_{n+2})^{M+N} c_3 \dots c_i$$

for all $N \in \mathbb{N}$ *and all* $j \in \{3, \dots, 1+n\}$ *for which* $c_j^* c_{j+1} \dots c_{n+2}(c_3 \dots c_{n+2})^{M+N} c_3 \dots c_i$ *is admissible, and*

$$\dots 0(c_3 \dots c_{n+2})^{M+2N'} c_3 \dots c_i \prec_L \dots 1(c_3 \dots c_{n+2})^{M+N'} c_3 \dots c_i,$$
$$\dots 0(c_3 \dots c_{n+2})^{M+2N''+1} c_3 \dots c_i \succ_L \dots 1(c_3 \dots c_{n+2})^{M+N''} c_3 \dots c_i$$

for infinitely many $N' \in \mathbb{N}$ *and all but finitely many* $N'' \in \mathbb{N}$, *or the whole statement with either only last two equalities reversed, or all the inequalities reversed, or only the first one reversed.*

Proof. Assume that $\#_1(c_3 \dots c_i)$ is odd and M is even. As in the proof of Proposition 7.20 we just need to note that

$$\dots c_j^* c_{j+1} \dots c_{n+2}(c_3 \dots c_{n+2})^{M+N} c_3 \dots c_i$$

are itineraries of all long basic arcs,

$$\dots 0(c_3 \dots c_{n+2})^{M+2N'} c_3 \dots c_i$$

are itineraries of all left basic arcs, and

$$\dots 0(c_3 \dots c_{n+2})^{M+2N''+1} c_3 \dots c_i$$

are itineraries of all right basic arcs (recall the notation long/left/right from that proof), and those are all basic arcs in a sufficiently small neighbourhood of $A(\overleftarrow{p})$. The conditions from the statement thus force all but finitely many long and left basic arcs below $A(\overleftarrow{p})$ and infinitely many right basic arcs above $A(\overleftarrow{p})$ (see Figure 7.5). Note that different parities of $\#_1(c_3 \dots c_i)$ and M determine which itineraries are right/left basic arcs and that the statement holds true if the equalities are reversed as described at the end of the statement. ∎

We give an example that satisfies the assumptions of Lemma 7.27.

Example 7.28 (Type 2, order reversing case). Take $\nu = 10(011101001)^\infty$ and $L = (011101001011110010)^\infty$ and $\overleftarrow{p} = (011101001)^\infty$. What follows is an easy computation:

$$\begin{aligned} &\dots 0(011101001)^{2M+1} \prec_L \overleftarrow{p}, && \dots 001001(011101001)^{M} \prec_L \overleftarrow{p},\\ &\dots 0(011101001)^{2M} \succ_L \overleftarrow{p}, && \dots 0101001(011101001)^{M} \prec_L \overleftarrow{p},\\ &\dots 11(011101001)^{M} \prec_L \overleftarrow{p}, && \dots 01101001(011101001)^{M} \prec_L \overleftarrow{p},\\ &\dots 101(011101001)^{M} \prec_L \overleftarrow{p}, && \dots 111101001(011101001)^{M} \prec_L \overleftarrow{p},\\ &\dots 11001(011101001)^{M} \prec_L \overleftarrow{p} \end{aligned}$$

for every $M \in \mathbb{N}$. So p is an accessible folding point of Type 2. Note that $\overleftarrow{l}(p) = (010010111)^\infty 01011 = S_{1011}$ (see Figure 7.5). By further symbolic calculations we again conclude that there is one simple dense canal for this embedding of X'.

7.2.2. Type 3.

From now on we study folding points of Type 3 (see Figure 7.6).

Remark 7.29. Let $\nu = 10(c_3 \dots c_{n+2})^\infty$ be such that $\#_1(c_3 \dots c_{n+2})$ is even and $c_{n+2} = 1$. Then X' does not contain folding points of Type 3.

The following lemma gives necessary and sufficient symbolic conditions for a folding point to be $\mathcal{E}$-embedded as Type 3.

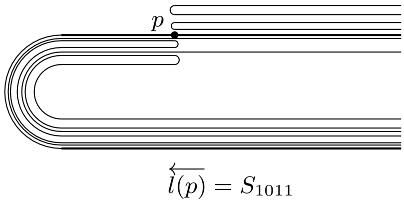


Fig. 7.5. Type 2 folding point from Example 7.28.

LEMMA 7.30 (Type 3). *Let $\nu = 10(c_3 \dots c_{n+2})^\infty$ with $c_{n+2} = 1$ and $\#_1(c_3 \dots c_{n+2})$ odd. Let $\bar{p} = (c_3 \dots c_{n+2})^\infty c_3 \dots c_i . c_{i+1} \dots c_{n+2}(c_3 \dots c_{n+2})^\infty$ be a symbolic description of a folding point $p \in X'$. Then p is a Type 3 folding point if and only if there exists $M > 0$ such that*

$$\dots c_j^* c_{j+1} \dots c_{n+2}(c_3 \dots c_{n+2})^{M+N} c_3 \dots c_i \prec_L \overleftarrow{p}$$

for all $N \in \mathbb{N}$ and all $j \in \{3, \dots, n+1\}$ for which $c_j^* c_{j+1} \dots c_{n+2}(c_3 \dots c_{n+2})^{M+N} c_3 \dots c_i$ is admissible, and

$$\dots 0(c_3 \dots c_{n+2})^{M+N'} c_3 \dots c_i \succ_L \overleftarrow{p}$$

for infinitely many of both even and odd $N' \in \mathbb{N}$, or with reversed inequalities (see Figure 7.6).

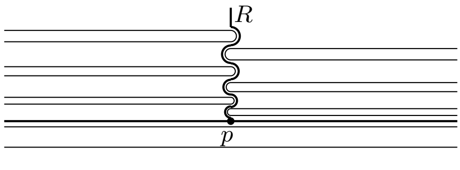


Fig. 7.6. Type 3 folding point. The folding point p is accessible from the complement by an arc $R \cup \{p\} \subset \mathbb{R}^2$, where R is a ray.

Proof. Note that

$$\dots c_j^* c_{j+1} \dots c_{n+2}(c_3 \dots c_{n+2})^{M+N} c_3 \dots c_i$$

are long basic arcs and

$$\dots 0(c_3 \dots c_{n+2})^{M+N'} c_3 \dots c_i$$

are right or left, depending on the parity of M and N'. In any case, the conditions force all long basic arcs below $A(\overleftarrow{p})$ and infinitely many right and left basic arcs above $A(\overleftarrow{p})$. ■

The following lemma gives conditions on preperiodic, order reversing ν such that no folding point can be $\mathcal{E}$-embedded as a Type 3 folding point (except possibly by the Brucks–Diamond embedding studied in detail in Chapter 11).

LEMMA 7.31. *Let $\nu = 10(c_3 \dots c_{n+2})^\infty$ be such that $\#_1(c_3 \dots c_{n+2})$ is odd, $c_{n+2} = 1$ and let $c_j^* c_{j+1} \dots c_{n+2}(c_3 \dots c_{n+2})^M c_3 \dots c_i$ be admissible for every $j \in \{3, \dots, 1+n\}$ and all $M \in \mathbb{N}$. If $c_{n+1} = 1$ then there exists no L such that a folding point $p \in X'$ is of Type 3.*

Proof. Take a folding point $p \in X'$ with the symbolic description

$$\bar{p} = (c_3 \dots c_{n+2})^\infty c_3 \dots c_i . c_{i+1} \dots c_{n+2}(c_3 \dots c_{n+2})^\infty$$

for some $i \in \{3, \dots, n+2\}$ and assume that $A(\bar{p})$ is not at the top or bottom of any cylinder in X'. Since $c_j^* c_{j+1} \dots c_{n+2}(c_3 \dots c_{n+2})^M c_3 \dots c_i$ is admissible for every $j \in \{3, \dots, n+1\}$ and all $M \in \mathbb{N}$, the same calculations as in the proof of Lemma 7.23 imply that the only L which satisfies all the conditions from Lemma 7.30 is

$$L = (c_3 \dots c_n 00)^\infty l_k \dots l_1$$

for some $l_k \dots l_1$. However, the word $00c_3 \dots c_n$ is not admissible, a contradiction. ∎

Example 7.32 (No Type 3 folding point). Note that $\nu = 10(0^\alpha 1^\beta)^\infty$ for $\beta \geq 2$ satisfies the assumptions of Lemma 7.31. Thus no folding point from the corresponding X' can be embedded as a Type 3 folding point using $\mathcal{E}$-embeddings (except maybe Brucks–Diamond). Note that this example also satisfies the assumptions of Lemma 7.23, so no folding point can be $\mathcal{E}$-embedded as Type 2 either. Thus in these cases a point from X' is accessible if and only if it is at the top or bottom of some cylinder. So there are $m \in \mathbb{N}$ simple dense canals in $\mathcal{E}$-embeddings of such X', where m is the number of fully accessible arc-components.

The following lemma gives sufficient symbolic conditions on a preperiodic ν such that every folding point can be $\mathcal{E}$-embedded as an accessible folding point of Type 3.

Lemma 7.33. *Let $\nu = 10(c_3 \dots c_{n+2})^\infty$ be such that $\#_1(c_3 \dots c_{n+2})$ is odd and $c_{n+2} = 1$. Assume that $c_{n+1} = 0$ and the tail $(10c_3 \dots c_n)^\infty$ is admissible. For every folding point $p \in X'$ there exists L such that p is of Type 3 in $\varphi_L(X')$.*

Proof. Take a folding point $p \in X'$ with the symbolic description

$$\bar{p} = (c_3 \dots c_{n+2})^\infty c_3 \dots c_i . c_{i+1} \dots c_{n+2}(c_3 \dots c_{n+2})^\infty$$

for some $i \in \{3, \dots, n+2\}$. Denote $\pi_0(A(\bar{p})) := [T^l(c), T^r(c)]$ for some $l, r \in \mathbb{N}$.

Let $L = (c_3 \dots c_n c_{n+1}^* c_{n+2}^*)^\infty c_3 \dots c_i$. Then

$$\dots 0(c_3 \dots c_{n+2})^m c_3 \dots c_i \succ_L \dots 1(c_3 \dots c_{n+2})^m c_3 \dots c_i,$$

$$\dots c_j^* c_{j+1} \dots c_{n+2}(c_3 \dots c_{n+2})^m c_3 \dots c_i \prec_L \dots c_j c_{j+1} \dots c_{n+2}(c_3 \dots c_{n+2})^m c_3 \dots c_i$$

for every $m \in \mathbb{N}$, $j \in \{3, \dots n+1\}$ and all admissible $c_j^* c_{j+1} \dots c_{n+2}(c_3 \dots c_{n+2})^m c_3 \dots c_i$ (see Figure 7.6). By the assumptions we conclude that $L = (10c_3 \dots c_n)^\infty 10c_3 \dots c_i$ is indeed admissible. Since $\#_1(c_3 \dots c_{n+2})$ is odd, we get pairs of basic arcs joined at a point which projects to $\pi_0(p)$, approaching $A(\bar{p})$ from above from both the left and right side of p, exactly as in Figure 7.6. ∎

Example 7.34 (Type 3 folding point). Take $\nu = 10(01101)^\infty$. If we embed X' with respect to the admissible $L = (01110)^\infty$, then $\bar{p} = (01101)^\infty.(01101)^\infty$ is an accessible folding point of Type 3, since it satisfies the conditions of Lemma 7.33. Note that only $\mathcal{U}_L$ can contain the extremum of a cylinder and it corresponds to the circle of prime ends minus a point. The remaining point is the second kind prime end corresponding to the accessible folding point p of Type 3. Specifically, there are no simple dense canals.

8. Extendability of the shift homeomorphism for $\mathcal{E}$-embeddings

Planar embeddings of X equivalent to those constructed in [13] ($L = 1^\infty$) and [11] ($L = 0^\infty 1$) make $\mathcal{R}$ & $\mathcal{C}$ and $\mathcal{C}$ respectively fully accessible as can be deduced from Proposition 6.8 and Remark 6.12. Additionally it can be deduced from Proposition 6.13 that only the remaining accessible points of embeddings of X (if any) need to be folding points. We denote the two special embeddings by $\varphi_{\mathcal{R}}$ and $\varphi_{\mathcal{C}}$ respectively (recall that φ_L denotes the planar embedding determined by the left-infinite sequence L) and refer to them as *standard embeddings*, for the reasons below.

Barge and Martin [4] show that every X (actually every interval inverse limit with a single bonding map) can be embedded in the plane as an attractor of a planar homeomorphism which is conjugate to σ on X. For unimodal inverse limits X, there are two ways to perform that construction: in an orientation preserving or orientation reversing way. Those embeddings are equivalent to $\varphi_{\mathcal{C}}$ and $\varphi_{\mathcal{R}}$, respectively. In particular, for $\varphi_{\mathcal{C}}(X)$ and $\varphi_{\mathcal{R}}(X)$, the homeomorphism σ is extendable to $\mathbb{R}^2$ (that is, to a planar homeomorphism). Bruin [13] showed directly that the shift homeomorphism can be extended to the plane for the embeddings $\varphi_{\mathcal{R}}$. Now we show that except for the two above-mentioned standard embeddings, σ is not extendable for any $\mathcal{E}$-embedding of X'.

Note that if $\sigma\colon \varphi_L(X) \to \varphi_L(X)$ is extendable to $\mathbb{R}^2$, then so is $\sigma|_{\varphi_L(X')}\colon \varphi_L(X') \to \varphi_L(X')$.

Let us recall that when it is clear from the context that we refer to the basic arc $A(\overleftarrow{s})$ we often abbreviate it to $\overleftarrow{s}$.

The following theorem partially answers the question, posed by Boyland, de Carvalho and Hall [8], whether for non-standard $\mathcal{E}$-embeddings the shift homeomorphism is extendable to the whole plane.

THEOREM 8.1. *If X' is embedded in the plane with respect to L, where $A(L) \not\subset \mathcal{C}, \mathcal{R}$, then the shift homeomorphism $\sigma\colon \varphi_L(X') \to \varphi_L(X')$ cannot be extended to a homeomorphism of the plane.*

Proof. Let $\nu = c_1c_2\ldots$ be a kneading sequence and $A(L) \not\subset \mathcal{C}, \mathcal{R}$, and assume for contradiction that $\sigma\colon \varphi_L(X') \to \varphi_L(X')$ is extendable to $\mathbb{R}^2$. Let $(n_i)_{i\in\mathbb{N}}$ be an increasing sequence in $\mathbb{N}$ such that $l_{n_i+3}l_{n_i+2} = 01$. Since $A(L) \not\subset \mathcal{C}, \mathcal{R}$, the sequence $(n_i)_{i\in\mathbb{N}}$ is indeed well defined. For $i \in \mathbb{N}$ define admissible tails

$$\overleftarrow{x}_i = 1^\infty 1011^{n_i}, \qquad \overleftarrow{y}_i = 1^\infty 0111^{n_i}, \qquad \overleftarrow{w}_i = 1^\infty 1101^{n_i}.$$

Note that $\overleftarrow{x}_i$ is between $\overleftarrow{y}_i$ and $\overleftarrow{w}_i$ in the ordering $\preceq_L$, and $\overleftarrow{x}_i1$ is the largest or the smallest (again in $\preceq_L$) among the admissible sequences $\overleftarrow{x}_i1$, $\overleftarrow{y}_i1$ and $\overleftarrow{w}_i1$ because of the choice $l_{n_i+3}l_{n_i+2} = 01$.

For i large enough, note that $\pi_0(\overleftarrow{x}_i1) = [T^2(c), T(c)]$ so $A(\overleftarrow{x}_i1)$ is a horizontal arc in the plane of length $|T(c) - T^2(c)| =: \delta > 0$. Note also that $\pi_0(\overleftarrow{x}_i) = \pi_0(\overleftarrow{y}_i) = \pi_0(\overleftarrow{w}_i) = [T^2(c), T(c)]$ for i large enough. Let $\overleftarrow{x}'_i = \pi_0^{-1}([c, T(c)]) \cap \overleftarrow{x}_i$, $\overleftarrow{y}'_i = \pi_0^{-1}([c, T(c)]) \cap \overleftarrow{y}_i$ and $\overleftarrow{w}'_i = \pi_0^{-1}([c, T(c)]) \cap \overleftarrow{w}_i$ (see Figure 8.1, left). Denote by $A_i \subset \mathbb{R}^2$ [$B_i \subset \mathbb{R}^2$] the vertical segment which joins the left [right] endpoints of $\overleftarrow{y}'_i$ and $\overleftarrow{w}'_i$. Note that $\operatorname{diam} A_i, \operatorname{diam} B_i \to 0$ as $i \to \infty$. Also $D = A_i \cup \overleftarrow{y}'_i \cup B_i \cup \overleftarrow{w}'_i$ separates the plane; denote the bounded component of $\mathbb{R}^2 \setminus D$ by $U \subset \mathbb{R}^2$. Note that $\operatorname{Int}(\overleftarrow{x}'_i) \subset U$.

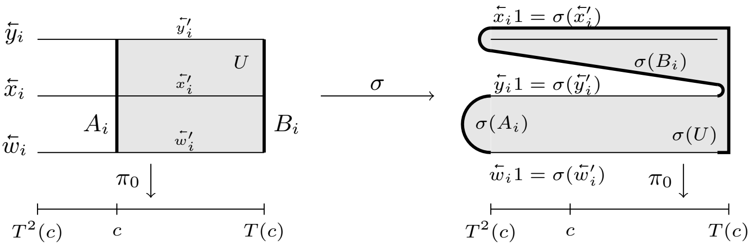


Fig. 8.1. Shuffling of basic arcs from the proof of Theorem 8.1.

Now note that $\sigma(\overleftarrow{x}'_i) = \overleftarrow{x}_i1$ and similarly for $\overleftarrow{y}'_i, \overleftarrow{w}'_i$. Since $\overleftarrow{x}_i1$ is the smallest or the largest among $\overleftarrow{x}_i1, \overleftarrow{y}_i1, \overleftarrow{w}_i1$ and σ is extendable, at least one $\sigma(A_i)$ or $\sigma(B_i)$ has length greater than δ (see Figure 8.1). This contradicts the continuity of σ. ∎

9. $\mathcal{E}$-embeddings of X' with more than one fully accessible arc-component

In this chapter we study $\mathcal{E}$-embeddings of an arbitrary X' that allow at least two fully accessible dense arc-components.

LEMMA 9.1. *Let $\nu = 10^{\kappa}1\dots$ and embed X' with respect to $L = (0^{\kappa}1)^{\infty}$. The smallest left-infinite tail with respect to $\prec_L$ is $A(S) = A(S_0) = A((10^{\kappa})^{\infty}) \not\subset \mathcal{U}_L$. Moreover, both $\mathcal{U}_L$ and $\mathcal{U}_S$ are fully accessible and dense in X'.*

Proof. First, let us comment that $L = (0^{\kappa}1)^{\infty}$ is admissible. Note that there exists a natural number $0 \le \kappa_2 < \kappa$ such that $\nu = 10^{\kappa}10^{\kappa_2}1\dots$, so the word $10^{\kappa}10^{\kappa}$ is indeed admissible.

It is straightforward to calculate S, and infinitely many changes occur because $0^{\kappa+1}$ is not admissible, i.e., the symbol 0 alters L (see Definition 6.22).

To prove that $\mathcal{U}_L$ and $\mathcal{U}_S$ are fully accessible, it is enough to show that every basic arc from $\mathcal{U}_L \cup \mathcal{U}_S$ is at the top or bottom of some cylinder.

Proposition 6.8 shows that $\mathcal{U}_L$ is fully accessible. Assume that $A(\overleftarrow{x}) \subset \mathcal{U}_S$ and take $k \in \mathbb{N}$ such that $x_{k+i} = s_{k+i}$ for every $i \in \mathbb{N}$ and $\kappa + 1$ divides k, where $S = \dots s_2 s_1$. Then $\overleftarrow{x} = \dots 10^{\kappa}10^{\kappa}x_k \dots x_1$. Note that if $\#_1(10^{\kappa}x_k \dots x_1)$ and $\#_1(l_{k+\kappa+1}\dots l_1)$ have the same parity, then $S_{10^{\kappa}x_k\dots x_1} = \overleftarrow{x}$, and $L_{10^{\kappa}x_k\dots x_1} = \overleftarrow{x}$ in the other case.

To show that $\mathcal{U}_L$ and $\mathcal{U}_S$ are dense, fix a point $x \in X'$ with backward itinerary $\overleftarrow{x} = \dots x_2 x_1$ and fix $n \in \mathbb{N}$. Denote $\nu = 10^{\kappa}10^{\kappa_2}10^{\kappa_3}10^{\kappa_4}1\dots$, where $0 \le \kappa_2 < \kappa$, $0 \le \kappa_3, \kappa_4 \le \kappa$.

If $\kappa_3 > 0$, then there exists $\gamma \ge 0$ such that $A((0^{\kappa}1)^{\infty}0^{\kappa_2}1^{\gamma}x_n \dots x_1) \subset \mathcal{U}_L$ is admissible.

Assume $\kappa_3 = 0$. If $\kappa_4 < \kappa$, there exists $\gamma' \ge 0$ so that $A((0^{\kappa}1)^{\infty}0^{\kappa_2}110^{\kappa_4+1}1^{\gamma'}x_n \dots x_1) \subset \mathcal{U}_L$ is admissible. If $\kappa_4 = \kappa$, then $\nu = 10^{\kappa}10^{\kappa_2}110^{\kappa}10^{\kappa_2}0\dots$. Therefore, there exists $\gamma'' \ge 0$ such that $A((0^{\kappa}1)^{\infty}0^{\kappa_2}110^{\kappa}10^{\kappa_2}11^{\gamma''}x_n \dots x_1) \subset \mathcal{U}_L$ is admissible. The proof for points from $\mathcal{U}_S$ is analogous. Therefore, there are points from both $\mathcal{U}_L$ and $\mathcal{U}_S$ which are arbitrarily close to any $x \in X'$ and thus $\mathcal{U}_L$ and $\mathcal{U}_S$ are dense in X'. ■

THEOREM 9.2. *For every X' there exists a planar embedding with at least two non-degenerate fully accessible dense arc-components.*

Proof. Let $\nu = 10^\kappa 1\ldots$ and construct $\varphi_L(X')$ with respect to $L = \ldots 0^\kappa 10^\kappa 10^\kappa 1$. Using Lemma 9.1 we conclude that $\mathcal{U}_S$ and $\mathcal{U}_L$ are fully accessible and dense and the claim follows. ■

In the special case when the orbit of c is finite and only $\mathcal{U}_L$ and $\mathcal{U}_S$ are fully accessible we obtain the following corollary.

COROLLARY 9.3. *Assume the orbit of the critical point is finite and X' is embedded as in Lemma 9.1. Moreover, assume that the set of accessible points consists of $\mathcal{U}_L$ and $\mathcal{U}_S$ only. Then there are exactly two simple dense canals.*

Proof. Take the embedding constructed in Lemma 9.1. So, X' with kneading sequence $\nu = 10^\kappa 1\ldots$ is embedded with respect to $L = (0^\kappa 1)^\infty$. Note that $\mathcal{U}_L$ and $\mathcal{U}_S$ do not contain endpoints for any chosen $\nu = 10^\kappa 1\ldots$ (since the kneading sequence $\nu = (10^\kappa)^\infty$ does not appear as a kneading sequence in the tent map family) and are thus dense lines.

If ν is periodic, the endpoints of X' are not accessible by Corollary 7.14. That in combination with Proposition 5.4 gives two simple dense canals. If ν is preperiodic and $T^3(c)$ is not periodic, the conclusion follows analogously to the above. What remains is to argue that when $T^3(c)$ is periodic, Type 3 folding points do not exist for the chosen L. Since L is periodic of period $\kappa + 1$, it follows that $\sigma^{\kappa+1}\colon \varphi_L(X') \to \sigma^{\kappa+1}(\varphi_L(X'))$ is extendable to the whole plane.

Assume $p \in X'$ is a Type 3 folding point. Thus so is $\sigma^{\kappa+1}(p)$. For $\nu = 10(c_3 \ldots c_{n+2})^\infty$, the itineraries of folding points are periodic of period $n \geq \kappa$. Combining the last two facts shows that $(\kappa + 1) \mid n$. If $\kappa + 1 = n$, since $c_{n+2} = 1$, we have $c_3 \ldots c_{n+2} = 0^{\kappa-1}11$, which is even, contradicting Remark 7.29. From the circle of prime ends we deduce that there can be at most two Type 3 accessible folding points and thus $n = 2(\kappa + 1)$. Since $\#_1(c_3 \ldots c_{n+2})$ is odd, it follows that $\ldots 0P^{2k+1} \succ_L \ldots 1P^{2k+1}$ and $\ldots 0P^{2k} \prec_L \ldots 1P^{2k}$ for all $k \in \mathbb{N}$, where $P = c_3 \ldots c_{n+2}$. That contradicts Lemma 7.30, and thus there are no accessible Type 3 folding points in these embeddings. ■

The following proposition shows that for L as in Lemma 9.1 and ν of a specific form there exist $\mathcal{E}$-embeddings of X' that permit more than two fully accessible arc-components dense in X'. Specifically, we improve the upper bound on the number of fully accessible non-degenerate arc-components in non-standard $\mathcal{E}$-embeddings from three to four (compare to Example 7.17).

PROPOSITION 9.4. *Assume ν is of the form $\nu = 10^\kappa 10^{\kappa-1}110\ldots$ with $\kappa > 1$. If $L = (0^\kappa 1)^\infty$, then $\varphi_L(X')$ has at least four fully accessible dense arc-components.*

Proof. Note that for these L and ν we have $S = (10^\kappa)^\infty$, and note that for κ even, $L_{1^{\kappa+1}} = (1110^{\kappa-1}10^{\kappa-1})^\infty 11^{\kappa+1}$ and $S_{01^{\kappa+1}} = (010^{\kappa-1}1110^{\kappa-2})^\infty 01^{\kappa+1}$. For κ odd we get $S_{1^{\kappa+1}} = (1110^{\kappa-1}10^{\kappa-1})^\infty 11^{\kappa+1}$ and $L_{01^{\kappa+1}} = (010^{\kappa-1}1110^{\kappa-2})^\infty 01^{\kappa+1}$. Thus there exist at least four different accessible left-infinite tails. For the rest of the proof we assume that κ is even.

To see that $\mathcal{U}_{L_{1^{\kappa+1}}}$ is fully accessible take $\overleftarrow{x} = \ldots x_2x_1 \subset \mathcal{U}_{L_{1^{\kappa+1}}}$ and $n \in \mathbb{N}$ such that $\ldots x_{n+2}x_{n+1} = (1110^{\kappa-1}10^{\kappa-1})^\infty$. Note that then $\overleftarrow{x}$ is either the largest or the smallest arc in the cylinder $[1110^{\kappa-1}10^{\kappa-1}x_n \ldots x_1]$, depending on the parity of $x_n \ldots x_1$. Similarly we show that $\mathcal{U}_{S_{01^{\kappa+1}}}$ is fully accessible.

To see that $\mathcal{U}_{L_{1^{\kappa+1}}}$ is dense in X', fix a point $x \in X'$ with backward itinerary $\overleftarrow{x} = \ldots x_2x_1$ and fix $n \in \mathbb{N}$. Note that $0^\kappa \not\subset L_{1^{\kappa+1}}$ and therefore there exists $\gamma \in \mathbb{N}$ such that $(1110^{\kappa-1}10^{\kappa-1})^\infty 1^\gamma x_n \ldots x_1 \subset \mathcal{U}_{L_{1^{\kappa+1}}}$ is admissible. We analogously prove that $\mathcal{U}_{S_{01^{\kappa+1}}}$ is dense in X'. Combining these facts with Lemma 9.1 we conclude the proof. ∎

The characterization of fully accessible arc-components of $\mathcal{E}$-embeddings of X' (excluding the standard embeddings; see Chapters 10 and 11) is still outstanding.

Question. Do there exist more than four fully accessible dense arc-components in non-standard (Chapters 10 and 11) $\mathcal{E}$-embeddings of X'? Specifically, what is the answer if c is periodic?

We lack the symbolic techniques to make a general construction that would answer the above question. Later we will see that for every $n \in \mathbb{N}$ there exists X' such that the Brucks–Diamond embedding of X' has n fully accessible dense arc-components.

10. Bruin's embeddings $\varphi_{\mathcal{R}}(X')$

In this chapter we study the core X' as a subset of the plane by Bruin's embedding constructed in [13], i.e., for $L = 1^\infty$. Recall that we denote these embeddings by $\varphi_{\mathcal{R}}(X')$. If the slope s equals 2 and thus $X' = X$, then the set of accessible points is exactly $\mathcal{R}$ and the endpoint $\bar{0}$ of $\mathcal{C}$; recall Remark 6.15. In particular, there are no simple dense canals.

From now on we restrict to cases when $X \neq X'$ (i.e., $s \neq 2$). Bruin [13] showed that $\sigma\colon \varphi_{\mathcal{R}}(X') \to \varphi_{\mathcal{R}}(X')$ is extendable to the plane and the extension is an orientation reversing planar homeomorphism.

Theorem 10.1. *In embeddings $\varphi_{\mathcal{R}}(X')$ the arc-component $\mathcal{R}$ is fully accessible and no other point from $\varphi_{\mathcal{R}}(X')$ is accessible. There exists one simple dense canal for every $\varphi_{\mathcal{R}}(X')$.*

Proof. Embeddings given by Bruin in [13] satisfy $L = 1^\infty$, and thus $\mathcal{U}_L = \mathcal{R}$.

We will explicitly calculate the top and bottom of an admissible cylinder $[a_n \ldots a_1]$ for $n \in \mathbb{N}$.

If $\#_1(a_n \ldots a_1)$ equals [does not equal] the parity of the natural number n, then $L_{a_n \ldots a_1} = 1^\infty a_n \ldots a_1$ $[S_{a_n \ldots a_1} = 1^\infty a_n \ldots a_1]$, since $1^\infty a_n \ldots a_1$ is always admissible by Lemma 6.4. Also, $S_{a_n \ldots a_1} = 1^\infty 01^k a_n \ldots a_1$ $[L_{a_n \ldots a_1} = 1^\infty 01^k a_n \ldots a_1]$, where $k \in \mathbb{N}_0$ is the smallest non-negative integer such that $01^k a_n \ldots a_1$ is admissible.

Assume for contradiction that such k do not exist. Then $01^i a_n \ldots a_1 \prec c_2c_3 \ldots c_{n+i+2}$ for every $i \in \mathbb{N}_0$. Since the word 01^i is always admissible, it follows that $c_2c_3 \ldots c_{i+2} = 01^i$ for every $i \in \mathbb{N}_0$, i.e., $\nu = 101^\infty$ and the unimodal interval map corresponding to this kneading sequence ν is renormalizable, which contradicts T being a tent map with slope $s \in (\sqrt{2}, 2]$.

Note that every $1^\infty a_n \dots a_1$ is realized as an extremum of a cylinder, namely $1^\infty a_n \dots a_1 = L_{a_n \dots a_1}$ if $\#_1(a_n \dots a_1)$ equals the parity of n and $1^\infty a_n \dots a_1 = S_{a_n \dots a_1}$ if $\#_1(a_n \dots a_1)$ and n are of different parity.

Note that if there is an accessible non-degenerate arc $Q \subset \varphi_{\mathcal{R}}(X')$ which is not the top or bottom of any cylinder, then, since σ is extendable, also every shift of Q is accessible. But σ extends arcs in X' (in the arc-length metric on X'), so there exists $i \in \mathbb{N}$ such that $\sigma^i(Q)$ contains a basic arc which is an extremum of a cylinder and thus $\sigma^i(Q)$ is a subset of $\varphi_{\mathcal{R}}(\mathcal{R})$. Therefore, also $Q \subset \varphi_{\mathcal{R}}(\mathcal{R})$. We conclude that $\varphi_{\mathcal{R}}(\mathcal{R})$ corresponds to the circle of prime ends minus a point. The remaining prime end P is either of the second, third, or fourth kind.

Assume first for contradiction that P is of the second kind, i.e., it corresponds to an accessible folding point. Since $\sigma\colon \varphi_{\mathcal{R}}(X') \to \varphi_{\mathcal{R}}(X')$ is extendable to the plane, it follows that P needs to correspond to an accessible point $\rho = (\dots, r, r, r)$, where r is the non-zero fixed point of T (since $\bar{\rho} = \dots 11.11 \dots$ is the only σ-invariant itinerary of a point in X'). However, $A(1^\infty)$ is the top or bottom of a cylinder, so ρ corresponds to a first kind prime end on the circle of prime ends, a contradiction.

Therefore, the remaining point P on the circle of prime ends is either of the third or the fourth kind. Since $\mathcal{R}$ is dense in X' (see [10, Proposition 1]) and $\varphi_{\mathcal{R}}(\mathcal{R})$ bounds the canal in $\varphi_{\mathcal{R}}(X')$ it follows that $\Pi(P) = \varphi_{\mathcal{R}}(X')$ and so $I(P) = \Pi(P) = \varphi_{\mathcal{R}}(X')$. Thus there exists one simple dense canal for every $\varphi_{\mathcal{R}}(X')$. ∎

11. Brucks–Diamond embeddings $\varphi_{\mathcal{C}}(X')$

In this chapter we study the core X' as a subset of the plane by the Brucks–Diamond embedding $\varphi_{\mathcal{C}}$ constructed in [11], i.e., for $L = 0^\infty 1$. If the slope s equals 2, i.e., $X = X'$ is the Knaster continuum, it follows from Corollary 6.14 and Remark 7.4 that $\mathcal{U}_L = \mathcal{C}$ is fully accessible and that no other point from $\varphi_{\mathcal{C}}(X')$ is accessible (observe the circle of prime ends). Specifically, there is no simple dense canal.

Thus we restrict to cases when $X \neq X'$ (i.e., $s \neq 2$). The embeddings $\varphi_{\mathcal{C}}(X')$ can be viewed as global attractors of an orientation preserving planar homeomorphism $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $h|_{\varphi_{\mathcal{C}(X')}} = \sigma$ as described by Barge and Martin [4]. In other words, the function $\sigma\colon \varphi_{\mathcal{C}}(X') \to \varphi_{\mathcal{C}}(X')$ can be extended to a planar homeomorphism h. For $\varphi_{\mathcal{C}}(X)$ the set of accessible points is $\mathcal{C}$ and it forms an infinite canal which is dense in the core. However, if $\mathcal{C}$ is stripped off, the set of accessible points and the prime ends of $\varphi_{\mathcal{C}}(X')$ become very interesting. Recently, Boyland, de Carvalho and Hall [8] gave a complete characterization of prime ends for embeddings $\varphi_{\mathcal{C}}$ of unimodal inverse limits satisfying certain regularity conditions which also hold for tent map inverse limits with indecomposable cores. In this chapter we obtain an analogous characterization of accessible points as in [8] using symbolic computations. What this chapter adds to the results from [8] is the characterization of types of accessible folding points, in particular in the irrational height case (see the definitions below). By knowing the exact symbolic descriptions of points in X' we can determine whether they are folding points or not, and if they are, whether they are endpoints of X'. The classification of accessible sets differs (as in [8]) according to the *height* of the kneading sequence which we introduce concisely in this chapter (for more details see [17]). Throughout this chapter the order $\prec_L$ corresponds to the standard parity-lexicographical order $\prec$.

We denote by $L' \in \{0,1\}^\infty$ the left-infinite itinerary which is the largest admissible sequence in the embedding X' for $L = 0^\infty 1$ (as in [11]) after $\mathcal{C}$ is removed. Therefore we need to find which basic arc of X' is the closest to the basic arc $A(0^\infty 1)$. This was calculated in [7] (see Lemma 11.8 below).

Definition 11.1. Let $q \in (0, 1/2)$. For $i \in \mathbb{N}$ define

$$\kappa_i(q) = \begin{cases} \lfloor 1/q \rfloor - 1 & \text{if } i = 1, \\ \lfloor i/q \rfloor - \lfloor (i-1)/q \rfloor - 2 & \text{if } i \geq 2. \end{cases}$$

If q is irrational, we say that the kneading sequence

$$\nu = 10^{\kappa_1(q)}110^{\kappa_2(q)}110^{\kappa_3(q)}11\dots$$

has *irrational height* q or that it is of *irrational type*. If $q = m/n$, where m and n are relatively prime, we define

$$c_q = 10^{\kappa_1(q)}110^{\kappa_2(q)}11\dots110^{\kappa_m(q)}1, \qquad w_q = 10^{\kappa_1(q)}110^{\kappa_2(q)}11\dots110^{\kappa_m(q)-1}.$$

We denote by $\hat{a}$ the reverse of a word a, so $\hat{w}_q = 0^{\kappa_m(q)-1}110^{\kappa_{m-1}(q)}11\dots110^{\kappa_1(q)}1$. We say that a kneading sequence has *rational height* q if $(w_q 1)^\infty \preceq \nu \preceq 10(\hat{w}_q 1)^\infty$. Denote $\mathrm{lhe}(q) := (w_q 1)^\infty$, $\mathrm{rhe}(q) := 10(\hat{w}_q 1)^\infty$. If $\mathrm{lhe}(q) \prec \nu \prec \mathrm{rhe}(q)$ we say that ν is of *rational interior type*, and of *rational endpoint type* otherwise. Every kneading sequence that appears in the tent map family is either of rational endpoint, rational interior or irrational type (see [7, Lemmas 8 and 9] for further information see also [17]).

Remark 11.2. The values of $\kappa_i(q)$ can be obtained in the following way (see [17, Lemma 2.5] for details). Draw the graph Γ_ζ of the function $\zeta \colon \mathbb{R} \to \mathbb{R}$, $\zeta(t) = qt$. Then $\kappa_i(q) = N_i - 2$, where N_i is the number of intersections of Γ_ζ with the vertical lines $t = N$, $N \in \mathbb{N}_0$, in the segment $[i-1, i]$; see Figure 11.1. Note that it automatically follows that the word $\kappa_1(q)\kappa_2(q)\dots\kappa_m(q)$ is a palindrome and thus c_q is a palindrome. Furthermore, for every $i \in \mathbb{N}$ either $\kappa_i(q) = \kappa_1(q)$ or $\kappa_i(q) = \kappa_1(q) - 1$.

Remark 11.3. Assume $q = m/n$ is rational with m and n being relatively prime. Take $k \in \{1, \dots, n-1\}$ such that $\lceil kq \rceil - kq$ attains the smallest value; such a k is unique, since m and n are relatively prime. Denote $K = \lceil kq \rceil$ and note that for every $i \in \{1, \dots, k\}$ the line that joins $(0,0)$ to (k, K) intersects a vertical line in $[i-1, i]$ if and only if qt intersects a vertical line in $[i-1, i]$. Thus $\kappa_1(q)\dots\kappa_K(q)$ is a palindrome; it is the longest palindrome among $\kappa_1(q)\dots\kappa_i(q)$ for $i < m$. By studying the line which joins (k, K) to (n, m) we conclude that $\kappa_{K+1}(q)\dots\kappa_{m-1}(q)(\kappa_m(q) - 1)$ is also a palindrome (see Figure 11.1). Thus for every rational q there exist palindromes Y, Z such that $c_q = Y1Z01$.

Remark 11.4. Note that $\{\kappa_i(q)\}_{i \geq 1}$ is a Sturmian sequence for irrational q and thus there exist infinitely many palindromic prefixes of increasing length (see e.g. [16, Theorem 5]) which are of even parity. This can also be concluded by studying the rational

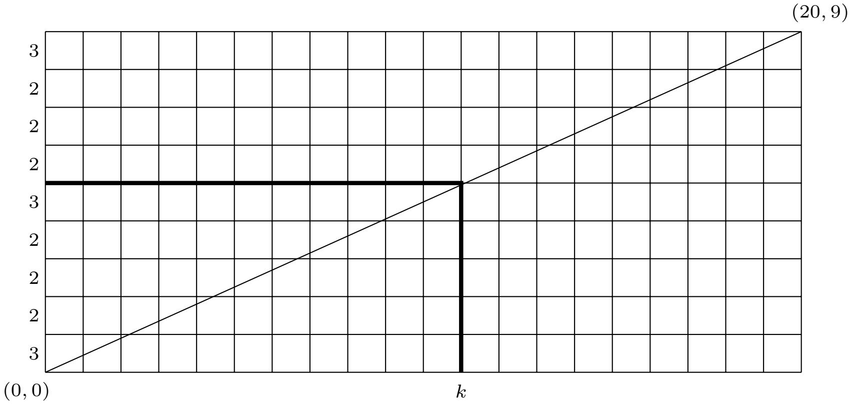


Fig. 11.1. Calculating $\kappa_i(q)$ by counting the intersections of the line qt with vertical lines over integers. The picture shows the values N_i for $q = 9/20$. It follows that $c_q = 1011111111011111111101 = (101111111101)1(111111)01 = Y1Z01$. The decomposition into palindromes Y, Z follows since $\lceil 9/20k \rceil - 9/20k$ attains its minimum for $k = 11 = \lfloor 5/q \rfloor$ (bold line in the figure).

approximations of q. Namely, if $k \in \mathbb{N}$ is such that $\lceil iq \rceil - iq$ achieves its minimum at $i = k$ for all $i \in \{1, \dots, k\}$, then the word $\kappa_1(q) \dots \kappa_k(q)$ is a palindrome. Note that $10^{\kappa_1(q)}11 \dots 110^{\kappa_k(q)}1$ is also a palindrome and it is an even word. By choosing better rational approximations of q from above, we see that k can be taken arbitrarily large, and thus the beginning of c_q consists of arbitrarily long even palindromes.

Lemma 11.5. *Let $q = m/n$. Then there exists $N \in \mathbb{N}$ such that $\sigma^N(\mathrm{rhe}(q)) = \mathrm{lhe}(q)$.*

Proof. Recall that $\mathrm{lhe}(q) = (w_q 1)^\infty$, $\mathrm{rhe}(q) = 10(\hat{w}_q 1)^\infty$, where $c_q = w_q 01$. By Remark 11.3, there exist palindromes Y, Z such that $c_q = Y1Z01$, so $w_q = Y1Z$. It follows that $\mathrm{lhe}(q) = (Y1Z1)^\infty$ and $\mathrm{rhe}(q) = 10(Z1Y1)^\infty$. ∎

Remark 11.6. The height of a kneading sequence is the rotation number of the natural mapping on the circle of prime ends. We will only need a symbolic representation of the height of a kneading sequence here; for a more detailed study of height see [17].

We also remark that the notation $\overleftarrow{x}$ is used in this chapter for a substantially different purpose than in the rest of the paper:

Definition 11.7. Given an infinite sequence $\vec{x} = x_1 x_2 x_3 \dots$, we denote in this chapter its reverse by $\overleftarrow{x} = \dots x_3 x_2 x_1$.

Lemma 11.8 ([7, Lemma 13]). *Let X' be embedded with $\varphi_{\mathcal{C}}$. Denote by L' the largest admissible basic arc in X' and by ν the kneading sequence corresponding to X'. Then*

$$L' = \begin{cases} \overleftarrow{\mathrm{rhe}(q)} & \text{if } \mathrm{lhe}(q) \prec \nu \preceq \mathrm{rhe}(q), \\ \overleftarrow{\nu} & \text{if } q \text{ is irrational or } \nu = \mathrm{lhe}(q). \end{cases}$$

11.1. Irrational height case. Assume that q is irrational and note that the map T is then long-branched (since the kneading map is bounded; see e.g. [14]). Therefore, every proper subcontinuum is a point or an arc [10, Proposition 3], and consequently, every composant is an arc-component and thus either a line or a ray (furthermore every composant of X' is dense in X' so an arc cannot be a composant of X'). We will show that the basic arc $A(L')$ (which is fully accessible) contains an endpoint of X'. Furthermore, we will prove that the basic arc adjacent to $A(L')$ is not an extremum of a cylinder, and thus contains a folding point which is not an endpoint. Moreover, the ray $\mathcal{U}_{L'}$ is partially accessible; only a compact arc $Q \subset \mathcal{U}_{L'}$ is fully accessible and $\mathcal{U}_{L'} \setminus Q$ is not accessible. Since σ is extendable, also $\sigma^i(Q)$ is accessible for every $i \in \mathbb{Z}$. Later in this section we show that no other non-degenerate arc except $\sigma^i(Q)$ for every $i \in \mathbb{Z}$ is fully accessible. From the circle of prime ends we then see that there is still a Cantor set of points remaining to be associated to either accessible points or infinite canals of $\varphi_{\mathcal{C}}(X')$. We prove that the remaining points on the circle of prime ends correspond to accessible endpoints of $\varphi_{\mathcal{C}}(X')$ and are thus second kind prime ends. Moreover, we prove that every endpoint from $\varphi_{\mathcal{C}}(X')$ is accessible. This is an extension of [8, Theorem 4.46]. In this section the usage of the variables m and n should not be confused with the values in the fraction $q = m/n$ which will be used in the rational height case later in the paper.

LEMMA 11.9. *If ν is of irrational type, then $\tau_R(L') = \infty$ and $A(L')$ is non-degenerate.*

Proof. If ν is of irrational type, then the bonding map T is long-branched, so every basic arc in X' is non-degenerate. To prove the first claim, first note that by Lemma 11.8 we have $L' = \overleftarrow{\nu}$. Remark 11.4 implies that there exist infinitely many even palindromes of increasing length at the beginning of ν. Thus there exists a strictly increasing sequence $(m_i)_{i\in\mathbb{N}}$ such that $l'_{m_i} \dots l'_1 = c_1 \dots c_{m_i}$ and $\#_1(c_1 \dots c_{m_i})$ is even for every i. Consequently, $\tau_R(L') = \infty$. ■

The following remark follows from [7, Remark 15] and the fact that we restrict our study only to the tent map family.

REMARK 11.10. If ν is of irrational or rational endpoint type, then $\overleftarrow{t} \in \{0,1\}^\infty$ is admissible (i.e., every subword of $\overleftarrow{t}$ is admissible) if and only if $\overrightarrow{t}$ is admissible (i.e., every subword of $\overrightarrow{t}$ is admissible).

LEMMA 11.11. *Let ν be of either irrational or rational endpoint type and X' be embedded with $\varphi_{\mathcal{C}}$. Then every extremum of a cylinder of $\varphi_{\mathcal{C}}(X')$ belongs to $\sigma^i(L')$ for some $i \in \mathbb{N}_0$.*

Proof. Take an admissible finite word $a_n \dots a_1 \in \{0,1\}^n$ and pick the smallest $k \in \{0,\dots,n-1\}$ such that $a_n \dots a_{k+1} = c_{n-k+1} \dots c_2$. If there is no such k we set $k = n$.

Assume first that $k > 1$ and note that $a_k = 1$; otherwise either $0^{\kappa+1} \subset a_n \dots a_1$ or k is not the smallest such number.

Assume that $\#_1(a_{k-1} \dots a_1)$ is even and let us calculate $L_{a_n \dots a_1}$. If admissible, the word $L' a_{k-1} \dots a_1$ is the largest in the cylinder $[a_n \dots a_1]$. Assume that $L' a_{k-1} \dots a_1$ is not admissible. By Remark 11.10, since both L' and $a_{k-1} \dots a_1$ are admissible, there exists $i \in \{1,\dots,k-1\}$ such that $a_i \dots a_{k-1} l'_1 \dots l'_j$ is not admissible for some $j \ge 1$. If $j \le n-k+1$, then $a_i \dots a_{k-1} l'_1 \dots l'_j$ is a subword of $a_1 \dots a_n$ which is not admissible, a contradiction. Assume that $j > n-k+1$. In this case the word $a_i \dots a_{k-1} l'_1 \dots l'_j \not\subseteq$

$a_1 \dots a_n$ is not admissible, but then $a_i \dots a_n = c_2 \dots c_{2+n-i}$, which contradicts k being the smallest such that $a_n \dots a_{k+1} = c_{n-k} \dots c_2$. If $\#_1(a_{k-1} \dots a_1)$ is odd, we conclude that $S_{a_n \dots a_1} = L' a_{k-1} \dots a_1$ using analogous arguments to those above.

Now assume $\#_1(a_{k-1} \dots a_1)$ is odd and let us calculate $L_{a_n \dots a_1}$. Say that $\#_1(a_n \dots a_k)$ is odd. Therefore, since we want to calculate the largest basic arc in the cylinder $[a_n \dots a_1]$, we need to set $L_{a_n \dots a_1} = \dots 1 a_n \dots a_1$, and note that $1 a_n \dots a_1$ is always admissible by Lemma 6.4. Then, knowing that $\#_1(a_n \dots a_k)$ is odd we deduce from the special structure of ν in the irrational height case that the kneading sequence starts as $a_k \dots a_n 11$ or $a_k \dots a_n 0$ and thus the word $a_k \dots a_n 10$ is admissible. It follows that $L' a_n \dots a_1$ is admissible and equals $L_{a_n \dots a_1}$. If $\#_1(a_n \dots a_k)$ is even, it follows from the structure of ν (blocks of ones in ν are of even length) that $a_n = 1$ and $a_k \dots a_n$ ends in an odd number of ones. The word $a_k \dots a_n 0^{\kappa_1(q)}$ is thus admissible and therefore $L_{a_n \dots a_1} = L' a_{n-1} \dots a_1$. Calculations for $S_{a_n \dots a_1}$ when $\#_1(a_{k-1} \dots a_1)$ is even proceed analogously.

Now assume that $k = 1$. Then $L_{a_n \dots a_1} = L'$. We conclude as in the preceding paragraph that if $\#_1(a_n \dots a_1)$ is even, then $S_{a_n \dots a_1} = L' a_{n-1} \dots a_1$, and if $\#_1(a_n \dots a_1)$ is odd, then $S_{a_n \dots a_1} = L' a_n \dots a_1$.

If $k = 0$, then $a_1 \dots a_n = c_2 \dots c_{n+1}$. So $S_{a_n \dots a_1} = S = \dots c_4 c_3 c_2$. To calculate $L_{a_n \dots a_1}$, let k' be the smallest natural number such that $a_n \dots a_{k'} = c_{n-k'+1} \dots c_1$. If k' does not exist, set $k' = n + 1$. From the structure of ν (blocks of ones in ν are of even length) it follows that $\#_1(a_{k'-1} \dots a_1)$ is odd. The rest of the proof in this case is the same as for $k > 1$. ■

LEMMA 11.12. *Assume ν is of irrational type and X' is embedded with $\varphi_{\mathcal{C}}$. Then the only basic arc from $\mathcal{U}_{L'}$ which is an extremum of a cylinder is $A(L')$.*

Proof. Let $a_n \dots a_1$ be an admissible word for some $n \in \mathbb{N}$. If $n = 1$, note that $L_1 = L' \subset \mathcal{U}_{L'}$ and $L_0, S_0, S_1 \not\subset \mathcal{U}_{L'}$, since ν is not (pre)periodic (note that $L' = \overleftarrow{\nu}$ and observe the structure of ν).

Now assume that $n \geq 2$. Since ν is not (pre)periodic, the proof of Lemma 11.11 shows that if $L_{a_n \dots a_1}$ or $S_{a_n \dots a_1}$ are contained in $\mathcal{U}_{L'}$, then $a_1 \dots a_n = c_1 \dots c_n$ (since otherwise $L_{a_n \dots a_1}$ or $S_{a_n \dots a_1}$ would be contained in $\sigma^i(\mathcal{U}_{L'})$ for some $i \in \mathbb{Z} \setminus \{0\}$). But then, following the proof of Lemma 11.11 we see that $L_{a_n \dots a_1} = L'$ and $S_{a_n \dots a_1} = L' a_n \dots a_1$ or $S_{a_n \dots a_1} = L' a_{n-1} \dots a_1$, depending on the parity of $\#_1(a_n \dots a_1)$. Since $L' a_n \dots a_1 \in \sigma^n(L')$ and $L' a_{n-1} \dots a_1 \in \sigma^{n-1}(L')$, the only extremum of a cylinder in $\mathcal{U}_{L'}$ is $A(L')$. ■

REMARK 11.13. It follows from Lemma 11.12 that when ν has irrational height, then $\mathcal{U}_{L'}$ is partially accessible. More precisely, from Proposition 6.17 it follows that $\overleftarrow{l(L')} = \dots 110^{\kappa_3(q)} 110^{\kappa_2(q)} 110^{\kappa_1(q)-1} 11$ contains a folding point p and $A(L') \cup [a, p]$ is fully accessible, where a denotes the left endpoint of $\overleftarrow{l(L')}$. It follows from Corollary 4.3 that no other point from $\mathcal{U}_{L'}$ (which is a ray) is accessible. Since $\sigma : \varphi_{\mathcal{C}}(X') \to \varphi_{\mathcal{C}}(X')$ is extendable to the plane, also $\sigma^i(A(L') \cup [a, p])$ is accessible for every $i \in \mathbb{Z}$. Moreover, those are the only accessible non-degenerate arcs, since σ is extendable to a planar homeomorphism and furthermore extends every arc in X' (see the discussion in the proof of Theorem 10.1). In the lemmas to follow we prove that the remaining Cantor set of points on the circle of prime ends corresponds to the endpoints of $\varphi_{\mathcal{C}}(X')$, and that all endpoints of $\varphi_{\mathcal{C}}(X')$ are accessible when ν is of irrational type.

The lemma below follows directly from the fact that $(\kappa_i(q))_{i\in\mathbb{N}}$ is Sturmian, but we prove it here for the sake of completeness. Say that $q \in (0,1/2)$ is irrational. Denote $\kappa = \kappa_1(q)$, so $\kappa_i(q) \in \{\kappa, \kappa-1\}$ for every $i \in \mathbb{N}$.

LEMMA 11.14. *Let $q \in (0,1/2)$ be irrational with the corresponding sequence $(\kappa_i(q))_{i\in\mathbb{N}}$. There exists $J \in \mathbb{N}$ such that if $\kappa_i(q)\kappa_{i+1}(q)\ldots\kappa_{i+N}(q)\kappa_{i+N+1}(q) = \kappa(\kappa-1)^N\kappa$, then $N \in \{J, J+1\}$.*

Proof. Let $J \in \mathbb{N}$ be such that $\kappa_2(q) = \cdots = \kappa_{J+1}(q) = \kappa - 1$ and $\kappa_{J+2}(q) = \kappa$ (such J indeed exists since q is irrational). So there exists a sequence of J consecutive $(\kappa-1)$s. Denote $H_n = \lfloor n/q \rfloor$ for $n \in \mathbb{N}$ and note that the function $g\colon \mathbb{N} \to \mathbb{R}$ given by $g(k) = \lceil kq \rceil - kq$ achieves its minimum on $[0, H_{J+2}]$ at H_{J+2} (since $J+2$ is the minimal index $a > 1$ for which $\kappa_a = \kappa$). If we translate the graph of the function $\zeta(t) = qt$ by $+\delta$ where $\delta \in (0, g(H_{J+2})]$, then the sequence of consecutive numbers of intersections with vertical lines over integers begins again with $(\kappa+2)(\kappa+1)^J(\kappa+2)$. Since g restricted to $[0, H_{J+2})$ achieves its minimum H_1, if $\delta \in (g(H_{J+2}), g(H_1))$, the sequence corresponding to the number of times the graph of $\zeta + \delta$ intersects vertical lines over integers begins with $(\kappa+2)(\kappa+1)^{J+1}(\kappa+2)$ (see Figure 11.2). Fix $i \geq 2$ such that $\kappa_i(q) = \kappa$. Note that then $g(H_{i-1}+1) < g(H_1)$ since otherwise $qH_{i-1} > i-1$, which is a contradiction. So the graph of ζ on $[H_{i-1}+1, \infty)$ can be obtained from the graph of ζ on $[0,\infty)$ by translating it by $+\delta$ for $\delta \in (0, g(H_1))$. ∎

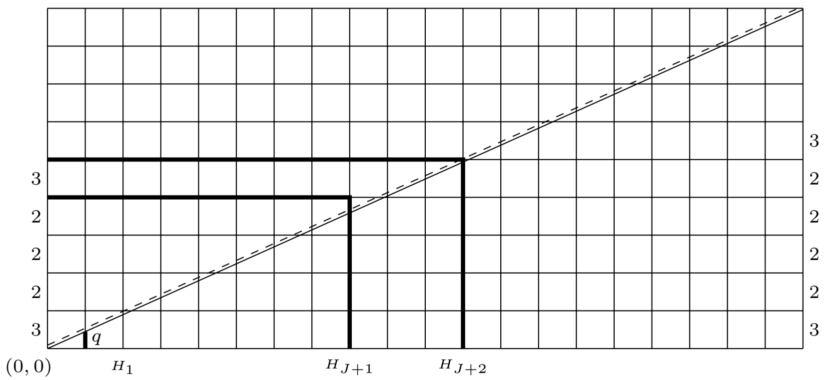


Fig. 11.2. The graph of qt for $q \approx 0.4483\ldots$ with the number of intersections with vertical integer lines on the left. The dashed line represents the graph of qt translated by $\delta \in (g(H_{J+2}), g(H_1))$. On the right we count the intersections of the translated graph with vertical integer lines.

LEMMA 11.15. *Let $q \in (0,1/2)$ be irrational with the corresponding sequence $(\kappa_i(q))_{i\in\mathbb{N}}$ and $i, N \in \mathbb{N}$ such that $\kappa_{i+1}(q)\ldots\kappa_{i+N}(q) = \kappa_1(q)\ldots\kappa_N(q)$ and $\kappa_{i+N+1}(q) \neq \kappa_{N+1}(q)$. Then $\kappa_1(q)\ldots\kappa_{N+1}(q)$ is a palindrome. Moreover, $\kappa_{i+N+2}(q) = \kappa_1(q)$. If $K \in \mathbb{N}$ is such that $\kappa_{i+N+2}(q)\ldots\kappa_{i+N+K+1}(q) = \kappa_1(q)\ldots\kappa_K(q)$ and $\kappa_{i+N+K+2}(q) \neq \kappa_{K+1}(q)$, then $\kappa_{K+1}(q)\ldots\kappa_1(q)\kappa_{i+N+1}(q)\ldots\kappa_{i+1}(q) = \kappa_1(q)\ldots\kappa_{K+N+1}(q)$.*

Proof. For $i \in \mathbb{N}$ denote $H_i = \lfloor i/q \rfloor$ and let $f\colon \mathbb{N} \to \mathbb{R}$ be given by $f(t) = tq - \lfloor tq \rfloor$. Note that the graph of $\zeta(t) = qt$ restricted to $[H_i+1, \infty)$ is a translation of the graph of ζ on $[0,\infty)$ by some $\delta>0$ (see e.g. Figure 11.2). The conditions $\kappa_{i+1}(q)\dots\kappa_{i+N}(q)=\kappa_1(q)\dots\kappa_N(q)$ and $\kappa_{i+N+1}(q)\neq\kappa_{N+1}(q)$ imply that the minimum of f on $[H_i,H_{i+N+1}+1]$ is $H_{i+N+1}+1$. So the graph of $\zeta-f(H_{i+N+1}+1)$ on $[H_i,H_{i+N+1}+1]$ intersects vertical lines over integers the same number of times as ζ except for the point $(H_{i+N+1}+1, i+N+1)$. We conclude that $(\kappa_{i+N+1}(q)+1)\kappa_{i+N}(q)\dots\kappa_{i+1}(q)=\kappa_1(q)\dots\kappa_{N+1}(q)$, which concludes the first part of the proof. To see that $\kappa_{i+N+2}(q)=\kappa_1(q)$ use Lemma 11.14.

For the last part of the proof assume $K\in\mathbb{N}$ is such that $\kappa_{i+N+2}(q)\dots\kappa_{i+N+K+1}(q)=\kappa_1(q)\dots\kappa_K(q)$ and $\kappa_{i+N+K+2}(q)\neq\kappa_{K+1}(q)$. This implies that the global minimum of f on $[H_i,H_{i+N+K+2}+1]$ is $H_{i+N+K+2}+1$. Again by translating the graph of ζ on $[H_i,H_{i+N+K+2}+1]$ by $-f(H_{i+N+K+2}+1)$ we conclude the second part of the proof (see Figure 11.3). ∎

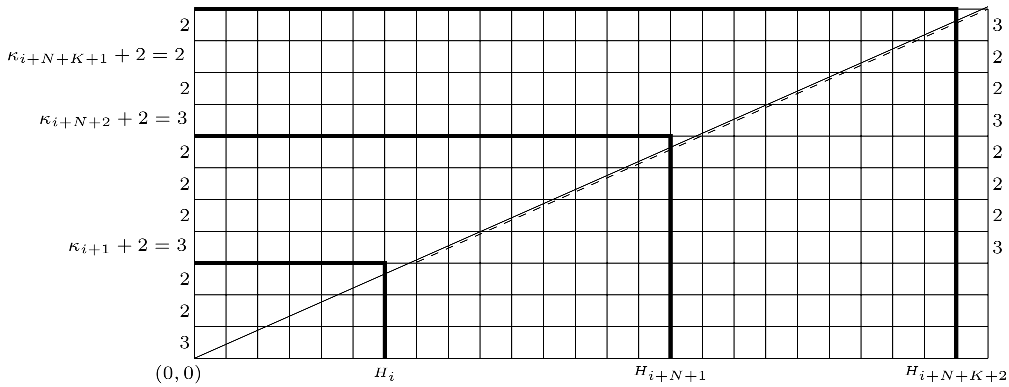


Fig. 11.3. Graphic representation of the proof of Lemma 11.15 for $q\approx 0.443\dots$. The dashed line represents the graph of $\zeta(t)=qt$ on $[H_i+1,H_{i+N+K+2}+1]$ translated by $-f(H_{i+N+K+2}+1)$. On the right side of the grid we count intersections of the dashed line with vertical integer lines.

LEMMA 11.16. *If ν is of irrational type or $\nu=\mathrm{lhe}(q)$, then every endpoint of $\varphi_{\mathcal{C}}(X')$ is accessible.*

Proof. Let $e\in X'$ be an endpoint and let $\overleftarrow{e}$ denote the left-infinite symbolic description of e.

Assume that $\tau_R(\overleftarrow{e})=\infty$ and thus there exists a strictly increasing sequence $(m_i)_{i\in\mathbb{N}}$ such that $c_1\dots c_{m_i}=e_{m_i}\dots e_1$ and $\#_1(e_{m_i}\dots e_1)$ is even. Assume $(m_i)_{i\in\mathbb{N}}$ is the complete sequence for e (see Definition 7.5).

Assume that for infinitely many $i\in\mathbb{N}$ there exist admissible left-infinite itineraries $\overleftarrow{x}^{O(i)}\prec_L\overleftarrow{e}\prec_L x^{I(i)}$ such that $\overleftarrow{x}^{O(i)},\overleftarrow{x}^{I(i)}\to\overleftarrow{e}$ as $i\to\infty$, and $\overleftarrow{x}^{O(i)},\overleftarrow{x}^{I(i)}$ differ only at the index m_i+1 and equal $c_1\dots c_{m_i}$ on the first m_i places (if we are able to construct such $\overleftarrow{x}^{O(i)},\overleftarrow{x}^{I(i)}$, the arcs will cap the endpoint e, which would thus be inaccessible—compare with the proof of Theorem 7.10). So, $\overleftarrow{x}^{O(i)}$ and $\overleftarrow{x}^{I(i)}$ are of the form

$$\overleftarrow{x}^{I(i)}=\dots110^{\kappa_1(q)}110^{\kappa_2(q)}11\dots110^{\kappa_j(q)}1,$$
$$\overleftarrow{e}=\dots110^{\kappa_1(q)}110^{\kappa_2(q)}11\dots110^{\kappa_j(q)}1,$$
$$\overleftarrow{x}^{O(i)}=\dots010^{\kappa_1(q)}110^{\kappa_2(q)}11\dots110^{\kappa_j(q)}1.$$

Note first that $0e_{m_i}\dots e_1$ is indeed admissible. Since $\#_1(e_{m_i}\dots e_1)$ is even, we see that $\overleftarrow{x}^{O(i)} \prec_L \overleftarrow{e}$ for every $i\in\mathbb{N}$. Thus we need to find $\overleftarrow{x}^{I(i)} \succ_L \overleftarrow{e}$ in order to cap e.

Denote by $J\in\mathbb{N}$ the smallest natural number such that

$$\overleftarrow{e} = \dots 110^{\kappa_J(q)-1}110^{\kappa_{J-1}(q)}11\dots 110^{\kappa_2(q)}110^{\kappa_1(q)}110^{\kappa_2(q)}11\dots 110^{\kappa_j(q)}1.$$

By Lemma 11.15 it follows that $\kappa_J(q)\dots\kappa_2(q)\kappa_1(q)$ is a palindrome and therefore $10^{\kappa_J(q)}110^{\kappa_{J-1}(q)}11\dots 110^{\kappa_2(q)}11$ equals the beginning of ν.

We want to find $\overleftarrow{x}^{I(i)} \succ \overleftarrow{e}$. To do so, we note first that none of $00^{\kappa_2(q)}110^{\kappa_1(q)},\dots,$ $00^{\kappa_{J-1}(q)}11\dots 110^{\kappa_1(q)}$ are admissible. If we set

$$\overleftarrow{x}^{I(i)} = \dots 00^{\kappa_J(q)-1}110^{\kappa_{J-1}(q)}11\dots 110^{\kappa_2(q)}110^{\kappa_1(q)}11\dots 110^{\kappa_j(q)}1,$$

then also

$$\overleftarrow{x}^{O(i)} = \dots 00^{\kappa_J(q)-1}110^{\kappa_{J-1}(q)}11\dots 110^{\kappa_2(q)}010^{\kappa_1(q)}11\dots 110^{\kappa_j(q)}1.$$

But since $100^{\kappa_J(q)-1}110^{\kappa_{J-1}(q)}11\dots 110^{\kappa_2(q)}1$ is equal to the beginning of ν, the word $00^{\kappa_J(q)-1}110^{\kappa_{J-1}(q)}11\dots 110^{\kappa_2(q)}0$ is not admissible, a contradiction.

Thus we have no other option but to set

$$\overleftarrow{x}^{I(i)} = \dots 110^{\kappa_J(q)-1}110^{\kappa_{J-1}(q)}11\dots 110^{\kappa_2(q)}110^{\kappa_1(q)}11\dots 110^{\kappa_j(q)}1.$$

By Lemma 11.14 it follows that

$$\overleftarrow{e} = \dots 110^{\kappa_1(q)}110^{\kappa_J(q)-1}110^{\kappa_{J-1}(q)}11\dots 110^{\kappa_2(q)}110^{\kappa_1(q)}110^{\kappa_2(q)}11\dots 110^{\kappa_j(q)}1.$$

Now take the smallest $K\in\mathbb{N}$ such that

$$\begin{aligned}\overleftarrow{e} = {}&\dots 110^{\kappa_{K+1}(q)-1}110^{\kappa_K(q)}11\dots 110^{\kappa_1(q)}110^{\kappa_J(q)-1}1\\ &10^{\kappa_{J-1}(q)}11\dots 110^{\kappa_2(q)}110^{\kappa_1(q)}110^{\kappa_2(q)}11\dots 110^{\kappa_j(q)}1.\end{aligned}$$

By Lemma 11.15 it follows that

$$10^{\kappa_{K+1}(q)}11\dots 110^{\kappa_1(q)}110^{\kappa_J(q)-1}110^{\kappa_{J-1}(q)}11\dots 110^{\kappa_2(q)}11$$

is the beginning of ν. Thus we analogously argue that

$$\begin{aligned}\overleftarrow{x}^{I(i)} = {}&\dots 110^{\kappa_{K+1}(q)-1}110^{\kappa_K(q)}11\dots 110^{\kappa_1(q)}110^{\kappa_J(q)-1}1\\ &10^{\kappa_{J-1}(q)}11\dots 110^{\kappa_2(q)}110^{\kappa_1(q)}110^{\kappa_2(q)}11\dots 110^{\kappa_j(q)}1,\end{aligned}$$

which agrees with $\overleftarrow{e}$. Continuing inductively we conclude that $\overleftarrow{x}^{I(i)} = \overleftarrow{e}$. Thus e is not capped and by Remark 7.3 it is accessible. ∎

REMARK 11.17. In this chapter we expand the definition of Type 3 folding point introduced in the preperiodic orbit case. A point p will be called a *Type* 3 *folding point* if it is not an endpoint, it is accessible, and there is an arc $p\in Q\subset\mathcal{U}_p$ such that $Q\setminus\{p\}$ is not accessible. See Figure 7.6.

LEMMA 11.18. *If ν is of irrational type or rational endpoint type and X' is embedded with φ_C, then there are no Type* 3 *folding points.*

Proof. Assume for contradiction that there is a basic arc $\overleftarrow{x} = \dots x_2x_1$ and an accessible folding point $p\in A(\overleftarrow{x})$ of Type 3. Since p is a folding point by Proposition 7.18 there exist blocks of symbols of ν of increasing length in $\overleftarrow{x}$.

We claim that if $c_n \dots c_{n+k} = c_m \dots c_{m+k}$ for some $m, n \in \mathbb{N}$ and there exists $i \in \{0, \dots, k\}$ such that $c_{n+i} = 0$, then the parities of $\#_1(c_1 \dots c_{n+k})$ and $\#_1(c_1 \dots c_{m+k})$ are the same (then all the wiggles will accumulate on $A(\overleftarrow{x})$ from exactly one side of p as in Figure 7.4). Indeed, take the largest such index i. Then it follows that $c_n \dots c_{n+i-1} = 1^i$. If i is even [odd], then $\#_1(c_1 \dots c_{n-1})$ is odd [even], which proves the claim.

Therefore, if for $\overleftarrow{x} = \dots x_2 x_1$ there exists $i \in \{0, \dots, k\}$ such that $c_{n+i} = 0$ and $x_{k+1} \dots x_1 = c_n \dots c_{n+k}$, it follows that $A(\overleftarrow{x})$ contains no Type 3 folding point.

Now assume that $\overleftarrow{x} = 1^\infty$. If $\kappa_1(q) > 1$, then $\dots 1101^\alpha \succ_L \overleftarrow{x} \succ_L \dots 1101^{\alpha+1}$ for every odd $\alpha \in \mathbb{N}$, and both $\dots 1101^\alpha$ and $\dots 1101^{\alpha+1}$ project to $[T^2(c), T(c)]$, which is again contrary to p being a Type 3 folding point.

If $\kappa_1(q) = 1$, then $\nu = 101^\beta 0 \dots$ for some even $\beta \in \mathbb{N}$. Then basic arcs with a symbolic description $1^\infty 01^\gamma$ for every $\gamma > \beta$ project to $[T^2(c), T(c)]$ and we get an analogous conclusion to that in the preceding paragraph. ■

LEMMA 11.19. *If ν is of irrational type, then there exist no third and no fourth kind prime ends corresponding to $\varphi_{\mathcal{C}}(X')$.*

Proof. Since the embedding $\varphi_{\mathcal{C}}(X')$ is realized as an alignment of basic arcs along a vertically embedded Cantor set connected with semicircles, we can study crosscuts which are vertical segments in the plane joining two adjacent cylinders (see Figure 5.1). Note that every infinite canal is realized by such vertical crosscuts. Take two n-cylinders $A = [a_n \dots a_1]$ and $B = [b_n \dots b_1]$ for some $n \in \mathbb{N}$, such that $A \succ_L B$ and A and B are *adjacent* n-cylinders, i.e., there is no n-cylinder D such that $A \succ_L D \succ_L B$. We will show that S_A and L_B have the same tail, i.e., they both belong to $\sigma^i(L')$ for some $i \in \mathbb{Z}$. Since the accessible subsets of $\sigma^i(L')$ are arcs of finite length, it follows immediately that there cannot exist infinite canals for $\varphi_{\mathcal{C}}(X')$.

Take A and B as above and let $m \in \{0, \dots, n-1\}$ be the smallest non-negative number such that $a_{m+1} \neq b_{m+1}$.

First assume that $\#_1(a_m \dots a_1)$ is odd. Then $S_A = S_{0a_m \dots a_1}$ and $L_B = L_{1a_m \dots a_1}$, since $A \succ_L B$ are adjacent. Let $k \in \{1, \dots, m-1\}$ be the smallest number such that $c_2 \dots c_{m-k+2} = a_{k+1} \dots a_m 1$ (compare with the proof of Lemma 11.11). Assume first that such k indeed exists. Since also $c_2 \dots c^*_{m-k+2} \subset S_A$ is admissible, it follows that $\#_1(a_{k+1} \dots a_m)$ is odd. Thus, $\#_1(a_k \dots a_1)$ is even, and since $a_k = 1$, we see that $\#_1(a_{k-1} \dots a_1)$ is odd. As in the proof of Lemma 11.11, we conclude that $L_{1a_m \dots a_1} = L' 1 a_m \dots a_1$. If k does not exist, then we again have $L_{1a_m \dots a_1} = L' 1 a_m \dots a_1$. Note that $k = m$ is not possible. Furthermore, since $\#_1(a_k \dots a_m)$ is odd, it follows from the specific form of ν that $S_{0a_m \dots a_1} = L' 0 a_m \dots a_1$, which is always admissible. Therefore, S_A and L_B have the same left-infinite tail.

Now assume that $\#_1(a_m \dots a_1)$ is even. Then $S_A = S_{1a_m \dots a_1}$ and $L_B = L_{0a_m \dots a_1}$, since $A \succ_L B$ are adjacent. Let $k \in \{1, \dots, m-1\}$ again be the smallest number such that $c_2 \dots c_{m-k+1} = a_{k+1} \dots a_m 1$. By analogous arguments to those in the preceding paragraph we deduce that $\#_1(a_{k-1} \dots a_1)$ is even and thus, as in the proof of Lemma 11.11, we conclude that $S_{1a_m \dots a_1} = L' 1 a_m \dots a_1$. Furthermore, $L_{0a_m \dots a_1} = L' 0 a_m \dots a_1$, which is always admissible. Again, S_A and L_B have the same left-infinite tail. Therefore, all the canals are finite, i.e., there exist no third and no fourth kind prime ends corresponding to $\varphi_{\mathcal{C}}(X')$. ∎

The theorem below follows directly from the preceding eight lemmas.

THEOREM 11.20. *If ν is of irrational type and X' is embedded with $\varphi_{\mathcal{C}}$, then there are countably infinitely many partially accessible rays of $\varphi_{\mathcal{C}}(X')$; these are the arc-components which are symbolically described by a tail which is a shift of $\overleftarrow{\nu}$. Each of them contains an endpoint of $\varphi_{\mathcal{C}}(X')$, and an accessible set is a compact arc which contains that endpoint. Furthermore, there exist uncountably many accessible arc-components which are accessible at a single point which is an endpoint of $\varphi_{\mathcal{C}}(X')$. All (uncountably many) endpoints of $\varphi_{\mathcal{C}}(X')$ are accessible.*

11.2. Rational endpoint case. Let $q = m/n$. In this section we study $\varphi_{\mathcal{C}}(X')$ when ν is either $\mathrm{rhe}(q)$ or $\mathrm{lhe}(q)$. We provide a symbolic proof of [8, Theorem 4.66].

When $\nu = \mathrm{lhe}(q) = (w_q 1)^\infty$, it follows that $L' = \overleftarrow{\mathrm{lhe}(q)}$. In Remark 11.3 we argued that there exist palindromes Y, Z such that $\mathrm{lhe}(q) = (Y1Z1)^\infty$, thus $\overleftarrow{\mathrm{lhe}(q)} = (1Z1Y)^\infty$. Note that both Y and Z are even, from which we conclude that $\tau_R(L') = \infty$. Thus the right endpoint of $A(L')$ is also an endpoint of X', and since there are no other folding points on $\mathcal{U}_{L'}$ except for this endpoint, the ray $\mathcal{U}_{L'}$ is fully accessible. Since σ is extendable to the plane it follows that $\sigma^i(\mathcal{U}_{L'})$ are fully accessible for every $i \in \{0, \dots, n-1\}$ (where n is the period of $\mathrm{lhe}(q)$). Lemma 11.11 ensures that the union of n rays is indeed the complete set of accessible points of $\varphi_{\mathcal{C}}(X')$ for $\nu = \mathrm{lhe}(q)$. Thus the circle of prime ends decomposes into n half-open intervals, where the endpoints represent the endpoints of X'. Summarizing, we have the following theorem:

THEOREM 11.21. *If $\nu = \mathrm{lhe}(q)$ for some $q = m/n$, where m and n are relatively prime, then in $\varphi_{\mathcal{C}}(X')$ there exist n fully accessible rays which are symbolically described by a tail which is a shift of $\overleftarrow{\nu}$ and no other point from $\varphi_{\mathcal{C}}(X')$ is accessible. Specifically, there exist no infinite canals in $\varphi_{\mathcal{C}}(X')$.*

When $\nu = \mathrm{rhe}(q)$, Lemma 11.8 shows that $L' = \overleftarrow{\mathrm{rhe}(q)} = (1Y1Z)^\infty 01$. Since Y starts with 1, there exists a folding point $p \in \mathcal{U}_{L'}$ on a basic arc with itinerary $\overleftarrow{l(L')} = (1Y1Z)^\infty 11$. Since $\mathrm{rhe}(q)$ is strictly preperiodic, the left tail of $\overleftarrow{l(L')}$ always differs from a positive shift of $\overleftarrow{\mathrm{rhe}(q)}$, so Lemma 11.11 implies that $\overleftarrow{l(L')}$ is not an extremum of any cylinder. Proposition 6.17 implies that p is a Type 2 folding point and consequently $\mathcal{U}_{L'}$ is partially accessible. Moreover, since $\mathcal{U}_{L'}$ contains no other folding points, we conclude that one component of $\mathcal{U}_{L'} \setminus \{p\}$ is fully accessible and the other is not accessible. Since σ is extendable, the $\sigma^i(\mathcal{U}_{L'})$ are also partially accessible. Lemma 11.11 implies that the circle of prime ends decomposes into n half-open intervals and their endpoints are accessible folding points of Type 2. Thus we obtain the following theorem:

THEOREM 11.22. *If $\nu = \mathrm{rhe}(q)$ for some $q = m/n$, where m and n are relatively prime, then in $\varphi_{\mathcal{C}}(X')$ there exist n partially accessible lines which are symbolically described by a tail which is a shift of $\overleftarrow{\nu}$, and no other point from $\varphi_{\mathcal{C}}(X')$ is accessible. Specifically, there exist no infinite canals in $\varphi_{\mathcal{C}}(X')$.*

11.3. Rational interior case. Assume $q = m/n$, where m and n are relatively prime. We will show that in the rational interior case there exist n fully accessible arc-components which are dense lines in X'. We show that folding points which are not lying in the extrema of cylinders are not accessible, so the remaining n points on the circle of prime ends are simple dense canals. That is an analogue of [8, Theorem 4.64] for tent inverse limits.

LEMMA 11.23 ([7, Theorem 16]). *Suppose that ν is of rational interior type for $q = m/n$, where m and n are relatively prime. Then a sequence $\overleftarrow{t} \in \{0,1\}^\infty$ which does not belong to $\mathcal{C}$ is admissible if and only if*

(a) $\sigma^i(\overleftarrow{t}) \preceq \mathrm{rhe}(q)$ *for all $i \in \mathbb{N}$, and*
(b) $\sigma^i(\overleftarrow{t}) \preceq \mathrm{lhe}(q)$ *for all $i \in \mathbb{N}$ for which $\sigma^i(\vec{t}) \succ \sigma^{n+1}(\nu)$.*

REMARK 11.24. Let $w \in \{0,1\}^\infty$ be an infinite sequence. In the rest of this section, for simplicity, we denote by $w[j]$ the first $j \in \mathbb{N}$ coordinates of w.

LEMMA 11.25. *Say that $q = m/n$, where m and n are relatively prime. If $\mathrm{lhe}(q) \prec \nu \prec \mathrm{rhe}(q)$, then all the extrema of cylinders of $\varphi_{\mathcal{C}}(X')$ have tails in $\sigma^i(L')$ for some $i \in \mathbb{Z}$.*

Proof. Fix an arbitrary admissible word $b_j \dots b_1 \in \{0,1\}^j$ for some $j \in \mathbb{N}$.

We will calculate the top/bottom of the cylinder $[b_j \dots b_1]$. Assume that $b_j \dots b_1 \succ \sigma^{n+1}(\nu)[j]$ and $\#_1(b_j \dots b_1)$ is even [odd]. We first show that if $\overleftarrow{\mathrm{lhe}(q)} b_j \dots b_1$ is admissible, then it equals $L_{b_j \dots b_1}$ $[S_{b_j \dots b_1}]$. Assume for contradiction that there exists an admissible $\dots x_2x_1b_j \dots b_1 \succ \overleftarrow{\mathrm{lhe}(q)} b_j \dots b_1$ $[\dots x_2x_1b_j \dots b_1 \prec \overleftarrow{\mathrm{lhe}(q)} b_j \dots b_1]$. Then $\dots x_2x_1 \succ \overleftarrow{\mathrm{lhe}(q)}$ $[\dots x_2x_1 \succ \overleftarrow{\mathrm{lhe}(q)}]$. But that combined with $b_j \dots b_1 \succ \sigma^{n+1}(\nu)[j]$ and Lemma 11.23(b) implies that $\dots x_2x_1b_j \dots b_1 \succ \overleftarrow{\mathrm{lhe}(q)} b_j \dots b_1$ is not admissible, a contradiction. Similarly we show that if $b_j \dots b_1 \preceq \sigma^{n+1}(\nu)[j]$, $\#_1(b_j \dots b_1)$ is even [odd] and $\overleftarrow{\mathrm{rhe}(q)} b_j \dots b_1$ is admissible, then it equals $L_{b_j \dots b_1}$ $[S_{b_j \dots b_1}]$.

In the next two paragraphs we prove that the sequences of the form $\overleftarrow{\mathrm{rhe}(q)} b_j \dots b_1$ and $\overleftarrow{\mathrm{lhe}(q)} b_j \dots b_1$ in the special case to which we restrict later in the proof satisfy conditions (a) and (b) from Lemma 11.23 and are thus admissible.

If $b_{i+1} \dots b_j$ is not the beginning of $\mathrm{rhe}(q)$ for any $i \in \{0, \dots, j-1\}$, then the sequences $\overleftarrow{\mathrm{rhe}(q)} b_j \dots b_1$ and $\overleftarrow{\mathrm{lhe}(q)} b_j \dots b_1$ satisfy *(a)*. Assume there is an $i \in \{0, \dots j-1\}$ such that $b_{i+1} \dots b_j$ is the beginning of $\mathrm{rhe}(q)$ and take the smallest such i. Assume $\#_1(b_{i+1} \dots b_j)$ is odd (later in the proof we need only this special case). If $b_{\alpha+1} \dots b_j$ is also the beginning of $\mathrm{rhe}(q)$ for some $\alpha \in \{0, \dots, j-1\}$, where $\alpha \geq i$, then $\#_1(b_{\alpha+1} \dots b_j)$ is also odd. Note that $b_{\alpha+1} \dots b_j 10 \prec \mathrm{rhe}(q)[j-\alpha+1]$ for every such α. Thus $\overleftarrow{\mathrm{rhe}(q)} b_j \dots b_1$ and $\overleftarrow{\mathrm{lhe}(q)} b_j \dots b_1$ satisfy (a).

If for every $i \in \{1, \dots, j\}$ either $b_i \dots b_1 \preceq \sigma^{n+1}(\nu)[i]$ or $b_{i+1} \dots b_j$ is not the beginning of $\mathrm{lhe}(q)$, then $\overleftarrow{\mathrm{rhe}(q)} b_j \dots b_1$ and $\overleftarrow{\mathrm{lhe}(q)} b_j \dots b_1$ satisfy (b). Assume there is $i < j$ such that $b_i \dots b_1 \succ \sigma^{n+1}(\nu)[i]$ and $b_{i+1} \dots b_j$ is the beginning of $\mathrm{lhe}(q)$ and take the smallest such index i. If $\#_1(b_{i+1} \dots b_j)$ is odd (as in the paragraph above, later in the proof we need only this special case) and there is $\beta \in \{i, \dots, j-1\}$ such that $b_{\beta+1} \dots b_j$ is also the beginning of $\mathrm{lhe}(q)$, then $\#_1(b_{\beta+1} \dots b_j)$ is also odd and thus $b_{\beta+1} \dots b_j 10 \prec \mathrm{lhe}(q)[j-\beta+1]$ for every such β. We conclude that $\overleftarrow{\mathrm{rhe}(q)} b_j \dots b_1$ and $\overleftarrow{\mathrm{lhe}(q)} b_j \dots b_1$ satisfy condition (b).

Recall that $L' = \overleftarrow{\mathrm{rhe}(q)} = (1w_q)^\infty 01$.

Fix an admissible word $a_N \dots a_1 \in \{0,1\}^N$ for some $N \in \mathbb{N}$. Let $k \in \{1, \dots, N\}$ be, if any, the smallest index such that $a_k \dots a_1 \succ \sigma^{n+1}(\nu)[k]$ and $a_{k+1} \dots a_N$ is the beginning of $\mathrm{lhe}(q)$. We set $k = N$ when $a_N \dots a_1 \succ \sigma^{n+1}(\nu)[N]$ (then $a_{k+1} \dots a_N = \emptyset$ is the beginning of $\mathrm{lhe}(q)$). Let $k' \in \{0, 1, \dots, N-1\}$ be the smallest index such that $a_{k'+1} \dots a_N$ equals the beginning of $\mathrm{rhe}(q)$. Note that if $a_i = 1$ for some $i \in \{1, \dots, N\}$, then such k' exists. If $a_N \dots a_1 = 0^N$, then $L_{a_N \dots a_1} = \overleftarrow{\mathrm{rhe}(q)} 0^N$ and $S_{a_N \dots a_1} = S = (1w_q)^\infty 0$.

If $a_i = 1$ for some $i \in \{1, \dots, N\}$, the diagram in Figure 11.4 provides an algorithm to calculate $L_{a_N \dots a_1}$ $[S_{a_N \dots a_1}]$.

To see that the sequences defined are indeed $L_{a_N \dots a_1}$ $[S_{a_N \dots a_1}]$ we use the first part of the proof. For example, consider the case where the algorithm gives $\overleftarrow{\mathrm{lhe}(q)} a_N \dots a_1$. Since $a_N \dots a_1 \succ \sigma^{n+1}(\nu)[N]$ and $\#_1(a_N \dots a_1 = a_N \dots a_{k'+1} a_{k'} \dots a_1)$ is even [odd], if $\overleftarrow{\mathrm{lhe}(q)} a_N \dots a_1$ is admissible, then it equals $L_{a_N \dots a_1}$ $[S_{a_N \dots a_1}]$. To see that it satisfies (a), note that $\#_1(a_N \dots a_{k'+1})$ is odd by assumption. To see that it satisfies (b), assume first that there exists k and $k \le k'$. Then $\#_1(a_{k+1} \dots a_{k'})$ is even and thus $\#_1(a_N \dots a_{k+1})$ is odd. If k does not exist, we are done. If $k > k'$, then since $a_{k'+1} \dots a_N$ is the beginning of $\mathrm{rhe}(q)$ and $a_{k+1} \dots a_N$ is the beginning of $\mathrm{lhe}(q)$, it follows that $\#_1(a_{k'+1} \dots a_k)$ is even and thus $\#_1(a_{k+1} \dots a_N)$ is of the same parity as $\#_1(a_{k'+1} \dots a_N)$, which is odd. That finishes the proof in this case. Other cases follow using analogous computations. Note that if $\#_1(a_N \dots a_{k'+1})$ is even, then since $a_{k'+1} \dots a_N$ is the beginning of $\mathrm{rhe}(q)$, it follows that $a_N = 1$ and thus $\#_1(a_{N-1} \dots a_{k'+1})$ is odd (this is needed in the proof of the two cases in the right branch of Figure 11.4). ■

Lemma 11.26. *Say that $q = m/n$, where m and n are relatively prime. If* $\mathrm{lhe}(q) \prec \nu \prec \mathrm{rhe}(q)$, *then every admissible itinerary in $\sigma^i(\mathcal{U}_{L'})$ is realized as an extremum of a cylinder.*

Proof. Assume that $\overleftarrow{x} = \dots x_2 x_1$ is an admissible tail and that there exists $K \in \mathbb{N}_0$ such that $\dots x_{K+2} x_{K+1} = \overleftarrow{\mathrm{lhe}(q)}$; take the smallest K with that property. Denote $\mathrm{lhe}(q) = (w_q 1)^\infty = (y_1 \dots y_n)^\infty$ and note that $\mathrm{rhe}(q) = 10(\hat{w}_q 1)^\infty$ and therefore $\sigma^{n+1}(\mathrm{rhe}(q)) = (1\hat{w}_q)^\infty = (y_n \dots y_1)^\infty$. Since $\mathrm{rhe}(q) \succ \nu$ and they agree on the first $n+1$ places (which equal c_q and which is a word of even parity; for details see e.g. [8]), it follows that $\sigma^{n+1}(\mathrm{rhe}(q)) \succ \sigma^{n+1}(\nu)$. Let $J \in \mathbb{N}$ be the smallest natural number such that $(y_n \dots y_1)^J \succ \sigma^{n+1}(\nu)[nJ]$. We study the cylinder $Y = [y_n \dots y_1 (y_n \dots y_1)^J x_K \dots x_1]$. Note that $x_i \dots x_K (y_1 \dots y_n)^{J+1}$ does not agree with the beginning of $\mathrm{lhe}(q)$ for any $i \in \{1, \dots, K\}$. Also $y_i \dots y_n (y_1 \dots y_n)^j$ does not agree with the beginning of $\mathrm{lhe}(q)$ for any $i \in \{2, \dots, n\}$ and any $j \in \mathbb{N}$. Denote $a_N \dots a_1 = y_n \dots y_1 (y_n \dots y_1)^J x_K \dots x_1$. Let $k \in \{1, \dots, N\}$ be, if any, the smallest index such that $a_k \dots a_1 \succ \sigma^{n+1}(\nu)[k]$ and $a_{k+1} \dots a_N$ is the beginning of $\mathrm{lhe}(q)$ (compare with the definition of k in the proof of Lemma 11.25). By the choice of J it follows that k indeed exists and $k \in \{K + Mn : M \in \{0, \dots, J\}\}$. So, if for any $i \in \{0, \dots, K-1\}$ the word $x_{i+1} \dots x_K (y_1 \dots y_n)^{J+1}$ does not equal the beginning of $\mathrm{rhe}(q)$, then Lemma 11.25 implies that $\overleftarrow{x} = L_Y$ or $\overleftarrow{x} = S_Y$, depending on the parity of $\#(x_K \dots x_1)$.

If there is $\alpha \in \{0, \dots, K-1\}$ such that $x_{\alpha+1} \dots x_K (y_1 \dots y_n)^{J+1}$ equals the beginning of $\mathrm{rhe}(q)$, then $x_\alpha \dots x_1 \preceq \sigma^{n+1}(\nu)[\alpha]$ (otherwise Y does not satisfy (b) of Lemma 11.23

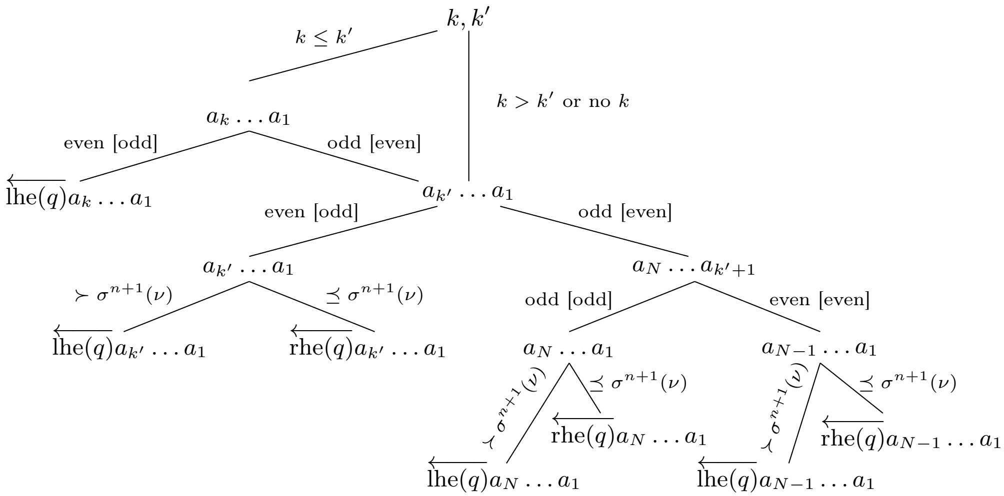


Fig. 11.4. Calculating the $L_{a_N \dots a_1}$ and $S_{a_N \dots a_1}$ in the rational interior case. The graph should be read as follows: If we want to calculate $L_{a_N \dots a_1}$ we read the terms outside of the brackets, and to calculate $S_{a_N \dots a_1}$ we read the terms inside the brackets. Say we want to calculate $L_{a_N \dots a_1}$ [$S_{a_N \dots a_1}$]. We first calculate k and k' and compare them. Say $k > k'$ or k does not exist. We move down the right branch. Next we calculate the parity of $a_{k'} \dots a_1$. Say it is even [odd]; then we move down the left branch. If $a_{k'} \dots a_1 \succ \sigma^{n+1}(\nu)[k']$ then $L_{a_N \dots a_1} = \overleftarrow{\mathrm{lhe}}\, a_{k'} \dots a_1$ [$S_{a_N \dots a_1} = \overleftarrow{\mathrm{lhe}}\, a_{k'} \dots a_1$], and if $a_{k'} \dots a_1 \preceq \sigma^{n+1}(\nu)[k']$ then $L_{a_N \dots a_1} = \overleftarrow{\mathrm{rhe}}\, a_{k'} \dots a_1$ [$S_{a_N \dots a_1} = \overleftarrow{\mathrm{rhe}}\, a_{k'} \dots a_1$]. For the sake of presentation we did not include $[\cdot]$ in the diagram.

and thus is not admissible). Lemma 11.25 implies that $\overleftarrow{\mathrm{rhe}(q)} x_\alpha \dots x_1$ equals L_Y or S_Y, depending on the parity of $\#(x_\alpha \dots x_1)$. Since the tails of $\mathrm{rhe}(q)$ and $\mathrm{lhe}(q)$ are shifts of one another and $J \ge 1$, it follows that $\overleftarrow{x} = \overleftarrow{\mathrm{rhe}(q)} x_\alpha \dots x_1$. ∎

Theorem 11.27. *Say that $q = m/n$, where m and n are relatively prime. If $\mathrm{lhe}(q) \prec \nu \prec \mathrm{rhe}(q)$, then in $\varphi_{\mathcal{C}}(X')$ there exist n fully accessible arc-components which are dense lines in X' and n simple dense canals. Moreover, a point from $\varphi_{\mathcal{C}}(X')$ is accessible if and only if it belongs to one of these n lines.*

Proof. Lemma 11.25 shows that all the extrema of cylinders have tails in $\sigma^i(L')$ for some $i \in \mathbb{Z}$, and Lemma 11.26 shows that every admissible itinerary in $\sigma^i(\mathcal{U}_{L'})$ is realized as an extremum of a cylinder. Since L' is preperiodic of preperiod n, we obtain n fully accessible lines in $\varphi_{\mathcal{C}}(X')$. Since σ is extendable, no other non-degenerate arc can be accessible. Thus the circle of prime ends can be decomposed into n open intervals and their n endpoints. We claim that the endpoints correspond to simple dense canals.

Assume for contradiction that a folding point $x \in \varphi_{\mathcal{C}}(X')$ is accessible. Then every shift $\sigma^j(x)$ has to be accessible for some natural number j which divides n (denoted from now on by $j \mid n$). We conclude that the tail corresponding to the point x must be periodic of period $j \mid n$, i.e., $\sigma^j(x) = x$. Note that there are no periodic kneading sequences ν of period $j \mid n$ for $\mathrm{lhe}(q) \prec \nu \prec \mathrm{rhe}(q)$ since $\mathrm{lhe}(q)$, $\mathrm{rhe}(q)$ and ν agree on the first $n-1$

places. Thus the basic arc $A(\overleftarrow{x})$ has $\tau_L(\overleftarrow{x})$ and $\tau_R(\overleftarrow{x})$ finite. In particular, $A(\overleftarrow{x})$ contains no endpoint of X', and x is the only accessible point in $A(\overleftarrow{x})$, and it thus has to be a Type 3 folding point. Write $\overleftarrow{x} = \dots x_3x_2x_1$. Since x is a folding point and not an endpoint, there exist arbitrarily large $M, k_i \in \mathbb{N}$ such that $x_M \dots x_1 = c_{k_i+1} \dots c_{k_i+M}$ and $x_{M+1} \neq c_{k_i}$. Now we proceed much as in the proof of Proposition 7.20. Fix a cylinder around $\overleftarrow{x}$ and assume that all long basic arcs in that cylinder lie below [above] $A(\overleftarrow{x})$. Here *long* basic arcs $\overleftarrow{y}$ are such that $\pi_0(x) \in \operatorname{Int}(\pi_0(\overleftarrow{y}))$.

In particular, for M large enough and when $c_{k_i}c_{k_i+1} \dots c_{k_i+M} \neq c_2 \dots c_{M+2}$, the basic arcs $1^\infty c_{k_i}c_{k_i+1} \dots c_{k_i+M}$ are long (if $M > \tau_L(x), \tau_R(x)$ then $\pi_0(1^\infty c_{k_i}c_{k_i+1} \dots c_{k_i+M}) = [T^{\tau_L(x)}, T^{\tau_R(x)}]$). Basic arcs in the chosen cylinder which do not project to $[T^{\tau_L(x)}, T^{\tau_R(x)}]$ are of the form $A\big(\dots {}^{0}_{1}c_1c_2 \dots c_{k_i}c_{k_i+1} \dots c_{k_i+M}\big)$, where ${}^{0}_{1}$ stands for either 0 or 1 in this entry. Since $c_{k_i} \neq x_{M+1}$, it follows that those arcs are on the same side of $\overleftarrow{x}$ as the long arcs $1^\infty c_{k_i}c_{k_i+1} \dots c_{k_i+M}$. Since we have assumed that all long basic arcs lie on the same side of $\overleftarrow{x}$, it follows that $\overleftarrow{x}$ is an extremum of a cylinder, a contradiction.

The remaining case is when $c_{k_i} \dots c_{k_i+M} = c_2 \dots c_{M+2}$ for all (but finitely many) $i \in \mathbb{N}$. That is, whenever $x_M \dots x_1$ appears in the kneading sequence, then $x_M \dots x_1 = c_3 \dots c_{M+2}$ and $x_{M+1} \neq c_2 = 0$. However, $\overleftarrow{x}$ is periodic of period $j \mid n$ and x is a folding point, from which we conclude that $T^3(c)$ is periodic of period $j \mid n$ and $\overleftarrow{x} = (c_3 \dots c_{n+2})^\infty$. Note that the only kneading sequence $\operatorname{lhe}(q) \prec \nu \prec \operatorname{rhe}(q)$ for which $T^3(c)$ is periodic of period $j \mid n$ is $10(\hat{w}_q0)^\infty$, which is actually periodic of period n. But there are no periodic kneading sequences ν of period n such that $\operatorname{lhe}(q) \prec \nu \prec \operatorname{rhe}(q)$, a contradiction. Thus no folding point $x \in \varphi_{\mathcal{C}}(X')$ is accessible.

We need to show that the n accessible lines $\mathcal{U}^i \subset X'$ for $i \in \{0, \dots, n-1\}$ are indeed dense in X'. It follows from Lemma 11.25 that the symbolic code of $\mathcal{U}^i$ is eventually $\sigma^i(\overleftarrow{\operatorname{lhe}(q)})$ for $i \in \{0, \dots, n-1\}$. Let $a \in X'$ be a point with the backward itinerary $\overleftarrow{a} = \dots a_2a_1$. Note that for every natural number β, every $i \in \{0, \dots, n-1\}$ and large enough natural number γ the left-infinite sequences $\sigma^i(\overleftarrow{\operatorname{lhe}(q)})1^\gamma a_\beta \dots a_1$ are admissible since they satisfy conditions (a) and (b) from Lemma 11.23. Thus, letting $\beta \to \infty$ we get a sequence of basic arcs from $\mathcal{U}^i$ converging to $A(\overleftarrow{a})$ such that their π_0th projections contain $\pi_0(a)$.

Therefore the n prime ends $P_1, \dots, P_n$ on the circle of prime ends are either of the third or the fourth kind. Since the shores of the canal are lines which are dense in both directions, it follows that $\Pi(P_i) = I(P_i) = \varphi_{\mathcal{C}}(X')$ for every $i \in \{1, \dots, n\}$. Consequently, there are n simple dense canals for every $\varphi_{\mathcal{C}}(X')$. ■

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