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Ioannis K. Argyros (Lawton, OK)
Santhosh George (Karnataka)

EXPANDING THE APPLICABILITY OF NEWTON'S METHOD AND OF A ROBUST MODIFIED NEWTON'S METHOD

Abstract. Newton's method cannot be used to approximate a solution of a nonlinear equation when the derivative of the function is singular or almost singular. To overcome this problem a modified Newton's method may be used. The Newton–Kantorovich theorem is used to show its convergence. The convergence domain of this method is small in general. In the present study, we show how to expand the convergence domain of Newton's method and the modified Newton's method by using the center Lipschitz condition and more precise convergence domains than in earlier studies. Numerical examples are also presented.

1. Introduction. Let $F : I \subset \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function. Newton's method defined for each $n = 0, 1, 2, \dots$ by

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \tag{1.1}$$

where $x_0 \in I$ is an initial point, is undoubtedly the most popular method for generating a sequence $\{x_n\}$ approximating a solution x^* of the equation

$$F(x) = 0. \tag{1.2}$$

Newton's method converges quadratically to x^* provided that x_0 is sufficiently close to the solution and $F'(x_n)$ is non-singular or almost non-singular at every step [1]–[20]. In cases where this is not true the following modified Newton's method has been suggested and studied in [19]:

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Algorithm 1.1

$$x_{n+1} = x_n - (F'(x_n) + \alpha_n F(x_n))^{-1} F(x_n),$$

where α_n, α are chosen to force convergence of the algorithm and

$$|\alpha_n| \leq \alpha < +\infty. \tag{1.3}$$

If $\alpha_n = 0$, then Algorithm 1.1 reduces to Newton's method. A semi-local convergence analysis was presented in [19] using the Newton–Kantorovich theorem [15] and the assumption

$$|F(x_0)| > \varepsilon, \tag{1.4}$$

where $\varepsilon > 0$ is the machine precision. It was also noted in [19] that the convergence domain of Algorithm 1.1 is small in general.

In the present paper we extend the convergence domain for both methods. Our new idea uses the center Lipschitz condition as well as more precise domains where the iterates lie. This way we obtain weaker Lipschitz conditions leading to weaker sufficient convergence criteria, tighter error bounds on the distances involved and an at least as precise information on the location of the solution x^*. These advantages are also obtained under the same or less computational cost on the Lipschitz constants involved. Moreover, we obtain a wider choice of sequences $\{\alpha_n\}$ and parameter α.

The rest of the paper is organized as follows. Section 2 contains the semi-local convergence analysis for both methods. The numerical examples are presented in Section 3.

2. Semi-local convergence. We first restate the classical form of the Newton–Kantorovich theorem to compare our results with the corresponding ones in [19].

Theorem 2.1 ([15]). *Let $F : I \subset \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function, $x_0 \in I$ with $F'(x_0) \neq 0$. Suppose that there exist $\beta > 0$, $\eta \geq 0$ and $L > 0$ such that*

$$|F'(x_0)^{-1}| \leq \beta, \tag{2.1}$$

$$|F'(x_0)^{-1} F(x_0)| \leq \eta, \tag{2.2}$$

$$|F'(x) - F'(y)| \leq L|x - y| \quad \text{for all } x, y \in I, \tag{2.3}$$

$$I^* := [x_0 - 2\eta, x_0 + 2\eta] \subseteq I. \tag{2.4}$$

Further, suppose that

$$H := \beta L \eta \leq 1/2. \tag{2.5}$$

Then the sequence $\{x_n\}$ generated by Newton's method is well defined in the interval I, remains in I^* for all $n = 0, 1, 2, \ldots$ and converges to a unique solution $x^* \in I$ of the equation $F(x) = 0$. Moreover, the following estimates *hold:*

$$|x_{n+1} - x_n| \le t_{n+1} - t_n, \tag{2.6}$$

$$|x^* - x_n| \le t^* - t_n, \tag{2.7}$$

for each $n = 0, 1, 2, \ldots,$ *where the sequence* $\{t_n\}$ *is defined by*

$$t_0 = 0, \quad t_1 = \eta, \quad t_{n+2} = t_{n+1} + \frac{\beta L(t_{n+1} - t_n)^2}{2(1 - \beta L t_{n+1})}, \quad n = 0, 1, 2, \ldots, \tag{2.8}$$

and $t^* = \lim_{n\to\infty} t_n = \frac{1-\sqrt{1-\beta L\eta}}{\beta L}$. ■

Relying on Theorem 2.1 the following result was shown in [19].

THEOREM 2.2. *Suppose that the hypotheses of Theorem* 2.1 *hold and there exists* $\alpha \ge 0$ *such that* (1.3) *holds and*

$$\alpha\eta \le \tfrac{1}{2} H. \tag{2.9}$$

Then the sequence $\{x_n\}$ *generated by Algorithm* 1.1 *converges to a solution* $x^* \in I$ *of* $F(x) = 0$.

Next, we show how both the preceding results can be improved without additional cost on the parameters β, L and η. Indeed in view of condition (2.3) there exists $L_0 > 0$ such that

$$|F'(x) - F'(x_0)| \le L_0|x - x_0| \quad \text{for each } x \in I. \tag{2.10}$$

Notice that

$$L_0 \le L \tag{2.11}$$

in general and L/L_0 can be arbitrarily large [4, 6]. Define

$$I_0 = I \cap [x_0 - 1/(\beta L_0), x_0 + 1/(\beta L_0)]. \tag{2.12}$$

Clearly,

$$I_0 \subset I. \tag{2.13}$$

Then, in view of (2.3), (2.12) and (2.13), there exists $L_1 > 0$ such that

$$|F'(x) - F'(y)| \le L_1|x - y| \quad \text{for all } x, y \in I_0. \tag{2.14}$$

We also have

$$L_1 \le L \tag{2.15}$$

in general, since (2.13) is true. Also, condition (2.3) implies (2.13) but not necessarily vice versa unless

$$2\beta L_0\eta \le 1. \tag{2.16}$$

The introduction of condition (2.13) was not possible in the proof of the Newton–Kantorovich theorem, since only condition (2.3) was used. Notice that (2.3) always implies (2.9) but not necessarily vice versa. Moreover, (2.9) (or (2.13)) is not an additional condition to (2.3), since in practice the computation of L requires the computation of L_0 (or L_1) as a special case. Furthermore, the introduction of condition (2.13) was not possible before in Theorem 2.1, since this condition depends on the interval I_0 which in turn depends on (2.9) (not introduced in Theorem 2.1). Finally, in the proof of the Newton–Kantorovich Theorem 2.1 or Theorem 2.2 the corresponding iterates remain in I_0, which is a more precise location than I^*.

For the convergence analysis that follows, it is convenient to introduce some parameters.

Let $\beta, L_0, L, L_1 > 0$ and $\eta \geq 0$. Suppose that $L_0\eta < 1$. Define

$$\delta = \frac{L_0}{L}, \qquad \delta_1 = \frac{L_0}{L_1}, \qquad q_0 = \frac{\beta L_0 \eta}{1 - \beta L_0 \eta},$$
$$\mu = \frac{2}{1 + \sqrt{1 + 8\delta}}, \qquad \mu_1 = \frac{2}{1 + \sqrt{1 + 8\delta_1}}, \qquad h = \beta L \eta, \qquad h_1 = \beta L_1 \eta.$$
$$p = \frac{1}{2}\left(1 + \frac{q_0}{2(1-\mu)}\right), \qquad p_1 = \frac{1}{2}\left(1 + \frac{q_0}{2(1-\mu_1)}\right), \qquad t^* = 2p\eta,\ t_1^* = 2p_1\eta,$$
$$\lambda = \frac{1}{8}(4\delta + \sqrt{\delta + 8\delta^2} + \sqrt{\delta}), \qquad \lambda_1 = \frac{1}{8}(4\delta_1 + \sqrt{\delta_1 + 8\delta_1^2} + \sqrt{\delta_1}).$$

The next two results were shown in [9] in the more general setting of a Banach space.

THEOREM 2.3. *Let $F : I \to \mathbb{R}$ be a continuously differentiable function, and let $x_0 \in I$ with $F'(x_0) \neq 0$. Suppose that* (2.1)–(2.3) *and* (2.10) *hold, $I_p := [x_0 - 2p\eta, x_0 + 2p\eta] \subseteq I$ and*

$$h \leq \frac{1}{2\lambda}. \tag{2.17}$$

Then the sequence $\{x_n\}$ generated by Newton's method is well defined in the interval I_p, remains in I_p for all $n = 0, 1, 2, \ldots$ and converges to a solution $x^* \in I_p$ of $F(x) = 0$. Moreover,

$$|x_{n+1} - x_n| \leq s_{n+1} - s_n, \tag{2.18}$$
$$|x_n - x^*| \leq s^* - s_n, \tag{2.19}$$

where the sequence $\{s_n\}$ is defined by

$$s_0 = 0, \quad s_1 = \eta, \quad s_2 = s_1 + \frac{\beta L_0 (s_1 - s_0)^2}{2(1 - \beta L_0 s_1)},$$
$$s_{n+2} = s_{n+1} + \frac{\beta L (s_{n+1} - s_n)^2}{2(1 - \beta L_0 s_{n+1})}, \qquad n = 0, 1, 2, \ldots,$$

and $s^* = \lim_{n\to\infty} s_n$.

THEOREM 2.4. *Let $F : I \to \mathbb{R}$ be a continuously differentiable function, and let $x_0 \in I$ with $F'(x_0) \neq 0$. Suppose that (2.1), (2.2), (2.10), and (2.14) hold, $I_{p_1} := [x_0 - 2p_1\eta, x_0 + 2p_1\eta] \subseteq I$ and*

$$h_1 \leq \frac{1}{2\lambda_1}. \tag{2.20}$$

Then the sequence $\{x_n\}$ generated by Newton's method is well defined in the interval I_{p_1}, remains in I_{p_1} for all $n = 0, 1, 2, \ldots$ and converges to a solution $x^* \in I_{p_1}$ of $F(x) = 0$. Moreover,

$$|x_{n+1} - x_n| \leq r_{n+1} - r_n, \tag{2.21}$$

$$|x_n - x^*| \leq r^* - r_n, \tag{2.22}$$

where the sequence $\{r_n\}$ is defined by

$$r_0 = 0, \quad r_1 = \eta, \quad r_2 = r_1 + \frac{\beta L_0 (r_1 - r_0)^2}{2(1 - \beta L_0 r_1)},$$

$$r_{n+2} = r_{n+1} + \frac{\beta L_1 (r_{n+1} - r_n)^2}{2(1 - \beta L_0 r_{n+1})}, \qquad n = 0, 1, 2, \ldots,$$

and $r^* = \lim_{n\to\infty} r_n$.

Proof. The conclusion is obtained from Theorem 2.3 (see [9]) by simply noting that (2.14) can be used instead of (2.3) (i.e., L_1 instead of L) in the proof and since $I_0 \subset I$ the iterates $\{x_n\}$ lie in I_0, which is more accurate location than I used in [19]. ■

REMARK 2.5. (a) It follows from (2.5), (2.15), (2.17) and (2.20) that

$$H \leq 1/2 \Longrightarrow h \leq 1/2 \Longrightarrow h_1 \leq 1/2. \tag{2.23}$$

(b) Concerning the error bounds, a simple inductive argument shows that

$$r_n \leq s_n \leq t_n, \tag{2.24}$$

$$r_{n+1} - r_n \leq s_{n+1} - s_n \leq t_{n+1} - t_n, \tag{2.25}$$

$$r^* \leq s^* \leq t^*. \tag{2.26}$$

(c) The uniqueness of the solution $x^* \in [x_0 - 2/(\beta L_0), x_0 + 2/(\beta L_0)]$ has been shown in [19]. We have improved so far the semi-local convergence criterion (2.5) for Newton's method (see also (2.23)). The semi-local convergence of Algorithm 1.1 is based on (2.5) and (2.9). Next, we show that (2.5) can be replaced by (2.17) (or (2.20)).

Let $q = 1 - 1/(2\lambda)$ and $q_1 = 1 - 1/(2\lambda_1)$. Consider the following *conditions* $(\mathcal{C})$:

$$\alpha\eta \leq \frac{h}{2\lambda} \quad \text{for } \lambda \geq 1, \tag{2.27}$$

or

$$\alpha\eta \le \frac{h}{2\lambda} \quad \text{and} \quad \alpha\eta \le \frac{1}{2}\left(\frac{1}{2\lambda} - 2\right) \quad \text{for } \lambda \le \frac{1}{4}, \tag{2.28}$$

$$\alpha\eta(1-2p) + h(3p-1) \le 1 + 2p, \tag{2.29}$$

$$2(1-\xi)(1+\alpha\eta)^2 + \left(\frac{\xi}{\lambda} - 2\right)(1+\alpha\eta) + \frac{1}{2\lambda} \le 0 \tag{2.30}$$

for some $\xi \ge 0$ such that

$$2\sqrt{\lambda(1-\lambda)} \le \xi < \min\{2\lambda, 1\} \quad \text{if } \lambda \le 1/4, \tag{2.31}$$

$$0 \le \xi < \min\{2\lambda, 1\} \quad \text{if } \lambda \ge 1, \tag{2.32}$$

or *conditions* $(\mathcal{C}_1)$:

$$\alpha\eta \le \frac{h_1}{2\lambda_1} \quad \text{for } \lambda_1 \ge 1, \tag{2.33}$$

$$\alpha\eta \le \frac{h_1}{2\lambda_1} \text{ and } \alpha\eta \le \frac{1}{2}\left(\frac{1}{2\lambda_1} - 2\right) \quad \text{for } \lambda_1 \le \frac{1}{4}, \tag{2.34}$$

$$\alpha\eta(1-2p_1) + h_1(3p_1-1) \le 1 + 2p_1, \tag{2.35}$$

$$2(1-\xi_1)(1+\alpha\eta)^2 + \left(\frac{\xi_1}{\lambda_1} - 2\right)(1+\alpha\eta) + \frac{1}{2\lambda_1} \le 0 \tag{2.36}$$

for some $\xi_1 \ge 0$ such that

$$2\sqrt{\lambda_1(1-\lambda_1)} \le \xi_1 < \min\{2\lambda_1, 1\} \quad \text{if } \lambda_1 \le 1/4, \tag{2.37}$$

or

$$0 \le \xi_1 < \min\{2\lambda_1, 1\} \quad \text{if } \lambda_1 \ge 1. \tag{2.38}$$

Then we can show the following semi-local convergence result for Algorithm 1.1 first under conditions $(\mathcal{C})$ corresponding to Theorem 2.2.

THEOREM 2.6. *Suppose that the hypotheses of Theorem* 2.3 *and conditions* $(\mathcal{C})$ *hold. Then the sequence* $\{x_n\}$ *generated by Algorithm* 1.1 *is well defined in* I_p*, remains in* I_p *for all* $n = 0, 1, 2, \dots$ *and converges to a solution* $x^* \in I_p$ *of* $F(x) = 0$.

Proof. We follow the proof in [19] until a certain point and then our proof changes. In view of the definition of Algorithm 1.1 and since x_1 is well defined by (2.1), by (2.2) we have, for $\eta_0 = \eta$, $\beta_0 = \beta$, $h_0 = h$,

$$|x_1 - x_0| \le \frac{|F'(x_0)^{-1}F(x_0)|}{1 + |\alpha_0|\,|F'(x_0)^{-1}F(x_0)|} \le \frac{\eta_0}{1+\alpha\eta_0}.$$

By the center-Lipschitz condition (2.10), (2.11) and (2.1) we get

$$|F'(x_1) - F'(x_0)| \le L_0|x_1 - x_0| \le \frac{L\eta_0}{1+\alpha\eta_0} < \frac{1}{\beta_0} = F'(x_0),$$

so

$$F'(x_0) - \frac{L\eta_0}{1+\alpha\eta_0} \leq F'(x_0) + \frac{L\eta_0}{1+\alpha\eta_0},$$

yielding $F'(x_1) \neq 0$ and

$$|F'(x_1)^{-1}| \leq \frac{1}{|F'(x_0)| - \frac{L\eta_0}{1+\alpha\eta_0}} \leq \frac{\beta(1+\alpha\eta_0)}{1+\alpha\eta_0 - \beta L\eta_0} =: \beta_1.$$

Next, we find an upper bound on $|F'(x_1)^{-1}F(x_1)|$. We obtain

$$\begin{aligned}|F(x_1)| &= |F(x_1) - F(x_0) - (F'(x_0) + \alpha_0 F(x_0))(x_1 - x_0)| \\ &\leq |F(x_1) - F(x_0) - F'(x_0)(x_1 - x_0)| + |\alpha_0|\,|F(x_0)|\,|x_1 - x_0|.\end{aligned}$$

But

$$\begin{aligned}&|F(x_1) - F(x_0) - F'(x_0)(x_1 - x_0)| \leq \frac{L}{2}|x_1 - x_0|^2 \leq \frac{L}{2}\frac{\eta_0^2}{(1+\alpha\eta_0)^2}, \\ &|F'(x_1)^{-1}|\frac{L}{2}\frac{\eta_0^2}{(1+\alpha\eta_0)^2} \leq \frac{\beta_0 L\eta_0^2}{2(1+\alpha\eta_0 - \beta_0 L\eta_0)(1+\alpha\eta_0)}, \\ &\alpha|F'(x_1)^{-1}|\,|F(x_0)|\,|x_1 - x_0| \leq \frac{(1+\alpha\eta_0)\alpha\eta_0^2}{(1+\alpha\eta_0 - h_0)(1+\alpha\eta_0)},\end{aligned}$$

leading to

$$|F'(x_1)^{-1}F(x_1)| \leq \frac{h_0\eta_0 + 2(1+\alpha\eta_0)\alpha\eta_0^2}{2(1+\alpha\eta_0 - h_0)(1+\alpha\eta_0)} =: \eta_1.$$

Then, we must show that

$$2p\eta_1 + |x_1 - x_0| \leq 2p\eta_0.$$

So, we need

$$2p\frac{h_0\eta_0 + 2(1+\alpha\eta_0)\alpha\eta_0^2}{2(1+\alpha\eta_0 - h_0)(1+\alpha\eta_0)} + \frac{\eta_0(1+\alpha\eta_0 - h_0)}{(1+\alpha\eta_0)(1+\alpha\eta_0 - h_0)} \leq 2p\eta_0,$$

or

$$\alpha\eta(1-2p) + h_0(3p-1) \leq 1 + 2p,$$

which is true by hypothesis (2.28). Hence, for any $x \in [x_1 - 2p\eta_1, x_1 + 2p\eta_1]$ we get

$$|x - x_0| \leq |x - x_1| + |x_1 - x_0| \leq 2p\eta_1 + |x_1 - x_0| \leq 2p\eta_0,$$

leading to

$$[x_1 - 2p\eta_1, x_1 + 2p\eta_1] \subseteq [x_0 - 2p\eta_0, x_0 + 2p\eta_0].$$

Next, we show that

$$h_1 := \beta_1 L\eta_1 \leq \frac{1}{2\lambda}.$$

We can write

$$h_1 \leq \frac{h_0 + 2(1+\alpha\eta_0)\alpha\eta_0}{2(1+\alpha\eta_0 - h_0)^2} h_0 \leq \frac{\frac{1}{2\lambda} + 2(1+\alpha\eta_0)\alpha\eta_0}{2\big(1 - \frac{1}{2\lambda} + \alpha\eta_0\big)^2} h_0 \leq h_0 \leq \frac{1}{2\lambda}$$

by (2.17), (2.27) and (2.28).

Hence, by induction,

$$[x_{k+1} - 2p\eta_{k+1}, x_{k+1} + 2p\eta_{k+1}] \subseteq [x_k - 2p\eta_k, x_k + 2p\eta_k],$$

so

$$\{x_{k+1}, x_{k+2}, \ldots\} \subseteq [x_k - 2p\eta_k, x_k + 2p\eta_k].$$

We must show

$$\alpha\eta_1 \leq \frac{1}{2\lambda} h_1.$$

This is indeed true, since

$$\alpha\eta_1 = \alpha \frac{[h_0 + 2(1+\alpha\eta_0)\alpha\eta_0]\eta_0}{2(1+\alpha\eta_0 - h_0)(1+\alpha\eta_0)} \leq \frac{\frac{h_0}{2\lambda}[h_0 + 2(1+\alpha\eta_0)\alpha\eta_0]}{2(1+\alpha\eta_0 - h_0)(1+\alpha\eta_0)} \leq \frac{1}{2\lambda} h_1.$$

Moreover, we must show that

$$\eta_1 \leq \xi\eta_0.$$

Evidently, this is true if

$$\eta_1 = \frac{[h_0 + 2(1+\alpha\eta_0)\alpha\eta_0]\eta_0}{2(1+\alpha\eta_) - h_0)(1+\alpha\eta_0)} \leq \xi\eta_0$$

or

$$\frac{1}{2\lambda} + 2(1+\alpha\eta_0)\alpha\eta_0 \leq 2\xi(1+\alpha\eta_0)\left(1 + \alpha\eta_0 - \frac{1}{2\lambda}\right)$$

which is true by hypothesis (2.29). Using induction, we have

$$\eta_{k+1} \leq \xi\eta_k \leq \cdots \leq \xi^{k+1}\eta_0$$

yielding $\lim_{k\to\infty} \eta_0 = 0$, since $\xi \in (0,1)$ by (2.30). Hence, $\{x_n\}$ is a Cauchy sequence in $\mathbb{R}$ and as such it converges to some $x^* \in I_p$ (since I_p is a closed set). ■

By replacing (2.17) and $(\mathcal{C})$ with (2.20) and $(\mathcal{C}_1)$ respectively in the preceding proof, we arrive at

Theorem 2.7. *Suppose that the hypotheses of Theorem* 2.4 *and conditions* $(\mathcal{C}_1)$ *hold. Then the sequence* $\{x_n\}$ *generated by Algorithm* 1.1 *is well defined in* I_{p_1}, *remains in* I_{p_1} *for all* $n = 0, 1, 2, \ldots$ *and converges to a solution* $x^* \in I_{p_1}$ *of* $F(x) = 0$.

Next, we present another semi-local convergence result for Algorithm 1.1.

Theorem 2.8. *Let* $F : I \subset \mathbb{R} \to \mathbb{R}$ *be a continuously differentiable function. Suppose that there exist* $\beta > 0$, $\eta \geq 0$, $L_0 > 0$, $L_1 > 0$, $\alpha \geq 0$,

$\{\alpha_n\} \subset \mathbb{R}$ *such that* (2.1), (2.2), (2.10), *and* (2.14) *hold, and*

$$H_1 = \beta L_1 \eta \le 1/2, \tag{2.39}$$

$$I^* \subseteq I, \tag{2.40}$$

$$|\alpha_n| \le |\alpha| \le \tfrac{1}{2}\beta L_1. \tag{2.41}$$

Then the sequence $\{x_n\}$ *converges to a solution* $x^* \in I^*$ *of* $F(x) = 0$. *Moreover,*

$$|x_{n+1} - x_n| \le \bar{t}_{n+1} - \bar{t}_n, \tag{2.42}$$

$$|x^* - x_n| \le \bar{t}^* - \bar{t}_n, \tag{2.43}$$

where the sequence $\{\bar{t}_n\}$ *is defined by*

$$\bar{t}_0 = 0, \quad \bar{t}_1 = \eta, \quad \bar{t}_{n+2} = \bar{t}_{n+1} + \frac{\beta L_1(\bar{t}_{n+1} - \bar{t}_n)^2}{2(1 - \beta L_0 \bar{t}_{n+1})}, \qquad n = 0, 1, 2, \dots, \tag{2.44}$$

where

$$\bar{t}^* = \lim_{n\to\infty} \bar{t}_n = \frac{1 - \sqrt{1 - \beta L_1 \eta}}{\beta L_1}$$

and

$$|x_{n+1} - x^*| \le \frac{1 + 2\alpha\eta}{2 + 2\alpha\eta}(2H_1)^{2^n - 1}. \tag{2.45}$$

Furthermore, x^* *is the only solution of* $F(x) = 0$ *in* $I_2 = I \cap U(x_0, 2/(\beta L_0))$.

Proof. Simply replace L by L_1 in the proof of Theorem 2.1 and notice that the iterate x_n lies in I_2, which is more accurate domain than I used in [19]. Hence, we arrive at (2.45). Concerning the uniqueness part (not studied in [19]), let $y^* \in I_2$ be such that $F(y^*) = 0$. Set

$$T = \int_0^1 F'(x^* + \theta(y^* - x^*))\, d\theta.$$

Then

$$\begin{aligned} |F'(x_0)^{-1}|\, |F'(x_0) - I| &\le \beta L_0 \int_0^1 [(1-\theta)|x^* - x_0| + \theta|y^* - x_0|]\, d\theta \\ &\le \beta L_0/2 < 1. \end{aligned}$$

Hence, T is non-singular. Then, from the identity $0 = F(y^*) - F(x^*) = T(y^* - x^*)$, we conclude that $x^* = y^*$. ∎

Remark 2.9. 1. Notice that

$$H \le 1/2 \Longrightarrow H_1 \le 1/2.$$

Concerning the error bounds, the new error bounds (2.42) and (2.43) are more precise than the old ones and the error bounds (2.45) are also more precise than the ones given in [19] for H replacing H_1 (since $H_1 \le H$).

2. It is worth noticing that Newton's method (1.1) does not change when we use the new conditions instead of the stronger conditions used in [19]. We can compute the computational order of convergence (COC) defined by

$$\xi = \ln\left(\frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|}\right) \Big/ \ln\left(\frac{\|x_n - x^*\|}{\|x_{n-1} - x^*\|}\right)$$

or the approximate computational order of convergence (ACOC)

$$\xi_1 = \ln\left(\frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}\right) \Big/ \ln\left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|}\right).$$

This way we obtain in practice the order of convergence in a way that avoids bounds involving estimates of higher than the first Fréchet derivative of the operator F.

3. Numerical examples. We present numerical examples for which the old convergence criteria in [19] are not satisfied but the new convergence criteria are satisfied.

Example 3.1. Let $X = Y = \mathbb{R}, x_0 = 1, I = [x_0 - (1-p), x_0 + (1-p)]$, $p \in (0, 0.5)$ and define a function F on I by

$$F(x) = x^3 - p.$$

Then $\beta = 1/3$, $L_0 = 3(3-p)$, $L = 6(2-p)$, $L_1 = 6\big(1 + \frac{1}{\beta L_0}\big)$, $\eta_0 = 1-p$, $t_1 = \frac{1-p}{4-p}$, $\eta = \frac{1}{3}(1-p)$. Choose $p = 0.49$. Notice that $L_0 < L_1 < L$. We can check the convergence criteria. By (2.5) we have

$$H = \tfrac{1}{3}6(2-p)\tfrac{1}{3}(1-p) = 0.513399996 > 0.5.$$

The Newton–Kantorovich Theorem 2.1 or Theorem 2.2 cannot guarantee that Newton's method (1.1) or Algorithm 1.1 starting from $x_0 = 1$ converge to $x^* = \sqrt[3]{p}$. However, condition (2.17) gives

$$H_1 = 0.475458167 < 0.5.$$

In view of (2.41) we must also choose

$$\alpha \le \tfrac{1}{2}\beta L_1 = 1.398406344.$$

Thus Theorem 2.8 guarantees the convergence of Newton's method (1.1) and Algorithm 1.1 to x^*. Hence, the applicability of these methods is expanded. Notice that this example is used as a motivational one.

The values of t_n, s_n and r_n for various values of p are compared in Table 1 (all computations have been carried out using MATLAB).

The sequence $\{\bar{\bar{t}}_n\}$ in Table 2 is the one used in [19] that can be obtained from (2.44) if $L_0 = L_1 = L$. Hence, the new error bounds are better than the old ones.

Table 1. Comparing t_n, s_n and r_n for various choices of p

p	n	t_n	s_n	r_n
0.55	1	0.15	0.15	0.15
	2	0.168433	0.193577	0.167433
	3	0.169396	0.198814	0.168159
	4	0.169399	0.198892	0.16816
	5	0.169399	0.198892	0.16816
	6	0.169399	0.198892	0.16816
	7	0.169399	0.198892	0.16816
0.6	1	0.133333	0.133333	0.133333
	2	0.14893	0.164706	0.133333
	3	0.149515	0.166985	0.14784
	4	0.149515	0.166997	0.14784
	5	0.149515	0.166997	0.148302
	6	0.149515	0.166997	0.148302
	7	0.149515	0.166997	0.148302
0.8	1	0.0666667	0.0666667	0.0666667
	2	0.0711467	0.0723958	0.0708385
	3	0.0711757	0.0724427	0.0708685
	4	0.0711757	0.0724427	0.0708685
	5	0.0711757	0.0724427	0.0708685
	6	0.0711757	0.0724427	0.0708685
	7	0.0711757	0.0724427	0.0708685

Table 2. Comparing $s_{n+1} - s_n, t_{n+1} - t_n$, $r_{n+1} - r_n$, $\bar{\bar{t}}_{n+1} - \bar{\bar{t}}_n$ and $\bar{t}_{n+1} - \bar{t}_n$ for various choices of p

p	α	n	$s_{n+1} - s_n$	$t_{n+1} - t_n$	$r_{n+1} - r_n$	$\bar{\bar{t}}_{n+1} - \bar{\bar{t}}_n$	$\bar{t}_{n+1} - \bar{t}_n$
0.55	1.4088	1	0.00523741	0.000963126	0.00072563	0.0121615	0.0500928
		2	0.0000775468	0.0000026438	0.00000126095	0.000591973	0.00693149
		3	$1.70067e-8$	$1.99217e-11$	$3.80773e-12$	$5.38982e-8$	0.000137291
		4	$8.04912e-16$	0	0	$7.98706e-12$	$8.29892e-15$
0.6	1.4167	1	0.00227868	0.000584181	0.000462082	0.0042842	0.037037
		2	0.000012131	$8.21823e-7$	$4.69645e-7$	0.0000510386	0.00328754
		3	$3.43829e-10$	$1.62645e-12$	$4.8514e-13$	$7.24568e-9$	0.0000262529
		5	0	0	0	$1.38778e-16$	$1.6743e-9$
0.8	1.4545	1	0.0000468498	0.0000290438	0.0000299891	0.0000586532	0.00757576
		2	$3.13325e-9$	$1.22078e-9$	$1.54976e-9$	$5.00622e-9$	0.0000997762
		3	$1.38778e-17$	0	0	$4.16334e-17$	$1.73118e-8$
		4	0	0	0	0	$5.27356e-16$

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Ioannis K. Argyros
Department of Mathematical Sciences
Cameron University
Lawton, OK 73505, U.S.A.
E-mail: iargyros@cameron.edu

Santhosh George
Department of Mathematical
and Computational Sciences
NIT Karnataka
Karnataka, India 575 025
E-mail: sgeorge@nitk.ac.in