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Continuity of non-regular pseudodifferential operators on variable Triebel–Lizorkin spaces

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Abstract. This paper is concerned with the boundedness of nonregular pseudodifferential operators with symbols belonging to certain vector-valued Besov space, on Triebel–Lizorkin spaces with variable smoothness and integrability. These symbols include the classical Hörmander classes. Our results cover the results on Triebel–Lizorkin spaces with fixed exponents.

1. Introduction. The pseudodifferential operators have been widely used in various areas of analysis. These generalizations of differential operators are defined by

$$\sigma(x,D)f(x)=(2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\sigma(x,\xi)\mathcal{F}f(\xi)\,d\xi,\qquad f\in\mathcal{S}(\mathbb{R}^n),\ x\in\mathbb{R}^n,$$

where σ is the symbol of $\sigma(x,D)$ and $\mathcal{F}f$ is the Fourier transform of f.

The boundedness of these operators on function spaces such as Lebesgue spaces, Hölder spaces, Besov spaces and Triebel–Lizorkin spaces has been considered in [3], [4], [5], [7], [13], [17], [22], [24], and in references therein.

In [18] J. Marschall introduced the class $SB^m_\delta(r,\mu,v;N,\lambda)$, which is defined by means of vector-valued Besov spaces, and proved the boundedness of the corresponding pseudodifferential operators on Besov spaces and Triebel–Lizorkin spaces.

Boundedness of pseudodifferential operators, with symbols in Hörmander classes, on weighted variable exponent Lebesgue and Bessel potential spaces was studied by V. S. Rabinovich and S. Samko [20, 21] and by A. Yu. Karlovich and I. M. Spitkovsky [14] (in variable Lebesgue spaces). Since Besov spaces can be written as (real) interpolation spaces between appro-

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priate Bessel potential spaces, Almeida and Hastö [2] extended the results of V. S. Rabinovich and S. Samko to Besov spaces with variable integrability $B^s_{p(\cdot),q}$. The boundedness of these operators on Besov spaces with variable smoothness and integrability has considered in [12]. Recently V. D. Kryakvin and V. S. Rabinovich [16] obtained the boundedness and compactness of pseudodifferential operators of variable order acting on Besov spaces of variable smoothness.

Our main result in this paper concerns the boundedness properties of pseudodifferential operators on Triebel–Lizorkin spaces with variable smoothness and integrability with symbols in the class $SB^m_\delta(r,\mu,v;N,\lambda)$.

The paper is arranged as follows. Section 2 is devoted to some preliminaries; in particular we recall the definition of Triebel–Lizorkin spaces with variable smoothness and integrability. In Section 3 we give some key technical lemmas needed in the proofs of the main statements. The main result of this paper is formulated in Section 4, where we prove the boundedness of nonregular pseudodifferential operators on variable Triebel–Lizorkin spaces.

2. Preliminaries. As usual, we denote by $\mathbb{N}$ the collection $\{1,2,\ldots\}$ of all natural numbers and $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. The expression $f\lesssim g$ means that $f\leq cg$ for some constant c and nonnegative functions f and g.

By $\operatorname{supp} f$ we denote the support of the function f, i.e., the closure of its nonzero set. If $E\subset\mathbb{R}^n$ is a measurable set, then $|E|$ stands for the (Lebesgue) measure of E.

The Hardy–Littlewood maximal operator $\mathcal{M}$ is defined on L^1_{loc} by

$$\mathcal{M}f(x)=\sup_{r>0}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)|\,dy,$$

and $\mathcal{M}_\tau f=(\mathcal{M}|f|^\tau)^{1/\tau}$, $0<\tau<\infty$. The symbol $\mathcal{S}(\mathbb{R}^n)$ is used for the space of Schwartz functions. We denote by $\mathcal{S}'(\mathbb{R}^n)$ the dual space of tempered distributions on $\mathbb{R}^n$. The Fourier transform of a tempered distribution f is denoted by $\mathcal{F}f$, and its inverse transform by $\mathcal{F}^{-1}f$.

2.1. Variable exponents. The variable exponents that we consider are always measurable functions p on $\mathbb{R}^n$ with range in $[c,\infty[$ for some $c>0$. We denote the set of such functions by $\mathcal{P}_0$. The subset of variable exponents with range $[1,\infty[$ is denoted by $\mathcal{P}$. We use the standard notation $p^-=\operatorname{ess\,inf}_{x\in\mathbb{R}^n}p(x)$ and $p^+=\operatorname{ess\,sup}_{x\in\mathbb{R}^n}p(x)$.

The *variable exponent Lebesgue space* $L^{p(\cdot)}$ is the class of all measurable functions f on $\mathbb{R}^n$ such that the modular

$$\varrho_{p(\cdot)}(f/\lambda)=\int_{\mathbb{R}^n}\left|\frac{f(x)}{\lambda}\right|^{p(x)}dx$$

is finite for some $\lambda > 0$. This is a quasi-Banach function space equipped with the quasi-norm

$$\|f\|_{p(\cdot)} = \inf\left\{\mu > 0 : \varrho_{p(\cdot)}\left(\frac{1}{\mu}f\right) \leq 1\right\}.$$

If $p(x) := p$ is constant, then $L^{p(\cdot)} = L^p$ is the classical Lebesgue space.

Let $p, q \in \mathcal{P}_0$. The space $L^{p(\cdot)}(\ell^{q(\cdot)})$ is defined to be the set of all sequences $(f_v)_{v\geq 0}$ of functions such that

$$\|(f_v)_{v\geq 0}\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} = \big\|\|(f_v(x))_{v\geq 0}\|_{\ell^{q(x)}}\big\|_{L^{p(\cdot)}} < \infty.$$

It is easy to show that $L^{p(\cdot)}(\ell^{q(\cdot)})$ is always a quasi-normed space, and it is a normed space if $\min(p(x), q(x)) \geq 1$ pointwise.

We say that $g : \mathbb{R}^n \to \mathbb{R}$ is *locally log-Hölder continuous*, abbreviated $g \in C^{\log}_{\mathrm{loc}}$, if there exists $c_{\log}(g) > 0$ such that

$$|g(x) - g(y)| \leq \frac{c_{\log}(g)}{\log(e + 1/|x - y|)} \quad \text{for all } x, y \in \mathbb{R}^n. \tag{2.1}$$

We say that g satisfies the *log-Hölder decay condition* if there exists $g_\infty \in \mathbb{R}$ and a constant $c_{\log} > 0$ such that

$$|g(x) - g_\infty| \leq \frac{c_{\log}}{\log(e + |x|)} \quad \text{for all } x \in \mathbb{R}^n.$$

The constants $c_{\log}(g)$ and $c_{\log}$ are called the *locally log-Hölder constant* and the *log-Hölder decay constant*, respectively. We note that all functions g in $C^{\log}_{\mathrm{loc}}$ belong to L^∞.

We say that g is *globally-log-Hölder continuous*, abbreviated $g \in C^{\log}$, if it is locally log-Hölder continuous and satisfies the log-Hölder decay condition.

We define the following class of variable exponents:

$$\mathcal{P}^{\log} = \{p \in \mathcal{P} : 1/p \in C^{\log}\},$$

which was introduced in [9, Section 2]. We define $1/p_\infty := \lim_{|x|\to\infty} 1/p(x)$ and we use the convention $1/\infty = 0$. Note that although $1/p$ is bounded, the variable exponent p itself can be unbounded. Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $p^- > 1$. Then there exists $K > 0$ only depending on the dimension n and $c_{\log}(p)$ such that

$$\|\mathcal{M}f\|_{p(\cdot)} \leq K\|f\|_{p(\cdot)} \quad \text{for all } f \in L^{p(\cdot)} \tag{2.2}$$

(see [8], [10, Theorem 4.3.8], and [9, Theorem 1.2]). Also if $p \in \mathcal{P}^{\log}$, then the convolution with a radially decreasing L^1-function is bounded on $L^{p(\cdot)}$:

$$\|\varphi * f\|_{p(\cdot)} \leq c\|\varphi\|_1\|f\|_{p(\cdot)}.$$

We also refer to [6] and [8], where various results on maximal functions in variable Lebesgue spaces were obtained. Recall that

$$\eta_{v,m}(x) := 2^{nv}(1 + 2^v|x|)^{-m}$$

for any $x \in \mathbb{R}^n$, $v \in \mathbb{N}_0$ and $m > 0$. Note that $\eta_{v,m} \in L^1$ when $m > n$, and $\|\eta_{v,m}\|_1 = c_m$ is independent of v.

2.2. Variable Triebel–Lizorkin spaces. In this subsection we give the Fourier-analytical definition of the spaces $F^{s(\cdot)}_{p(\cdot),q(\cdot)}$ and recall their basic properties. We first need the concept of a smooth dyadic resolution of unity. Let Ψ be a function in $\mathcal{S}'(\mathbb{R}^n)$ satisfying $\Psi(x) = 1$ for $|x| \leq 1$ and $\Psi(x) = 0$ for $|x| \geq 2$. We put $\mathcal{F}\varphi_0(x) = \Psi(x)$, $\mathcal{F}\varphi_1(x) = \Psi(x/2) - \Psi(x)$, and $\mathcal{F}\varphi_v(x) = \mathcal{F}\varphi_1(2^{1-v}x)$ for $v = 2, 3, \ldots$. Then $\{\mathcal{F}\varphi_v\}_{v\in\mathbb{N}_0}$ is a smooth dyadic resolution of unity, $\sum_{v=0}^{\infty} \mathcal{F}\varphi_v(x) = 1$ for all $x \in \mathbb{R}^n$. Thus we obtain the Littlewood–Paley decomposition

$$f = \sum_{v=0}^{\infty} \varphi_v * f \quad \text{for all } f \in \mathcal{S}'(\mathbb{R}^n)$$

(convergence in $\mathcal{S}'(\mathbb{R}^n)$).

We recall the definition and some properties of Triebel–Lizorkin spaces with variable smoothness and integrability.

Definition 2.1. Let $\{\mathcal{F}\varphi_v\}_{v\in\mathbb{N}_0}$ be as resolution of unity. For $s : \mathbb{R}^n \to \mathbb{R}$ and $p, q \in \mathcal{P}_0(\mathbb{R}^n)$ with $0 < p^+, q^+ < \infty$, the *Triebel–Lizorkin space* $F^{s(\cdot)}_{p(\cdot),q(\cdot)}$ consists of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}} = \|\{2^{vs(\cdot)}\varphi_v * f\}_v\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} < \infty.$$

We directly obtain the following simplification in the case when q is constant:

$$\|f\|_{F^{s(\cdot)}_{p(\cdot),q}} = \big\|\|\{2^{vs(\cdot)}\varphi_v * f\}_v\|_{\ell^q}\big\|_{p(\cdot)}.$$

The Triebel–Lizorkin spaces with variable smoothness were first introduced in [11]. If $p(\cdot)$, $q(\cdot)$, and $s(\cdot)$ are constants, then we recover the standard Triebel–Lizorkin spaces. For any $p, q \in C^{\log}$, with $0 < p^+, q^+ < \infty$ and $s \in C^{\log}_{\mathrm{loc}}$, the space $F^{s(\cdot)}_{p(\cdot),q(\cdot)}$ does not depend on the chosen smooth dyadic resolution of unity $\{\mathcal{F}\varphi_v\}_{v\in\mathbb{N}_0}$ (in the sense of equivalent quasi-norms) and

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow F^{s(\cdot)}_{p(\cdot),q(\cdot)} \hookrightarrow \mathcal{S}'(\mathbb{R}^n).$$

We next consider embeddings of Sobolev type. For constant exponents it is well known that

$$F^{\alpha_0}_{p_0,\infty} \hookrightarrow F^{\alpha_1}_{p_1,q}$$

if $\alpha_0 - n/p_0 = \alpha_1 - n/p_1$, where $0 < p_0 \leq p_1 < \infty$, $0 < q \leq \infty$, $-\infty < \alpha_1 \leq \alpha_0 < \infty$ (see e.g. [25, Theorem 2.7.1]). For the variable case we have the following results (see [26]).

Theorem 2.2. *Let $p_0, p_1, q_0, q_1 \in C^{\log}$ with $0 < p_0^+, p_1^+, q_0^+, q_1^+ < \infty$ and $\alpha_0, \alpha_1 \in C^{\log}_{\mathrm{loc}}$ with $\alpha_0 \geq \alpha_1$. If*

$$\alpha_0(x) - \frac{n}{p_0(x)} = \alpha_1(x) - \frac{n}{p_1(x)} \quad \text{and} \quad \inf(\alpha_0 - \alpha_1) > 0,$$

then

$$F^{\alpha_0(\cdot)}_{p_0(\cdot),q_0(\cdot)} \hookrightarrow F^{\alpha_1(\cdot)}_{p_1(\cdot),q(\cdot)}.$$

More information about these function spaces can be found in [1], [11], [15], and in references therein.

3. Basic tools. In this section we present some results to be used later. The first lemma is a Hardy-type inequality which is easy to prove.

Lemma 3.1. *Let $0 < a < 1$ and $0 < q \leq \infty$. Let $\{\varepsilon_k\}_{k\in\mathbb{N}_0}$ be a sequence of positive real numbers such that $\|\{\varepsilon_k\}_k\|_{\ell^q} = I < \infty$. The sequence*

$$\Big\{\delta_k : \delta_k = \sum_{j=0}^{\infty} a^{|k-j|}\varepsilon_j\Big\}_{k\in\mathbb{N}_0}$$

is in ℓ^q with $\|\{\delta_k\}_k\|_{\ell^q} \leq cI$, where c depends only on a and q.

The next lemma often allows us to deal with exponents which are smaller than 1.

Lemma 3.2. *Let $r > 0$, $v \in \mathbb{N}_0$, and $m > n$. Then there exists $c = c(r, m, n) > 0$ such that for all $g \in \mathcal{S}'(\mathbb{R}^n)$ with $\operatorname{supp} \mathcal{F}g \subset \{\xi \in \mathbb{R}^n : |\xi| \leq 2^{v+1}\}$, we have*

$$|g(x)| \leq c(\eta_{v,m} * |g|^r(x))^{1/r}, \qquad x \in \mathbb{R}^n.$$

The following lemma comes from [15, Lemma 19] (see also [11, Lemma 6.1]).

Lemma 3.3. *Let $\alpha \in C^{\log}_{\mathrm{loc}}$ and let $R \geq c_{\log}(\alpha)$, where $c_{\log}(\alpha)$ is the constant from* (2.1) *for α. Then*

$$2^{v\alpha(x)}\eta_{v,m+R}(x-y) \leq c2^{v\alpha(y)}\eta_{v,m}(x-y)$$

with $c > 0$ independent of $x, y \in \mathbb{R}^n$ and $v, m \in \mathbb{N}_0$.

The following lemma is from [11, Theorem 3.2] (we use it, since the maximal operator is in general not bounded on $L^{p(\cdot)}(\ell^{q(\cdot)})$).

Lemma 3.4. *Let $p, q \in C^{\log}$ with $1 < p^- \leq p^+ < \infty$ and $1 < q^- \leq q^+ < \infty$. For $m > n$, there exists $c > 0$ such that*

$$\|\{\eta_{v,m} * f_v\}_v\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} \leq c\|\{f_v\}_v\|_{L^{p(\cdot)}(\ell^{q(\cdot)})}$$

for every sequence $\{f_v\}_v$ of L^1_{loc}-functions.

Lemma 3.5. *Let $A, B > 0$, $p, q \in C^{\log}$ with $p^+, q^+ < \infty$, and $s \in C^{\log}_{\mathrm{loc}}$. Let $\{f_k\}_{k\in\mathbb{N}_0}$ be a sequence of functions such that*

$$\operatorname{supp}\mathcal{F}f_0 \subseteq \{\xi \in \mathbb{R}^n : |\xi| \le A\},$$
$$\operatorname{supp}\mathcal{F}f_k \subseteq \{\xi \in \mathbb{R}^n : B2^{k+1} \le |\xi| \le A2^{k+1}\}.$$

Then

$$\Big\|\sum_{k=0}^{\infty} f_k\Big\|_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}} \le c\|\{2^{ks(\cdot)} f_k\}_k\|_{L^{p(\cdot)}(\ell^{q(\cdot)})}.$$

Proof. Let $\{\mathcal{F}\varphi_j\}_{j\in\mathbb{N}_0}$ be a resolution of unity. In view of the support properties of $\mathcal{F}f_k$ and $\mathcal{F}\varphi_j$, the sum $\varphi_j * \sum_{k=0}^{\infty} f_k$ becomes

$$\sum_{l=-\kappa_1}^{\kappa_2} \varphi_j * f_{j+l}$$

for some $\kappa_1, \kappa_2 \in \mathbb{N}_0$. By Lemma 3.2,

$$|\varphi_j * f_{j+l}| \lesssim (\eta_{j,m} * |f_{j+l}|^t)^{1/t}, \qquad l = -\kappa_1, \ldots, \kappa_2,$$

for any $m > n + c_{\log}(s)$ and any $t > 0$. Therefore, with $t \in (0, \min\{1, p^-, q^-\})$,

$$\begin{aligned}\Big\|\sum_{k=0}^{\infty} f_k\Big\|_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}} &\lesssim \sum_{l=-\kappa_1}^{\kappa_2} \|\{(\eta_{j,m-c_{\log}(\alpha)} * 2^{js(\cdot)t}|f_{j+l}|^t)^{1/t}\}_k\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} \\ &\lesssim \sum_{l=-\kappa_1}^{\kappa_2} \|\{2^{js(\cdot)} f_{j+l}\}_k\|_{L^{p(\cdot)}(\ell^{q(\cdot)})},\end{aligned}$$

by Lemmas 3.3 and 3.4. ∎

The following proposition plays a fundamental role in this paper.

Proposition 3.6. *Let $\omega \in \mathcal{S}(\mathbb{R}^n)$, $s \in C^{\log}_{\mathrm{loc}}$, $m > n + c_{\log}(s)$, $c > 0$, $1 \le \lambda \le \infty$, $1/\tau > \max(1/2, 1/\lambda)$, and $N > \frac{m-c_{\log}(s)}{\tau} + c_{\log}(s)$. Let $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ be a bounded and measurable symbol such that*

$$\operatorname{supp} a(x, \cdot) \subseteq \{\xi \in \mathbb{R}^n : |\xi| \le c2^k\}.$$

(i) *If $\tau \ge 1$, then*

$$\begin{aligned}&|2^{ks(x)} a(x, D) f_k(x)| \\ &\qquad \lesssim \|a(x, 2^k \cdot)\|_{B^N_{\lambda,\infty}} (\eta_{k,m-c_{\log}(s)} * 2^{ks(\cdot)\tau}|f|^{\tau}(x))^{1/\tau}, \qquad x \in \mathbb{R}^n, \qquad (3.1)\end{aligned}$$

for any $k \in \mathbb{N}_0$, where $f_k = 2^{kn}\omega(2^k \cdot) * f$, with the implicit constant not depending on k.

(ii) *If $0 < \tau < 1$ and*

$$\operatorname{supp}\mathcal{F}f \subseteq \{\xi \in \mathbb{R}^n : |\xi| \le c2^k\},$$

then (3.1) holds with f in place of f_k, $k \in \mathbb{N}_0$.

Proof. The proof follows the ideas of [12, Proposition 1] and [19, Proposition 4].

(i) Let

$$K(x,x-y)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(x-y)\xi}a(x,\xi)\,d\xi=\mathcal{F}_\xi a(x,\,\cdot)(x-y)$$

be the kernel of $a(x,D)$. It follows from the Hölder inequality that

$$|2^{ks(x)}a(x,D)f_k(x)|$$

can be estimated by

$$\begin{aligned}\sum_{v=-\infty}^{\infty}\int_{\mathbb{R}^n}|2^{ks(x)}K(x,x-y)\mathcal{F}\varphi_v(y-x)f_k(y)|\,dy&\\ \le\sum_{v=-\infty}^{\infty}\Big(\int_{\mathbb{R}^n}&|K(x,x-y)\mathcal{F}\varphi_v(y-x)|^{\tau'}\,dy\Big)^{1/\tau'}\\ &\times\Big(\int_{|x-y|\le 2^{v+1}}|2^{ks(x)}f_k(y)|^{\tau}\,dy\Big)^{1/\tau}.\end{aligned}$$

Observe that

$$K(x,x-y)\mathcal{F}\varphi_v(y-x)=\mathcal{F}_\xi^{-1}\big(\mathcal{F}_\xi^{-1}(\mathcal{F}\varphi_v\mathcal{F}_\xi a(x,\,\cdot))\big)(x-y).$$

Hence by the Hausdorff–Young inequalities, since $1\le\tau\le 2$,

$$\begin{aligned}&|2^{ks(x)}a(x,D)f_k(x)|\\ &\quad\le c\sum_{v=-\infty}^{\infty}\|\mathcal{F}_\xi^{-1}(\mathcal{F}\varphi_v\mathcal{F}_\xi a(x,\,\cdot))\|_\tau\Big(\int_{|x-y|\le 2^{v+1}}|2^{ks(x)}f_k(y)|^{\tau}\,dy\Big)^{1/\tau}.\end{aligned}$$

Hölder's inequality and the fact that

$$2^{-kn}\eta_{k,d}(y-z)\le 2^{-2kn}\eta_{k,-d}(x-y)\eta_{k,d}(x-z),\qquad x,y,z\in\mathbb{R}^n,\ d>0,$$

imply

$$|f_k(y)|\lesssim(\eta_{k,m}*|f|^\tau(x))^{1/\tau}$$

for any $x,y\in\mathbb{R}^n$ such that $|x-y|\le 2^{v+1}$ and $v\le -k-1$. Since s is log-Hölder continuous we obtain

$$2^{ks(x)}\le 2^{-kn}\eta_{k,-c_{\log}(s)}(x-y)2^{ks(y)}\le(1+2^{k+v+1})^{c_{\log}(s)}2^{ks(y)}$$

and

$$2^{-kn}\eta_{k,-m}(x-y)\le(1+2^{k+v+1})^m$$

for any $x,y\in\mathbb{R}^n$ such that $|x-y|\le 2^{v+1}$. Therefore

$$|2^{ks(x)}a(x,D)f(x)| \leq c\sum_{v=-\infty}^{-k-1} 2^{(v+k)n/\tau} H_{v,k}(x) + c\sum_{v=-k}^{\infty} 2^{(v+k)((m-c_{\log}(s))/\tau+c_{\log}(s))} H_{v,k}(x),$$

where

$$H_{v,k}(x) = \|\mathcal{F}_\xi^{-1}(\mathcal{F}\varphi_{v+k}\mathcal{F}_\xi a(x,2^k\cdot))\|_\tau \big(\eta_{k,m-c_{\log}(s)} * 2^{ks(\cdot)\tau}|f|^\tau(x)\big)^{1/\tau}.$$

The second sum is clearly bounded by

$$c\|a(x,2^k\cdot)\|_{B^N_{\tau,\infty}} \big(\eta_{k,m-c_{\log}(s)} * 2^{ks(\cdot)\tau}|f|^\tau(x)\big)^{1/\tau}, \quad x\in\mathbb{R}^n.$$

The first sum is bounded by

$$c\sup_{i\leq 0} \|\mathcal{F}_\xi^{-1}(\varphi_i\mathcal{F}_\xi a(x,2^k\cdot))\|_\tau (\eta_{k,m-c_{\log}(s)} * 2^{ks(\cdot)\tau}|f|^\tau(x))^{1/\tau} \leq c\|a(x,2^k\cdot)\|_\tau (\eta_{k,m-c_{\log}(s)} * 2^{ks(\cdot)\tau}|f|^\tau(x))^{1/\tau}, \quad x\in\mathbb{R}^n,$$

where we have used the fact that

$$\|\mathcal{F}_\xi^{-1}(\varphi_i\mathcal{F}_\xi a(x,2^k\cdot))\|_\tau \leq \|\mathcal{F}^{-1}\varphi_i\|_1 \|a(x,2^k\cdot)\|_\tau \lesssim \|a(x,2^k\cdot)\|_\tau, \quad i\leq 0.$$

Our estimate follows from the fact that

$$\|a(x,2^k\cdot)\|_\tau + \|a(x,2^k\cdot)\|_{B^N_{\tau,\infty}} \lesssim \|a(x,2^k\cdot)\|_{B^N_{\lambda,\infty}}.$$

(ii) Applying the Plancherel–Pólya–Nikol'skii inequality, we obtain

$$|2^{ks(x)}a(x,D)f(x)| \lesssim 2^{kn(1/\tau-1)}\Big(\int_{\mathbb{R}^n} 2^{ks(x)\tau}|K(x,x-y)|^\tau|f(y)|^\tau\,dy\Big)^{1/\tau}$$

for any $x\in\mathbb{R}^n$. This expression raised to the power τ is bounded by

$$\begin{aligned}
&c2^{kn(1-\tau)}\sum_{v\in\mathbb{Z}} \sup_y |K(x,x-y)\mathcal{F}\varphi_v(x-y)|^\tau \int_{|x-y|<2^{v+1}} |2^{ks(x)}f(y)|^\tau\,dy\\
&\quad\lesssim \sum_{v\in\mathbb{Z}} 2^{(c_{\log}(s)\tau+m-c_{\log}(s))\max(0,k+v)-kn\tau} \sup_y |K(x,x-y)\mathcal{F}\varphi_v(x-y)|^\tau\\
&\qquad\times \int_{|x-y|<2^{v+1}} 2^{ks(y)\tau}\eta_{k,m-c_{\log}(s)}(x-y)|f(y)|^\tau\,dy.
\end{aligned}$$

As before,

$$\begin{aligned}
\|K(x,x-\cdot)\mathcal{F}\varphi_v(x-\cdot)\|_\infty &\leq \|\mathcal{F}_\xi^{-1}(\mathcal{F}\varphi_v\mathcal{F}_\xi a(x,\cdot))\|_1\\
&= 2^{kn}\|\mathcal{F}_\xi^{-1}(\mathcal{F}\varphi_{v+k}\mathcal{F}_\xi a(x,2^k\cdot))\|_1.
\end{aligned}$$

Hence

$$|2^{ks(x)}a(x,D)f(x)|^\tau \lesssim \|a(\cdot,2^k\cdot)\|^\tau_{B^{(m-c_{\log}(s))/\tau+c_{\log}(s)}_{1,\tau}} \eta_{k,m-c_{\log}(s)} * 2^{ks(\cdot)}|f|^\tau(x), \quad x\in\mathbb{R}^n.$$

The proof is complete. ∎

We need Marschall's inequalities—see [18, Proposition 1.3] and [28, Proposition 6.1] for the case of constant exponent.

LEMMA 3.7. *Let $A > 0, R \geq 1$, and $s \in C^{\log}_{\mathrm{loc}}$. Let $b \in \mathcal{D}(\mathbb{R}^n)$ and $f \in C^\infty(\mathbb{R}^n)$ be such that*

$$\operatorname{supp} \mathcal{F}f \subseteq \{\xi \in \mathbb{R}^n : |\xi| \leq AR\} \quad \text{and} \quad \operatorname{supp} b \subseteq \{\xi \in \mathbb{R}^n : |\xi| \leq A\}.$$

Then

$$2^{js(x)}|\mathcal{F}^{-1}b * f(x)| \leq c2^{(1-1/t)jn}(AR)^{n/t-n}\|b(2^j\cdot)\|_{B^{n/t+c_{\log}(s)}_{1,t}}\big(\mathcal{M}(2^{js(\cdot)t}|f|^t)(x)\big)^{1/t}$$

for any $0 < t \leq 1$ and any $x \in \mathbb{R}^n$, where c is independent of A, R, b, j, and f.

Proof. With x fixed, $y \mapsto \mathcal{F}^{-1}b(x-y)f(y)$ has its spectrum in the ball $B(0,(R+1)A)$. Applying the Plancherel–Pólya–Nikol'skii inequality, we obtain

$$\begin{aligned}|\mathcal{F}^{-1}b * f(x)| &\leq \int_{\mathbb{R}^n} |\mathcal{F}^{-1}b(x-y)f(y)|\,dy = \|\mathcal{F}^{-1}b(x-\cdot)f\|_1 \\ &\leq c(AR)^{n/t-n}\|\mathcal{F}b(x-\cdot)f\|_t.\end{aligned}$$

Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$ be supported in $\{\xi \in \mathbb{R}^n : 1/2 \leq |\xi| \leq 2\}$ and such that

$$\sum_{k=-\infty}^{\infty} \varphi(2^{-k}\xi) = 1, \qquad \xi \neq 0.$$

We have

$$\begin{aligned}\|\mathcal{F}^{-1}b(x-\cdot)f\|_t^t &= \int_{\mathbb{R}^n} |\mathcal{F}^{-1}b(x-y)f(y)|^t\,dy \\ &\leq \sum_{k=-\infty}^{\infty}\int_{\mathbb{R}^n} |\mathcal{F}^{-1}b(x-y)\varphi(2^{-k}(x-y))f(y)|^t\,dy \\ &\leq \sum_{k=-\infty}^{\infty}\sup_y |\mathcal{F}^{-1}b(x-y)\varphi(2^{-k}(x-y))|^t \int_{B(x,2^{k+1})} |f(y)|^t\,dy \\ &\leq \sum_{k=-\infty}^{\infty}\sup_z |\mathcal{F}^{-1}b(z)\varphi(2^{-k}(z))|^t \int_{B(x,2^{k+1})} |f(y)|^t\,dy.\end{aligned}$$

Since

$$\mathcal{F}^{-1}b(z)\varphi(2^{-k}z) = \mathcal{F}^{-1}\mathcal{F}(\mathcal{F}^{-1}b\varphi(2^{-k}\cdot))(z),$$

we obtain

$$|\mathcal{F}^{-1}b(z)\varphi(2^{-k}(z))| \leq \int_{\mathbb{R}^n} |\mathcal{F}(\mathcal{F}^{-1}b\varphi(2^{-k}\cdot))(y)|\,dy = \|\mathcal{F}(\varphi(2^{-k}\cdot)) * b\|_1.$$

From the fact that

$$2^{js(x)} \leq (1+2^j|x-y|)^{c_{\log}(s)} 2^{js(y)}, \qquad x,y \in \mathbb{R}^n,$$

by Lemma 3.3 we obtain

$$\begin{aligned} & 2^{js(x)t}\|\mathcal{F}^{-1}b(x-\cdot)f\|_t^t \\ & \leq \sum_{k=-\infty}^{\infty} 2^{\max(0,j+k)c_{\log}(s)t}\|\mathcal{F}(\varphi(2^{-k}\cdot)) * b\|_1^t \int\limits_{B(x,2^{k+1})} 2^{js(y)t}|f(y)|^t\,dy \\ & \leq \sum_{k=-\infty}^{\infty} 2^{\max(0,j+k)c_{\log}(s)t+kn}\|\mathcal{F}(\varphi(2^{-k}\cdot)) * b\|_1^t \mathcal{M}(2^{js(\cdot)t}|f|^t)(x), \qquad x \in \mathbb{R}^n. \end{aligned}$$

Observe that

$$\begin{aligned} \mathcal{F}(\varphi(2^{-k}\cdot)) * b(x) &= \int\limits_{\mathbb{R}^n} \mathcal{F}(\varphi(2^{-k}\cdot))(z)b(x-z)\,dz \\ &= 2^{-jn}\int\limits_{\mathbb{R}^n} \mathcal{F}(\varphi(2^{-k-j}\cdot))(2^{-j}z)b(x-z)\,dz \\ &= \mathcal{F}(\varphi(2^{-k-j}\cdot)) * b(2^j\cdot)(2^{-j}x). \end{aligned}$$

Therefore

$$\|\mathcal{F}(\varphi(2^{-k}\cdot)) * b\|_1 \leq 2^{jn}\|\mathcal{F}(\varphi(2^{-k-j}\cdot)) * b(2^j\cdot)\|_1.$$

Write

$$\sum_{k=-\infty}^{\infty} 2^{\max(0,j+k)c_{\log}(s)t+(j+k/t)nt}\|\mathcal{F}(\varphi(2^{-k-j}\cdot)) * b(2^j\cdot)\|_1^t = \sum_{k+j\geq 0}\cdots + \sum_{k+j<0}\cdots.$$

The first sum is bounded by

$$2^{(1-1/t)jnt}\|b(2^j\cdot)\|^t_{B_{1,t}^{n/t+c_{\log}(s)}}$$

and since

$$\|\mathcal{F}(\varphi(2^{-k-j}\cdot)) * b(2^j\cdot)\|_1 \lesssim \|b(2^j\cdot)\|_1 \lesssim \|b(2^j\cdot)\|_{B_{1,t}^{n/t+c_{\log}(s)}},$$

we obtain the desired estimate. ∎

Lemma 3.8. *Let $A > 0$, $p \in C^{\log}$, and $s \in C^{\log}_{\mathrm{loc}}$ be such that $p^+ < \infty$ and $s^- > n(\max\{1,1/p^-\}-1)$. Let $\{f_k\}_{k\in\mathbb{N}_0}$ be a sequence of functions such that* $\operatorname{supp}\mathcal{F}f_k \subseteq \{\xi \in \mathbb{R}^n : |\xi| \leq A2^{k+1}\}$. *Then*

$$\Big\|\sum_{k=0}^{\infty} f_k\Big\|_{F^{s(\cdot)}_{p(\cdot),\infty}} \leq c\|\{2^{ks(\cdot)}f_k\}_k\|_{L^{p(\cdot)}(\ell^\infty)}.$$

Proof. Let $\{\mathcal{F}\varphi_j\}_{j\in\mathbb{N}_0}$ be a resolution of unity. Using the support properties of $\{f_k\}_{k\in\mathbb{N}_0}$, we obtain

$$\sum_{k=0}^{\infty}\varphi_j * f_k = \sum_{k=j+\sigma}^{\infty}\varphi_j * f_k = \sum_{i=\sigma}^{\infty}\varphi_j * f_{j+i}, \qquad \sigma\in\mathbb{R}.$$

Let $\beta=\min\{1,p^-\}$. We have

$$\begin{aligned}\Big\|\sum_{k=0}^{\infty} f_k\Big\|^{\beta}_{F^{s(\cdot)}_{p(\cdot),\infty}} &\le \Big\|\Big\{2^{js(\cdot)}\sum_{i=\sigma}^{\infty}|\varphi_j * f_{j+i}|\Big\}_j\Big\|^{\beta}_{L^{p(\cdot)}(\ell^\infty)}\\ &= \Big\|\Big\{\Big(2^{js(\cdot)}\sum_{i=\sigma}^{\infty}|\varphi_j * f_{j+i}|\Big)^{\beta}\Big\}_j\Big\|_{L^{p(\cdot)/\beta}(\ell^\infty)}\\ &\le \sum_{i=\sigma}^{\infty}\|\{2^{js(\cdot)}|\varphi_j * f_{j+i}|\}_j\|^{\beta}_{L^{p(\cdot)}(\ell^\infty)}.\end{aligned}$$

Observe that $\varphi_j=\mathcal{F}^{-1}\mathcal{F}\varphi_j$ and $\operatorname{supp}\mathcal{F}\varphi_j\subset\{\xi:|\xi|\le 2^{j+1}\}$. Therefore we can use Lemma 3.7 and (2.2):

$$\begin{aligned}&2^{js(x)}|\varphi_j * f_{j+i}(x)|\\ &\quad\lesssim 2^{(n-n/t)j+(j+i)(n/t-n)}\|\mathcal{F}\varphi_j(2^j\,\cdot)\|_{B^{n/t+c_{\log}(s)}_{1,t}}(\mathcal{M}(2^{js(\cdot)}|f_{j+i}|^t)(x))^{1/t}\\ &\quad\lesssim 2^{i(n/t-n)}\|\varphi_1\|_{B^{n/t+c_{\log}(s)}_{1,t}}(\mathcal{M}(2^{js(\cdot)}|f_{j+i}|^t)(x))^{1/t}, \qquad x\in\mathbb{R}^n,\ 0<t\le 1.\end{aligned}$$

The proof is complete in view of the fact that $s^->n(\max\{1,1/p^-\}-1)$. ∎

4. Main results. Let $\{\mathcal{F}\varphi_j\}_{j\in\mathbb{N}_0}$ be a resolution of unity. For a function $a:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{C}$, we write

$$a_j(x,\xi)=\mathcal{F}^{-1}_{y\to x}(\mathcal{F}\varphi_j(y)\mathcal{F}a(y,\xi)).$$

Let $0<\mu\le\infty$, $1\le\lambda\le\infty$, $r\ge n/\mu$, and $N>n/\lambda$. The space $B^r_{\mu,v}(B^N_{\lambda,\infty})$ consists of all distributions $a\in\mathcal{S}'(\mathbb{R}^n\times\mathbb{R}^n)$ such that

$$\|a\|_{B^r_{\mu,v}(B^N_{\lambda,\infty})}=\big\|\{2^{jr}\|a_j(x,\,\cdot)\|_{B^N_{\lambda,\infty}}\}_j\big\|_{\ell^v(L^\mu)}<\infty.$$

Notice that these are just the spaces $SB^{\bar r}_{\bar p,\bar q}$ with $\bar r=(N,r)$, $\bar p=(\lambda,\mu)$, and $\bar q=(\infty,v)$; see [23] for further properties of these function spaces. Let $m,r,N\in\mathbb{R}$, $0\le\delta\le1$, $0<\mu\le\infty$, $r>n/\mu$, and $N>n/\lambda$. We say that a symbol a belongs to $SB^m_\delta(r,\mu,v;N,\lambda)$ if

$$\begin{aligned}&\sup_k 2^{-km}\big\|\|a(x,2^k\,\cdot)\mathcal{F}\varphi_k(2^k\,\cdot)\|_{B^N_{\lambda,\infty}}\big\|_{L^\infty(dx)}<\infty,\\ &\sup_k 2^{-k(m+\delta r)}\|a(x,2^k\cdot)\mathcal{F}\varphi_k(2^k\,\cdot)\|_{B^r_{\mu,v}(B^N_{\lambda,\infty})}<\infty;\end{aligned}$$

this notion was introduced by J. Marschall [18], [19]. Choosing $\mu = v = N = \lambda = \infty$ we see that these symbols include the classical Hörmander classes $S^m_{1,\delta}$. Moreover the class $SB^m_0(r, \mu, v; \infty, 1)$ equals the class $S'(B^{(1,\dots,1),r}_{\mu,v})^m$ of M. Yamazaki [27].

Also J. Marschall [18] gave new symbol classes which may roughly speaking be viewed as leading to vector-valued Triebel spaces. Let $0 < \mu < \infty$, $0 < v \le \infty$, $1 \le \lambda \le \infty$, $0 \le \delta \le 1$, $(1-\delta)r \ge n/\mu$, and $N > n/\lambda$. The space $F^r_{\mu,v}(B^N_{\lambda,\infty})$ consists of all distributions $a \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$\|a\|_{F^r_{\mu,v}(B^N_{\lambda,\infty})} = \big\|\{2^{jr}\|a_j(x,\cdot)\|_{B^N_{\lambda,\infty}}\}_j\big\|_{L^\mu(\ell^v)} < \infty.$$

Let $m, r, N \in \mathbb{R}$, $0 \le \delta \le 1, 0 < \mu < \infty, 0 < v \le \infty, 1 \le \lambda \le \infty$, $(1-\delta)r \ge n/\mu$, and $N > n/\lambda$. We say that a symbol a belongs to $SF^m_\delta(r, \mu, v; N, \lambda)$ if

$$\sup_k 2^{-km}\big\|\|a(x, 2^k\,\cdot)\mathcal{F}\varphi_k(2^k\,\cdot)\|_{B^N_{\lambda,\infty}}\big\|_{L^\infty(dx)} < \infty,$$
$$\sup_k 2^{-k(m+\delta r)}\|a(x, 2^k\cdot)\mathcal{F}\varphi_k(2^k\,\cdot)\|_{F^r_{\mu,v}(B^N_{\lambda,\infty})} < \infty$$

(see J. Marschall [18]). Notice that

$$SB^m_\delta(r, \mu, p; N, \lambda) \hookrightarrow SF^m_{\delta_1}(r_1, p, q; N, \lambda) \tag{4.1}$$

if $0 < \mu < p < \infty$, $0 < v \le \infty$, $r - n/\mu = r_1 - n/p$ and $\delta r = \delta_1 r_1$.

The next theorem concerns the boundedness of pseudodifferential operators in $F^{s(\cdot)}_{p(\cdot),q(\cdot)}$ spaces.

Theorem 4.1. *Let $s \in C^{\log}_{\mathrm{loc}}$ and p, $q \in C^{\log}$ with $0 < p^+, q^+ < \infty$. Let $a \in SB^m_\delta(r, \mu, v; N, \lambda)$ with $0 < \mu < \infty$, $0 < v \le \infty$, $r > 0$, $(1-\delta)r \ge n/\mu$, $0 \le \delta < 1$, and $1 \le \lambda \le \infty$. Let $N > n\max\{1/2, 1/\lambda, 1/p^-, 1/q^-\} + c_{\log}(s)$ and*

$$n\max\Big\{1, \frac{1}{\mu} + \frac{1}{p^-}\Big\} - n - (1-\delta)r < s^- \le s^+ < r - n\max\Big\{\frac{1}{\mu} - \frac{1}{p^+}, 0\Big\}.$$

Then $a(x, D)$ is a continuous linear mapping from $F^{s(\cdot)+m}_{p(\cdot),q(\cdot)}$ to $F^{s(\cdot)}_{p(\cdot),q(\cdot)}$.

Proof. Let $\{\mathcal{F}\varphi_k\}_k$ be a resolution of unity. We set

$$a_{j,k}(x,\xi) = \mathcal{F}^{-1}(\mathcal{F}\varphi_j(\eta)\mathcal{F}_x a(\cdot,\xi))\mathcal{F}\varphi_k(\xi).$$

We decompose the symbol into three parts:

$$a(x,\xi) = a^{(1)}(x,\xi) + a^{(2)}(x,\xi) + a^{(3)}(x,\xi),$$

where

$$a^{(1)}(x,\xi) = \sum_{k=4}^{\infty}\sum_{j=0}^{k-4} a_{j,k}(x,\xi),$$
$$a^{(2)}(x,\xi) = \sum_{k=0}^{\infty}\sum_{j=k-3}^{k+3} a_{j,k}(x,\xi),$$
$$a^{(3)}(x,\xi) = \sum_{k=0}^{\infty}\sum_{j=k+4}^{\infty} a_{j,k}(x,\xi).$$

Accordingly, the proof is divided into three steps.

Step 1. *There is a constant $c > 0$ such that for every $f \in \mathcal{S}(\mathbb{R}^n)$,*

$$\|a^{(1)}(x,D)f\|_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}} \le c\|f\|_{F^{s(\cdot)+m}_{p(\cdot),q(\cdot)}}. \tag{4.2}$$

Indeed, $\sum_{j=0}^{k-4} a_{j,k}(x,D)f_k$has its spectrum in $\{\xi \in \mathbb{R}^n : c_1 2^k \le |\xi| \le c_2 2^k\}$ where $c_1, c_2 > 0$ are independent of k. Then we can apply Lemma 3.5 to obtain

$$\|a^{(1)}(x,D)f\|_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}} \lesssim \Big\|\Big\{2^{ks(\cdot)}\sum_{j=0}^{k-4} a_{j,k}(x,D)f_k\Big\}_k\Big\|_{L^{p(\cdot)}(\ell^{q(\cdot)})},$$

where $f_k := \varphi_k * f$. Let us show that the last quasi-norm is bounded by $c\|f\|_{F^{s(\cdot)+m}_{p(\cdot),q(\cdot)}}$. By Proposition 3.6, the left-hand side does not exceed

$$\begin{aligned}\|\{(\eta_{k,\sigma} * 2^{k(s(\cdot)+m)\tau}|f_k|^\tau)^{1/\tau}\}_k\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} &\lesssim \|\{2^{k(s(\cdot)+m)}f_k\}_k\|_{L^{p(\cdot)}(\ell^{q(\cdot)})}\\ &\lesssim \|f\|_{F^{s(\cdot)+m}_{p(\cdot),q(\cdot)}},\end{aligned}$$

by Lemma 3.4, with $1/\tau > \max(1/2, 1/\lambda, 1/p^-, 1/q^-)$ and $\sigma > n$. This proves (4.2).

Step 2. *There is a constant $c > 0$ such that for every $f \in \mathcal{S}(\mathbb{R}^n)$,*

$$\|a^{(2)}(x,D)f\|_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}} \le c\|f\|_{F^{s(\cdot)+m}_{p(\cdot),q(\cdot)}}. \tag{4.3}$$

Here $\sum_{j=k-3}^{k+3} a_{j,k}(x,D)f_k$ has its spectrum in $\{\xi \in \mathbb{R}^n : |\xi| \le c_2 2^k\}$ where $c_2 > 0$ is independent of k. Let $1/p_1(\cdot) = 1/\mu + 1/p(\cdot)$. Since $(1-\delta)r \ge n/\mu$, we have the Sobolev embedding

$$F^{s(\cdot)+(1-\delta)r}_{p_1(\cdot),\infty} \hookrightarrow F^{s(\cdot)+(1-\delta)r-n/\mu}_{p(\cdot),q(\cdot)} \hookrightarrow F^{s(\cdot)}_{p(\cdot),q(\cdot)}$$

(see Theorem 2.2). By Lemma 3.8, which can be used since $s^- > n\max\{1,$

$1/\mu+1/p^-\}-n-(1-\delta)r$,

$$\|a^{(2)}(x,D)f\|_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}} \lesssim \Big\|\Big\{2^{k(s(\cdot)+(1-\delta)r)}\sum_{j=k-3}^{k+3} a_{j,k}(x,D)f_k\Big\}_k\Big\|_{L^{p_1(\cdot)}(\ell^\infty)}.$$

We have

$$\Big|2^{k(s(x)+(1-\delta)r)}\sum_{j=k-3}^{k+3} a_{j,k}(x,D)f_k(x)\Big|$$

$$\lesssim \Big(\sup_k 2^{((1-\delta)r-m)}\Big\|\sum_{j=k-3}^{k+3} a_{j,k}(x,2^k\,\cdot)\Big\|_{B^N_{\lambda,\infty}}\Big)(\eta_{k,\sigma}*2^{k(s(\cdot)+m)\tau}|f_k|^\tau(x))^{1/\tau},$$

by Proposition 3.6 with $\sigma>n$. Taking the ℓ^∞-norm and then the $L^{p_1(\cdot)}$-norm we obtain, by Hölder's inequality,

$$\begin{aligned}\|a^{(2)}(x,D)f\|_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}} &\lesssim \|\{(\eta_{k,\sigma}*2^{k(s(\cdot)+m)\tau}|f_k|^\tau)^{1/\tau}\}_k\|_{L^{p(\cdot)}(\ell^\infty)}\\ &\lesssim \|\{(2^{k(s(\cdot)+m)\tau}|f_k|^\tau)^{1/\tau}\}_k\|_{L^{p(\cdot)}(\ell^\infty)} \lesssim \|f\|_{F^{s(\cdot)+m}_{p(\cdot),q(\cdot)}},\end{aligned}$$

by Lemma 3.4, with $1/\tau>\max(1/2,1/\lambda,1/p^-,1/q^-)$.

STEP 3. *There is a constant $c>0$ such that for every $f\in\mathcal{S}(\mathbb{R}^n)$,*

$$\|a^{(3)}(x,D)f\|_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}} \le c\|f\|_{F^{s(\cdot)+m}_{p(\cdot),q(\cdot)}}.$$

Now, since $s^+<r-n\max\{1/\mu-1/p^+,0\}$, we can apply Lemma 3.5 to obtain

$$\|a^{(3)}(x,D)f\|_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}} \le c\Big\|\Big\{2^{js(\cdot)}\sum_{k=0}^{j-4} a_{j,k}(x,D)f_k\Big\}_j\Big\|_{L^{p(\cdot)}(\ell^{q(\cdot)})}.$$

We set $\mu_1=\max(\mu,p^+)$. Let $r_1>0$ and $\delta_1>0$ be such that $r-n/\mu=r_1-n/\mu_1$ and $\delta r=\delta_1 r_1$. Then

$$r_1=r-n\max\{1/\mu-1/p^+,0\}$$

and

$$SB^m_\delta(r,\mu,v;N,\lambda)\hookrightarrow SF^m_{\delta_1}(r_1,\mu_1,v;N,\lambda)$$

(see (4.1)). By Proposition 3.6,

$$\begin{aligned}&|2^{ks(x)}a_{j,k}(x,D)f_k(x)|\\ &\qquad\le C_N\|a_{j,k}(x,2^k\,\cdot)\|_{B^N_{\lambda,\infty}}\big(\eta_{k,\sigma}*2^{ks(\cdot)\tau}|f_k|^\tau(x)\big)^{1/\tau}\\ &\qquad\le C_N 2^{-r_1 j}\sup_i\|2^{r_1 i}a_{i,k}(x,2^k\,\cdot)\|_{B^N_{\lambda,\infty}}\big(\eta_{k,\sigma}*2^{ks(\cdot)\tau}|f_k|^\tau(x)\big)^{1/\tau}\end{aligned}$$

for any $x\in\mathbb{R}^n$. Let

$$g_k=(\eta_{k,\sigma}*2^{ks(\cdot)\tau}|f_k|^\tau)^{1/\tau},\qquad x\in\mathbb{R}^n,\ k\in\mathbb{N}.$$

Applying Lemma 3.1 we find that $\|a^{(3)}(x,D)f\|_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}}$ can be estimated from above by

$$c\Big\|\Big\{\sup_i \|2^{r_1 i}a_{i,k}(x,2^k\cdot)\|_{B^N_{\lambda,\infty}}2^{-kr_1}g_k\Big\}_k\Big\|_{L^{p(\cdot)}(\ell^{q(\cdot)})}$$
$$\lesssim \Big\|\Big\{2^{-k(m+\delta_1 r_1)}\sup_i \|2^{r_1 i}a_{i,k}(x,2^k\cdot)\|_{B^N_{\lambda,\infty}}2^{k(m-(1-\delta_1)r_1)}g_k\Big\}_k\Big\|_{L^{p(\cdot)}(\ell^{q(\cdot)})}.$$

Putting $1/p(\cdot)=1/\mu_1+1/p_2(\cdot)$ and applying Hölder's inequality we estimate the last term by

$$c\|\{(\eta_{k,\sigma}*2^{k(s(\cdot)+m-(1-\delta_1)r_1)\tau}|f_k|^\tau)^{1/\tau}\}_k\|_{L^{p_2(\cdot)}(\ell^{q(\cdot)})}$$
$$\lesssim \|\{(2^{k(s(\cdot)+m-(1-\delta_1)r_1)\tau}|f_k|^\tau)^{1/\tau}\}_k\|_{L^{p_2(\cdot)}(\ell^{q(\cdot)})}$$
$$\lesssim \|f\|_{F^{s(\cdot)+m-(1-\delta_1)r_1}_{p_2(\cdot),q(\cdot)}}\lesssim \|f\|_{F^{s(\cdot)+m}_{p(\cdot),q(\cdot)}},$$

where we have used Lemma 3.4 with $\sigma>n$ and the embedding

$$F^{s(\cdot)+m}_{p(\cdot),q(\cdot)}\hookrightarrow F^{s(\cdot)+m-(1-\delta_1)r_1}_{p_2(\cdot),q(\cdot)}.$$

The proof is complete. ■

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