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Existence of solutions to some fractional equations involving the Bessel operator in $\mathbb{R}^N$

Nguyen Van Thin (Jinan City and Thai Nguyen City)

Abstract. The aim of this paper is to study the existence of solutions to concave-convex nonlinear equations involving the Bessel operator in $\mathbb{R}^N$:

$$M\Big(\int_{\mathbb{R}^N} |(I-\Delta)^{\alpha/2}u|^2\,dx\Big)(I-\Delta)^{\alpha}u+\lambda V(x)u=\gamma f(x,u)+\mu\xi(x)|u|^{p-2}u,$$

$$M\Big(\int_{\mathbb{R}^N} |(I-\Delta)^{\alpha/2}u|^2\,dx+\lambda\int_{\mathbb{R}^N} V(x)|u|^2dx\Big)((I-\Delta)^{\alpha}u+\lambda V(x)u)$$
$$=\gamma f(x,u)+\mu\xi(x)|u|^{p-2}u,$$

where λ,γ,μ are positive parameters, $\xi:\mathbb{R}^N\to(0,\infty)$ belongs to $L^{2/(2-p)}(\mathbb{R}^N)$, $1<p<2$, $M:[0,\infty)\to(0,\infty)$ is a continuous function, $V:\mathbb{R}^N\to\mathbb{R}^+$ is a continuous function, $0<\alpha<1$ with $2\alpha<N$, and f is a continuous function on $\mathbb{R}^N\times\mathbb{R}$ which does not satisfy the Ambrosetti–Rabinowitz condition. By using the Mountain Pass Theorem and the variational method, we obtain the existence of solutions to the above equations. Furthermore, if M is degenerate ($M(0)=0$) and f satisfies the Ambrosetti–Rabinowitz condition, we investigate the existence of solutions of that equation without the concave-convex nonlinearity in case its right side contains the critical exponent $2^*_\alpha=2N/(N-2\alpha)$. The difficulty lies in the lack of compactness and the degeneracy of M.

1. Introduction. In 2017, S. Secchi [20] studied the existence of solutions to a class of nonlinear fractional equation of the form

$$(I-\Delta)^{\alpha}u+\lambda V(x)u=f(x,u)+\mu\xi(x)|u|^{p-2}u\quad\text{ in }\mathbb{R}^N,\tag{1.1}$$

where $0<\alpha<1$, $V:\mathbb{R}^N\to\mathbb{R}$ and $f:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ are continuous functions, $\xi:\mathbb{R}^N\to(0,\infty)$ belongs to $L^{2/(2-p)}(\mathbb{R}^N)$, $\lambda,\mu>0$ and $1<p<2$. The nonlinear term $\xi(x)|u|^{p-2}u$ has a concave shape. Before, P. Felmer and I. Vergara [13] gave some results on regularity, exponential decay and

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symmetrix properties for solutions to the equation

$$(I-\Delta)^\alpha u = f(x,u) \quad \text{in } \mathbb{R}^N.$$

Equations involving the nonlocal operator $(I-\Delta)^\alpha$ arise in the study of standing waves $\Psi = \Psi(t,x)$ for the Schrödinger–Klein–Gordon equation of the form

$$i\frac{\partial \Psi}{\partial t} = (I-\Delta)^\alpha \Psi - f(x,\Psi), \qquad (t,x) \in \mathbb{R}^N \times \mathbb{R},$$

which describes the behavior of bosons. The operator $(I-\Delta)^\alpha$ is related to the pseudo-relativistic Schrödinger operator $(m^2-\Delta)^{1/2} - m > 0$ and much attention has been paid to equations involving it. We refer to [6, 7, 8, 9, 11, 13, 5] and the references therein for more details and physical motivation for the operator $(I-\Delta)^\alpha$. Note that the hardest issue in dealing with this operator is the lack of scaling properties: there is no standard group action under which $(I-\Delta)^\alpha$ behaves like a local differential operator.

In order to state the results of S. Secchi [20], we recall some notations and definitions. For $\alpha > 0$, we introduce the *Bessel function space*

$$L^{\alpha,2}(\mathbb{R}^N) = \{f : f = G_\alpha * g \text{ for some } g \in L^2(\mathbb{R}^N)\},$$

where the *Bessel convolution kernel* is defined by

$$G_\alpha(x) = \frac{1}{(4\pi)^{\alpha/2}\Gamma(\alpha/2)} \int_0^\infty \exp\Big(-\frac{\pi}{t}|x|^2\Big) \exp\Big(-\frac{t}{4\pi}\Big) t^{(\alpha-N)/2-1}\, dt.$$

The norm of this Bessel space is $\|f\|_{L^{\alpha,2}(\mathbb{R}^N)} = \|g\|_{L^2(\mathbb{R}^N)}$ if $f = G_\alpha * g$. The operator $(I-\Delta)^\alpha u$ is defined by its Fourier transform:

$$\mathcal{F}((I-\Delta)^\alpha u)(\zeta) = (1+|\zeta|^2)^\alpha \mathcal{F}u(\zeta), \tag{1.2}$$

where $\mathcal{F}u(\zeta) = (2\pi)^{-N/2}\int_{\mathbb{R}^N} e^{-i\zeta\cdot x}u(x)\,dx$. From Fubini's Theorem, (1.2) is equivalent to

$$\begin{aligned}(I-\Delta)^\alpha u(x) &= \mathcal{F}^{-1}(\mathcal{F}((I-\Delta)^\alpha u))(x) = \mathcal{F}^{-1}((1+|\zeta|^2)^\alpha \mathcal{F}u(\zeta))(x) \\ &= \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{i\zeta\cdot x}\big((1+|\zeta|^2)^\alpha \mathcal{F}u(\zeta)\big)\, d\zeta \\ &= \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{i\zeta\cdot x}(1+|\zeta|^2)^\alpha \Big(\int_{\mathbb{R}^N} e^{-i\zeta.y}u(y)\,dy\Big)\, d\zeta \\ &= \frac{1}{(2\pi)^N} \iint_{\mathbb{R}^N\times\mathbb{R}^N} e^{i\zeta\cdot(x-y)}(1+|\zeta|^2)^\alpha u(y)\,dy\,d\zeta.\end{aligned} \tag{1.3}$$

We know that the inverse operator of $(I-\Delta)^\alpha u$ satisfies $(I-\Delta)^{-\alpha}u = G_\alpha * u$, and $(I-\Delta)^{-\alpha}u$ is usually called the *Bessel operator* of order α. In transform variables this operator reads

$$G_\alpha = \mathcal{F}^{-1} \circ \big((1+|\zeta|^2)^{-\alpha/2} \circ \mathcal{F}\big)$$

so that

$$\|f\|_{L^{\alpha,2}(\mathbb{R}^N)} = \|(I-\Delta)^{\alpha/2} f\|_{L^2(\mathbb{R}^N)}.$$

We refer the reader to [1, 21] and references therein for more information. In [11], a pointwise formula for $(I-\Delta)^\alpha$ is given:

$$(I-\Delta)^\alpha u(x) = c_{N,\alpha}\,\mathrm{PV}\int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{(N+2\alpha)/2}} K_{(N+2\alpha)/2}(|x-y|)\,dy + u(x)$$

for all $u \in C_c^2(\mathbb{R}^N)$. Here $c_{N,\alpha}$ is a positive constant depending only on N and α, PV denotes the principal value of the singular integral, and $K_{(N+2\alpha)/2}$ is the modified Bessel function of the second kind with order $(N+2\alpha)/2$ (see [11, Remark 7.3] for more details).

Consider the weighted Sobolev space

$$H = \Big\{ u \in L^{\alpha,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u(x)|^2\,dx < \infty \Big\}$$

equipped with the norm

$$\|u\|_H^2 = \int_{\mathbb{R}^N} |(I-\Delta)^{\alpha/2} u|^2\,dx + \int_{\mathbb{R}^N} V(x)|u(x)|^2\,dx = \|u\|^2_{L^{\alpha,2}(\mathbb{R}^N)} + \|u\|^2_{2,V},$$

where $\|u\|^2_{2,V} = \int_{\mathbb{R}^N} V(x)|u(x)|^2\,dx$.

For each $\lambda > 0$, we also define the space

$$H_\lambda = \Big\{ u \in L^{\alpha,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u(x)|^2\,dx < \infty \Big\}$$

with norm

$$\|u\|^2_{H_\lambda} = \|u\|^2_{L^{\alpha,2}(\mathbb{R}^N)} + \lambda \|u\|^2_{2,V}.$$

Note that H_λ and H coincide as sets, but their norms are different, and $H_1 = H$. We can show that H_λ is a Hilbert space (see Lemma 2.4) with inner product

$$\langle u, v\rangle = \int_{\mathbb{R}^N} (I-\Delta)^{\alpha/2} u (I-\Delta)^{\alpha/2} v\,dx + \lambda \int_{\mathbb{R}^N} V(x)u(x)v(x)\,dx.$$

Definition 1.1. We say that $u \in H_\lambda$ is a *weak solution* to (1.1) if

$$\int_{\mathbb{R}^N} (I-\Delta)^{\alpha/2} u (I-\Delta)^{\alpha/2} v\,dx + \lambda \int_{\mathbb{R}^N} V(x)u(x)v(x)\,dx$$
$$= \int_{\mathbb{R}^N} f(x,u(x))v(x)\,dx + \int_{\mathbb{R}^N} \xi(x)|u|^{p-2}uv\,dx$$

for all $v \in H_\lambda$, or equivalently

$$\int\limits_{\mathbb{R}^N} \mathcal{F}^{-1}((1+|\zeta|^2)^{\alpha/2}\mathcal{F}u(\zeta))(x)\mathcal{F}^{-1}((1+|\zeta|^2)^{\alpha/2}\mathcal{F}v(\zeta))(x)\,dx$$
$$+\lambda\int\limits_{\mathbb{R}^N} V(x)u(x)v(x)\,dx = \int\limits_{\mathbb{R}^N} f(x,u(x))v(x)\,dx + \int\limits_{\mathbb{R}^N} \xi(x)|u|^{p-2}uv\,dx.$$

We recall the following assumptions on f and V from [20]:

(f_1) $|f(x,u)| \le c(1+|u|^{q-1})$ for some $q \in (2, 2_\alpha^*)$, where
$$2_\alpha^* = 2N/(N-2\alpha);$$
(f_2) $f(x,u) = o(|u|)$ as $u \to 0$ uniformly with respect to $x \in \mathbb{R}^N$;
(f_3) there exists a constant $\nu > 2$ such that $0 < \nu F(x,u) \le uf(x,u)$ for all $x \in \mathbb{R}^N$ and $u \ne 0$, where $F(x,u) = \int_0^u f(x,s)\,ds$;
(V_1) $\operatorname{ess\,inf}_{x\in\mathbb{R}^N} V(x) > 0$;
(V_2) $\lim_{|y|\to\infty} \int_{B(y,1)} dx/V(x) = 0$, where $B(y,1) = \{x \in \mathbb{R}^N : |y-x| < 1\}$.

A function V satisfying conditions (V_1) and (V_2) is called a *coercive electric potential*. In [2], it is shown that the usual coercivity condition
$$\lim_{|x|\to\infty} V(x) = \infty$$
immediately implies (V_2). Further, V is a coercive electric potential if and only if

$(V_2)'$ $\lim_{|y|\to\infty} \operatorname{meas}\{x \in B(y,1) : V(x) \le c\} = 0$ for all $c > 0$.

With the above notations and definitions, S. Secchi proved

THEOREM 1.2 ([20]). *Assume that* (f_1)–(f_3), (V_1) *and* (V_2) *hold. Then for every* $\lambda > 0$, *there exists* $\mu_0 > 0$ *such that for every* $\mu \in (0,\mu_0)$ *there exists a nontrivial solution of* (1.1).

2. Some preliminary results. We summarize the embedding properties of Bessel spaces. For the proofs, we refer to [12, Theorem 3.1], [21, Chapter V, Section 3], [17] and [22, Section 4].

THEOREM 2.1.

(1) $L^{\alpha,2}(\mathbb{R}^N) = W^{\alpha,2}(\mathbb{R}^N) = H^\alpha(\mathbb{R}^N)$.
(2) *If* $\alpha \ge 0$ *and* $2 \le q \le 2_\alpha^*$, *then* $L^{\alpha,2}(\mathbb{R}^N)$ *is continuously embedded into* $L^q(\mathbb{R}^N)$; *if* $2 \le q < 2_\alpha^*$, *then the embedding is locally compact.*
(3) *Assume that* $0 \le \alpha \le 2$ *and* $\alpha > N/2$. *If* $\alpha - N/2 > 1$ *and* $0 < \delta \le \alpha - N/2 - 1$, *then* $L^{\alpha,2}(\mathbb{R}^N)$ *is continuously embedded into* $C^{1,\delta}(\mathbb{R}^N)$. *If* $\alpha - N/2 < 1$ *and* $0 < \delta \le \alpha - N/2$, *then* $L^{\alpha,2}(\mathbb{R}^N)$ *is continuously embedded into* $C^{0,\delta}(\mathbb{R}^N)$.

If V satisfies (V_1), we have

$$\begin{aligned}\|u\|^2_{L^{\alpha,2}(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} |(I-\Delta)^{\alpha/2}u|^2\,dx \\ &\le \int_{\mathbb{R}^N} |(I-\Delta)^{\alpha/2}u|^2\,dx + \lambda \int_{\mathbb{R}^N} V(x)|u(x)|^2\,dx = \|u\|^2_{H_\lambda}.\end{aligned}$$

Thus H_λ embeds continuously into $L^{\alpha,2}(\mathbb{R}^N)$. Therefore, we get

LEMMA 2.2. *H_λ embeds continuously into $L^q(\mathbb{R}^N)$ for all $q \in [2, 2^*_\alpha]$, and there exists a best constant*

$$S_{q,\lambda} = \sup_{u\in H_\lambda,\, u\neq 0} \frac{\|u\|_{L^q(\mathbb{R}^N)}}{\|u\|_{H_\lambda}}. \tag{2.1}$$

REMARK 2.3. Although the Bessel space $L^{\alpha,2}(\mathbb{R}^N)$ is topologically indistinguishable from the Sobolev fractional space $H^\alpha(\mathbb{R}^N)$, we will not confuse them, since our equations involve the Bessel norm.

LEMMA 2.4. *H_λ is a Hilbert space with inner product*

$$\langle u, v\rangle = \int_{\mathbb{R}^N} (I-\Delta)^{\alpha/2}u(I-\Delta)^{\alpha/2}v\,dx + \lambda \int_{\mathbb{R}^N} V(x)u(x)v(x)\,dx.$$

Proof. We know that the Bessel space $L^{\alpha,2}(\mathbb{R}^N)$ is a Hilbert space with inner product

$$\langle f_1, f_2\rangle = \int_{\mathbb{R}^N} (I-\Delta)^{\alpha/2}f_1 \cdot (I-\Delta)^{\alpha/2}f_2\,dx,$$

where $f_1 = G_\alpha * g_1 \in L^{\alpha,2}(\mathbb{R}^N)$, $f_2 = G_\alpha * g_2 \in L^{\alpha,2}(\mathbb{R}^N)$ and $g_1, g_2 \in L^2(\mathbb{R}^N)$. For any $f \in L^{\alpha,2}(\mathbb{R}^N)$, we have $\|f\|_{L^{\alpha,2}(\mathbb{R}^N)} = \|g\|_{L^2(\mathbb{R}^N)}$ if $f = G_\alpha * g$, and also $\|f\|_{L^{\alpha,2}(\mathbb{R}^N)} = \|(I-\Delta)^{\alpha/2}f\|_{L^2(\mathbb{R}^N)}$.

First, we see that $\langle\cdot,\cdot\rangle : H_\lambda \times H_\lambda \to \mathbb{R}$ as in the statement is an inner product. Let $\{f_n\}$ be any Cauchy sequence in H_λ. We show that there exists $f \in H_\lambda$ such that $f_n \to f$ in H_λ. For any $\varepsilon > 0$, there exists $n_0(\varepsilon) \in \mathbb{N}$ such that $\|f_n - f_m\|_{H_\lambda} < \varepsilon$ for all $m, n \ge n_0$. That implies $\|f_n - f_m\|_{L^{\alpha,2}(\mathbb{R}^N)} < \varepsilon$ and $\|f_n - f_m\|_{2,V} < \varepsilon$ for all $m, n \ge n_0$. Since $L^{\alpha,2}(\mathbb{R}^N)$ is a Banach space, there exists $f \in L^{\alpha,2}(\mathbb{R}^N)$ such that

$$\|f_n - f\|_{L^{\alpha,2}(\mathbb{R}^N)} \to 0 \quad \text{as } n \to \infty. \tag{2.2}$$

By Theorem 2.1, we have $f_n \to f$ in $L^q(\mathbb{R}^N)$ for all $q \in [2, 2^*_\alpha)$. Therefore, there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that

$$f_{n_k} \to f \quad \text{in } \mathbb{R}^N. \tag{2.3}$$

Since $\{f_n\}$ is a Cauchy sequence in $L^2_V(\mathbb{R}^N)$, $\{f_{n_k}\}$ is a Cauchy sequence in $L^2_V(\mathbb{R}^N)$. Hence there exists $h \in L^2(\mathbb{R}^N)$ such that $V(x)^{1/2}f_{n_k} \to h$

in $L^2(\mathbb{R}^N)$. Up to a subsequence, we may assume that $V(x)^{1/2} f_{n_k} \to h$ in $\mathbb{R}^N$. From (2.3), we get $h = V^{1/2} f$. Then

$$\|f_{n_k} - f\|_{2,V} \to 0 \quad \text{as } n_k \to \infty \quad \text{and} \quad f \in L^2_V(\mathbb{R}^N). \tag{2.4}$$

By combining (2.2) with (2.4), we deduce that f_{n_k} converges to f in H_λ. Furthermore, $\{f_n\}$ is a Cauchy sequence, and we have $f_n \to f$ in H_λ and $f \in H_\lambda$. ■

LEMMA 2.5 ([20]). *If V is a coercive electric potential and V satisfies (V_1), then for all $\lambda > 0$, the space H_λ is continuously embedded into $L^q(\mathbb{R}^N)$ for all $2 \le q \le 2^*_\alpha$, and compactly embedded into $L^q(\mathbb{R}^N)$ for all $2 \le q < 2^*_\alpha$.*

Now, we state some assumptions on the potential V:

$(V_1)'$ $V(x) \ge V_0 > 0$ for all $x \in \mathbb{R}^N$;
$(V_3)'$ $\text{meas}\{x \in \mathbb{R}^N : V(x) \le c\} < \infty$ for all $c > 0$.

From Lemmas 2.2 and 2.5, we get

LEMMA 2.6. *If V satisfies $(V_1)'$, then for all $\lambda > 0$, the space H_λ is continuously embedded into $L^q(\mathbb{R}^N)$ for all $2 \le q \le 2^*_\alpha$. Furthermore, if V satisfies $(V_1)'$ and $(V_2)'$, then H_λ is compactly embedded into $L^q(\mathbb{R}^N)$ for all $2 \le q < 2^*_\alpha$.*

LEMMA 2.7. *If V satisfies $(V_1)'$ and $(V_3)'$, then for all $\lambda > 0$, the space H_λ is compactly embedded into $L^q(\mathbb{R}^N)$ for all $2 \le q < 2^*_\alpha$.*

Proof. If u_n converges weakly to u in H_λ, then $u_n - u_0$ converges weakly to 0 in H_λ. Therefore, we only need to prove that if $u_n \to 0$ weakly in H_λ then $u_n \to 0$ in L^q. From $(V_1)'$, H_λ embeds continuously into $L^{\alpha,2}(\mathbb{R}^N)$. By Theorem 2.1, $u_n \to 0$ in $L^q_{\text{loc}}(\mathbb{R}^N)$ for all $2 \le q < 2^*_s$.

First, we claim that

$$u_n \to 0 \quad \text{strongly in } L^2(\mathbb{R}^N). \tag{2.5}$$

To see this, we only have to show that for any $\varepsilon > 0$, there exists $R > 0$ such that $\int_{\mathbb{R}^N \setminus B_R} |u_n(x)|^2\,dx < \varepsilon$, where $B_R = \{x \in \mathbb{R}^N : |x| < R\}$. Set

$$A(R,c) = \{x \in \mathbb{R}^N \setminus B_R : V(x) \ge c\},$$
$$B(R,c) = \{x \in \mathbb{R}^N \setminus B_R : V(x) < c\},$$

where $c > 0$. We see that

$$\int_{A(R,c)} |u_n(x)|^2\,dx \le \int_{\mathbb{R}^N} \frac{V(x)}{c} |u_n(x)|^2\,dx \le \frac{\|u\|^2_{H_\lambda}}{\lambda c} \tag{2.6}$$

for each $\lambda > 0$. Choose $\sigma \in (1, 2^*_\alpha/2)$ and $\sigma' > 0$ such that $1/\sigma + 1/\sigma' = 1$, and apply the Hölder inequality to get

$$\int_{B(R,c)} |u_n(x)|^2\,dx \le \Big(\int_{B(R,c)} |u_n(x)|^{2\sigma}\,dx\Big)^{1/\sigma} \operatorname{meas}(B(R,c))^{1/\sigma'} \le C_\lambda \|u_n\|_{H_\lambda}^2 \operatorname{meas}(B(R,c))^{1/\sigma'}, \tag{2.7}$$

where $C_\lambda > 0$ is a suitable constant. Here, we use the continuous embedding $H_\lambda \to L^{2\sigma}(\mathbb{R}^N)$. Since $\|u_n\|_{H_\lambda}$ is bounded and $(V_3)'$ holds, we may choose R, c large enough such that $\|u_n\|_{H_\lambda}^2/c$ and $\operatorname{meas}(B(R,c))$ are small enough. Therefore, for all $\varepsilon > 0$, from (2.6) and (2.7) we get

$$\int_{\mathbb{R}^N\setminus B_R} |u_n(x)|^2\,dx = \int_{A(R,c)} |u_n(x)|^2\,dx + \int_{B(R,c)} |u_n(x)|^2\,dx < \varepsilon,$$

which implies (2.5).

For any $q \in (2, 2_\alpha^*)$, there exists $\sigma_1 \in (0,1)$ such that $1/q = \sigma_1/2 + (1-\sigma_1)/2_\alpha^*$. By the Hölder inequality,

$$\int_{\mathbb{R}^N} |u_n(x)|^q\,dx = \int_{\mathbb{R}^N} |u_n(x)|^{q\sigma_1} |u_n(x)|^{(1-\sigma_1)q}\,dx \le \Big(\int_{\mathbb{R}^N} |u_n(x)|^2\,dx\Big)^{\sigma_1 q/2} \Big(\int_{\mathbb{R}^N} |u_n(x)|^{2_\alpha^*}\,dx\Big)^{(1-\sigma_1)q/2_\alpha^*}. \tag{2.8}$$

From (2.1), we have

$$\|u_n\|_{L^2(\mathbb{R}^N)} \le S_{2,\lambda}\|u_n\|_{H_\lambda} \quad \text{and} \quad \|u_n\|_{L^{2_\alpha^*}(\mathbb{R}^N)} \le S_{2_\alpha^*,\lambda}\|u_n\|_{H_\lambda}.$$

By combining these two inequalities with (2.8), we deduce

$$\int_{\mathbb{R}^N} |u_n(x)|^q\,dx \le S_{2,\lambda}^{\sigma_1 q}\|u_n\|_{L^2(\mathbb{R}^N)}^{\sigma_1 q} \cdot S_{2_\alpha^*,\lambda}^{(1-\sigma_1)q}\|u_n\|_{H_\lambda}^{(1-\sigma_1)q}. \tag{2.9}$$

Combining this with (2.5), we get $u_n \to 0$ in $L^q(\mathbb{R}^N)$ for all $q \in [2, 2_\alpha^*)$ since $\|u_n\|_{H_\lambda}$ is a bounded sequence in H_λ. ∎

The compactness assumption required by the Mountain Pass Theorem is the well known *Palais–Smale condition* (see, for instance, [23, 25] and references therein), which in our framework reads:

Palais–Smale condition. Let $\Phi \in C^1(H_\lambda, \mathbb{R})$. The functional Φ satisfies the *Palais–Smale compactness condition* at level $c \in \mathbb{R}$ if any sequence $\{u_j\}_{j\in\mathbb{N}}$ in H_λ such that $\Phi(u_j) \to c$ and

$$\sup_{\|v\|_{H_\lambda}=1} |\langle \Phi'(u_j), v\rangle| \to 0$$

(called a (PS) *sequence at level* $c \in \mathbb{R}$) has a subsequence strongly convergent in H_λ.

To prove our theorems, we need the Mountain Pass Theorem:

Lemma 2.8 ([19]). *Let X be a real Banach space and let $\Phi \in C^1(X, \mathbb{R})$ satisfy Palais–Smale condition. Suppose that $\Phi(0) = 0$ and*

(I_1) there exist constants $\beta_0, \alpha_0 > 0$ such that $\Phi(u) \geq \alpha_0$ for all $\|u\| = \beta_0$;
(I_2) there is $u_1 \in X$ with $\|u_1\| \geq \beta_0$ such that $\Phi(u_1) \leq 0$.

Then I has a critical value $c \geq \alpha_0$. Moreover, c can be characterized as

$$c = \inf_{\gamma^* \in \Gamma} \max_{t \in [0,1]} \Phi(\gamma^*(t)),$$

where $\Gamma = \{\gamma^* \in C([0,1], X) : \gamma^*(0) = 0,\ \gamma^*(1) = u_1\}$.

3. Main results and their proofs. The work of Secchi motivates considering the following fractional equation involving the Bessel operator:

$$M\Big(\int_{\mathbb{R}^N} |(I - \Delta)^{\alpha/2} u|^2\, dx\Big)(I - \Delta)^\alpha u + \lambda V(x) u = \gamma f(x, u) + \mu \xi(x) |u|^{p-2} u \tag{3.1}$$

in $\mathbb{R}^N$, where $1 < p < 2$ and $\lambda, \gamma, \mu > 0$.

Let us first state some assumptions on M and f:

(M_1) $M \in C(\mathbb{R}_0^+)$ satisfies $\inf_{t \in \mathbb{R}_0^+} M(t) \geq a > 0$, where $a > 0$ is a constant.
(M_2) There exists $\theta \in [1, N/(N - 2\alpha))$ such that

$$\theta \mathcal{M}(t) = \theta \int_0^t M(\tau) d\tau \geq M(t) t \quad \text{for all } t \in \mathbb{R}_0^+.$$

$(f_3)'$ There exist $\nu \in (2\theta, 2_\alpha^*)$ and $r \geq 0$ such that

$$f(x,t)t - \nu F(x,t) \geq -\rho |t|^2 - \varphi_*(x) \quad \text{for all } x \in \mathbb{R}^N \text{ and } |t| \geq r,$$

where $\rho \geq 0$ and $0 \leq \varphi_* \in L^1(\mathbb{R}^N)$.

(f_4) There exists $\beta > 0$ such that $F(x,t) \geq \beta |t|^\nu - P(|t|)$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, where

$$P(t) = a_n |t|^{\alpha_n} + \cdots + a_1 |t|^{\alpha_1} + a_0 \quad \text{for all } t \in \mathbb{R},$$

for some $n \in \mathbb{N}$, $a_i \in \mathbb{R}$, $i = 0, \ldots, n$, $a_n \neq 0$ and $\infty > \alpha_n > \cdots > \alpha_1 > 0$, $\nu > \alpha_n \geq 2$, $a_0 \leq 0$, and if $n \geq 2$ then for $i \in \{1, \ldots, n-1\}$ we have $a_i < 0$ if $\alpha_i \in (0, 2)$.

Conditions (M_1) and (M_2) were introduced by Pucci–Xiang–Zhang [18]. Condition $(V_1)'$ implies (V_1). In [24], we give functions satisfying (f_1), (f_2), $(f_3)'$ and (f_4). For example, $f(x,t) = \sin|t| - |t|\cos|t| + |t|^\nu/\nu$ ($\nu > 2\theta$) satisfies (f_1), (f_2), $(f_3)'$ with $\varphi_* = 0$, and (f_4). In [20], conditions (f_1)–(f_3) are shown to imply

$$F(x,t) \geq c_F(|t|^\nu - |t|^2) \tag{3.2}$$

for some constant $c_F > 0$. Then F satisfies (f_4) with $P(|t|) = |t|^2$. Hence, the class of functions satisfying (f_1), (f_2), $(f_3)'$ and (f_4) is larger than for (f_1)–(f_3).

For (3.1), we use the energy function

$$\begin{aligned}\mathbb{I}(u) = \frac{1}{2}\mathcal{M}\Big(\int_{\mathbb{R}^N} |(I-\Delta)^{\alpha/2}u|^2\,dx\Big) + \frac{\lambda}{2}\int_{\mathbb{R}^N} V(x)|u(x)|^2\,dx \\ - \gamma\int_{\mathbb{R}^N} F(x,u)\,dx - \frac{\mu}{p}\int_{\mathbb{R}^N}\xi(x)|u|^p\,dx.\end{aligned}$$

Then $\mathbb{I} \in C^1(H_\lambda, \mathbb{R})$ and

$$\begin{aligned}\langle \mathbb{I}'(u), v\rangle = M\Big(\int_{\mathbb{R}^N} |(I-\Delta)^{\alpha/2}u|^2\,dx\Big)\int_{\mathbb{R}^N}(I-\Delta)^{\alpha/2}u(I-\Delta)^{\alpha/2}v\,dx \\ + \lambda\int_{\mathbb{R}^N} V(x)u(x)v(x)\,dx - \gamma\int_{\mathbb{R}^N} f(x,u(x))v(x)\,dx \\ - \mu\int_{\mathbb{R}^N}\xi(x)|u|^{p-2}uv\,dx\end{aligned}$$

for all $v \in H_\lambda$. A function $u \in H_\lambda$ is called a *weak solution* to (3.1) if $\langle \mathbb{I}'(u), v\rangle = 0$ for all $v \in H_\lambda$. In detail, we can state the definition of weak solution as follows:

DEFINITION 3.1. We say that $u \in H_\lambda$ is a *weak solution* to (3.1) if

$$\begin{aligned}M\Big(\int_{\mathbb{R}^N} |(I-\Delta)^{\alpha/2}u|^2\,dx\Big)\int_{\mathbb{R}^N}(I-\Delta)^{\alpha/2}u(I-\Delta)^{\alpha/2}v\,dx + \lambda\int_{\mathbb{R}^N} V(x)u(x)v(x)\,dx \\ = \gamma\int_{\mathbb{R}^N} f(x,u(x))v(x)\,dx + \mu\int_{\mathbb{R}^N}\xi(x)|u|^{p-2}uv\,dx\end{aligned}$$

for any $v \in H_\lambda$.

Assume that ν, γ and ρ satisfy the condition

$$\frac{\gamma\rho}{\nu V_0} < \Big(1 - \frac{1}{n_0}\Big)\Big(\frac{1}{2} - \frac{1}{\nu}\Big), \tag{3.3}$$

where n_0 is a large enough positive integer.

Our first result is:

THEOREM 3.2. *Let* (M_1) *and* (M_2) *hold for* $a \geq \theta\frac{\nu-2}{\nu-2\theta}$, *and suppose that* $(V_1)'$, *either* $(V_2)'$ *or* $(V_3)'$, (f_1), (f_2), $(f_3)'$ *and* (f_4) *hold. Then for any* $\lambda > 0$, *there exists* $\mu_{\lambda,\gamma} > 0$ *such that* (3.1) *has at least two nontrivial solutions in* H_λ *with* γ *satisfying* (3.3) *and* $\mu \in (0, \mu_{\lambda,\gamma})$.

In [20], condition (f_3) is very important, and S. Secchi showed that for every $\tau \in \big(\max\{1, \frac{N}{2\alpha}\}, \frac{q}{q-2}\big)$ there exists $R > 0$ such that $|u| \geq R$ implies

$$\frac{|f(x,u)|^\tau}{|u|^\tau} \le \mathfrak{F}(x,u), \quad \text{where} \quad \mathfrak{F}(x,u) = \frac{1}{2} f(x,u) - F(x,u). \tag{3.4}$$

In Theorem 3.2, the class of nonlinear functions is larger than in the result of S. Secchi. Namely, we cannot show that f satisfies (3.4). Therefore, we must propose some new estimate to get a bound on a (PS) sequence. Furthermore, V satisfies the new condition $(V_3)'$. Then we need to show H_λ is compactly embedded into $L^q(\mathbb{R}^N)$ in Lemma 2.7.

In order to prove Theorem 3.2, we need a few lemmas.

LEMMA 3.3. *Suppose that* (M_1) *and* (M_2) *hold for* $a \ge \theta\frac{\nu-2}{\nu-2\theta}$, *and* $(V_1)'$, (f_1) *and* (f_2) *hold. Then for all* $\lambda, \gamma > 0$, *there exist positive constants* t_0, α_0, $\mu_{\lambda,\gamma}$ *such that* $\mathbb{I}(u) \ge \alpha_0$ *for all* $u \in H_\lambda$ *with* $\|u\|_{H_\lambda} = t_0$, *and* $\mu \in (0, \mu_{\lambda,\gamma})$.

Proof. From (f_1) and (f_2), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$F(x,t) \le \varepsilon |t|^2 + C_\varepsilon |t|^q$$

for any $x \in \mathbb{R}^N, t \in \mathbb{R}$, and for some $C_\varepsilon > 0$. By (2.1) and the Hölder inequality,

$$\begin{aligned} \int_{\mathbb{R}^N} \xi(x)|u(x)|^p\,dx &\le \Big(\int_{\mathbb{R}^N} |\xi(x)|^{2/(2-p)}\,dx\Big)^{(2-p)/2}\Big(\int_{\mathbb{R}^N} |u(x)|^2\,dx\Big)^{p/2} \\ &= \|\xi\|_{L^{2/(2-p)}(\mathbb{R}^N)}\|u\|^p_{L^2(\mathbb{R}^N)} \\ &\le \|\xi\|_{L^{2/(2-p)}(\mathbb{R}^N)} S^p_{2,\lambda}\|u\|^p_{H_\lambda}. \end{aligned} \tag{3.5}$$

Then from (M_1), (M_2) and (3.5), we obtain

$$\begin{aligned} \mathbb{I}(u) &= \frac{1}{2}\Big(\mathcal{M}\Big(\int_{\mathbb{R}^N} |(I-\varDelta)^{\alpha/2}u|^2\,dx\Big) + \lambda\int_{\mathbb{R}^N} V(x)|u|^2\,dx\Big) \\ &\quad - \gamma\int_{\mathbb{R}^N} F(x,u)\,dx - \frac{\mu}{p}\int_{\mathbb{R}^N}\xi(x)|u|^p\,dx \\ &\ge \frac{1}{2}\Big(\frac{a}{\theta}\|u\|^2_{L^{\alpha,2}(\mathbb{R}^N)} + \lambda\|u\|^2_{2,V}\Big) - \varepsilon\gamma\|u\|^2_{L^2(\mathbb{R}^N)} \\ &\quad - \gamma C_\varepsilon\|u\|^q_{L^q(\mathbb{R}^N)} - \frac{\mu}{p}\|\xi\|_{L^{2/(2-p)}(\mathbb{R}^N)} S^p_{2,\lambda}\|u\|^p_{H_\lambda}. \end{aligned} \tag{3.6}$$

From (2.1), (3.6) and $a \ge \theta\frac{\nu-2}{\nu-2\theta} \ge \theta$, we have

$$\begin{aligned} \mathbb{I}(u) &\ge \frac{1}{2}\|u\|^2_{H_\lambda} - \varepsilon\gamma S^2_{2,\lambda}\|u\|^2_{H_\lambda} - \gamma C_\varepsilon S^q_{q,\lambda}\|u\|^q_{H_\lambda} \\ &\quad - \frac{\mu}{p}\|\xi\|_{L^{2/(2-p)}(\mathbb{R}^N)} S^p_{2,\lambda}\|u\|^p_{H_\lambda} \\ &= \|u\|^p_{H_\lambda}\Big[\Big(\frac{1}{2} - \varepsilon\gamma S^2_{2,\lambda}\Big)\|u\|^{2-p}_{H_\lambda} - \gamma C_\varepsilon S^q_{q,\lambda}\|u\|^{q-p}_{H_\lambda} - \frac{\mu}{p}\|\xi\|_{L^{2/(2-p)}(\mathbb{R}^N)} S^p_{2,\lambda}\Big]. \end{aligned} \tag{3.7}$$

We choose $\varepsilon(\gamma) > 0$ small enough such that $1/2 - \varepsilon\gamma S_{2,\lambda}^2 > 0$ for any $\gamma > 0$. Let $h(t) = (1/2 - \varepsilon\gamma S_{2,\lambda}^2)t^{2-p} - \gamma C_\varepsilon S_{q,\lambda}^q t^{q-p}$ for $t \geq 0$. We see that there exists $t_0 > 0$ such that $h(t_0) > 0$ and the maximum of h is attained at t_0. Then we can choose $\mu_{\lambda,\gamma}$ small such that

$$h(t_0) - \frac{\mu}{p}\|\xi\|_{L^{2/(2-p)}(\mathbb{R}^N)} S_{2,\lambda}^p > 0. \tag{3.8}$$

From (3.7) and (3.8), for $\|u\|_{H_\lambda} = t_0$, we have

$$\mathbb{I}(u) \geq t_0^p\Big(h(t_0) - \frac{\mu}{p}\|\xi\|_{L^{2/(2-p)}(\mathbb{R}^N)} S_{2,\lambda}^p\Big) = \alpha_0 > 0. \ \blacksquare$$

Lemma 3.4. *Suppose that* (M_1), (M_2), $(V_1)'$ *and* (f_4) *hold. Then there exists a function* $v \in H_\lambda$ *with* $\|v\|_{H_\lambda} > t_0$ *such that* $\mathbb{I}(v) < 0$, *where* $t_0 > 0$ *is defined in Lemma* 3.3.

Proof. From (M_2), we have

$$\mathcal{M}(\zeta) \leq \mathcal{M}(1)\zeta^\theta \quad \text{for all } \zeta \geq 1.$$

For all $u \in H_\lambda$ with $\|u\|_{H_\lambda} = 1$, from (f_4), we obtain

$$\begin{aligned}\mathbb{I}(tu) &= \frac{1}{2}\big(\mathcal{M}(\|tu\|_{L^{\alpha,2}(\mathbb{R}^N)}^2) + \lambda t^2\|u\|_{2,V}^2\big) \\ &\quad - \gamma\int_{\mathbb{R}^N} F(x,tu)\,dx - \frac{\mu}{p}\int_{\mathbb{R}^N}\xi(x)|u|^p\,dx \\ &\leq \frac{1}{2}\big(\mathcal{M}(1)\|u\|_{L^{\alpha,2}(\mathbb{R}^N)}^{2\theta} t^{2\theta} + \lambda\|u\|_{2,V}^2 t^2\big) - \gamma\beta t^\nu\int_{\mathbb{R}^N}|u(x)|^\nu\,dx \\ &\quad + \gamma\int_{\mathbb{R}^N} P(t|u|)\,dx - \frac{\mu}{p}t^p\int_{\mathbb{R}^N}\xi(x)|u|^p\,dx\end{aligned}$$

for all t large enough. Then for all $t \geq 1$ large enough,

$$\begin{aligned}\mathbb{I}(tu) \leq\ &\max\{1,\mathcal{M}(1)\}t^{2\theta} - \gamma\beta t^\nu\|u\|_{L^\nu(\mathbb{R}^N)}^\nu \\ &+ \gamma\int_{\mathbb{R}^N} P(t|u|)\,dx - \frac{\mu}{p}t^p\int_{\mathbb{R}^N}\xi(x)|u|^p\,dx.\end{aligned}$$

Using the Hölder inequality, we get

$$\begin{aligned}\int_{\mathbb{R}^N}\xi(x)|u(x)|^p\,dx &\leq \|\xi\|_{L^{2/(2-p)}(\mathbb{R}^N)} S_{2,\lambda}^p\|u\|_{H_\lambda}^p \\ &= \|\xi\|_{L^{2/(2-p)}(\mathbb{R}^N)} S_{2,\lambda}^p < \infty.\end{aligned} \tag{3.9}$$

We can write $P = P_1 + Q_1$, where P_1 is a real polynomial with degree in $[2,\nu)$ and Q_1 is a polynomial with degree in $[0,2)$. Since all coefficients of Q_1 are negative, $\int_{\mathbb{R}^N} Q_1(|tu|)\,dx < 0$. By Lemma 2.2, for all $\alpha \in [2,\nu)$,

$$\|u\|_{L^\alpha(\mathbb{R}^N)} \leq S_\alpha\|u\|_{H_\lambda} = S_{\alpha,\lambda} < \infty.$$

Then the growth rate of $\int_{\mathbb{R}^N} P_1(|tu|)\,dx$ is smaller than t^ν as $t\to\infty$. Since $\nu>2\theta>p$ and $\nu>\deg P$, we have $\mathbb{I}(tu)\to-\infty$ as $t\to\infty$. Taking $v=\rho_0 u$ with $\rho_0>t_0>0$ large enough, we have $\mathbb{I}(v)<0$ and $\|v\|_{H_\lambda}>t_0$. ∎

LEMMA 3.5. *Suppose that* (M_1) *and* (M_2) *hold for* $a\ge\theta\frac{\nu-2}{\nu-2\theta}$, *and* $(V_1)'$, *either* $(V_2)'$ *or* $(V_3)'$, (f_1), (f_2) *and* $(f_3)'$ *hold. Then* $\mathbb{I}$ *satisfies the Palais–Smale condition at level* $c\in\mathbb{R}$ *for all* λ *satisfying* (3.3).

Proof. Let $\{u_n\}$ be a (PS) sequence at level $c\in\mathbb{R}$ in H_λ. This means

$$\mathbb{I}(u_n)\to c\quad\text{and}\quad\sup_{\|\varphi\|_{H_\lambda}=1}|\langle\mathbb{I}'(u_n),\varphi\rangle|\to 0 \tag{3.10}$$

as $n\to\infty$. We split the proof into two steps. First, we show that $\{u_n\}$ is bounded in H_λ and has a subsequence strongly convergent in H_λ. From (3.10), for any $n\in\mathbb{N}$, there exists $C>0$ such that

$$\left|\left\langle\mathbb{I}'(u_n),\frac{u_n}{\|u_n\|_{H_\lambda}}\right\rangle\right|\le K\quad\text{and}\quad|\mathbb{I}(u_n)|\le C$$

so that

$$\mathbb{I}(u_n)-\frac{1}{\nu}\langle\mathbb{I}'(u_n),u_n\rangle\le C(1+\|u_n\|_{H_\lambda}), \tag{3.11}$$

where ν is the parameter in $(f_3)'$. Set $A_n=\{x\in\mathbb{R}^N:|u_n(x)|\le r\}$. We have

$$\begin{aligned}\mathbb{I}(u_n)-\frac{1}{\nu}\langle\mathbb{I}'(u_n),u_n\rangle&=\frac12\big(\mathcal{M}(\|u_n\|^2_{L^{\alpha,2}(\mathbb{R}^N)})+\lambda\|u_n\|^2_{2,V}\big)\\&\quad-\gamma\int_{\mathbb{R}^N}F(x,u_n)\,dx-\frac{\mu}{p}\int_{\mathbb{R}^N}\xi(x)|u_n(x)|^p\,dx\\&\quad-\frac{1}{\nu}\Big[M(\|u_n\|^2_{L^{\alpha,2}(\mathbb{R}^N)})\|u_n\|^2_{L^{\alpha,2}(\mathbb{R}^N)}+\lambda\|u_n\|^2_{2,V}\\&\quad-\gamma\int_{\mathbb{R}^N}f(x,u_n)u_n\,dx-\mu\int_{\mathbb{R}^N}\xi(x)|u_n(x)|^p\,dx\Big].\end{aligned}$$

Therefore,

$$\begin{aligned}&\mathbb{I}(u_n)-\frac{1}{\nu}\langle\mathbb{I}'(u_n),u_n\rangle\\&\quad=\left(\frac12\mathcal{M}(\|u_n\|^2_{L^{\alpha,2}(\mathbb{R}^N)})-\frac{1}{\nu}M(\|u_n\|^2_{L^{\alpha,2}(\mathbb{R}^N)})\|u_n\|^2_{L^{\alpha,2}(\mathbb{R}^N)}\right)\\&\qquad+\lambda\left(\frac12-\frac1\nu\right)\|u\|^2_{2,V}+\frac{\gamma}{\nu}\int_{\mathbb{R}^N}(f(x,u_n)u_n-\nu F(x,u_n))\,dx\\&\qquad-\mu\left(\frac1p-\frac1\nu\right)\int_{\mathbb{R}^N}\xi(x)|u_n(x)|^p\,dx.\end{aligned}$$

Since $a\ge\theta\frac{\nu-2}{\nu-2\theta}$, we have $\min\big\{\big(\frac{1}{2\theta}-\frac1\nu\big)a,\frac12-\frac1\nu\big\}=\frac12-\frac1\nu$, and we get

$$\mathbb{I}(u_n) - \frac{1}{\nu}\langle \mathbb{I}'(u_n), u_n\rangle \geq \left(\frac{1}{2} - \frac{1}{\nu}\right)\|u_n\|^2_{H_\lambda} - \mu\left(\frac{1}{p} - \frac{1}{\nu}\right) \int\limits_{\mathbb{R}^N} \xi(x)|u_n(x)|^p\, dx$$
$$+ \frac{\gamma}{\nu} \int\limits_{A_n} \big(f(x,u_n)u_n - \nu F(x,u_n)\big)\, dx$$
$$- \frac{\gamma\rho}{\nu} \int\limits_{\mathbb{R}^N} |u_n(x)|^2\, dx - \frac{\gamma}{\nu} \int\limits_{\mathbb{R}^N} \varphi_*(x)\, dx.$$

Therefore, by (3.5),

$$\begin{aligned}(3.12)\qquad & \mathbb{I}(u_n) - \frac{1}{\nu}\langle \mathbb{I}'(u_n), u_n\rangle \\ &\geq \left(\frac{1}{2} - \frac{1}{\nu}\right)\|u_n\|^2_{H_\lambda} - \mu\left(\frac{1}{p} - \frac{1}{\nu}\right)\|\xi\|_{L^{2/(2-p)}(\mathbb{R}^N)} S^p_{2,\lambda}\|u\|^p_{H_\lambda} \\ &\quad + \frac{\gamma}{\nu} \int\limits_{A_n} \big(f(x,u_n)u_n - \mu F(x,u_n)\big)\, dx - \frac{\gamma\rho}{\nu V_0} \int\limits_{\mathbb{R}^N} V(x)|u_n(x)|^2\, dx + O(1).\end{aligned}$$

From the assumption (3.3), there exists n_0 such that

$$\frac{\gamma\rho}{\nu V_0} < \left(1 - \frac{1}{n_0}\right)\left(\frac{1}{2} - \frac{1}{\nu}\right).$$

We denote $\kappa = (1 - \frac{1}{n_0})(\frac{1}{2} - \frac{1}{\nu}) - \frac{\gamma\rho}{\nu V_0}$. From (f_1) and (f_2), we obtain

$$(3.13)\qquad |f(x,t)| \leq 2\varepsilon|t| + qC_\varepsilon|t|^{q-1}$$

for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$. Then, on A_n, we have

$$\frac{\gamma}{\nu}|f(x,u_n)u_n - \nu F(x,u_n)| \leq \frac{\gamma}{\nu}\big(\varepsilon(2+\nu) + C_\varepsilon(q+\nu)r^{q-2}\big)|u_n|^2 = C_1|u_n|^2.$$

Take $M > \max\{C_1/\kappa, V_0\}$, and let $B_1 = \{x \in B_h(y) : V(x) \leq M\}$ if V satisfies $(V_2)'$, and $B_2 = \{x \in \mathbb{R}^N : V(x) \leq M\}$ if V satisfies $(V_3)'$. By $(V_2)'$ and $(V_3)'$, we have $\operatorname{meas} B_1 < \infty$ for all $y \in \mathbb{R}^N$ and $\operatorname{meas} B_2 < \infty$, respectively. Then

$$\begin{aligned}(3.14)\qquad & \kappa \int\limits_{\mathbb{R}^N} V(x)|u_n|^2\, dx + \frac{\gamma}{\nu} \int\limits_{A_n} (f(x,u_n)u_n - \mu F(x,u_n))\, dx \\ &\geq \kappa \int\limits_{\mathbb{R}^N} V(x)|u_n|^2\, dx - C_1 \int\limits_{A_n} |u_n|^2\, dx \\ &\geq \kappa \int\limits_{\mathbb{R}^N} V(x)|u_n|^2\, dx - M\kappa \int\limits_{A_n} |u_n|^2\, dx \\ &= \kappa \int\limits_{A_n} (V(x) - M)|u_n|^2\, dx \geq \kappa \int\limits_{A_n\cap B_i} (V(x) - M)r^2\, dx \\ &\geq \kappa(V_0 - M)r^2 \operatorname{meas}(A_n \cap B_i) \geq \kappa(V_0 - M)r^2 \operatorname{meas} B_i,\end{aligned}$$

where $i = 1$ or $i = 2$. On combining (3.11) to (3.14), we obtain

$$C(1 + \|u_n\|_{H_\lambda}) \geq \frac{1}{n_0}\left(\frac{1}{2} - \frac{1}{\nu}\right)\|u_n\|^2_{H_\lambda} - \mu\left(\frac{1}{p} - \frac{1}{\nu}\right)\|\xi\|_{L^{2/(2-p)}(\mathbb{R}^N)} S^p_{2,\lambda}\|u\|^p_{H_\lambda} + \kappa(V_0 - M)r^2 \operatorname{meas} B_i. \tag{3.15}$$

By the hypothesis $1 < p < 2$, from (3.15), we conclude that the sequence $\{u_n\}$ is bounded in H_λ. Going to a subsequence if necessary, thanks to Lemmas 2.5, 2.7 and [25, Lemma A.1], we have

$$u_n \to u \text{ weakly in } H_\lambda, \quad u_n \to u \text{ in } L^2(\mathbb{R}^N) \text{ and } L^q(\mathbb{R}^N),$$
$$u_n \to u \text{ in } \mathbb{R}^N, \quad |u_n| \leq h \text{ in } \mathbb{R}^N \text{ for some } h \in L^2(\mathbb{R}^N) \cap L^q(\mathbb{R}^N).$$

We denote by B_ϕ by the following linear function on H_λ:

$$B_\phi(\varphi) = \int_{\mathbb{R}^N} (I - \Delta)^{\alpha/2}\phi(I - \Delta)^{\alpha/2}\varphi \, dx.$$

Clearly, by the Hölder inequality, B_ϕ is a continuous linear mapping on H_λ and

$$|B_\phi(\varphi)| \leq \|\phi\|_{H_\lambda}\|\varphi\|_{H_\lambda} \quad \text{for all } \varphi \in H_\lambda.$$

We have

$$\lim_{n\to\infty} \big(M(\|u_n\|^2_{L^{\alpha,2}(\mathbb{R}^N)}) - M(\|u\|^2_{L^{\alpha,2}(\mathbb{R}^N)})\big)B_u(u_n - u) = 0 \tag{3.16}$$

since $\{M(\|u_n\|^2_{L^{\alpha,2}(\mathbb{R}^N)}) - M(\|u\|^2_{L^{\alpha,2}(\mathbb{R}^N)})\}$ is bounded in $\mathbb{R}$. From (3.13) and using the Hölder inequality, we get

$$\begin{aligned}\int_{\mathbb{R}^N} |(f(x, u_n) - f(x, u))(u_n - u)| \, dx &\leq \int_{\mathbb{R}^N} [2\varepsilon|u_n| + 2\varepsilon|u| + qC_\varepsilon(|u_n|^{q-1} + |u|^{q-1})]|u_n - u| \, dx \\ &\leq 2\varepsilon(\|u_n\|_{L^2(\mathbb{R}^N)} + \|u\|_{L^2(\mathbb{R}^N)})\|u_n - u\|_{L^2(\mathbb{R}^N)} \\ &\quad + qC_\varepsilon(\|u_n\|^{q-1}_{L^q(\mathbb{R}^N)} + \|u\|^{q-1}_{L^q(\mathbb{R}^N)})\|u_n - u\|_{L^q(\mathbb{R}^N)}.\end{aligned}$$

This implies

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (f(x, u_n) - f(x, u))(u_n - u) \, dx = 0. \tag{3.17}$$

From (3.5), we have

$$\begin{aligned}\int_{\mathbb{R}^N} \xi(x)|\psi(x)|^p \, dx &\leq \|\xi\|_{L^{2/(2-p)}(\mathbb{R}^N)}\|\psi\|^p_{L^2(\mathbb{R}^N)} \\ &\leq S^p_{2,\lambda}\|\xi\|_{L^{2/(2-p)}(\mathbb{R}^N)}\|\psi\|^p_{H_\lambda}\end{aligned} \tag{3.18}$$

for all $\psi \in H_\lambda$.

Now, we prove

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \xi(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)\, dx = 0. \tag{3.19}$$

Applying (3.18) for $\psi = u_n - u$, we have

$$\int_{\mathbb{R}^N} \xi(x)|u_n - u|^p\, dx \le \|\xi\|_{L^{2/(2-p)}(\mathbb{R}^N)} \|u_n - u\|^p_{L^2(\mathbb{R}^N)} \to 0$$

as $n \to \infty$ since $u_n \to u \in L^2(\mathbb{R}^N)$. Using the Hölder inequality, (3.18) and noting that $\{\|u_n\|_{H_\lambda}\}$ is bounded, we get

$$\begin{aligned}\Big|\int_{\mathbb{R}^N} \xi(x)|u_n|^{p-2}u_n(u_n-u)\, dx\Big| &\le \int_{\mathbb{R}^N} \xi(x)|u_n|^{p-1}|u_n-u|\, dx \\ &= \int_{\mathbb{R}^N} \xi(x)^{1/p}|u_n-u|\xi(x)^{(p-1)/p}|u_n|^{p-1}\, dx \\ &\le \Big(\int_{\mathbb{R}^N} \xi(x)|u_n-u|^p\, dx\Big)^{1/p}\Big(\int_{\mathbb{R}^N} \xi(x)|u_n(x)|^p\, dx\Big)^{p/(p-1)} \\ &\le \Big(\int_{\mathbb{R}^N} \xi(x)|u_n-u|^p\, dx\Big)^{1/p}(S^p_{2,\lambda}\|\xi\|_{L^{2/(2-p)}(\mathbb{R}^N)}\|u_n\|^p_{H_\lambda})^{p/(p-1)} \to 0\end{aligned} \tag{3.20}$$

as $n \to \infty$. Similarly,

$$\int_{\mathbb{R}^N} \xi(x)|u|^{p-2}u(u_n-u)\, dx \to 0 \tag{3.21}$$

as $n \to \infty$. On combining (3.20) with (3.21), we obtain (3.19).

Obviously, $\langle \mathbb{I}'(u_n) - \mathbb{I}'(u), u_n - u\rangle \to 0$ since $u_n \to u$ weakly in W and $\mathbb{I}'(u_n) \to 0$. Therefore, from (3.17) and (3.19), we get

$$\begin{aligned} o(1) &= \langle \mathbb{I}'(u_n) - \mathbb{I}'(u), u_n - u\rangle \\ &= M(\|u_n\|^2_{L^{\alpha,2}(\mathbb{R}^N)})(B_{u_n}(u_n-u) - B_u(u_n-u)) \\ &\quad + (M(\|u_n\|^2_{L^{\alpha,2}(\mathbb{R}^N)}) - M(\|u\|^2_{L^{\alpha,2}(\mathbb{R}^N)}))B_u(u_n-u) \\ &\quad + \lambda\int_{\mathbb{R}^N} V(x)|u_n-u|^2\, dx - \gamma\int_{\mathbb{R}^N}(f(x,u_n) - f(x,u))(u_n-u)\, dx \\ &\quad - \mu\int_{\mathbb{R}^N} \xi(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n-u)\, dx \\ &= M(\|u_n\|^p)\big(B_{u_n}(u_n-u) - B_u(u_n-u)\big) \\ &\quad + \lambda\int_{\mathbb{R}^N} V(x)|u_n-u|^2\, dx + o(1)\end{aligned}$$

for all n large enough. Since $(I-\Delta)^{\alpha/2}$ is a linear map, we have

$$(I-\Delta)^{\alpha/2}u_n - (I-\Delta)^{\alpha/2}u = (I-\Delta)^{\alpha/2}(u_n-u).$$

This implies

$$(3.22)\quad B_{u_n}(u_n-u)-B_u(u_n-u)=\int\limits_{\mathbb{R}^N}(I-\Delta)^{\alpha/2}u_n(I-\Delta)^{\alpha/2}(u_n-u)\,dx - \int\limits_{\mathbb{R}^N}(I-\Delta)^{\alpha/2}u(I-\Delta)^{\alpha/2}(u_n-u)\,dx$$
$$= \int\limits_{\mathbb{R}^N}\big((I-\Delta)^{\alpha/2}u_n-(I-\Delta)^{\alpha/2}u\big)(I-\Delta)^{\alpha/2}(u_n-u)\,dx$$
$$= \int\limits_{\mathbb{R}^N}|(I-\Delta)^{\alpha/2}(u_n-u)|^2\,dx=\|u_n-u\|^2_{L^{\alpha,2}(\mathbb{R}^N)}.$$

Noting that $M(t)\ge a>0$ and $V(x)|u_n-u|^2\ge 0$, we obtain

$$(3.23)\qquad \lim_{n\to\infty}B_{u_n}(u_n-u)-B_u(u_n-u)=0,\qquad \lim_{n\to\infty}\int\limits_{\mathbb{R}^N}V(x)|u_n-u|^2\,dx=0.$$

Combining (3.22) and (3.23), we get

$$\|u_n-u\|^2_{H_\lambda}=\|u_n-u\|^2_{L^{\alpha,2}(\mathbb{R}^N)}+\lambda\|u_n-u\|^2_{2,V}\to 0$$

as $n\to\infty$. Hence $u_n\to u$ strongly in H_λ. ■

From the above lemmas, we can now prove Theorem 3.2 as follows:

First step. There exists a function $\varphi_0\in H_\lambda$ such that [20]

$$\int\limits_{\mathbb{R}^N}\xi(x)|\varphi_0(x)|^p\,dx>0.$$

Then

$$\mathbb{I}(t\varphi)\le\frac12\Big(\mathcal{M}(\|t\varphi_0\|^2_{L^{\alpha,2}(\mathbb{R}^N)})+\lambda\int\limits_{\mathbb{R}^N}V(x)|t\varphi_0|^2\,dx\Big)-\gamma\beta t^\nu\|\varphi_0\|^\nu_{L^\nu(\mathbb{R}^N)}$$
$$+\gamma\int\limits_{\mathbb{R}^N}P(t|\varphi_0|)\,dx-t^p\frac{\mu}{p}\int\limits_{\mathbb{R}^N}\xi(x)|\varphi_0(x)|^p\,dx.$$

Hence,

$$\mathbb{I}(t\varphi)\le\frac12\sup_{\zeta\in[0,t_0^2]}M(\zeta)\|t\varphi_0\|^2_{L^{\alpha,2}(\mathbb{R}^N)}+\frac{\lambda}{2}\int\limits_{\mathbb{R}^N}V(x)|t\varphi_0|^2\,dx$$
$$-\gamma\beta t^\nu\|\varphi_0\|^\nu_{L^\nu(\mathbb{R}^N)}+\gamma\int\limits_{\mathbb{R}^N}P(t|\varphi_0|)\,dx-t^p\frac{\mu}{p}\int\limits_{\mathbb{R}^N}\xi(x)|\varphi_0(x)|^p\,dx<0$$

for all $t\in(0,1)$ small enough, where t_0 is given in Lemma 3.3. Indeed, we can write $P=P_1+Q_1$, where P_1 is a polynomial with degree in $[2,\nu)$ and Q_1 is a polynomial with degree in $[0,2)$. Since all coefficients of Q_1 are negative, $\int_{\mathbb{R}^N}Q_1(|t\varphi_0|)\,dx<0$. By Lemma 2.5, for all $\alpha\in[2,\nu)$,

$$\|\varphi_0\|_{L^\alpha(\mathbb{R}^N)}\le S_{\alpha,\lambda}\|\varphi_0\|_{H_\lambda}<\infty.$$

Then the growth rate of $\int_{\mathbb{R}^N} P_1(|t\varphi_0|)\,dx$ is smaller than t^p $(1 < p < 2)$ as $t \to 0$. Since $\deg P_1 \geq 2 > p$, we have $\mathbb{I}(t\varphi_0) < 0$ for all $t \in (0,1)$ small enough.

This implies

$$c_0 = \inf\{\mathbb{I}(u) : u \in \overline{B}_{t_0}\} < 0,$$

where $B_{t_0} = \{u \in H_\lambda : \|u\|_{H_\lambda} < t_0\}$, and $t_0 > 0$ is given by Lemma 3.3. By Lemma 3.3,

$$\inf_{u \in \partial B_{t_0}} \mathbb{I}(u) \geq \alpha_0 > 0.$$

For all n large enough, we have $1/n \in (0, \inf_{u\in\partial B_{t_0}} \mathbb{I}(u) - \inf_{u\in\overline{B}_{t_0}} \mathbb{I}(u))$. Then there exists a sequence $w_n \in \overline{B}_{t_0}$ such that

$$\inf_{u\in\overline{B}_{t_0}} \mathbb{I}(u) \leq \mathbb{I}(w_n) \leq \inf_{u\in\overline{B}_{t_0}} \mathbb{I}(u) + 1/n. \tag{3.24}$$

By the Ekeland variational principle [10], there exists a sequence $\{v_n\} \subset \overline{B}_{t_0}$ such that

$$\inf_{u\in\overline{B}_{t_0}} \mathbb{I}(u) \leq \mathbb{I}(v_n) \leq \mathbb{I}(w_n) \leq \inf_{u\in\overline{B}_{t_0}} \mathbb{I}(u) + \frac{1}{n}, \qquad \|w_n - v_n\|_{H_\lambda} \leq 1, \tag{3.25}$$

and

$$\mathbb{I}(v_n) \leq \mathbb{I}(u) + \frac{1}{n}\|u - v_n\|_{H_\lambda} \quad \text{for all } u \in \overline{B}_{t_0}, \tag{3.26}$$

hence

$$\mathbb{I}(v_n) \leq \inf_{u\in\overline{B}_{t_0}} \mathbb{I}(u) + \frac{1}{n} < \inf_{\partial B_{t_0}} \mathbb{I}(u).$$

So $v_n \in B_{t_0}$. We consider the function

$$\varphi_n(u) = \mathbb{I}(u) + \frac{1}{n}\|v_n - u\|_{H_\lambda}.$$

By (3.26), $v_n \in B_{t_0}$ minimizes φ_n on $\overline{B}_{t_0}$. Then for all $\psi \in H_\lambda$ with $\|\psi\|_{H_\lambda} = 1$, taking $t > 0$ such that $v_n + t\psi \in \overline{B}_{t_0}$, we get

$$\frac{\varphi_n(v_n + t\psi) - \varphi_n(v_n)}{t} \geq 0.$$

Hence

$$\frac{\mathbb{I}(v_n + t\psi) - \mathbb{I}(v_n)}{t} + \frac{1}{n} \geq 0.$$

Letting $t \to 0$, we get $\langle \mathbb{I}'(v_n), \psi\rangle \geq -1/n$. We replace ψ by $-\psi$ to obtain

$$|\langle \mathbb{I}'(v_n), \psi\rangle| \leq 1/n. \tag{3.27}$$

Passing to the limit in (3.25) and (3.27), we conclude that

$$\mathbb{I}(v_n) \to \inf_{u\in\overline{B}_{t_0}} \mathbb{I}(u) = c_0$$

and $\mathbb{I}'(v_n) \to 0$. By Lemma 3.5, there exists $u_0 \in B_{t_0}$ such that $\mathbb{I}'(u_0) = 0$, $\mathbb{I}(u_0) = c_0 < 0$.

Second step. By Lemmas 3.3 and 3.4 and the Mountain Pass Theorem, there exists a sequence $\{u_n\} \subset H_\lambda$ such that

$$\mathbb{I}(u_n) \to c_1 > 0 \quad \text{and} \quad \mathbb{I}'(u_n) \to 0$$

as $n \to \infty$. By Lemma 3.5, there exists $u_1 \in H_\lambda$ such that $\mathbb{I}(u_1) = c_1 > 0$ and $\mathbb{I}'(u_1) = 0$. Then $u_1 \neq 0, u_1 \neq u_0$. ■

Note that if $V(x)$ is bounded from below by a negative constant, then there is a constant $C > 0$ such that $\overline{V}(x) = V(x) + C \geq V_0 > 0$ for all $x \in \mathbb{R}^N$. We denote

$$\overline{f}(x,t) = f(x,t) + Cu.$$

Then (3.1) is equivalent to

$$M\Big(\int_{\mathbb{R}^N} |(I-\Delta)^{\alpha/2}u|^2\,dx\Big)(I-\Delta)^\alpha u + \lambda\overline{V}(x)u = \gamma\overline{f}(x,u) + \mu\xi(x)|u|^{p-2}u. \tag{3.28}$$

We assume the conditions on $\overline{f}$ are the same as on f. Hence, (3.28) is studied similarly to (3.1), and we get

Theorem 3.6. *Suppose that (M_1) and (M_2) hold for $a \geq \theta\frac{\nu-2}{\nu-2\theta}$, $(V_1)'$ and either $(V_2)'$ or $(V_3)'$ hold for $\overline{V}$, and (f_1), (f_2), $(f_3)'$ and (f_4) hold for $\overline{f}$. Then for any $\lambda > 0$, there exists $\mu_{\lambda,\gamma} > 0$ such that (3.28) has at least two nontrivial solutions in H_λ with γ satisfying (3.3) and $\mu \in (0, \mu_{\lambda,\gamma})$.*

Next, we study the equation

$$M(\|u\|^2_{H_\lambda})((I-\Delta)^\alpha u + \lambda V(x)u) = \gamma f(x,u) + \mu\xi(x)|u|^{p-2}u, \tag{3.29}$$

where γ, λ, μ are real positive parameters. For this, we use the energy function

$$J(u) = \frac{1}{2}\mathcal{M}(\|u\|^2_{H_\lambda}) - \gamma\int_{\mathbb{R}^N} F(x,u)\,dx - \frac{\mu}{p}\int_{\mathbb{R}^N} \xi(x)|u|^p\,dx.$$

Then

$$\begin{aligned}\langle J'(u), v\rangle &= M(\|u\|^2_{H_\lambda})\Big(\int_{\mathbb{R}^N} (I-\Delta)^{\alpha/2}u(I-\Delta)^{\alpha/2}v\,dx + \lambda\int_{\mathbb{R}^N} V(x)u(x)v(x)\,dx\Big)\\ &\quad - \gamma\int_{\mathbb{R}^N} f(x,u(x))v(x)\,dx - \mu\int_{\mathbb{R}^N}\xi(x)|u|^{p-2}uv\,dx \quad \text{for all } v \in H_\lambda.\end{aligned}$$

DEFINITION 3.7. We say that $u \in H_\lambda$ is a *weak solution* to (3.29) if

$$M(\|u\|^2_{H_\lambda})\Big(\int_{\mathbb{R}^N}(I-\Delta)^{\alpha/2}u(I-\Delta)^{\alpha/2}v\,dx+\lambda\int_{\mathbb{R}^N}V(x)u(x)v(x)\,dx\Big)$$
$$=\gamma\int_{\mathbb{R}^N}f(x,u(x))v(x)\,dx+\mu\int_{\mathbb{R}^N}\xi(x)|u|^{p-2}uv\,dx\qquad\text{for any } v\in H_\lambda.$$

Assume that ν, γ and ρ satisfy the condition

$$\frac{\lambda\rho}{\nu V_0}<\Big(1-\frac{1}{n_0}\Big)\Big(\frac{1}{p\theta}-\frac{1}{\nu}\Big)a, \tag{3.30}$$

where n_0 is a large enough positive integer. We obtain

THEOREM 3.8. *Suppose that (M_1), (M_2), $(V_1)'$, either $(V_2)'$ or $(V_3)'$, (f_1), (f_2), $(f_3)'$ and (f_4) hold. Then for any $\lambda>0$, there exists $\mu_{\lambda,\gamma}>0$ such that (3.29) has at least two nontrivial solutions in H_λ with γ satisfying (3.30) and $\mu\in(0,\mu_{\lambda,\gamma})$.*

Proof. We need some lemmas:

LEMMA 3.9. *Suppose that (M_1), (M_2), $(V_1)'$, (f_1) and (f_2) hold. Then for all $\lambda,\gamma>0$, there exist positive constants $t_0,\alpha_0,\mu_{\lambda,\gamma}$ such that $J(u)\ge\alpha_0$ for all $u\in W$ with $\|u\|_W=t_0$ and $\mu\in(0,\mu_{\lambda,\gamma})$.*

Proof. From (f_1) and (f_2), for any $\varepsilon>0$, there exists $C_\varepsilon>0$ such that

$$F(x,t)\le\varepsilon|t|^2+C_\varepsilon|t|^q$$

for any $x\in\mathbb{R}^N$, $t\in\mathbb{R}$, and for some $C_\varepsilon>0$. Then from (M_1) and (M_2), using the Hölder inequality we get

$$\begin{aligned}J(u)&=\frac{1}{2}\mathcal{M}(\|u\|^2_{H_\lambda})-\gamma\int_{\mathbb{R}^N}F(x,u)\,dx-\frac{\mu}{p}\int_{\mathbb{R}^N}\xi(x)|u|^{p-2}u\,dx\\&\ge\frac{a}{2\theta}\|u\|^2_{H_\lambda}-\varepsilon\gamma\|u\|^2_{L^2(\mathbb{R}^N)}-\gamma C_\varepsilon\|u\|^q_{L^q(\mathbb{R}^N)}\\&\quad-\frac{\mu}{p}\|\xi\|_{L^{2/(2-p)}(\mathbb{R}^N)}S^p_{2,\lambda}\|u\|^p_{H_\lambda}.\end{aligned}\tag{3.31}$$

From (2.1) and (3.31), we have

$$\begin{aligned}J(u)&\ge\frac{a}{2\theta}\|u\|^2_{H_\lambda}-\varepsilon\gamma S^2_{2,\lambda}\|u\|^2_{H_\lambda}-\gamma C_\varepsilon S^q_{q,\lambda}\|u\|^q_{H_\lambda}\\&\quad-\frac{\mu}{p}\|\xi\|_{L^{2/(2-p)}(\mathbb{R}^N)}S^p_{2,\lambda}\|u\|^p_{H_\lambda}\\&=\|u\|^p_{H_\lambda}\bigg[\Big(\frac{a}{2\theta}-\varepsilon\gamma S^2_{2,\lambda}\Big)\|u\|^{2-p}_{H_\lambda}-\gamma C_\varepsilon S^q_{q,\lambda}\|u\|^{q-p}_{H_\lambda}\\&\qquad\qquad-\frac{\mu}{p}\|\xi\|_{L^{2/(2-p)}(\mathbb{R}^N)}S^p_{2,\lambda}\bigg].\end{aligned}\tag{3.32}$$

The number $\varepsilon(\gamma) > 0$ is chosen small enough such that $\frac{a}{2\theta} - \varepsilon\gamma S_{2,\lambda}^2 > 0$ for any $\gamma > 0$. Let $h(t) = \big(\frac{a}{2\theta} - \varepsilon\gamma S_{2,\lambda}^2\big)t^{2-p} - \gamma C_\varepsilon S_{q,\lambda}^q t^{q-p}$ for $t \geq 0$. We see that there exists $t_0 > 0$ such that $h(t_0) > 0$ and the maximum of h is attained at $t_0 > 0$. Then we can choose $\mu_{\lambda,\gamma}$ small such that

$$h(t_0) - \frac{\mu}{p}\|\xi\|_{L^{2/(2-p)}(\mathbb{R}^N)} S_{2,\lambda}^p > 0. \tag{3.33}$$

Combining (3.32) and (3.33), for $\|u\|_{H_\lambda} = t_0$ we have

$$J(u) \geq t_0^p\bigg(h(t_0) - \frac{\mu}{p}\|\xi\|_{L^{2/(2-p)}(\mathbb{R}^N)} S_{2,\lambda}^p\bigg) = \alpha_0 > 0. \ ■$$

LEMMA 3.10. *Suppose that* (M_1), (M_2), $(V_1)'$ *and* (f_4) *hold. Then there exists a function* $v \in H_\lambda$ *with* $\|v\|_{H_\lambda} > t_0$ *such that* $J(v) < 0$, *where* $t_0 > 0$ *is defined in Lemma* 3.9.

Proof. From (M_2), we have $\mathcal{M}(\zeta) \leq \mathcal{M}(1)\zeta^\theta$ for all $\zeta \geq 1$. For all $u \in H_\lambda$ with $\|u\|_{H_\lambda} = 1$, from (f_4), arguing as for Lemma 3.4, for all $t \geq 1$ we obtain

$$\begin{aligned} J(tu) &= \frac{1}{2}\mathcal{M}(t^2\|u\|_{H_\lambda}^2) - \gamma \int_{\mathbb{R}^N} F(x,tu)\,dx - \frac{\mu}{p}t^p \int_{\mathbb{R}^N} \xi(x)|u|^p\,dx \\ &\leq \frac{\mathcal{M}(1)}{2}\|u\|_{H_\lambda}^{2\theta} t^{2\theta} - \gamma\beta t^\nu \int_{\mathbb{R}^N} |u(x)|^\nu\,dx \\ &\quad + \gamma \int_{\mathbb{R}^N} P(t|u|)\,dx - \frac{\mu}{p}t^p \int_{\mathbb{R}^N} \xi(x)|u|^p\,dx \\ &= \frac{\mathcal{M}(1)}{2}t^{2\theta} - \gamma\beta t^\nu \|u\|_{L^\nu(\mathbb{R}^N)}^\nu + \gamma \int_{\mathbb{R}^N} P(t|u|)\,dx - \frac{\mu}{p}t^p \int_{\mathbb{R}^N} \xi(x)|u|^p\,dx \\ &\to -\infty \end{aligned}$$

as $t \to \infty$, since $\nu > 2\theta > p$ and $\nu > \deg P$. Taking $v = \rho_0 u$ with $\rho_0 > t_0 > 0$ large enough, we have $J(v) < 0$ and $\|v\|_{H_\lambda} > t_0$. ■

LEMMA 3.11. *Assume that* (M_1), (M_2), $(V_1)'$, *either* $(V_2)'$ *or* $(V_3)'$, (f_1), (f_2) *and* $(f_3)'$ *hold. Then* J *satisfies the Palais–Smale condition at level* $c \in \mathbb{R}$ *for all* λ *satisfying* (3.30).

Proof. Let $\{u_n\}$ be a (PS) sequence at level $c \in \mathbb{R}$ in H_λ. This means

$$J(u_n) \to c \quad \text{and} \quad \sup_{\|\varphi\|_{H_\lambda}=1} |\langle J'(u_n), \varphi\rangle| \to 0 \tag{3.34}$$

as $n \to \infty$. We split the proof into two steps. First, we show that the sequence $\{u_n\}$ is bounded in H_λ and that it has a subsequence strongly convergent in H_λ. From (3.34), for any $n \in \mathbb{N}$, there exists $C > 0$ such that

$$|\langle J'(u_n), u_n/\|u_n\|_{H_\lambda}\rangle| \leq K \quad \text{and} \quad |J(u_n)| \leq C$$

so that

$$J(u_n) - \frac{1}{\nu}\langle J'(u_n), u_n\rangle \le C(1 + \|u_n\|_{H_\lambda}), \tag{3.35}$$

where ν is the parameter in $(f_3)'$. Set $A_n = \{x \in \mathbb{R}^N : |u_n(x)| \le r\}$. We have

$$\begin{aligned} J(u_n) - \frac{1}{\nu}\langle J'(u_n), u_n\rangle = \frac{1}{2}\mathcal{M}(\|u_n\|^2_{H_\lambda}) - \gamma \int_{\mathbb{R}^N} F(x, u_n)\,dx \\ - \frac{\mu}{p}\int_{\mathbb{R}^N} \xi(x)|u_n(x)|^p\,dx - \frac{1}{\nu}M(\|u_n\|^2_{H_\lambda})\|u_n\|^2_{H_\lambda} \\ + \frac{1}{\nu}\Big[\gamma\int_{\mathbb{R}^N} f(x, u_n)u_n\,dx + \mu\int_{\mathbb{R}^N} \xi(x)|u_n(x)|^p\,dx\Big]. \end{aligned}$$

This implies

$$\begin{aligned} J(u_n) - \frac{1}{\nu}\langle J'(u_n), u_n\rangle = \left(\frac{1}{2}\mathcal{M}(\|u_n\|^2_{H_\lambda}) - \frac{1}{\nu}M(\|u_n\|^2_{H_\lambda})\|u_n\|^2_{W}\right) \\ + \frac{\gamma}{\nu}\int_{\mathbb{R}^N} (f(x, u_n)u_n - \nu F(x, u_n))\,dx - \mu\left(\frac{1}{p} - \frac{1}{\nu}\right)\int_{\mathbb{R}^N} \xi(x)|u_n(x)|^p\,dx. \end{aligned}$$

Hence,

$$\begin{aligned} J(u_n) - \frac{1}{\mu}\langle J'(u_n), u_n\rangle \ge \left(\frac{1}{2\theta} - \frac{1}{\nu}\right)a\|u_n\|^2_{H_\lambda} \\ - \mu\left(\frac{1}{p} - \frac{1}{\nu}\right)\int_{\mathbb{R}^N} \xi(x)|u_n(x)|^p\,dx \\ + \frac{\gamma}{\nu}\int_{A_n} (f(x, u_n)u_n - \nu F(x, u_n))\,dx \\ - \frac{\gamma\rho}{\nu}\int_{\mathbb{R}^N} |u_n(x)|^2\,dx - \frac{\gamma}{\nu}\int_{\mathbb{R}^N} \varphi_*(x)\,dx. \end{aligned}$$

By (3.18),

$$\int_{\mathbb{R}^N} \xi(x)|u_n(x)|^p\,dx \le S^p_{2,\lambda}\|\xi\|_{L^{2/(2-p)}(\mathbb{R}^N)}\|u_n\|^p_{H_\lambda}.$$

Therefore,

$$\begin{aligned} J(u_n) - \frac{1}{\mu}\langle J'(u_n), u_n\rangle \\ \ge \left(\frac{1}{2\theta} - \frac{1}{\nu}\right)a\|u_n\|^2_{H_\lambda} - \mu\left(\frac{1}{p} - \frac{1}{\nu}\right)S^p_{2,\lambda}\|\xi\|_{L^{2/(2-p)}(\mathbb{R}^N)}\|u_n\|^p_{H_\lambda} \\ + \frac{\gamma}{\nu}\int_{A_n} (f(x, u_n)u_n - \nu F(x, u_n))\,dx - \frac{\gamma\rho}{\nu V_0}\int_{\mathbb{R}^N} V(x)|u_n(x)|^2\,dx + O(1). \end{aligned} \tag{3.36}$$

From the assumption (3.30), there exists n_0 such that

$$\frac{\gamma\rho}{\nu V_0} < \left(1 - \frac{1}{n_0}\right)\left(\frac{1}{2\theta} - \frac{1}{\nu}\right)a.$$

We denote $\kappa = (1 - \frac{1}{n_0})(\frac{1}{2\theta} - \frac{1}{\nu})a - \frac{\gamma\rho}{\nu V_0}$. Then, on A_n, we have

$$\frac{\gamma}{\nu}|f(x,u_n)u_n - \mu F(x,u_n)| \le \frac{\gamma}{\nu}(\varepsilon(2+\nu) + C_\varepsilon(q+\nu)r^{q-2})|u_n|^2 = C_1|u_n|^2.$$

Take $M > \max\{C_1/\kappa, V_0\}$, and let $B_1 = \{x \in B_h(y) : V(x) \le M\}$ if V satisfies $(V_2)'$, and $B_2 = \{x \in \mathbb{R}^N : V(x) \le M\}$ if V satisfies $(V_3)'$. By $(V_2)'$ and $(V_3)'$, we have $\operatorname{meas} B_1 < \infty$ for all $y \in \mathbb{R}^N$ and $\operatorname{meas} B_2 < \infty$, respectively. From (3.14), we have

$$\kappa \int_{\mathbb{R}^N} V(x)|u_n|^2\,dx + \frac{\gamma}{\nu}\int_{A_n} (f(x,u_n)u_n - \nu F(x,u_n))\,dx \ge \kappa(V_0 - M)r^2 \operatorname{meas}(A_n \cap B_i) \ge \kappa(V_0 - M)r^2 \operatorname{meas} B_i, \tag{3.37}$$

where $i = 1$ or $i = 2$. Combining (3.35)–(3.37), we have

$$C(1 + \|u_n\|_{H_\lambda}) \ge \frac{1}{n_0}\left(\frac{1}{2\theta} - \frac{1}{\nu}\right)a\|u_n\|^2_{H_\lambda} - \mu\left(\frac{1}{p} - \frac{1}{\nu}\right)S^p_{2,\lambda}\|\xi\|_{L^{2/(2-p)}(\mathbb{R}^N)}\|u_n\|^p_{H_\lambda} + \kappa(V_0 - M)r^p \operatorname{meas} B_i. \tag{3.38}$$

By hypothesis $1 < p < 2$, from (3.38), we conclude that the sequence $\{u_n\}$ is bounded in H_λ. Going to a subsequence if necessary, thanks to Lemmas 2.5 and 2.7 and [25, Lemma A.1] we have

$$\begin{aligned} &u_n \to u \text{ weakly in } H_\lambda, \quad u_n \to u \text{ in } L^2(\mathbb{R}^N) \text{ and } L^q(\mathbb{R}^N),\\ &u_n \to u \text{ in } \mathbb{R}^N, \quad |u_n| \le h \text{ in } \mathbb{R}^N \text{ for some } h \in L^2(\mathbb{R}^N) \cap L^q(\mathbb{R}^N). \end{aligned} \tag{3.39}$$

We denote by B_ϕ the following linear function on H_λ:

$$B_\phi(\varphi) = \int_{\mathbb{R}^N} (I - \Delta)^{\alpha/2}\phi(I - \Delta)^{\alpha/2}\varphi\,dx + \lambda\int_{\mathbb{R}^N} V(x)\phi(x)\varphi(x)\,dx.$$

Clearly, by the Hölder and Cauchy–Schwarz inequalities, B_ϕ is continuous on H_λ and

$$\begin{aligned} |B_\phi(\varphi)| &\le \|\phi\|_{L^{\alpha,2}(\mathbb{R}^N)}\|\varphi\|_{L^{\alpha,2}(\mathbb{R}^N)} + \sqrt{\lambda}\,\|\phi\|_{2,V}\sqrt{\lambda}\,\|\varphi\|_{2,V}\\ &\le \|\phi\|_{H_\lambda}\|\varphi\|_{H_\lambda} \end{aligned}$$

for all $\varphi \in H_\lambda$. Obviously, $\langle J'(u_n) - J'(u), u_n - u\rangle \to 0$ since $u_n \to u$ weakly in H_λ and $J'(u_n) \to 0$. Therefore, from (3.17) to (3.19), we get

$$\begin{aligned} o(1) &= \langle J'(u_n) - J'(u), u_n - u \rangle \\ &= M(\|u_n\|^2_{H_\lambda})\big(B_{u_n}(u_n - u) - B_u(u_n - u)\big) \\ &\quad + \big(M(\|u_n\|^2_{H_\lambda}) - M(\|u\|^2_{H_\lambda})\big)B_u(u_n - u) \\ &\quad - \int_{\mathbb{R}^N} \big(f(x,u_n) - f(x,u)\big)(u_n - u)\,dx \\ &\quad - \mu \int_{\mathbb{R}^N} \xi(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)\,dx \\ &= M(\|u_n\|^2_{H_\lambda})\big(B_{u_n}(u_n - u) - B_u(u_n - u)\big) + o(1). \end{aligned}$$

From $M(t) \ge a > 0$, we obtain

$$\lim_{n\to\infty} \big(B_{u_n}(u_n - u) - B_u(u_n - u)\big) = 0.$$

Note that

$$B_{u_n}(u_n - u) - B_u(u_n - u) = \|u_n - u\|^2_{H_\lambda}.$$

Hence $\|u_n - u\|_{H_\lambda} \to 0$, so $u_n \to u$ strongly in H_λ. ■

Theorem 3.8 is proved similarly to Theorem 3.2 by using Lemmas 3.9–3.11 and the Mountain Pass Theorem. ■

Finally, we study the following degenerate equation involving the critical exponent:

$$M(\|u\|^2_{H_\lambda})((I - \varDelta)^\alpha u + \lambda V(x)u) = \gamma f(x,u) + |u|^{2^*_\alpha - 2}u \tag{3.40}$$

in $\mathbb{R}^N$, where λ and γ are real positive parameters.

Now, we state some assumptions on the functions M and f:

- (M_3) $M : [0,\infty) \to [0,\infty)$ is a continuous function, and there exists $\eta \in (1, 2^*_\alpha/2)$ such that $M(t)t \le \eta\mathcal{M}(t)$ for all $t \ge 0$, where $\mathcal{M}(t) = \int_0^t M(\tau)\,d\tau$;
- (M_4) for all $\sigma > 0$, there exists $\mathfrak{r} = \mathfrak{r}(\sigma)$ such that $M(t) \ge \mathfrak{r}$ for all $t \ge \mathfrak{r}$;
- (M_5) $\liminf_{t\to 0^+} M(t)/t^{\eta-1} > 0$.
- (f_6) $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying
$$|f(x,u)| \le g(x)|u|^{q-1},$$
where $q \in (2\eta, 2^*_\alpha)$, g is a bounded function on $\mathbb{R}^N$ and $0 \le g \in L^{2^*_\alpha/(2^*_\alpha - q)}(\mathbb{R}^N)$.
- (f_7) there exists $\nu \in (2\eta, 2^*_\alpha)$ such that $0 < \nu F(x,u) \le uf(x,u)$ for all $x \in \mathbb{R}^N$ and $u \in \mathbb{R} \setminus \{0\}$.

Obviously, $M(t) = a_1 + b_1\eta t^{\eta-1}$ satisfies (M_3)–(M_5), where $a_1, b_1 \ge 0$, $a_1 + b_1 > 0$. This function covers the degenerate case $M(0) = 0$ if we take

$a_1 = 0$. The conditions (M_3)–(M_5), (f_6) and (f_7) have been used by Xiang–Zhang–Zhang [26] in the critical Kirchhoff problem involving the fractional Laplacian in $\mathbb{R}^N$. The difficult problem is when $M(0) = 0$; then the nonlinear function must satisfy conditions like (f_6) and (f_7). We see that (f_6) implies (f_1), and (f_7) implies (f_3). For the Kirchhoff degenerate problem involving fractional Laplacian, we refer the reader to [4, 13] for further comments. In order to study solutions of (3.40), we consider the energy function

$$\mathcal{I}(u) = \frac{1}{2}\mathcal{M}(\|u\|^2_{H_\lambda}) - \gamma \int_{\mathbb{R}^N} F(x,u)\,dx - \frac{1}{2^*_\alpha}\int_{\mathbb{R}^N} |u|^{2^*_\alpha}\,dx.$$

From (f_6) and since g is a bounded function on $\mathbb{R}^N$, using compact embedding from H_λ into $L^q(\mathbb{R}^N)$ for $q \in [2, 2^*_\alpha)$ we get $D(u) = \int_{\mathbb{R}^N} f(x,u)\,dx \in C^1(H_\lambda, \mathbb{R})$ and

$$\langle D'(u), v\rangle = \int_{\mathbb{R}^N} f(x,u)v\,dx, \quad \text{and } D' \text{ is weakly continuous in } H_\lambda. \tag{3.41}$$

Hence $\mathcal{I} \in C^1(H_\lambda, \mathbb{R})$, and we get

$$\begin{aligned}
&\langle \mathcal{I}'(u), v\rangle \\
&= M(\|u\|^2_{H_\lambda})\Big(\int_{\mathbb{R}^N} (I-\Delta)^{\alpha/2}u(I-\Delta)^{\alpha/2}v\,dx + \lambda\int_{\mathbb{R}^N} V(x)u(x)v(x)\,dx\Big) \\
&\qquad - \gamma\int_{\mathbb{R}^N} f(x,u(x))v(x)\,dx - \int_{\mathbb{R}^N} |u|^{2^*_\alpha-2}uv\,dx \quad \text{for all } v \in H_\lambda.
\end{aligned}$$

Definition 3.12. We say that $u \in H_\lambda$ is a *weak solution* to (3.40) if

$$\begin{aligned}
&M(\|u\|^2_{H_\lambda})\Big(\int_{\mathbb{R}^N} (I-\Delta)^{\alpha/2}u(I-\Delta)^{\alpha/2}v\,dx + \lambda\int_{\mathbb{R}^N} V(x)u(x)v(x)\,dx\Big) \\
&\qquad = \gamma\int_{\mathbb{R}^N} f(x,u(x))v(x)\,dx + \int_{\mathbb{R}^N} |u|^{2^*_\alpha-2}uv\,dx \quad \text{for any } v \in H_\lambda.
\end{aligned}$$

Theorem 3.13. *Assume that conditions* (M_3), (M_4), (M_5), (V_1), (V_2), (f_6) *and* (f_7) *hold. Then for all* $\lambda > 0$, *there exists* $\gamma_* > 0$ *such that for all* $\gamma \ge \gamma_*$, *equation* (3.40) *has a nontrivial solution* $u_\gamma \in H_\lambda$ *satisfying* $\lim_{\gamma\to\infty} \|u_\gamma\|_{H_\lambda} = 0$.

Proof. We need some lemmas:

Lemma 3.14. *Suppose that* (M_3), (V_1) *and* (f_6) *hold. Then for all* $\lambda, \gamma > 0$, *there exist positive constants* $\mathfrak{t}_0, \alpha_0$ *such that* $\mathcal{I}(u) \ge \alpha_0$ *for all* $u \in H_\lambda$ *with* $\|u\|_{H_\lambda} = \mathfrak{t}_0$.

Proof. By (2.1) and using the Hölder inequality, we have

$$\begin{aligned}(3.42)\qquad &\int_{\mathbb{R}^N} g(x)|u(x)|^q\,dx\\ &\le \Big(\int_{\mathbb{R}^N} |g(x)|^{2_\alpha^*/(2_\alpha^*-q)}\,dx\Big)^{(2_\alpha^*-q)/2_\alpha^*}\Big(\int_{\mathbb{R}^N} |u(x)|^{2_\alpha^*}\,dx\Big)^{q/2_\alpha^*}\\ &= \|g\|_{L^{2_\alpha^*/(2_\alpha^*-q)}(\mathbb{R}^N)}\|u\|^q_{L^{2_\alpha^*}(\mathbb{R}^N)} \le \|g\|_{L^{2_\alpha^*/(2_\alpha^*-q)}(\mathbb{R}^N)} S^q_{2_\alpha^*,\lambda}\|u\|^q_{H_\lambda}.\end{aligned}$$

Then from (M_1), we have $\mathcal{M}(t) \ge \mathcal{M}(1)t^\eta$ for all $t \in [0,1]$. From (2.1) and (3.42), we obtain

$$\begin{aligned}(3.43)\qquad \mathcal{I}(u) &= \frac{1}{2}\mathcal{M}(\|u\|^2_{H_\lambda}) - \gamma\int_{\mathbb{R}^N} F(x,u)\,dx - \frac{1}{2_\alpha^*}\int_{\mathbb{R}^N}|u|^{2_\alpha^*}\,dx\\ &\ge \frac{\mathcal{M}(1)}{2}\|u\|^{2\eta}_{H_\lambda} - \gamma\|g\|_{L^{2_\alpha^*/(2_\alpha^*-q)}(\mathbb{R}^N)}S^q_{2_\alpha^*,\lambda}\|u\|^q_{H_\lambda} - \frac{S^{2_\alpha^*}_{2_\alpha^*,\lambda}}{2_\alpha^*}\|u\|^{2_\alpha^*}_{H_\lambda}\\ &= \|u\|^{2\eta}_{H_\lambda}\Big(\frac{\mathcal{M}(1)}{2} - \gamma\|g\|_{L^{2_\alpha^*/(2_\alpha^*-q)}(\mathbb{R}^N)}S^q_{2_\alpha^*,\lambda}\|u\|^{q-2\eta}_{H_\lambda} - \frac{S^{2_\alpha^*}_{2_\alpha^*,\lambda}}{2_\alpha^*}\|u\|^{2_\alpha^*-2\eta}_{H_\lambda}\Big)\end{aligned}$$

for all $\|u\|_{H_\lambda} \le 1$. Let

$$h(t) = \frac{\mathcal{M}(1)}{2} - \gamma\|g\|_{L^{2_\alpha^*/(2_\alpha^*-q)}(\mathbb{R}^N)}S^q_{2_\alpha^*,\lambda}t^{q-2\eta} - \frac{S^{2_\alpha^*}_{2_\alpha^*,\lambda}}{2_\alpha^*}t^{2_\alpha^*-2\eta},\qquad t\ge 0.$$

We see that $\lim_{t\to 0^+} h(t) = \mathcal{M}(1)/2 > 0$ exists, hence there exists $\mathfrak{t}_0 > 0$ small enough such that $h(t) \ge \mathcal{M}(1)/4 > 0$ for all $0 \le t \le \mathfrak{t}_0$. Taking $\|u\|_{H_\lambda} = \mathfrak{t}_0$, we obtain

$$\mathcal{I}(u) \ge \frac{\mathcal{M}(1)}{4}{\mathfrak{t}_0}^{2\eta} = \alpha_0 > 0. \ \blacksquare$$

LEMMA 3.15. *Suppose that (M_3), (V_1) and (f_7) hold. Then there exists $v \in H_\lambda$ with $\|v\|_{H_\lambda} > \mathfrak{t}_0$ such that $\mathcal{I}(v) < 0$, where $\mathfrak{t}_0 > 0$ is defined in Lemma* 3.14.

Proof. From (M_3), we have $\mathcal{M}(\zeta) \le \mathcal{M}(1)\zeta^\eta$ for all $\zeta \ge 1$. For all $u \in H_\lambda$ with $\|u\|_{H_\lambda} = 1$, from (f_7) we have $F \ge 0$, and then for all $t \ge 1$,

$$(3.44)\qquad \mathcal{I}(tu) \le \frac{1}{2}\mathcal{M}(t^2\|u\|^2_{H_\lambda}) - \frac{t^{2_\alpha^*}}{2_\alpha^*}\int_{\mathbb{R}^N}|u|^{2_\alpha^*}\,dx.$$

By the definition of the best constant,

$$S_{2_\alpha^*,\lambda} = \sup_{u\in H_\lambda,\,u\ne 0}\frac{\|u\|_{L^{2_\alpha^*}(\mathbb{R}^N)}}{\|u\|_{H_\lambda}}.$$

For any $0 < \varepsilon < S_{2^*_\alpha,\lambda}$, we choose $u \in H_\lambda$ such that $\|u\|_{H_\lambda} = 1$ and $\|u\|_{L^{2^*_\alpha}(\mathbb{R}^N)} > S_{2^*_\alpha,\lambda} - \varepsilon > 0$. Then from (3.44) we get

$$\mathcal{I}(tu) \le \frac{\mathcal{M}(1)}{2} t^{2\eta} - \frac{(S_{2^*_\alpha,\lambda} - \varepsilon)^{2^*_\alpha}}{2^*_\alpha} t^{2^*_\alpha} \to -\infty$$

as $t \to \infty$, since $2^*_\alpha > 2\eta$. Setting $v = \rho_0 u$ with $\rho_0 > \mathfrak{t}_0 > 0$ large enough, we have $\mathcal{I}(v) < 0$ and $\|v\|_{H_\lambda} > \mathfrak{t}_0$. ∎

Now, we discuss the compactness property for the function $\mathcal{I}$, given by a (PS) sequence at a suitable level. We fix $\gamma > 0$ and set

$$c_\gamma = \inf_{\gamma^* \in \Gamma} \max_{t \in [0,1]} \mathcal{I}(\gamma^*(t)), \tag{3.45}$$

where $\Gamma = \{\gamma^* \in C([0,1], X) : \gamma^*(0) = 0,\ \mathcal{I}(\gamma^*(1)) < 0\}$. Then $c_\gamma > 0$ by Lemma 3.14. Moreover, we have

LEMMA 3.16. *Suppose that M satisfies (M_3) and f satisfies (f_6) and (f_7). Then $\lim_{\gamma\to\infty} c_\gamma = 0$, where c_γ is given in* (3.45).

Proof. For v in Lemma 3.15, we have $\lim_{t\to\infty} \mathcal{I}(tv) = -\infty$. Then by the maximum principle, there exists $t_\gamma > 0$ such that $\mathcal{I}(t_\gamma v) = \max_{t\ge 0} \mathcal{I}(tv)$, and so $\langle \mathcal{I}'(t_\gamma v), t_\gamma v\rangle = 0$. Therefore,

$$M(\|t_\gamma v\|^2_{H_\lambda})\|t_\lambda v\|^2_{H_\lambda} = \gamma t_\gamma \int_{\mathbb{R}^N} f(x, t_\gamma v) v\,dx + t_\gamma^{2^*_\alpha} \int_{\mathbb{R}^N} |v|^{2^*_\alpha}\,dx. \tag{3.46}$$

We show that $\{t_\gamma\}_\gamma$ is bounded. Without loss of generality, we assume that $t_\gamma \ge 1$ for all $\gamma > 0$. By (M_3), $\mathcal{M}(t) \le \mathcal{M}(1)t^\eta$ for all $t \ge 1$. Then from (3.46) we get

$$\begin{aligned}\eta\mathcal{M}(1)\|t_\lambda v\|^{2\eta}_{H_\lambda} &\ge \eta\mathcal{M}(\|t_\lambda v\|^2_{H_\lambda}) \ge M(\|t_\gamma v\|^2_{H_\lambda})\|t_\lambda v\|^2_{H_\lambda}\\ &= \gamma t_\gamma \int_{\mathbb{R}^N} f(x, t_\gamma v) v\,dx + t_\gamma^{2^*_\alpha} \int_{\mathbb{R}^N} |v|^{2^*_\alpha}\,dx \ge t_\gamma^{2^*_\alpha} \int_{\mathbb{R}^N} |v|^{2^*_\alpha}\,dx,\end{aligned}$$

since $uf(x,u) > 0$ for all $u \in \mathbb{R} \setminus \{0\}$. From

$$0 < (\rho_0(S_{2^*_\alpha,\lambda} - \varepsilon))^{2^*_\alpha} < \int_{\mathbb{R}^N} |v|^{2^*_\alpha}\,dx \le (\rho_0 S_{2^*_\alpha,\lambda})^{2^*_\alpha} < \infty$$

and $2^*_\alpha > 2\eta$, $\{t_\gamma\}_\gamma$ must be bounded.

Fix any sequence $\{\gamma_n\}_n$ with $\gamma_n \to \infty$ as $n \to \infty$; then there exists a subsequence, still denoted by $\{\gamma_n\}_n$, and $t_0 \ge 0$ such that $t_{\gamma_n} \to t_0$. Hence, there is a constant $C > 0$ such that

$$M(\|t_{\gamma_n} v\|^2_{H_\lambda})\|t_{\gamma_n} v\|^2_{H_\lambda} \le C \quad \text{for all } n. \tag{3.47}$$

We claim $t_0 = 0$. Indeed, if $t_0 > 0$, by Lebesgue's dominated convergence theorem we have

$$\gamma_n t_{\gamma_n} \int_{\mathbb{R}^N} f(x, t_{\gamma_n} v) v\,dx + t_{\gamma_n}^{2^*_\alpha} \int_{\mathbb{R}^N} |v|^{2^*_\alpha}\,dx \to \infty$$

as $n \to \infty$. This contradicts (3.47), and we get $t_\gamma \to 0$ as $\gamma \to \infty$. Let $\overline{\gamma^*} = tv$. Then $\overline{\gamma^*} \in \Gamma$, and we have

$$0 < c_\gamma \le \max_{t\ge 0} \mathcal{I}(\overline{\gamma^*}(t)) = \mathcal{I}(t_\gamma v) \le \tfrac{1}{2}\mathcal{M}(\|t_\gamma v\|^2_{H_\lambda}).$$

Since $\mathcal{M}$ is a continuous function and $\|t_\gamma v\|_{H_\lambda} \to 0$ as $\gamma \to \infty$, we have $c_\gamma \to 0$ as $\gamma \to \infty$. ∎

LEMMA 3.17. *Assume that f satisfies (f_6) and (f_7) and M satisfies (M_3)–(M_5). Then there exists $\gamma_* > 0$ such that $\mathcal{I}$ satisfies the (PS) condition on H_λ at level c_γ for all $\gamma \ge \gamma_*$ and all $\lambda > 0$.*

Proof. For any sequence $\{u_n\}_n \subset H_\lambda$ such that $\mathcal{I}(u_n) \to c_\gamma$ and $\mathcal{I}'(u_n) \to 0$ as $n \to \infty$ in H^*_λ, where H^*_λ is the dual space of H_λ, there exists $C > 0$ such that

$$|\langle \mathcal{I}'(u_n), u_n\rangle| \le C\|u_n\|_{H_\lambda} \quad \text{and} \quad |\mathcal{I}(u_n)| \le C \quad \text{for all } n. \tag{3.48}$$

Now, we consider two cases:

CASE 1: $\inf_n \|u_n\|_{H_\lambda} = d > 0$. We first show that $\{u_n\}$ is bounded. Denote by $\mathfrak{r} = \mathfrak{r}(d)$ the number corresponding to $\sigma = d^2$ in (M_4), so that

$$M(\|u_n\|^2_{H_\lambda}) \ge \mathfrak{r} \quad \text{for all } n. \tag{3.49}$$

By (M_3), (M_4) and (f_7), and since $2\eta < \nu < 2^*_\alpha$, we have

$$\begin{aligned} C + C\|u_n\|_{H_\lambda} &\ge \mathcal{I}(u_n) - \frac{1}{\nu}\langle \mathcal{I}'(u_n), u_n\rangle \\ &= \frac{1}{2}\mathcal{M}(\|u_n\|^2_{H_\lambda}) - \frac{1}{\nu}M(\|u_n\|^2_{H_\lambda})\|u_n\|^2_{H_\lambda} \\ &\quad + \frac{\gamma}{\nu}\int_{\mathbb{R}^N} (f(x,u_n)u_n - \nu F(x,u_n))\,dx + \left(\frac{1}{\nu} - \frac{1}{2^*_\alpha}\right)\int_{\mathbb{R}^N} |u_n|^{2^*_\alpha}\,dx \\ &\ge \left(\frac{1}{2\eta} - \frac{1}{\nu}\right)\mathfrak{r}\|u_n\|^2_{H_\lambda}. \end{aligned} \tag{3.50}$$

This implies $\{u_n\}$ is a bounded sequence. As H_λ is Hilbert, it is reflexive, so by [3, Theorem 4.9] there exist $u_\gamma \in H_\lambda$ and $\alpha_\gamma \ge 0$ such that up to a subsequence, still denoted by $\{u_n\}_n$, we have

$$\begin{aligned} &u_n \to u_\gamma \quad \text{weakly in } H_\lambda \cap L^{2^*_\alpha}(\mathbb{R}^N), \\ &u_n \to u_\gamma \quad \text{a.e. in } \mathbb{R}^N, \\ &\|u_n\|_{H_\lambda} \to \alpha_\gamma, \quad \|u_n - u_\gamma\|_{L^{2^*_\alpha}(\mathbb{R}^N)} \to \delta_\gamma, \\ &|u_n|^{2^*_\alpha - 2}u_n \to |u_\gamma|^{2^*_\alpha-2}u_\gamma \quad \text{weakly in } L^{(2^*_\alpha)'}(\mathbb{R}^N), \end{aligned} \tag{3.51}$$

where $(2^*_\alpha)' = 2^*_\alpha/(2^*_\alpha - 1)$. Since $\inf_n \|u_n\|_{H_\lambda} = d > 0$, we have $\alpha_\gamma > 0$, and $M(\|u_n\|^2_{H_\lambda}) \to M(\alpha^2_\gamma) > 0$ as $n \to \infty$, thanks to (M_4) and continuity of M.

We claim that $\lim_{\gamma\to\infty}\alpha_\gamma = 0$. Suppose that, on the contrary, there exists a sequence $\gamma_k \to \infty$ as $k \to \infty$ such that $\alpha_{\gamma_k} \to \alpha > 0$ as $k \to \infty$. From (3.50), we get

$$c_{\gamma_k} \geq \left(\frac{1}{2\eta} - \frac{1}{\nu}\right) M(\alpha_{\gamma_k}^2)\alpha_{\gamma_k}^2.$$

Taking the limit of both sides and using Lemma 3.16, we get

$$0 \geq \left(\frac{1}{2\eta} - \frac{1}{\nu}\right) M(\alpha^2)\alpha^2 > 0.$$

This is impossible. Thus

$$\lim_{\gamma\to\infty}\alpha_\gamma = 0. \tag{3.52}$$

Since $u_n \to u_\gamma$ weakly in H_λ, we get

$$\|u_\gamma\|_{H_\lambda} \leq \lim_{n\to\infty}\|u_n\|_{H_\lambda} = \alpha_\gamma. \tag{3.53}$$

Note that H_λ embeds continuously into $L^{2^*_\alpha}(\mathbb{R}^N)$. Then from (3.52) and (3.53), we deduce

$$\lim_{\gamma\to\infty}\|u_\gamma\|_{L^{2^*_\alpha}(\mathbb{R}^N)} = \lim_{\gamma\to\infty}\|u_\gamma\|_{H_\lambda} = 0. \tag{3.54}$$

Fix $\phi \in H_\lambda$, and denote by B_ϕ the following linear function on H_λ:

$$B_\phi(\varphi) = \int_{\mathbb{R}^N}(I-\Delta)^{\alpha/2}\phi(I-\Delta)^{\alpha/2}\varphi\,dx + \lambda\int_{\mathbb{R}^N}V(x)\phi(x)\varphi(x)\,dx.$$

By the Hölder inequality, B_ϕ is continuous on H_λ and

$$|B_\phi(\varphi)| \leq \|\phi\|_{H_\lambda}\|\varphi\|_{H_\lambda} \quad \text{for all } \varphi \in H_\lambda. \tag{3.55}$$

Since u_n converges weakly to u_γ in H_λ, we have

$$\lim_{n\to\infty}B_{u_\gamma}(u_n - u_\gamma) = 0. \tag{3.56}$$

From (3.55) and since $\{u_n\}$ is a bounded sequence in H_λ, the sequence B_{u_n} is bounded in H_λ^*. As H_λ is reflexive, so is H_λ^*. Thus, up to taking a subsequence, we can assume that there exists $\mathbb{B} \in H_\lambda^*$ such that $B_{u_n} \to \mathbb{B}$ weakly in H_λ^*. As $H_\lambda^{**} = H_\lambda$, where H_λ^{**} is the dual space of H_λ^*, we get

$$\lim_{n\to\infty}\mathbb{B}_{u_n}(\varphi) = \mathbb{B}(\varphi) \quad \text{for all } \varphi \in H_\lambda.$$

From (3.41), we have $\lim_{n\to\infty}\int_{\mathbb{R}^N} f(x,u_n)\varphi\,dx = \int_{\mathbb{R}^N} f(x,u_\gamma)\varphi\,dx$ for all $\varphi \in H_\lambda$. Since $\mathcal{I}'(u_n) \to 0$ in H_λ^*, we get $\lim_{n\to\infty}\langle\mathcal{I}'(u_n),\varphi\rangle = 0$ for all $\varphi \in H_\lambda$. Combining this with (3.51), we obtain

$$M(\alpha_\gamma^2)\mathbb{B}(\varphi) = \gamma\int_{\mathbb{R}^N} f(x,u_\gamma)\varphi\,dx + \int_{\mathbb{R}^N}|u_\gamma|^{2^*_\alpha-2}u_\gamma\varphi\,dx \tag{3.57}$$

for all $\varphi \in H_\lambda$. Taking $\varphi = u_\gamma$ in (3.57), we get

$$M(\alpha_\gamma^2)\mathbb{B}(u_\gamma) = \gamma \int_{\mathbb{R}^N} f(x, u_\gamma)u_\gamma\, dx + \int_{\mathbb{R}^N} |u_\gamma|^{2^*_\alpha}\, dx. \tag{3.58}$$

By (f_7) and (3.58), we have $\mathbb{B}(u_\gamma) \geq 0$. Since $\{u_n\}$ is a (PS) sequence, we deduce from (3.51) that

$$\begin{aligned}
o(1) &= \langle \mathcal{I}'(u_n) - \mathcal{I}'(u_\gamma), u_n - u_\gamma\rangle \\
&= M(\|u_n\|^2_{H_\lambda})B_{u_n}(u_n - u_\gamma) - M(\|u\|^2_{H_\lambda})B_{u_\gamma}(u_n - u_\gamma) \\
&\quad - \gamma \int_{\mathbb{R}^N} [f(x, u_n) - f(x, u_\gamma)](u_n - u_\gamma)\, dx \\
&\quad - \int_{\mathbb{R}^N} (|u_n|^{2^*_\alpha - 2}u_n - |u_\gamma|^{2^*_\alpha - 2}u_\gamma)(u_n - u_\gamma)\, dx \\
&= M(\|u_n\|^2_{H_\lambda})\big(B_{u_n}(u_n - u_\gamma) - B_{u_\gamma}(u_n - u_\gamma)\big) \\
&\quad + (M(\|u_n\|^2_{H_\lambda}) - M(\|u\|^2_{H_\lambda}))B_{u_\gamma}(u_n - u_\gamma) \\
&\quad - \gamma \int_{\mathbb{R}^N} [f(x, u_n) - f(x, u_\gamma)](u_n - u_\gamma)\, dx \\
&\quad - \int_{\mathbb{R}^N} (|u_n|^{2^*_\alpha - 2}u_n - |u_\gamma|^{2^*_\alpha - 2}u_\gamma)(u_n - u_\gamma)\, dx \\
&= M(\|u_n\|^2_{H_\lambda})\big(B_{u_n}(u_n - u_\gamma) - B_{u_\gamma}(u_n - u_\gamma)\big) \\
&\quad - \|u_n - u_\gamma\|^{2^*_\alpha}_{L^{2^*_\alpha}(\mathbb{R}^N)} + o(1),
\end{aligned} \tag{3.59}$$

for n large enough. Here, we use the following properties:

(i) $\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{2^*_\alpha - 2}u_n u_\gamma\, dx = \int_{\mathbb{R}^N} |u_\gamma|^{2^*_\alpha}\, dx$ since $|u_n|^{2^*_\alpha - 2}u_n \to |u_\gamma|^{2^*_\alpha - 2}u_\gamma$ weakly in $L^{(2^*_\alpha)'}(\mathbb{R}^N)$;

(ii) $\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_\gamma|^{2^*_\alpha - 2}u_\gamma u_n\, dx = \int_{\mathbb{R}^N} |u_\gamma|^{2^*_\alpha}\, dx$ since $u_n \to u_\gamma$ weakly in $L^{2^*_\alpha}(\mathbb{R}^N)$ and $|u_\gamma|^{2^*_\alpha - 2}u_\gamma \in L^{(2^*_\alpha)'}(\mathbb{R}^N)$;

(iii) by the Brezis–Lieb Lemma,

$$\|u_n\|^{2^*_\alpha}_{L^{2^*_\alpha}(\mathbb{R}^N)} = \|u_n - u_\gamma\|^{2^*_\alpha}_{L^{2^*_\alpha}(\mathbb{R}^N)} + \|u_\gamma\|^{2^*_\alpha}_{L^{2^*_\alpha}(\mathbb{R}^N)} + o(1)$$

when n large enough;

(iv) $\lim_{n\to\infty} \int_{\mathbb{R}^N} [f(x, u_n) - f(x, u_\gamma)](u_n - u_\gamma)\, dx = 0$.

Now, we prove (iv). By the Hölder inequality,

$$\begin{aligned}
\Big|\int_{\mathbb{R}^N} f(x, u_n)(u_n - u_\gamma)\, dx\Big| &\leq \int_{\mathbb{R}^N} g(x)|u_n|^{q-1}|u_n - u_\gamma|\, dx \\
&\leq \Big(\int_{\mathbb{R}^N} g(x)|u_n|^q\, dx\Big)^{(q-1)/q}\Big(\int_{\mathbb{R}^N} g(x)|u_n - u_\gamma|^q\, dx\Big)^{1/q}.
\end{aligned} \tag{3.60}$$

Again by the Hölder inequality,

$$\int_{\mathbb{R}^N} g(x)|u_n|^q\,dx \le S^q_{2^*_\alpha,\lambda}\|g\|_{L^{2^*_\alpha/(2^*_\alpha-q)}(\mathbb{R}^N)}\|u_n\|^q_{H_\lambda} < \infty. \tag{3.61}$$

Now, we prove

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} g(x)|u_n-u_\gamma|^q\,dx = 0. \tag{3.62}$$

Since $g \in L^{\frac{2^*_\alpha}{2^*_\alpha-q}}(\mathbb{R}^N)$, for any $\varepsilon > 0$ there exists $R_0 > 0$ such that

$$\int_{\mathbb{R}^N\setminus B_R} |g(x)|^{\frac{2^*_\alpha}{2^*_\alpha-q}}\,dx < \varepsilon \quad \text{for all } R \ge R_0, \tag{3.63}$$

where B_R is an open ball in $\mathbb{R}^N$. By compact embedding on the bounded domain B_R, $\{u_n\}_n$ has a subsequence, still denoted by $\{u_n\}$, which converges to u_γ in $L^q(B_R)$. Up to a subsequence, we may assume that $u_n \to u$ a.e. in B_R. Note that $\{u_n\}$ is bounded in $L^{p^*_s}(\mathbb{R}^N)$. For any measurable $U \subset B_R$, by the Hölder inequality,

$$\begin{aligned}\int_U g(x)|u_n-u_\gamma|^q\,dx &\le \|g\|_{L^{2^*_\alpha/(2^*_\alpha-q)}(U)}\|u_n-u_\gamma\|^q_{L^{2^*_\alpha}(\mathbb{R}^N)}\\ &\le \kappa_*\|g\|_{L^{2^*_\alpha/(2^*_\alpha-q)}(U)},\end{aligned}$$

where κ_* is a suitable constant. Therefore, $\{g(x)|u_n-u_\gamma|^q\}_n$ is equi-integrable in B_R, thanks to $g \in L^{2^*_\alpha/(2^*_\alpha-q)}(\mathbb{R}^N)$. By the Vitali convergence theorem,

$$\lim_{n\to\infty}\Big|\int_{B_R} g(x)|u_n-u_\gamma|^q\,dx\Big| = 0. \tag{3.64}$$

and

$$\begin{aligned}&\Big|\int_{\mathbb{R}^N} g(x)|u_n-u_\gamma|^q\,dx\Big|\\ &\le \Big|\int_{B_R} g(x)|u_n-u_\gamma|^q\,dx\Big| + \Big|\int_{\mathbb{R}^N\setminus B_R} g(x)|u_n-u_\gamma|^q\,dx\Big|\\ &\le \Big|\int_{B_R} g(x)|u_n-u_\gamma|^q\,dx\Big|\\ &\quad + \Big(\int_{\mathbb{R}^N\setminus B_R}|g|^{2^*_\alpha/(2^*_\alpha-q)}\,dx\Big)^{(2^*_\alpha-q)/2^*_\alpha}\Big(\int_{\mathbb{R}^N\setminus B_R}|u_n-u_\gamma|^{2^*_\alpha}\,dx\Big)^{q/2^*_\alpha}.\end{aligned} \tag{3.65}$$

From (3.63) to (3.65), we obtain (3.62).

Collecting (3.60)–(3.62) and the fact that $\{\|u_n\|_{H_\lambda}\}$ is a bounded sequence, we deduce

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} f(x,u_n)(u_n-u_\gamma)\,dx = 0.$$

Similarly,

$$\lim_{n\to\infty}\int\limits_{\mathbb{R}^N} f(x,u_\gamma)(u_n-u_\gamma)\,dx=0.$$

Thus we obtain (iv).

From (3.59), we see that

(3.66)

$$\lim_{n\to\infty} M(\|u_n\|^2_{H_\lambda})\big(B_{u_n}(u_n-u_\gamma)-B_{u_\gamma}(u_n-u_\gamma)\big)=\lim_{n\to\infty}\|u_n-u_\gamma\|^{2^*_\alpha}_{L^{2^*_\alpha}(\mathbb{R}^N)}.$$

Apply the Brezis–Lieb Lemma and (3.51), we obtain

$$\begin{aligned} c_\gamma+o(1)&=\mathcal{I}(u_n)-\frac{1}{\nu}\langle\mathcal{I}'(u_n),u_n-u_\gamma\rangle\geq\left(\frac{1}{\nu}-\frac{1}{2^*_\alpha}\right)\|u_n-u_\gamma\|^{2^*_\alpha}_{L^{2^*_\alpha}(\mathbb{R}^N)}\\ &=\left(\frac{1}{\nu}-\frac{1}{2^*_\alpha}\right)(\delta_\gamma^{2^*_\alpha}+\|u_\gamma\|^{2^*_\alpha}_{L^{2^*_\alpha}(\mathbb{R}^N)})+o(1),\end{aligned}$$

which together with Lemma 3.16 and (3.54) implies that

$$\lim_{\gamma\to\infty}\delta_\gamma=0. \tag{3.67}$$

We have

$$B_{u_n}(u_n-u_\gamma)-B_{u_\gamma}(u_n-u_\gamma)=\|u_n-u_\gamma\|^2_{H_\lambda}\geq S^{-2}_{2^*_\alpha}\|u_n-u_\gamma\|^2_{L^{2^*_\alpha}(\mathbb{R}^N)}. \tag{3.68}$$

On combining (3.51) with (3.66) and (3.68), we obtain

$$\delta_\gamma^{2^*_\alpha}\geq S^{-2}_{2^*_\alpha}M(\alpha_\gamma^2)\delta_\gamma^2. \tag{3.69}$$

We claim that there exists $\gamma_*>0$ such that $\delta_\gamma=0$ for all $\gamma\geq\gamma_*$. Indeed, otherwise there exists a sequence $\{\gamma_k\}_k$ with $\gamma_k\to\infty$ as $k\to\infty$ such that $\delta_{\gamma_k}>0$ for all $k\in\mathbb{N}$. From (3.66), noting that $B_{u_\gamma}(u_n-u_\gamma)\to 0$ as $n\to\infty$, and

$$B_{u_n}(u_n)=\|u_n\|^2_{H_\lambda}\to\alpha_\gamma^2$$

as $n\to\infty$, and $B_{u_n}(u_\gamma)\to\mathbb{B}(u_\gamma)\geq 0$, we have

$$M(\alpha^2_{\gamma_k})(\alpha^2_{\gamma_k}-\mathbb{B}(u_{\gamma_k}))=\delta_\gamma^{2^*_\alpha}. \tag{3.70}$$

Collecting (3.69) and (3.70), we get

$$(\delta_{\gamma_k}^{2^*_\alpha})^{(2^*_\alpha-2)/2^*_\alpha}=\big(M(\alpha^2_{\gamma_k})(\alpha^2_{\gamma_k}-\mathbb{B}(u_{\gamma_k}))\big)^{(2^*_\alpha-2)/2^*_\alpha}\geq S^{-2}_{2^*_\alpha}M(\alpha^2_\gamma). \tag{3.71}$$

This implies

$$\alpha_{\gamma_k}^{2(2^*_\alpha-2)/2^*_\alpha}\geq(\alpha^2_{\gamma_k}-\mathbb{B}(u_{\gamma_k}))^{(2^*_\alpha-2)/2^*_\alpha}\geq S^{-2}_{2^*_\alpha}M(\alpha^2_\gamma)^{2/2^*_\alpha}.$$

Thus,

$$\alpha_{\gamma_k}^{2^*_\alpha-2\eta}\geq(S^{-2}_{2^*_\alpha})^{2^*_\alpha/2}\frac{M(\alpha^2_{\gamma_k})}{\alpha_{\gamma_k}^{2(\eta-1)}}. \tag{3.72}$$

From (3.54), (3.72) and (M_5), we get a contradiction.

Thus there exists $\gamma_* > 0$ such that $\delta_\gamma = 0$ for all $\gamma \geq \gamma_*$. This implies that

$$\lim_{n\to\infty} \|u_n - u_\gamma\|^{2^*_\alpha}_{L^{2^*_\alpha}(\mathbb{R}^N)} = 0$$

for all $\gamma \geq \gamma_*$. We have

$$B_{u_n}(u_n - u_\gamma) - B_{u_\gamma}(u_n - u_\gamma) = \|u_n - u_\gamma\|^2_{H_\lambda}. \tag{3.73}$$

Combining (3.73) with (3.66), (3.59) and (M_4), we get

$$\lim_{n\to\infty} \|u_n - u_\gamma\|_{H_\lambda} = 0,$$

which implies $u_n \to u_\gamma$ strongly in H_λ for all $\gamma \geq \gamma_*$.

Case 2: $\inf_n \|u_n\|_{H_\lambda} = 0$. Either 0 is an accumulation point of the sequence $\{u_n\}_n$ and so there exists a subsequence of $\{u_n\}_n$ strongly converging to 0, or 0 is an isolated point of the sequence $\{u_n\}_n$ and so there exists a subsequence, still denoted by $\{u_n\}_n$, such that $\inf_n \|u_n\|_{H_\lambda} > 0$. In the former case we are done, while in the latter we can proceed as in Case 1. ∎

By Lemmas 3.14–3.17 and the Mountain Pass Theorem, there exist $\gamma_* > 0$ such that problem (3.40) has a solution $u_\gamma \in H_\lambda$ for all $\gamma \geq \gamma_*$ and all $\lambda > 0$. Now we show that $\lim_{\gamma\to\infty} \|u_\gamma\|_{H_\lambda} = 0$. Indeed, from $\mathcal{I}(u_\gamma) = c_\gamma$ and $\mathcal{I}'(u_\gamma) = 0$ in H^*_λ, we have

$$\begin{aligned} c_\gamma &= \mathcal{I}(u_\gamma) - \frac{1}{\nu}\langle \mathcal{I}'(u_\gamma), u_\gamma\rangle \\ &\geq \left(\frac{1}{2\eta} - \frac{1}{\nu}\right) M(\|u_\gamma\|^2_{H_\lambda})\|u_\gamma\|^2_{H_\lambda}. \end{aligned} \tag{3.74}$$

If $\lim_{\gamma\to\infty} \|u_\gamma\|_{H_\lambda} = a_0 > 0$, letting $\gamma \to \infty$ on both sides of (3.74) yields

$$0 \geq \left(\frac{1}{2\eta} - \frac{1}{\nu}\right) M(a_0)a_0 > 0,$$

a contradiction, so $\lim_{\gamma\to\infty} \|u_\gamma\|_{H_\lambda} = 0$. ∎

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Nguyen Van Thin
Shandong University
Department of Mathematics
Jinan City, Shandong, P.R. China
and
Thai Nguyen University of Education
Department of Mathematics
Thai Nguyen City, Thai Nguyen, Viet Nam
E-mail: thinmath@gmail.com