The order topology on duals of $C^*$-algebras and von Neumann algebras

by

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Abstract. For a von Neumann algebra $\mathcal{M}$, we study the order topology associated to the hermitian part $\mathcal{M}_h^*$, and to intervals of the predual $\mathcal{M}_*$. It is shown that the order topology on $\mathcal{M}_h^*$ coincides with the topology induced by the norm. In contrast, it is proved that the condition of having the order topology, associated to the interval $[0, \phi]$, equal to the topology induced by the norm, for every $\phi \in \mathcal{M}_h^*$, is necessary and sufficient for the commutativity of $\mathcal{M}$. It is also proved that if $\phi$ is a positive bounded functional on a $C^*$-algebra $\mathcal{A}$, then the norm-null sequences in $[0, \phi]$ coincide with the null sequences, with respect to the order topology on $[0, \phi]$, if and only if the von Neumann algebra $\pi_\phi(\mathcal{A})'$ is of finite type (where $\pi_\phi$ denotes the corresponding GNS representation). This fact allows us to give a new topological characterization of finite von Neumann algebras. Moreover, we demonstrate that convergence to zero for norm and order topology, on order-bounded parts of dual spaces, are inequivalent for all $C^*$-algebras that are not of type I.

1. Introduction and preliminaries. The aim of the paper is to present new results on the interplay between the norm topology and the order topology on dual spaces of $C^*$-algebras and von Neumann algebras.

The following variant of the squeezing lemma is well known to every student of calculus. Let $(y_n)_{n \in \mathbb{N}}$ be an increasing sequence of real numbers and $(z_n)_{n \in \mathbb{N}}$ be a decreasing sequence of real numbers such that $y_n \leq z_n$ for all $n$. Suppose that the supremum of $(y_n)_{n \in \mathbb{N}}$ is the same as the infimum of $(z_n)_{n \in \mathbb{N}}$ and equals $\alpha$. Then, for any other sequence $(x_n)_{n \in \mathbb{N}}$ squeezed between $(y_n)_{n \in \mathbb{N}}$ and $(z_n)_{n \in \mathbb{N}}$ (i.e. $y_n \leq x_n \leq z_n$ for all $n$), $(x_n)_{n \in \mathbb{N}}$ converges to $\alpha$. This simple principle lies behind the definition of order convergence, and subsequently of the order topology $\tau_0(P)$ on a general poset $P$, which is defined as the finest topology on $P$ for which the squeezing lemma holds. Order convergence has been studied in the context of posets and lattices by various authors [2, 3, 15] (see also [13, 16, 17]). If the order structure of $P$
is functional-analytic, it is apparent that the order topology captures some analytic features of $P$. We mention, for example, the beautiful result of Floyd and Klee [11] saying that a normed space $X$ is reflexive if and only if the order topology on the lattice of all closed subspaces of $X$, ordered by set-theoretic inclusion, is Hausdorff. Continuing this line of research, the order topology on the lattice of closed subspaces of a Hilbert space was studied in [18, 10].

A systematic treatment of the order topology on various structures associated with a von Neumann algebra was carried out in [10]. In that case, the order topology comes from the standard operator order defined on the self-adjoint operators acting on a Hilbert space $H$; namely, for self-adjoint operators $T$ and $S$, we have $T \leq S$ if the operator $S - T$ has a positive spectrum. The work in [10] showed that the order topology on the self-adjoint part of a von Neumann algebra, albeit far from being a linear topology, on bounded parts coincides—in a surprising way—with the strong operator topology. Moreover, finite von Neumann algebras were characterized in terms of the order topology on its projection lattice. A study of the order topology on von Neumann algebras, induced by the star order, was given in an interesting recent work [9]. In another recent paper [8], coauthored by the first author, the importance of studying order topologies for structure of pre-Hilbert spaces was shown.

In the present paper, we study the order topology induced by the canonical order on duals, and preduals, of $C^*$-algebras. The order is defined as follows: For functionals $\varphi$ and $\psi$ on a $C^*$-algebra $\mathcal{A}$, we have $\varphi \leq \psi$ if $\varphi(x) \leq \psi(x)$ for all positive $x \in \mathcal{A}$. Our exposition is organized as follows. In Section 2 we extend the result of our previous work [10] by showing that a $\sigma$-finite von Neumann algebra $\mathcal{M}$ is finite if and only if the strong operator topology and the order topology on the effect algebra $E(\mathcal{M})$ have the same null sequences. In Section 3, we demonstrate that the order topology on the hermitian part of the predual of a von Neumann algebra is precisely the norm topology. This is in contrast with the order topology on operator algebras—in general the latter is not linear. Therefore, in dual spaces, the order determines the norm topology. This explicitly means that if $T : \mathcal{A}_s^* \to \mathcal{B}_s^*$ is a bijection (not necessarily linear) between the hermitian parts of the duals of $C^*$-algebras $\mathcal{A}$ and $\mathcal{B}$ preserving the order in both directions, then $T$ is a homeomorphism with respect to the norm topologies.

Our main results are given in Section 4. Let $\varphi$ be a state on a $C^*$-algebra $\mathcal{A}$. We investigate the order topology of the interval $[0, \varphi]$, and show that this topology differs from the subspace topology induced by the order topology $\tau_o(\mathcal{A}_s^*)$. Indeed, the norm-null sequences coincide with the sequences that are null with respect to the order topology $\tau_o[0, \varphi]$ exactly when the commutant $\pi_\varphi(\mathcal{A})'$ of the GNS algebra $\pi_\varphi(\mathcal{A})$ is a finite von Neumann
algebra. Using modular theory, we then show that a σ-finite von Neumann algebra is finite if and only if the null sequences with respect to the norm, and the order topology \(\tau_0[0, \varphi]\) coincide for some (and hence every) normal faithful state \(\varphi\) on \(\mathcal{M}\) (see Corollary 4.10). Pursuing the matter further, we show that on \(C^*\)-algebras that are not of type I, one can always find an interval \([0,\varphi]\) in the dual space on which convergence to zero in norm and order topology are not equivalent. In contrast, we show that a \(C^*\)-algebra \(\mathcal{A}\) is abelian if and only if for each state \(\varphi\) on \(\mathcal{A}\), the norm and the order topology on \([0,\varphi]\) coincide. On the one hand, this extends classical results on order topology on measurable functions. On the other hand, it is a topological characterization of commutativity in \(C^*\)-algebras. As an important technical tool, we are using the map that identifies positive functionals that are dominated by a given state with operators in the effect algebra of the commutant algebra associated with the GNS representation. This can be viewed as an analog of the Radon–Nikodym Theorem. We are also giving some analysis of the continuity properties of this map which may be of independent interest.

Let us recall a few notions and fix the notation. If \(\tau_1\) and \(\tau_2\) are two topologies on a set \(X\) such that \(\tau_1 \subset \tau_2\) then we say that \(\tau_2\) is finer than \(\tau_1\).

Let \((P,\leq)\) be a poset. Given \(a, b \in P\) we shall use the symbol \([a, b]\) for the set \(\{x \in P : a \leq x \leq b\}\). A subset \(X \subset P\) is called convex if \([a, b] \subset X\) whenever \(a, b \in X\). If \(S \subset P\), then \(\bigvee S\) and \(\bigwedge S\) will denote the supremum and the infimum of \(S\), respectively (on condition that they exist). Let \((x_\gamma)_{\gamma \in \Gamma}\) be a net in \(P\). We shall say that this net is increasing (resp. decreasing) if \(x_{\gamma_1} \leq x_{\gamma_2}\) (resp. \(x_{\gamma_1} \geq x_{\gamma_2}\)) whenever \(\gamma_1 \leq \gamma_2\). The symbol \(x_\gamma \uparrow x\) means that \((x_\gamma)_{\gamma \in \Gamma}\) is increasing and \(x = \bigvee \{x_\gamma : \gamma \in \Gamma\}\). Similarly, \(x_\gamma \downarrow x\) means that the net \((x_\gamma)_{\gamma \in \Gamma}\) is decreasing and \(x = \bigwedge \{x_\gamma : \gamma \in \Gamma\}\). A net \((x_\gamma)_{\gamma \in \Gamma}\) is said to order converge to \(x \in P\) if there exist nets \((y_\gamma)_{\gamma \in \Gamma}\) and \((z_\gamma)_{\gamma \in \Gamma}\) such that

\[
y_\gamma \leq x_\gamma \leq z_\gamma \quad \text{and} \quad y_\gamma \uparrow x, \quad z_\gamma \downarrow x.
\]

A subset \(X \subset P\) is called order-bounded if \(X \subset [a, b]\) for some \(a, b \in P\). It is clear that an order-convergent net is eventually order-bounded. A subset \(X \subset P\) is called order-closed (resp. sequentially order-closed) if there is no net in \(X\) (resp. no sequence) order converging to a point outside \(X\). The collection of order-closed sets (resp. sequentially order-closed sets) determines a topology, called the order topology (resp. sequential order topology) on \(P\). Let \(\tau_0(P)\) and \(\tau_{os}(P)\) denote the order topology and the sequential order topology on \(P\), respectively. Clearly, \(\tau_0(P) \subset \tau_{os}(P)\). We shall say that \(P\) is monotone order separable if for every increasing (and resp. decreasing) net \((x_\gamma)_{\gamma \in \Gamma}\) there is an increasing (resp. decreasing) sequence \((\gamma_n)_{n \in \mathbb{N}}\) in \(\Gamma\) such that \(\bigvee_{n \in \mathbb{N}} x_{\gamma_n} = \bigvee_{\gamma \in \Gamma} x_\gamma\) (resp. \(\bigwedge_{n \in \mathbb{N}} x_{\gamma_n} = \bigwedge_{\gamma \in \Gamma} x_\gamma\)). It has been proved...
[7] Proposition 3] that \( \tau_o(P) = \tau_{os}(P) \) if and only if \( P \) is monotone order separable. We shall frequently use the following result proved in [7, Proposition 2]: The sequence \((x_n)_{n \in \mathbb{N}}\) in \( P \) converges with respect to the sequential order topology to \( x \in P \) if, and only if, from each subsequence of \((x_n)_{n \in \mathbb{N}}\) we can extract a further subsequence that order converges to \( x \).

Let us remark that for a nonempty subset \( P_0 \) of \( P \), the subspace topology \( \tau_o(P) \mid P_0 \) and the intrinsic order topology \( \tau_o(P_0) \) are not comparable, in general. (See, for example, [8, Theorem 9]. This will also follow from the results of the present paper.) We shall, however, make use of the following observations, the proofs of which are included for completeness. We recall that \((P, \leq)\) is conditional monotone complete if every increasing net having an upper bound has a supremum and every decreasing net having a lower bound has an infimum.

**Proposition 1.1.** Let \((P, \leq)\) be a partially ordered set and let \( P_0 \) be a subset of \( P \).

(i) If every increasing net in \( P_0 \) having a supremum in \( P_0 \) has the same supremum in \( P \), and every decreasing net in \( P_0 \) having an infimum in \( P_0 \) has the same infimum in \( P \), then \( \tau_o(P) \mid P_0 \subseteq \tau_o(P_0) \).

(ii) Suppose that \((P, \leq)\) is conditional monotone complete. If \( P_0 \) is \( \tau_o(P) \)-closed or convex, then \( \tau_o(P) \mid P_0 \subseteq \tau_o(P_0) \).

(iii) If \( P_0 \) is \( \tau_o(P) \)-open, then \( \tau_o(P) \mid P_0 \supseteq \tau_o(P_0) \).

**Proof.** (i) If a subset \( X \subseteq P_0 \) is not \( \tau_o(P_0) \)-closed one can find nets \((a_\gamma)_{\gamma \in \Gamma}, (b_\gamma)_{\gamma \in \Gamma}\) and \((x_\gamma)_{\gamma \in \Gamma}\) in \( P_0 \) such that \( x_\gamma \in X \cap [a_\gamma, b_\gamma] \) for every \( \gamma \), \( a_\gamma \uparrow, b_\gamma \downarrow \) and \( \bigvee_\gamma a_\gamma = \bigwedge_\gamma b_\gamma \) does not belong to \( X \), where the supremum and infimum are here taken in \( P_0 \). So the hypothesis precisely asserts that \((x_\gamma)_{\gamma \in \Gamma}\) is o-convergent to \( x \) in the larger poset \( P \), i.e. \( X \) is not closed with respect to the subspace topology \( \tau_o(P) \mid P_0 \). Hence \( \tau_o(P) \mid P_0 \not\subseteq \tau_o(P_0) \).

(ii) Suppose that \((a_\gamma)_{\gamma \in \Gamma}\) increases to \( a \) in \( P_0 \). Since \( P \) is conditional monotone complete, the supremum of \( \{a_\gamma : \gamma \in \Gamma\} \) exists in \( P \); call it \( a' \). Clearly, \( a_\gamma \rightarrow a' \) with respect to \( \tau_o(P) \) and \( a_\gamma \leq a' \leq a \), for every \( \gamma \in \Gamma \). Thus, under any of the two conditions of the hypothesis, it follows that \( a' \in P_0 \), and so \( a' = a \). An identical argument would establish the case for decreasing nets, and therefore the assertion follows by (i).

(iii) If \( X \subseteq P_0 \) is not closed with respect to \( \tau_o(P) \mid P_0 \) one can find nets \((a_\gamma)_{\gamma \in \Gamma}, (b_\gamma)_{\gamma \in \Gamma}\) and \((x_\gamma)_{\gamma \in \Gamma}\) in \( P \) such that \( x_\gamma \in X \cap [a_\gamma, b_\gamma] \) for every \( \gamma \), \( a_\gamma \uparrow, b_\gamma \downarrow \) and \( \bigvee_\gamma a_\gamma = \bigwedge_\gamma b_\gamma \) belongs to \( P_0 \) but not to \( X \), where the supremum and infimum are this time taken in \( P \). So, \((a_\gamma)_{\gamma \in \Gamma}\) and \((b_\gamma)_{\gamma \in \Gamma}\) are \( \tau_o(P) \)-convergent to a point in \( P_0 \), and therefore, since \( P_0 \) is \( \tau_o(P) \)-open, \((a_\gamma)_{\gamma \in \Gamma}\) and \((b_\gamma)_{\gamma \in \Gamma}\) are eventually in \( P_0 \). Hence, \( X \) is not \( \tau_o(P_0) \)-closed. □
For a Banach space $X$, we let $X_1$ denote its closed unit ball and $X^*$ its dual space. For a set $Y \subset X$, we shall denote by $[Y]$ the closed linear span of $Y$. The norm topology on $X$ will be denoted by $\tau_{\|\cdot\|}(X)$.

Throughout the paper, we let $\mathcal{A}$ be a unital $C^*$-algebra with a unit $1$. For all unmentioned details on operator algebras, we refer the reader to the monographs [5, 23, 14, 19]. Let $\mathcal{A}_s = \{ x \in \mathcal{A} : x = x^* \}$ and $\mathcal{A}_+ = \{ x^* x : x \in \mathcal{A} \}$. Then $\mathcal{A}_s$ is a real vector space and $\mathcal{A}_+$ is a cone in it. When equipped with the partial order $\leq$ induced by the cone $\mathcal{A}_+$, the self-adjoint elements $\mathcal{A}_s$ form an ordered vector space with order unit $1$.

The effect algebra $\mathcal{E}(\mathcal{A})$ of $\mathcal{A}$ is defined by $\mathcal{E}(\mathcal{A}) := \{ x \in \mathcal{A} : 0 \leq x \leq 1 \}$. The term ‘effect’ comes from the operator-algebraic approach to quantum theory, where the structure of positive contractive operators is connected with quantum measurement. The projection poset $\mathcal{P}(\mathcal{A})$ is the set $\{ p \in \mathcal{A} : p = p^2 = p^* \}$ equipped with the partial order inherited from $\mathcal{A}_s$. Let $\mathcal{A}^*_s$ be the set $\{ \varphi \in \mathcal{A}^* : \varphi(x) \in \mathbb{R} \text{ for all } x = x^* \}$.

A positive functional $\varphi$ on $\mathcal{A}$ is a functional such that $\varphi(x) \geq 0$ for every $x \geq 0$ in $\mathcal{A}$. A positive functional having unit norm is called a state. A positive functional $\varphi$ is called faithful if $\varphi(a^* a) = 0$ implies $a = 0$. The set of positive functionals on $\mathcal{A}$, denoted by $\mathcal{A}^*_+$, is a cone in $\mathcal{A}^*_s$ and thus it induces a partial order on $\mathcal{A}^*_s$.

The symbol $B(H)$ will denote the $C^*$-algebra of all bounded operators acting on a Hilbert space $H$. For a set $X \subset B(H)$ we shall denote its commutant by $X' = \{ y \in B(H) : y x = x y \text{ for all } x \in X \}$. Given $\xi \in H$, the vector functional $\omega_\xi$ is defined by $\omega_\xi(a) = \langle a \xi, \xi \rangle$. It is a positive functional when restricted to any $C^*$-subalgebra of $B(H)$. Let $\mathcal{A} \subset B(H)$ and $\xi \in H$. The vector $\xi$ is called separating for $\mathcal{A}$ if $a \xi = 0$ implies $a = 0$ for all $a \in \mathcal{A}$. The vector $\xi$ is called cyclic for $\mathcal{A}$ if $[\mathcal{A} \xi] = H$. Finally, $\xi$ is said to be bicyclic for $\mathcal{A}$ if it is both cyclic and separating.

Given a $C^*$-algebra $\mathcal{A}$ we shall denote by $M_k(\mathcal{A})$ the matrix algebra of all $k \times k$ matrices over $\mathcal{A}$.

Given a positive functional $\varphi$ on $\mathcal{A}$, there is a Hilbert space $H_\varphi$, a $*$-representation $\pi_\varphi : \mathcal{A} \to B(H_\varphi)$, and a vector $\xi_\varphi \in H_\varphi$ such that $\varphi(a) = \langle \pi_\varphi(a) \xi_\varphi, \xi_\varphi \rangle \quad (a \in \mathcal{A})$ and $[\pi_\varphi(\mathcal{A}) \xi_\varphi] = H_\varphi$. The triple $(H_\varphi, \pi_\varphi, \xi_\varphi)$ is called the GNS representation of $\varphi$. It is unique up to unitary equivalence. A $C^*$-algebra $\mathcal{A}$ is called type I if each nonzero quotient $\mathcal{B}$ of $\mathcal{A}$ contains a nonzero positive element $x$ such that the closure of $x \mathcal{B} x$ is an abelian $C^*$-algebra.

A von Neumann algebra is defined as a $C^*$-algebra that is a dual Banach space. Throughout the paper, $\mathcal{M}$ will represent a von Neumann algebra. Elements of the predual of $\mathcal{M}$ are called normal functionals. A $C^*$-algebra $\mathcal{A} \subset B(H)$ containing the identity operator on $H$ is a von Neumann algebra if and only if $\mathcal{A} = \mathcal{A}''$. If $\varphi$ is a normal positive functional on $\mathcal{M}$
and \( \psi \) is a positive functional with \( \psi \leq \varphi \), then \( \psi \) is normal as well. For a functional \( \varphi \) in \( \mathcal{M}_* \) we shall denote by \( |\varphi| \) its absolute value, that is, \( |\varphi| \) is a positive functional in \( \mathcal{M}_* \) that has the same norm as \( \varphi \) and satisfies \( |\varphi(x)|^2 \leq \|\varphi\| \cdot |\varphi|(xx^*) \) for all \( x \in \mathcal{M} \). When \( \varphi \) is hermitian we have \( \varphi \leq |\varphi| \).

Let us recall the basic topologies associated with von Neumann algebras. The \( \sigma \)-strong topology on \( \mathcal{M} \) is the locally convex topology generated by the seminorms \( \{q_\psi : \psi \in \mathcal{M}_++\} \), where \( q_\psi(x) = \sqrt{\psi(x^*x)} \). It will be denoted by \( s(\mathcal{M},\mathcal{M}_*) \). If \( \mathcal{M} \) acts on a Hilbert space \( H \), we denote by \( \tau_s(\mathcal{M}) \) the \( \sigma \)-strong operator topology, i.e. the topology determined by the seminorms \( \{p_\xi : \xi \in H\} \), where \( p_\xi(x) = \|x\xi\| \). We have \( \tau_s(\mathcal{M}) \subset s(\mathcal{M},\mathcal{M}_*) \). These two topologies coincide on bounded subsets of \( B(H) \).

A von Neumann algebra is called \( \sigma \)-finite if it does not admit any uncountable system of pairwise orthogonal nonzero projections. A state \( \varphi \) on a \( C^* \)-algebra is called a trace if \( \varphi(x^*x) = \varphi(xx^*) \) for all \( x \in A \). For a normal trace \( \psi \) on a von Neumann algebra \( \mathcal{M} \) we have the subadditivity property with respect to projections: \( \psi(\bigvee_n p_n) \leq \sum_n \psi(p_n) \) where \( (p_n)_{n \in \mathbb{N}} \) is a sequence of projections and the supremum is computed in the projection lattice.

A \( C^* \)-algebra \( A \) is called finite if \( aa^* = 1 \) whenever \( a \in A \) and \( a^*a = 1 \). A projection \( p \) in \( A \) is called finite if the algebra \( pAp \) is finite.

A von Neumann algebra \( \mathcal{M} \) is called type II\(_1\) if it is finite and contains no nonzero projection \( p \) such that \( pAp \) is abelian. \( \mathcal{M} \) is called type III if it does not contain any nonzero finite projection.

2. The order topology on the effects. In this section, we consider the order topology on the effect structure \( \mathcal{E}(\mathcal{M}) \) and prove a theorem that gives a new topological characterization of finite von Neumann algebras. This theorem is a key result for our further investigation presented in this paper. Although the proof is a variant of [10, proof of Theorem 5.3], we prefer to give a self-contained argument for completeness.

**Theorem 2.1.** Let \( \mathcal{M} \) be a \( \sigma \)-finite von Neumann algebra. Then the following statements are equivalent:

(i) \( \mathcal{M} \) is finite.

(ii) Every sequence \( (x_n)_{n \in \mathbb{N}} \) in \( \mathcal{E}(\mathcal{M}) \) converging \( \sigma \)-strongly to zero converges to zero with respect to \( \tau_0(\mathcal{E}(\mathcal{M})) \).

**Proof.** (i)\( \Rightarrow \) (ii). Let \( (x_n)_{n \in \mathbb{N}} \) be a sequence in \( \mathcal{E}(\mathcal{M}) \) converging \( \sigma \)-strongly to \( 0 \). To prove the convergence in the order topology we need to find a subsequence \( (x_{n_i})_{i \in \mathbb{N}} \) of \( (x_n)_{n \in \mathbb{N}} \) that order converges to \( 0 \). Since \( \mathcal{M} \) is finite and \( \sigma \)-finite, \( \mathcal{M} \) admits a faithful, normal, tracial state \( \psi \). By the assumption
we can extract a subsequence \((x_{n_i})_{i \in \mathbb{N}}\) of \((x_n)_{n \in \mathbb{N}}\) such that
\[
\psi(x_{n_i}) \leq \sqrt{\psi(x_{n_i}^2)} < 4^{-i}.
\]
For each \(i \in \mathbb{N}\) and \(\lambda \in \mathbb{R}\), let \(e^i(\lambda)\) be the spectral projection in \(\mathcal{M}\), of \(x_{n_i}\), corresponding to \(\lambda\), that is, \(e^i(\lambda) = \chi_{(-\infty, \lambda]}(x_{n_i})\). Then
\[
0 \leq x_{n_i} \leq 2^{-i} e^i(2^{-i}) + (1 - e^i(2^{-i}))
\]
\[
= 2^{-i} \left( \bigwedge_{j \geq i} e^j(2^{-j}) + e^i(2^{-i}) - \bigwedge_{j \geq i} e^j(2^{-j}) \right) + (1 - e^i(2^{-i}))
\]
\[
\leq 2^{-i} \bigwedge_{j \geq i} e^j(2^{-j}) + e^i(2^{-i}) - \bigwedge_{j \geq i} e^j(2^{-j}) + 1 - e^i(2^{-i})
\]
\[
= 2^{-i} \bigwedge_{j \geq i} e^j(2^{-j}) + \bigvee_{j \geq i} (1 - e^j(2^{-j})).
\]
Let
\[
y_i = 2^{-i} \bigwedge_{j \geq i} e^j(2^{-j}) + \bigvee_{j \geq i} (1 - e^j(2^{-j})).
\]
Then \(0 \leq x_{n_i} \leq y_i \leq 1\). Let us verify that the sequence \((y_i)_{i \in \mathbb{N}}\) is decreasing (see also [10, Lemma 5.2]). Indeed,
\[
y_i - y_{i+1} = 2^{-i} \bigwedge_{j \geq i} e^j(2^{-j}) + \bigvee_{j \geq i} (1 - e^j(2^{-j}))
\]
\[- 2^{-i-1} \bigwedge_{j \geq i+1} e^j(2^{-j}) - \bigvee_{j \geq i+1} (1 - e^j(2^{-j}))
\]
\[
= 2^{-i} \bigwedge_{j \geq i} e^j(2^{-j}) - 2^{-i-1} \bigwedge_{j \geq i+1} e^j(2^{-j}) + \bigvee_{j \geq i+1} e^j(2^{-j}) - \bigwedge_{j \geq i} e^j(2^{-j})
\]
\[
= (2^{-i} - 1) \bigwedge_{j \geq i} e^j(2^{-j}) + (1 - 2^{-i-1}) \bigwedge_{j \geq i+1} e^j(2^{-j})
\]
\[
\geq (2^{-i} - 2^{-i-1}) \bigwedge_{j \geq i} e^j(2^{-j}) \geq 0.
\]

Thus, \(\bigwedge_{i \in \mathbb{N}} y_i\) exists in \(\mathcal{E}(\mathcal{M})\). The normality of \(\psi\) entails that
\[
\psi\left( \bigwedge_{i \in \mathbb{N}} y_i \right) = \lim_{i \to \infty} \psi(y_i).
\]
Since \(2^{-j}(1 - e^j(2^{-j})) \leq x_{n_j}\), it follows that
\[
\psi(1 - e^j(2^{-j})) \leq 2^j \psi(x_{n_j}) < 2^{-j}.
\]
Since \(\psi\) is \(\sigma\)-subadditive, we can estimate
\[
\psi(y_i) \leq 2^{-i} + \sum_{j \geq i} \psi(1 - e^j(2^{-j})) < 2^{-i} + \sum_{j \geq i} 2^{-j} = 3 \cdot 2^{-i}.
\]
Thus, \( \psi(\bigwedge_{i\in\mathbb{N}} y_i) = 0 \) and therefore, since \( \psi \) is faithful, it follows that \( \bigwedge_{i\in\mathbb{N}} y_i = 0 \). Consequently, \( (x_{ni})_{i\in\mathbb{N}} \) is order-convergent to 0.

(ii)\(\Rightarrow\)(i). If \( \mathcal{M} \) is not finite, then it has a properly infinite direct summand \( \mathcal{N} \). By structure theory, \( \mathcal{N} \) contains a unital von Neumann subalgebra \( \mathcal{R} \) that is isomorphic to \( B(H) \), for some infinite-dimensional separable Hilbert space \( H \). We shall identify \( \mathcal{R} \) with \( B(H) \). Fix an orthonormal basis \( (\xi_n)_{n\in\mathbb{N}} \) of \( H \) and let \( p_n \) be the projection of \( H \) onto the space \( \{\xi_n + \frac{1}{n}\xi_1, \xi_{n+1}, \xi_{n+2}, \ldots \} \). For every \( \eta \in H \) let us denote by \( \alpha_n = \langle \eta, \xi_n \rangle \) the coordinates of \( \eta \) with respect to \( (\xi_n) \). Then an easy computation gives

\[
\|p_n\eta\|^2 = \left| \frac{1}{n} \alpha_1 + \alpha_n \right|^2 + \sum_{i=n+1}^{\infty} |\alpha_i|^2,
\]

showing that the sequence \( (p_n)_{n\in\mathbb{N}} \) goes to zero in the strong operator topology and therefore in \( s(\mathcal{R}, \mathcal{R}^*) \). Since the restriction to \( \mathcal{R} \) of the \( \sigma \)-strong topology \( s(\mathcal{M}, \mathcal{M}^*) \) is equal to \( s(\mathcal{R}, \mathcal{R}^*) \), it follows that \( (p_n)_{n\in\mathbb{N}} \) is a sequence in \( E(\mathcal{M}) \) that is \( s(\mathcal{M}, \mathcal{M}^*) \)-null. Let us show that this sequence cannot be \( \tau_0(\mathcal{E}(\mathcal{M})) \)-null. Suppose, for a contradiction, that there is a subsequence \( (p_{ni})_{i\in\mathbb{N}} \) that order converges to zero. Let \( (a_i)_{i\in\mathbb{N}} \) be a decreasing sequence in \( \mathcal{E}(\mathcal{M}) \) with zero infimum that witnesses this fact. Denote by \( q \) the projection of \( H \) onto \( \{\xi_1\} \). Then

\[
p_{ni} \leq a_i \quad \text{for every } i \in \mathbb{N}
\]

implies that \( a_i \geq p_{nj} \) for all \( j \geq i \) and therefore, in view of [10] Lemma 2.8,

\[
a_i \geq \bigvee_{j \geq i} p_{nj} \geq q,
\]

i.e. the infimum of the sequence \( (a_n) \) cannot be zero—a contradiction.

3. Order topology on dual spaces. In this section, we show that the order topology on the dual space is equal to the norm topology. We will make use of the following lemma, which could, in fact, be interpreted as a noncommutative Monotone Convergence Theorem. Although it is well known, we prefer to give a brief argument for the sake of completeness.

At this point, we deem fit to underline the difference between the notions of order-boundedness and norm-boundedness in the set-up of dual spaces. This is in contrast to the case of \( C^* \)-algebras, where it is known that the norm-bounded subsets are precisely the order-bounded ones. It is easy to see that every order-bounded subset of \( \mathcal{A}_s^* \) is norm-bounded. On the other hand, if \( \mathcal{A} \) is infinite-dimensional, it contains a sequence \( (a_n)_{n\in\mathbb{N}} \) of positive elements satisfying \( \|a_ia_j\| = \delta_{ij} \). So, if \( \varphi_i \) is a state on \( \mathcal{A} \) satisfying \( \varphi_i(a_j) = \delta_{ij} \), it is easily seen that \( \{\varphi_i : i \in \mathbb{N}\} \) is a norm-bounded subset of \( \mathcal{A}_s^* \) that is not order-bounded.
Lemma 3.1. Let \((\varphi_\gamma)_{\gamma \in \Gamma}\) be an increasing and norm-bounded net of hermitian elements in \(M^*_s\). Then \((\varphi_\gamma)_{\gamma \in \Gamma}\) is norm-convergent, say to \(\varphi\), and
\[
\varphi(x) = \sup_\gamma \varphi_\gamma(x)
\]
for all positive elements \(x \in M\).

Proof. The net of real numbers \((\varphi_\gamma(x))_{\gamma \in \Gamma}\) is increasing and bounded for every \(x \in M_+\). Since \(M\) is linearly generated by its positive elements, it follows that there is a unique linear form \(\varphi\) on \(M\) satisfying (3.1). Moreover, \(\varphi\) is bounded on the positive part of the unit ball and so it is a bounded functional. Using the fact that \(\varphi - \varphi_\gamma\) is positive for each \(\gamma \in \Gamma\), we have
\[
\|\varphi - \varphi_\gamma\| = \varphi(1) - \varphi_\gamma(1),
\]
which goes to zero. Therefore, \(\varphi\) is the norm limit of \((\varphi_\gamma)_{\gamma \in \Gamma}\). Employing the fact that the space of normal functionals is norm-closed, we conclude that \(\varphi\) is a normal functional. 

Clearly, the dual assertion holds for a decreasing net. Thus, the lemma implies that \((M^*_s, \leq)\) is conditional monotone complete.

Lemma 3.2. Suppose that a net \((\varphi_\gamma)_{\gamma \in \Gamma}\) order converges to \(\varphi\) in \(M^*_s\). Then \((\varphi_\gamma)_{\gamma \in \Gamma}\) converges to \(\varphi\) in norm.

Proof. Suppose that
\[
\psi'_\gamma \leq \varphi_\gamma \leq \psi_\gamma \quad \text{in} \quad M^*_s
\]
and \(\psi_\gamma \downarrow \varphi\), \(\psi'_\gamma \uparrow \varphi\). Then by Lemma 3.1 \(\psi_\gamma, \psi'_\gamma \to \varphi\) in norm. For every \(x \in M^*_1\) and for every \(\gamma\) we have
\[
\psi'_\gamma(x) \leq \varphi_\gamma(x) \leq \psi_\gamma(x) \quad \text{and} \quad \psi'_\gamma(x) \leq \varphi(x) \leq \psi_\gamma(x).
\]
This implies that
\[
|\varphi_\gamma(x) - \varphi(x)| \leq |\psi_\gamma(x) - \psi'_\gamma(x)|.
\]
Thus,
\[
\sup_{x \in M^*_1} |\varphi_\gamma(x) - \varphi(x)| \leq \sup_{x \in M^*_1} |\psi_\gamma(x) - \psi'_\gamma(x)| \to 0.
\]
Since every element in the unit ball can be written as \(x_1 - x_2 + i(x_3 - x_4)\), where \(x_i \in \mathcal{E}(M)\) \((i = 1, 2, 3, 4)\), it follows that \(\varphi\) is the norm limit of \((\varphi_\gamma)_{\gamma \in \Gamma}\). 

The function \(f\) mapping a poset \((P, \leq)\) into another poset \((Q, \leq)\) is order-continuous if \((f(x_\gamma))_{\gamma \in \Gamma}\) order converges to \(f(x)\) whenever \((x_\gamma)_{\gamma \in \Gamma}\) is a net in \(P\) that is order-convergent to \(x\). It is easy to verify that if \(f\) is order-continuous, then \(f\) is continuous with respect to \(\tau_0(P)\) and \(\tau_0(Q)\). When \(f(a) \leq f(b)\) for every \(a, b \in P\) satisfying \(a \leq b\), we say that \(f\) is order-preserving or isotone.
**Proposition 3.3.** Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras, and let $A$ and $B$ be closed subsets of $\mathcal{M}_s^*$ and $\mathcal{N}_s^*$, respectively. The following assertions hold:

(i) $\tau_o(\mathcal{M}_s^*)|A \subset \tau_o(A)$.

(ii) If $f : A \to B$ is order-preserving and norm-continuous, then it is continuous with respect to $\tau_o(A)$ and $\tau_o(B)$.

**Proof.** By Lemma 3.2, it follows that $A$ is $\tau_o(\mathcal{M}_s^*)$-closed. Since $\mathcal{M}_s^*$ is conditional monotone complete, (i) follows from Proposition 1.1(ii).

For (ii), it suffices to show that if $\varphi \uparrow \varphi$ and $\psi \downarrow \varphi$ in $A$, then $f(\varphi \uparrow \varphi)$ and $f(\psi \downarrow \varphi)$ in $B$. We shall prove the assertion for $(\varphi \gamma)\gamma \in \Gamma$; the other follows by a dual argument. By (i) we know that $\varphi$ is the supremum of $(\varphi \gamma)\gamma \in \Gamma$, taken in $\mathcal{M}_s^*$, and so, $(\varphi \gamma)\gamma \in \Gamma$ norm converges to $\varphi$. Since $f$ is order-preserving, it follows that $(f(\varphi \gamma))\gamma \in \Gamma$ is increasing and bounded above by $f(\varphi)$. By Lemma 3.1, we know that $(f(\varphi \gamma))\gamma \in \Gamma$ is norm-convergent to its supremum taken in $\mathcal{N}_s^*$. Our assumption on the continuity of $f$ (and the Hausdorffness of the norm topology) implies that this supremum must be equal to $f(\varphi)$. Since $B$ is assumed to be closed, $f(\varphi)$ must be in $B$, and thus $f(\varphi)$ must be equal to the supremum of $(f(\varphi \gamma))\gamma \in \Gamma$ taken in $B$.

**Theorem 3.4.** Let $\mathcal{M}$ be a von Neumann algebra. Then

$$\tau_{||\cdot||}(\mathcal{M}_s^*) = \tau_o(\mathcal{M}_s^*) = \tau_{os}(\mathcal{M}_s^*).$$

In particular, $\mathcal{M}_s^*$ is monotone order separable.

**Proof.** By Lemma 3.2, we know that the order topology $\tau_o(\mathcal{M}_s^*)$ is finer than the norm topology. We show that $\tau_{||\cdot||}(\mathcal{M}) \supset \tau_{os}(\mathcal{M}_s^*)$. In fact, we show a bit more. We prove that if $(\varphi_n)_{n \in \mathbb{N}}$ is a sequence in $\mathcal{M}_s^*$ that norm converges to $\varphi$, then it $\tau_{os}(\mathcal{M}_s^*)$-converges to $\varphi$. To this end, we need to show that $(\varphi_n)_{n \in \mathbb{N}}$ contains a subsequence that order converges to $\varphi$. But passing to a suitable subsequence and with a little abuse of notation, we can suppose that

$$||\varphi_n - \varphi|| \leq 2^{-n} \quad \text{for every } n \in \mathbb{N}.$$

Put

$$v_i = \sum_{n \geq i} |\varphi_n - \varphi|.$$

Clearly,

$$-v_n \leq -|\varphi_n - \varphi| \leq \varphi_n - \varphi \leq |\varphi_n - \varphi| \leq v_n$$

and so

$$\varphi - v_n \leq \varphi_n \leq \varphi + v_n.$$

Since $v_n \downarrow 0$ we have $\varphi + v_n \downarrow \varphi$ and $\varphi - v_n \uparrow \varphi$. Thus $(\varphi_n)_{n \in \mathbb{N}}$ order converges to $\varphi$. That $\mathcal{M}_s^*$ is monotone order separable follows from [7, Proposition 3].

The dual space of a $C^*$-algebra is a special case of a predual of a von Neumann algebra. Therefore, the norm topology coincides with the order topology on the hermitian part of the dual of every $C^*$-algebra.

**Corollary 3.5.** Let $\mathcal{A}$ be a $C^*$-algebra. Then

$$\tau_{\|\cdot\|}(\mathcal{A}) = \tau_0(\mathcal{A}_s^*) = \tau_{os}(\mathcal{A}_s^*),$$

and $\mathcal{A}_s^*$ is monotone order separable.

### 4. Order topology on bounded parts of dual spaces.

In this section, we shall consider intervals of $M^s_*$ as the underlying poset. In $M^s_*$, translation by a fixed element is an order-isomorphism, i.e. given $\varphi$ and $\psi$ in $M^s_*$, the topological spaces $([\varphi, \psi], \tau_0[\varphi, \psi])$ and $([0, \varphi - \psi], \tau_0[0, \varphi - \psi])$ are homeomorphic. That is why we shall consider only intervals of the type $[0, \varphi]$ for $\varphi \in M^+_*$. We shall start by showing, in Theorem 4.4, that the restriction of the norm topology to $[0, \varphi]$ is equal to $\tau_0[0, \varphi]$ for every $\varphi \in M^+_*$ if and only if $M$ is commutative.

Recall the easily verifiable fact that every closed interval in a poset $P$ is $\tau_0(P)$-closed. If the poset is linearly ordered, one clearly also has the dual statement: the set $]a, b[ = \{x \in P : a < x < b, x \neq a, x \neq b\}$ is open with respect to $\tau_0(P)$. This, however, fails to hold even when one considers $\mathbb{R}^2$ with the pointwise order. What holds true in the set-up of matrix algebras is the following ‘variant’ of this property (see Lemma 4.2 below) that is used in the proof of Proposition 4.3.

Let $M^s_m$ denote the set of self-adjoint $m \times m$ complex matrices. For $a, b \in M^s_m$ let $]a, b[$ denote the set of all $x \in M^s_m$ such that $x - a$ and $b - x$ are positive definite.

**Remark 4.1.** Via the duality $(a, b) \mapsto \text{tr}(ab)$, the predual of $M_m$ can be identified (as a Banach space) with $(M_m, \|\cdot\|_1)$ where $\|a\|_1 = \text{tr}(|a|)$. If we denote by $\varphi_a$ the functional on $M_m$ associated with $a \in M_m$ (i.e. $\varphi_a : b \mapsto \text{tr}(ab)$) then $a \geq 0$ if and only if $\varphi_a \geq 0$. Thus, in virtue of Theorem 3.4, we immediately infer that $\tau_0(M^s_m)$ equals the restriction of the norm topology. In particular, it follows that $\tau_0(M^s_m)$ is a linear topology. This follows also by [10, Theorem 4.8].

With the identification of the order topology $\tau_0(M^s_m)$ with the norm topology in hand, one can easily deduce the claim of the following lemma, using the well-known fact that the positive definite cone of a $C^*$-algebra is norm-open. However, we prefer to give a direct proof.

**Lemma 4.2.** Let $a, b \in M^s_m$ and let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $M^s_m$, order-convergent to some $x \in ]a, b[$. Then $(x_n)_{n \in \mathbb{N}}$ is eventually in $]a, b[$.
Proof. Let \((a_n)_{n \in \mathbb{N}}\) be a sequence in \(M_m^s\) satisfying \(a_n \uparrow x\). We show that \(a_n\) is eventually in \(\|a,b\|\). Since \(a_n \leq x\) and \(b - x\) is positive definite, it follows that \(b - a_n\) is positive definite for every \(n \in \mathbb{N}\). The sequence \(a_n - a\) norm converges to \(x - a\), and so \(a_n - a\) is eventually invertible. If, for contradiction, we suppose that \(a_n - a\) is not eventually positive, there exists a subsequence \((a_{n_i} - a)_{i \in \mathbb{N}}\) and a sequence of unit vectors \((\xi_i)_{i \in \mathbb{N}}\) in \(\mathbb{C}^m\) such that \(\langle (a_{n_i} - a)\xi_i, \xi_i \rangle \leq 0\) for every \(i \in \mathbb{N}\). Compactness of the unit sphere of \(\mathbb{C}^m\) implies that \((\xi_i)_{i \in \mathbb{N}}\) clusters, say, to some unit vector \(\xi\). The norm convergence of \((a_{n_i} - a)_{i \in \mathbb{N}}\) to \(x - a\) leads to the contradiction \(\langle (x - a)\xi, \xi \rangle \leq 0\). Thus, \((a_n - a)_{n \in \mathbb{N}}\) is eventually positive definite. Similarly, one shows that if \((b_n)_{n \in \mathbb{N}}\) is a sequence in \(M_m^s\) satisfying \(b_n \downarrow x\), then \((b_n)_{n \in \mathbb{N}}\) is eventually in \(\|a,b\|\). The result follows because if \(a_n, b_n \in \|a,b\|\), then \([a_n, b_n] \subseteq \|a, b\|\). □

Proposition 4.3.

(i) Let \(m > 1\). The restriction of the norm topology to the effect algebra \(\mathcal{E}(M_m)\) is not equal to the order topology \(\tau_o(\mathcal{E}(M_m))\).

(ii) The following topologies on \(\|0, 1\|\) are equal:

- the restriction of the norm topology;
- the restriction of the order topology \(\tau_o(\mathcal{E}(M_m))\);
- the order topology \(\tau_o]\|0, 1\|\).

Proof. (i) Let \(\{\xi_1, \ldots, \xi_m\}\) denote the canonical basis in \(\mathbb{C}^m\) and define

\[\eta_n := \cos \theta_n \xi_1 + \sin \theta_n \xi_2,\]

where \((\theta_n)_{n \in \mathbb{N}}\) is a strictly increasing sequence in \((\pi/4, \pi/2)\) with limit \(\pi/2\). Then \((\eta_n)_{n \in \mathbb{N}}\) converges to \(\xi_2\) and any two members of this sequence are linearly independent. Denote by \(p_n\) the projection of \(\mathbb{C}^m\) onto \(\eta_n\), by \(p\) the projection onto \(\xi_2\), and by \(q\) the projection onto \(\{\xi_1, \xi_2\}\). Then \((p_n)_{n \in \mathbb{N}}\) is norm-convergent to \(p\). We shall show, however, that \((p_n)_{n \in \mathbb{N}}\) does not \(\tau_o(\mathcal{E}(M_m))\)-converge to \(p\). In the light of \([10]\) Proposition 2.1, it suffices to show that no subsequence of \((p_n)_{n \in \mathbb{N}}\) is order-convergent to \(p\) in \(\mathcal{E}(M_m)\). The latter assertion follows by \([10]\) Lemma 2.8(ii) because if \(1 \geq x\) satisfies \(x \geq p_n\) and \(x \geq p_m\) for some \(n \neq m\), then \(x \geq q\). So, if it exists, the order limit in \(\mathcal{E}(M_m)\) of any subsequence of \((p_n)\) must dominate \(q\).

(ii) The following string of inclusions follow from Proposition 1.1:

\[\tau_o]\|0, 1\|\subseteq \tau_o(M_m^s) \subseteq \tau_o(\mathcal{E}(M_m)) \subseteq \tau_o]\|0, 1\|\subseteq \tau_o]\|0, 1\|\).

The first inclusion follows from Proposition 1.1(iii), as \(\|0, 1\|\) is open. The second and the third inclusions follow from Proposition 1.1(ii); the second because \(\mathcal{E}(M_m) = [0, 1]\) is \(\tau_o(M_m^s)\)-closed, and the third because \(\|0, 1\|\) is convex. □
Theorem 4.4. For a von Neumann algebra $\mathcal{M}$ the following statements are equivalent:

(i) $\mathcal{M}$ is abelian.
(ii) For every $\varphi \in \mathcal{M}_+^*$ the restriction of the norm topology to $[0, \varphi]$ is equal to $\tau_0[0, \varphi]$.

Proof. (i)⇒(ii). If $\mathcal{M}$ is abelian, then $\mathcal{M}_+^*$ is isometrically isomorphic (as an ordered normed space) with $L^1(X, \Sigma, \mu, \mathbb{R})$ for some localizable measure space $(X, \Sigma, \mu)$. So it suffices to show that if $u \in L^1(X, \Sigma, \mu)$ and $u \geq 0$, then the $L^1$-norm topology on $[0, u]$ coincides with the order topology $\tau_0[0, u]$. Since $L^1(X, \Sigma, \mu, \mathbb{R})$ is a Dedekind complete lattice (see for example [12, 244L, p. 167]), and since the interval $[0, u]$ is an order-closed sublattice of $L^1(X, \Sigma, \mu, \mathbb{R})$, the final assertion follows by [10, Proposition 2.3(ii)] and by Theorem 3.4.

(ii)⇒(i). If $\mathcal{M}$ is not abelian, then there exists a nonzero central projection $z$ and an integer $m \geq 2$ such that the direct summand $z\mathcal{M}$ contains a unital subalgebra $\mathcal{N}$ that is $\ast$-isomorphic to $M_m$.

Let us first suppose that $z = 1$. For every $a \in \mathcal{N}$ let $\hat{\varphi}_a$ denote the functional on $\mathcal{N}$ described in Remark 4.1. Then $[0, \hat{\varphi}_1]$ is order isomorphic to $\mathcal{E}(M_m)$. Let $p$ and $p_n$ ($n \in \mathbb{N}$) be the one-dimensional projections constructed in the proof of Proposition 4.3. Let us write $\hat{\varphi}_n$ instead of $\hat{\varphi}_{p_n}$, and $\hat{\varphi}$ instead of $\hat{\varphi}_p$, i.e.

(i) $\hat{\varphi}_n$ and $\hat{\varphi}$ belong to $[0, \hat{\varphi}_1]$, 
(ii) $(\hat{\varphi}_n)$ goes in norm to $\hat{\varphi}$, 
(iii) $(\hat{\varphi}_n)$ is not $\tau_0[0, \hat{\varphi}_1]$-convergent to $\hat{\varphi}$.

By [4, IV 2.2.4, p. 355] we know that there is a normal conditional expectation $\Phi$ of $\mathcal{M}$ onto $\mathcal{N}$. For a general functional $\hat{\omega}$ on $\mathcal{N}$ denote by $\omega$ the normal functional on $\mathcal{M}$ given by 

$$\omega = \hat{\omega} \circ \Phi.$$ 

By the properties of normal conditional expectation, it follows that the linear map $\hat{\omega} \mapsto \omega$ is order- and norm-preserving. Thus, $\varphi_1$ is a positive functional on $\mathcal{M}$, the $\varphi_n$’s and $\varphi$ belong to $[0, \varphi_1]$, and $\|\varphi_n - \varphi\| \to 0$.

The function $f : [0, \varphi_1] \to [0, \varphi_1]$ defined by $f(\varphi) = \varphi|\mathcal{N}$ is order-preserving and norm-continuous. Moreover, since $[0, \varphi_1]$ and $[0, \hat{\varphi}_1]$ are closed with respect to $\tau_0(\mathcal{M}_+^*)$ and $\tau_0(\mathcal{N}_+^*)$, respectively, Proposition 3.3(ii) implies that $f$ is continuous with respect to $\tau_0[0, \varphi_1]$ and $\tau_0[0, \hat{\varphi}_1]$. So, if $\varphi_n \to \varphi$ with respect to $\tau_0[0, \varphi_1]$, then $\hat{\varphi}_n \to \hat{\varphi}$ with respect to $\tau_0[0, \hat{\varphi}_1]$, which we know is not true.

Now consider the case $z \neq 1$. Observe that if $\tau$ is a normal positive functional on $\mathcal{M}$ such that $\tau(z) = \|\tau\|$, then any positive functional majorized by $\tau$ is zero on $(1 - z)\mathcal{M}$. Therefore the interval $[0, \tau] \subset \mathcal{M}_+$ is
order-isomorphic to the same interval taken in \((z\mathcal{M})^*_r\). So the result follows from the previous part of the proof. ■

Clearly, a \(C^*\)-algebra \(\mathcal{A}\) is abelian if and only if the enveloping von Neumann algebra \(\mathcal{A}^{**}\) is abelian. Therefore we get the following corollary.

**Corollary 4.5.** For a \(C^*\)-algebra \(\mathcal{A}\) the following are equivalent:

(i) \(\mathcal{A}\) is abelian.

(ii) For every nonzero \(\varphi \in \mathcal{A}^*_+\) the restriction of the norm topology to \([0, \varphi]\) is equal to \(\tau_0[0, \varphi]\).

We shall now compare the sets of null sequences in intervals of the type \([0, \varphi]\), \(\varphi \in \mathcal{A}^*\), with respect to the norm and order topology \(\tau_0[0, \varphi]\). For this investigation, we shall make an essential use of the properties of the GNS representation, and the map \(\theta_{\varphi}\) that identifies the effect algebra of the commutant of the GNS algebra with the interval \([0, \varphi]\) in the dual space. Even though this map is important in many aspects of operator algebra theory, we did not find a proper reference for the continuity properties of this map (and its inverse). We therefore present this analysis that may be of independent interest. Let us fix the notation.

For a nonzero positive element \(\varphi\) in the dual \(\mathcal{A}^*\) let \(C_{\varphi}\) denote the linear span of \([0, \varphi]\). Via the GNS representation of \(\mathcal{A}\), induced by \(\varphi\), it is possible to define a map

\[
\theta_{\varphi} : \pi_{\varphi}(\mathcal{A}^\prime) \to \mathcal{A}^* : x \mapsto \theta_{\varphi}(x),
\]

where

\[
\theta_{\varphi}(x) : \mathcal{A} \to \mathbb{C} : a \mapsto \langle \pi_{\varphi}(a)x\xi_\varphi, \xi_\varphi \rangle.
\]

We recall (see for e.g. [23, Proposition 3.10, p. 201]) that \(\theta_{\varphi}\) is a linear-isomorphism of \(\pi_{\varphi}(\mathcal{A}^\prime)\) onto \(C_{\varphi}\), and that if we denote its inverse by \(\Phi_{\varphi}\), then both \(\theta_{\varphi}\) and \(\Phi_{\varphi}\) are completely positive. In particular, this implies that the restriction of \(\theta_{\varphi}\) is an order isomorphism between \(E(\pi_{\varphi}(\mathcal{A}^\prime))\) and \([0, \varphi]\). A generalization of this result was given by Arveson [1], who studied the interval \([0, \varphi]\) when \(\varphi\) is a completely positive map on a \(C^*\)-algebra, and not only a state. This has interesting applications in the study of extremal completely positive maps and elsewhere.

Let us mention that in [23, Remark 3.11(i), p. 202], it is erroneously stated that \(\theta_{\varphi}\) need not be continuous. Although it is true that a completely positive map, from a subspace of \(\mathcal{A}\) into another subspace of \(\mathcal{A}^*\), need not be continuous, the continuity of \(\theta_{\varphi}\) follows easily from the estimation

\[
\|\theta_{\varphi}(x)\| \leq \|x\| \|\xi_{\varphi}\|^2.
\]

On the other hand, it is the inverse map \(\Phi_{\varphi}\) that need not be continuous, as the following example shows.
EXAMPLE 4.6. Let $\zeta = (\alpha_n)_{n \in \mathbb{N}}$ be a vector in $\ell^1$ such that $\alpha_n > 0$ for every $n \in \mathbb{N}$. Then $\eta = (\sqrt{\alpha_n})_{n \in \mathbb{N}}$ belongs to $\ell^2$. Let $\mathcal{A}$ be the $C^*$-algebra $\ell^\infty$ acting on $\ell^2$ and let $\varphi := \omega_\eta$ be the positive functional on $\mathcal{A}$ defined by

$$\varphi : a = (\beta_n)_{n \in \mathbb{N}} \mapsto \sum_{n=1}^\infty \beta_n \alpha_n = \langle a \eta, \eta \rangle.$$ 

Denote by $\ell^p_\zeta$ the sequence space $L^p(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_\zeta)$ where $\mu_\zeta$ is the $\sigma$-additive measure defined by $\mu_\zeta(\{n\}) = \alpha_n$. Let $\{H_\varphi, \pi_\varphi, \xi_\varphi\}$ be the GNS construction of $\mathcal{A}$ with respect to $\varphi$. Then $H_\varphi = \ell^2_\zeta$, the constant unit sequence equals $\xi_\varphi$, and $\pi_\varphi(a)$ equals the operator on $\ell^2_\zeta$ given by multiplication by $a$. Our assumption on the $\alpha_n$'s implies that $\pi_\varphi$ is an isomorphism and therefore an isometry. Let $h = (\beta_n)_{n \in \mathbb{N}} \in \mathcal{A}$ and let $\psi_h$ denote the functional $\theta_\varphi(\pi_\varphi(h))$. We show that $\|\psi_h\| = \|h\|_{\ell^1_\zeta}$. First observe that for every $a = (\lambda_n)_{n \in \mathbb{N}} \in \mathcal{A}_1$ we have

$$|\psi_h(a)| = |\langle \pi_\varphi(ah)\xi_\varphi, \xi_\varphi \rangle| = \sum_{n=1}^\infty \lambda_n \beta_n \alpha_n \leq \sum_{n=1}^\infty |\beta_n| \alpha_n = \|h\|_{\ell^1_\zeta}.$$ 

On the other hand, if we let $u$ denote the unitary element in $\mathcal{A}$ such that $uh = |h|$, then

$$\|h\|_{\ell^1_\zeta} = \sum_{n=1}^\infty |\beta_n| \alpha_n = |\langle \pi_\varphi(uh)\xi_\varphi, \xi_\varphi \rangle| = |\psi_h(u)|.$$ 

Thus, we have $\|\psi_h\| = \|h\|_{\ell^1_\zeta}$ and since the inverse $\Phi_\varphi$ of $\theta_\varphi$ maps $\psi_h$ to $\pi_\varphi(h)$, i.e. $\Phi_\varphi$ maps $(\ell^\infty, \|\cdot\|_{\ell^1_\zeta})$ into $(\ell^\infty, \|\cdot\|_\infty)$ it follows that $\Phi_\varphi$ is not continuous. Indeed, consider the sequence $(x_n)_{n \in \mathbb{N}}$ in $\ell^\infty$, where $x_n = (\delta_{jn})_{j \in \mathbb{N}}$. We can see that $\|x_n\|_\infty = 1$, while $\|x_n\|_{\ell^1_\zeta} = \alpha_n \to 0$. Note that in this case the space $C_\varphi$ is not complete since by the Open Mapping Theorem this would imply continuity of $\Phi_\varphi$.

PROPOSITION 4.7. Let $\varphi \in \mathcal{A}^*_+$. Then $\|\theta_\varphi(x)\| = \|x^{1/2}\xi_\varphi\|^2$ for every $x \in \mathcal{E}(\pi_\varphi(\mathcal{A})')$.

Proof. For any $x \in \mathcal{E}(\pi_\varphi(\mathcal{A})')$ and $a \in \mathcal{A}_1$ we have

$$|\langle \theta_\varphi(x)(a) \rangle| = |\langle \pi_\varphi(a)x\xi_\varphi, \xi_\varphi \rangle| = |\langle \pi_\varphi(a)x^{1/2}\xi_\varphi, x^{1/2}\xi_\varphi \rangle| \leq \|x^{1/2}\xi_\varphi\|^2.$$ 

To establish the equality one simply needs to consider an approximate identity $(u_\gamma)$ in $\mathcal{A}$:

$$\langle \theta_\varphi(x)(u_\gamma) \rangle = \langle \pi_\varphi(u_\gamma)x\xi_\varphi, \xi_\varphi \rangle = \langle \pi_\varphi(u_\gamma)x^{1/2}\xi_\varphi, x^{1/2}\xi_\varphi \rangle + \langle x\xi_\varphi, \xi_\varphi \rangle \to \|x^{1/2}\xi_\varphi\|^2.$$ 

COROLLARY 4.8. Let $\varphi \in \mathcal{A}^*_+$. A net $(x_\gamma)_{\gamma \in \Gamma}$ in $\mathcal{E}(\pi_\varphi(\mathcal{A})')$ is null with respect to the strong operator topology if and only if $(\theta_\varphi(x_\gamma))_{\gamma \in \Gamma}$ is norm-null.
Proof. Since $\xi_\varphi$ is cyclic for $\pi_\varphi(\mathcal{A})$, a bounded net $(x_\gamma)_{\gamma \in I}$ in $\pi_\varphi(\mathcal{A})'$ is null with respect to the strong operator topology if and only if $x_\gamma \xi_\varphi \to 0$. Moreover, since $x_\gamma \geq 0$ for every $\gamma$, the condition $x_\gamma \xi_\varphi \to 0$ is equivalent to $x_\gamma^{1/2} \xi_\varphi \to 0$. So the corollary follows from Proposition 4.7.

**Theorem 4.9.** Let $\varphi$ be a state on a $C^*$-algebra $\mathcal{A}$. The following statements are equivalent:

(i) The von Neumann algebra $\pi_\varphi(\mathcal{A})'$ is finite.
(ii) The interval $[0, \varphi]$ has the same null sequences with respect to the norm topology and the order topology $\tau_{[0, \varphi]}$.

Proof. First let us observe that $\pi_\varphi(A)'$ is $\sigma$-finite as $\omega_\xi_\varphi$ is a faithful normal state on it. Then the result follows immediately by Corollary 4.8 together with Theorem 2.1, and the fact that $[0, \varphi]$ and $E(\pi_\varphi(\mathcal{A})')$ are order-isomorphic.

Let us remark that if a state $\varphi$ is faithful, then the vector $\xi_\varphi$ is bicyclic. In this case, deep modular theory (see e.g. [5]) tells us that $\pi_\varphi(\mathcal{A})''$ and $\pi_\varphi(\mathcal{A})'$ are anti-isomorphic. It follows that $\pi_\varphi(\mathcal{A})''$ is finite if and only if $\pi_\varphi(\mathcal{A})'$ is finite (see e.g. [14, Theorem 9.1.3, p. 588]). Therefore we have the following corollary.

We say that a state $\varphi$ on a $C^*$-algebra $\mathcal{A}$ is **smooth** if it satisfies (ii) of Theorem 4.9.

**Corollary 4.10.** For a faithful state $\varphi$ on a $C^*$-algebra $\mathcal{A}$ the following conditions are equivalent:

(i) $\pi_\varphi(\mathcal{A})''$ is finite.
(ii) $\varphi$ is smooth.

Moreover, if $\mathcal{M}$ is a von Neumann algebra and $\varphi$ is a faithful normal state on $\mathcal{M}$, then $\varphi$ is smooth if and only if $\mathcal{M}$ is finite.

**Proof.** The ‘moreover’ part follows from the fact that if $\varphi$ is faithful and normal, then $\pi_\varphi$ is a normal faithful representation and so $\pi_\varphi(\mathcal{M})$ is isomorphic, as a von Neumann algebra, to $\mathcal{M}$.

**Theorem 4.11.** Let $\mathcal{A}$ be a $C^*$-algebra.

(i) If every state on $\mathcal{A}$ is smooth then $\mathcal{A}$ is of type I.
(ii) In particular, if $\mathcal{A}$ is a von Neumann algebra and every normal state on $\mathcal{A}$ is smooth, then $\mathcal{A}$ is isomorphic to a finite direct sum of matrix algebras over abelian von Neumann algebras.

**Proof.** (i) If $\mathcal{A}$ is not of type I, then by the celebrated Glimm–Sakai Theorem—established for the nonseparable case in [20, 21]—we know that $\mathcal{A}$ has a type III representation, say $(\pi, H)$. Let $\xi$ be a unit vector in $H$ and let $\varphi$ denote the state on $\mathcal{A}$ defined by $\varphi(a) = \langle \pi(a)\xi, \xi \rangle$. Then $H_\varphi$
is unitarily equivalent to the subspace \( H_\xi = [\pi(\mathcal{A})\xi] \) of \( H \) and the GNS representation \( \pi_\varphi \) is equivalent to the compression of \( \pi \) to \( H_\xi \). Denote by \( p_\xi \) the projection of \( H \) onto \( H_\xi \) and let \( \mathcal{M} = \pi(\mathcal{A})'' \). Since \( \mathcal{M} \) is of type III, the same holds for its commutant \( \mathcal{M}' \) (see e.g. [14, Theorem 9.1.3, p. 558]). This property passes to the hereditary subalgebra \( p_\xi \mathcal{M} p_\xi \) as well. Working on the Hilbert space \( H_\xi \) we have

\[
(p_\xi \mathcal{M})' = (p_\xi \mathcal{M} p_\xi)' = p_\xi \mathcal{M}' p_\xi.
\]

So \( (p_\xi \mathcal{M})' \)—and therefore \( p_\xi \mathcal{M} \) (by [14, Theorem 9.1.3, p. 558])—is of type III. Since \( p_\xi \mathcal{M} \) is isomorphic to \( \pi_\varphi(\mathcal{A})'' \) it follows that \( \pi_\varphi(\mathcal{A})' \) is of type III and therefore infinite. Hence \( \varphi \) is not smooth by Theorem 4.9.

(ii) This follows from the fact that any type I \( C^* \)-algebra is nuclear (see e.g. [6, Proposition 2.7.3]). Moreover, it is known that a von Neumann algebra \( \mathcal{A} \) is nuclear if and only if it is a finite direct sum of finite homogeneous algebras, that is, exactly when \( \mathcal{A} = M_{k_1}(\mathcal{A}_1) \oplus M_{k_2}(\mathcal{A}_2) \oplus \cdots \oplus M_{k_j}(\mathcal{A}_j) \), where \( \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_j \) are abelian von Neumann algebras.

**Acknowledgements.** The authors are very grateful to the referee for careful reading of the paper and helpful suggestions.

The work of Jan Hamhalter was supported by the following two projects:

- CZ.02.1.01/0.0/0.0/16_019/00007,
- “Czech Science Foundation” [grant number 17-00941S, “Topological and geometrical properties of Banach spaces and operator algebras II”].

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