Some new results about the ubiquitous semigroup $\mathbb{H}$

by

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Abstract. The semigroup $\mathbb{H}$ is defined as $\bigcap_{n=1}^{\infty} \mathcal{c}\ell_{\beta\mathbb{N}}(2^n\mathbb{N})$, and it has the algebraic structure (and topology) inherited from the right topological semigroup $(\beta\mathbb{N}, \cdot)$. Topological and algebraic copies of $\mathbb{H}$ are found in $(\beta S, \cdot)$ for any discrete semigroup $S$ which has some sequence with distinct finite products. And any compact Hausdorff right topological semigroup which has a countable dense set contained in its topological center is an image of $\mathbb{H}$ under a continuous homomorphism. (Thus the term “ubiquitous” in the title.) Much is already known about the structure of $\mathbb{H}$. In this paper we present several new results. Included are the following facts. (1) For any $n \in \mathbb{N}$, $\mathbb{H}$ is the union of $n$ pairwise disjoint clopen copies of itself, each of which is a right ideal of $\mathbb{H}$, and $\mathbb{H}$ is the union of $n$ pairwise disjoint clopen copies of itself, each of which is a left ideal of $\mathbb{H}$. (2) $\mathbb{H}$ contains $c$ pairwise disjoint clopen copies of itself, each of which is a right ideal of $\mathbb{H}$, and $\mathbb{H}$ contains $c$ pairwise disjoint clopen copies of itself, each of which is a left ideal of $\mathbb{H}$. (3) If $S$ is a countable dense subgroup of $(\mathbb{R}, +)$ and $S_d$ is $S$ with the discrete topology, then the set of ultrafilters in $\beta S_d$ that converge to $0$ (in the usual topology on $S$) is a copy of $\mathbb{H}$. (4) If $S$ is the direct sum of countably many countable partial semigroups each of which has an identity and at least two elements, then the set of ultrafilters in $\beta S_d$ that converge to the identity in the product topology on $S$ is a copy of $\mathbb{H}$.

1. Introduction. Given a discrete space $S$, we take the Stone–Čech compactification $\beta S$ of $S$ to be the set of ultrafilters on $S$, the principal ultrafilters being identified with the points of $S$. Given $A \subseteq S$, let $\bar{A} = \{ p \in \beta S : A \in p \}$. Then $\{ \bar{A} : A \subseteq S \}$ is a base for the open sets and a base for the closed sets of $\beta S$. If $(S, \cdot)$ is a discrete semigroup, the operation extends to $\beta S$ so that $(\beta S, \cdot)$ becomes a right topological semigroup with $S$ contained in its topological center. That is, for any $p \in \beta S$, the function $q \mapsto q \cdot p$ from $\beta S$ to itself is continuous, and for any $x \in S$, the function $q \mapsto x \cdot q$ from $\beta S$ to itself is continuous. It follows that, for any $p, q \in \beta S$, $p \cdot q = \lim_{s \to p} \lim_{t \to q} st$, where $s$ and $t$ denote elements of $S$.

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For \( p, q \in \beta S \) and \( A \subseteq S, A \in p \cdot q \) if and only if \( \{ x \in S : x^{-1}A \notin q \} \in p \) where \( x^{-1}A = \{ y \in S : x \cdot y \in A \} \). If the operation on \( S \) is written additively, we write that \( A \in p + q \) if and only if \( \{ x \in S : -x + A \notin q \} \in p \) where \( -x + A = \{ y \in S : x + y \in A \} \). Any compact Hausdorff right topological semigroup has an idempotent. See [4, Part I] for much more information about the structure of \( \beta S \).

We take \( \mathbb{N} \) to be the set of positive integers and let \( \omega = \mathbb{N} \cup \{ 0 \} \). Given a set \( X \), we let \( P_f(X) = \{ F \subseteq X : F \) is finite and nonempty\}. We define \( \mathbb{H} = \bigcap_{n=1}^{\infty} \mathbb{2}^n \mathbb{N} \). Then by [4] Lemma 6.8, \( \mathbb{H} \) is a compact subsemigroup of \( (\beta \mathbb{N}, +) \) which contains all of the idempotents of \( \beta \mathbb{N} \).

Let \((S, \cdot)\) be a discrete semigroup and let \( \langle x_n \rangle_{n=1}^{\infty} \) be a sequence in \( S \) such that if \( F \) and \( H \) are distinct members of \( P_f(\mathbb{N}) \), then \( \prod_{t \in F} x_t \neq \prod_{t \in H} x_t \), where the products are computed in increasing order of indices. For \( m \in \mathbb{N} \), let \( FP(\langle x_n \rangle_{n=1}^{\infty - m}) = \{ \prod_{t \in F} x_t : F \in P_f(\mathbb{N}) \) and \( \min F \geq m \}. \)

It was shown in [2] Theorem 5.6 that \( \bigcap_{m=1}^{\infty} FP(\langle x_n \rangle_{n=1}^{\infty - m}) \) is topologically isomorphic to \( \mathbb{H} \). By “topologically isomorphic” we mean there is a function \( \varphi : \bigcap_{m=1}^{\infty} FP(\langle x_n \rangle_{n=1}^{\infty - m}) \rightarrow \mathbb{H} \) which is both an isomorphism and a homeomorphism. In this case, that function is particularly simple. Define \( f : FP(\langle x_n \rangle_{n=1}^{\infty}) \rightarrow \mathbb{N} \) by \( f(\prod_{t \in F} x_t) = \sum_{t \in F} 2^{\ell(t)} \), let \( \bar{f} : \beta S \rightarrow \beta \mathbb{N} \) be the continuous extension of \( f \), and let \( \varphi \) be the restriction of \( \bar{f} \) to \( \bigcap_{m=1}^{\infty} FP(\langle x_n \rangle_{n=1}^{\infty - m}) \).

In [7] Theorem 1], John Pym showed that one does not need a full semigroup structure to produce copies of \( \mathbb{H} \). In fact, the only algebraic information needed to produce a copy of \( \mathbb{H} \) is the rule for multiplying by an identity. (We shall have more to say about Pym’s construction in Section [4]) In [6] Theorem 3.1], Talin Papazyan showed that if \( S \) is a discrete right cancellative and weakly left cancellative semigroup, then every \( G_\delta \) subset of \( S^* = \beta S \setminus S \) which contains an idempotent contains a copy of \( \mathbb{H} \). (A semigroup \( S \) is weakly left cancellative provided that for all \( a, b \in S \), \( \{ x \in S : ax = b \) is finite.\)

The fact cited in the abstract that any compact Hausdorff right topological semigroup which has a countable dense set contained in its topological center is an image of \( \mathbb{H} \) under a continuous homomorphism is established in [4] Theorem 6.4].

It is also known that \( \mathbb{H} \) has a substantial algebraic structure. For example, by [4] Corollary 7.36], \( \mathbb{H} \) contains copies of the free group on \( 2^\omega \) generators.

Recall that if \( X \) is a topological space, \( x \in X \), and \( p \) is an ultrafilter on \( X \), that is, \( p \in \beta X_d \) where \( X_d \) is the set \( X \) with the discrete topology, one says that \( p \) converges to \( x \) if \( p \) contains the neighborhood filter of \( x \). If \( S \) is a semigroup (not necessarily commutative) with identity which is also a topological space, we let \( O(S) \) be the set of ultrafilters on \( S \) that converge to the identity. In the event that \((S, \cdot)\) is a left topological semigroup with identity \( e \), one finds that \( O(S) \) is a compact subsemigroup of \((\beta S_d, \cdot)\).
In [3, Theorem 2.6(b)] it was shown that 

\[ x \in (0, \epsilon) \]

for some \( \epsilon > 0 \). That is, \( (0, \epsilon) \) is a dense subsemigroup of \( (\mathbb{R}, +) \). In [3] several facts about \( O(S) \) were obtained. We let \( O^+(S) = \{ p \in O(S) : (0, 1) \cap S \in p \} \) and let \( O^-(S) = \{ p \in O(S) : (-1, 0) \cap S \in p \} \). If \( S \subseteq (0, \infty) \), then of course \( O^-(S) = \emptyset \), but in any event, \( O^+(S) \) and \( O^-(S) \) have no isolated points. Note also that, given \( p \in \beta S_d \), we have \( p \in O^+(S) \) if and only if for every \( \epsilon > 0 \), \( (0, \epsilon) \cap S \in p \), and \( p \in O^-(S) \) if and only if for every \( \epsilon > 0 \), \( (-\epsilon, 0) \cap S \in p \). If \( 0 \not\in S \), then \( O(S) = O^-(S) \cup O^+(S) \); if \( 0 \in S \), then \( O(S) = O^-(S) \cup \{ 0 \} \cup O^+(S) \).

Let \( B(S) \) be the set of bounded ultrafilters on \( S \). That is,

\[ B(S) = \{ p \in \beta S_d : (\exists n \in \mathbb{N})(\lfloor -n, n \rfloor \cap S \in p) \}. \]

In [3, Theorem 2.6(b)] it was shown that \( O(\mathbb{R}) \) has all of the algebraic structure of \( B(\mathbb{R}) \) not already revealed by the algebra of \( \mathbb{R} \). That is, \( \mathbb{R} \times O(\mathbb{R}) \) is isomorphic to \( B(\mathbb{R}) \). At the conclusion of this introduction we will show that a similar, though weaker, conclusion applies to any dense subgroup \( S \) of \( (\mathbb{R}, +) \), and \( O^+(S) \) and \( O^-(S) \) are isomorphic and homeomorphic. Consequently, there is additional interest in the algebraic structure of \( O^+(S) \).

There are also applications of the algebraic structure of \( O(S) \) to Ramsey theory. See for example [1] and [8].

Some of our results have elementary proofs in the sense that we can explicitly describe the functions demonstrating that \( O^+(S) \) or \( O(S) \setminus \{ 0 \} \) is topologically isomorphic to \( \mathbb{H} \). We present these results in Section 2.

Our remaining results are based on [4, Theorem 7.24] and a modification thereof. In Section 3 we obtain assertions (1)–(3) stated in the abstract. In Section 4 we obtain a very general result about direct sums of sets with much less algebraic structure than a semigroup (the \( \text{ids} \) of [7]) and derive consequences of that result.

In Section 5 we present the proof of our modification of [4, Theorem 7.24].

As promised above, we show now that if \( S \) is a dense subgroup of \( (\mathbb{R}, +) \), then the ultrafilters converging to \( 0 \) contain much of the algebraic structure of \( \beta S_d \).

**Definition 1.1.** Let \( S \) be a dense subsemigroup of \( (\mathbb{R}, +) \).

(a) Let \( \iota : S \to [-\infty, \infty] \) be the inclusion function and let \( \alpha : \beta S_d \to [-\infty, \infty] \) be its continuous extension.

(b) Let \( B'(S) = \{ p \in \beta S_d : \alpha(p) \in S \} \).

Notice that if \( S \neq \mathbb{R} \), then \( B'(S) \neq B(S) \) because, given \( x \in \mathbb{R} \setminus S \), the set \( A = \{ S \cap (x - \epsilon, x + \epsilon) : \epsilon > 0 \} \) consists of subsets of \( S \) with
the finite intersection property. If \( p \in \beta S_d \) and \( A \subseteq p \), then \( \alpha(p) = x \) so \( p \in B(S) \setminus B'(S) \).

**Theorem 1.2.** Let \( S \) be a dense subgroup of \((\mathbb{R}, +)\). Define \( \varphi : S_d \times O(S) \to B'(S) \) by \( \varphi(x, p) = x + p \). Then \( \varphi \) is a continuous isomorphism onto \( B'(S) \).

**Proof.** Let \( T = [-\infty, \infty] \) with the usual topology, and for \( x, y \in T \) define

\[
x \ast y = \begin{cases} 
  x + y & \text{if } x, y \in \mathbb{R}, \\
  y & \text{if } y = \infty \text{ or } y = -\infty, \\
  x & \text{if } y \in \mathbb{R} \text{ and either } x = \infty \text{ or } x = -\infty.
\end{cases}
\]

It is routine to establish that \((T, \ast)\) is a compact right topological semigroup and the topological center \( A(T) \) equals \( \mathbb{R} \). Note that \( O(S) = \alpha^{-1}[[0]] \).

To see that \( \varphi \) is a homomorphism, let \((x, p), (y, q) \in S_d \times O(S)\). By [4] Theorem 4.23, \( p + y = y + p \) so \( \varphi((x, p) + (y, q)) = \varphi(x + y, p + q) = x + y + p + q = x + p + y + q = \varphi(x, p) + \varphi(y, q) \).

To see that \( \varphi \) is injective, assume that \((x, p), (y, q) \in S_d \times O(S)\) and \( \varphi(x, p) = \varphi(y, q) \). Then \( x + p = y + q \). By [4] Corollary 4.22, \( \alpha \) is a homomorphism so \( \alpha(x + p) = \alpha(x) + \alpha(p) = x + 0 = x \) and \( \alpha(y + q) = \alpha(y) + \alpha(q) = y + 0 = y \) so \( x = y \). Then by [4] Lemma 8.1, \( p = q \).

Notice that we have also established that \( \alpha[S_d \times O(S)] \subseteq B'(S) \).

To see that \( \varphi \) is continuous, let \((x, p) \in S_d \times O(S)\) and let \( U \) be a neighborhood of \( \varphi(x, p) \) in \( B'(S) \). Pick \( A \in x + p \) such that \( \overline{A} \cap B'(S) \subseteq U \). Since \( A \in x + p \), \(-x + A \in p \) so \( [x] \times (-x + A \cap O(S)) \) is a neighborhood of \((x, p) \) in \( S_d \times O(S) \). If \( q \in -x + A \cap O(S) \), then \( A \in x + q \) and \( \alpha(x + q) = \alpha(x) + \alpha(q) = x + 0 \in S \). Therefore \( \varphi([x] \times (\overline{-x + A} \cap O(S))) \subseteq \overline{A} \cap B'(S) \).

To see that \( \varphi \) is surjective, let \( p \in B'(S) \). Let \( x = \alpha(p) \). Then \( \alpha(-x + p) = \alpha(-x) + \alpha(p) = -x + x = 0 \), so \((x, -x + p) \in S_d \times O(S)\) and \( \varphi(x, -x + p) = p \). \( \blacksquare \)

Note that even if \( S = \mathbb{R} \), in which case \( B'(S) = B(S) \), the function \( \varphi \) is not a homeomorphism. To see this, we claim that \( \varphi[[0] \times O(S)] = O(S) \) is not open in \( B'(S) \). Indeed, pick \( p \in O(S) \setminus \{0\} \) and suppose we have a neighborhood \( U \) of \( p = 0 + p \) in \( B'(S) \) such that \( U \subseteq O(S) \). Pick \( A \in p \) such that \( \overline{A} \cap B'(S) \subseteq U \). Since \( p \in \beta S_d \), \( A \subseteq S \). Since \( p \) is not isolated, pick \( x \in A \setminus \{0\} \). Then \( x \in \overline{A} \cap B'(S) \subseteq O(S) \), while \( \alpha(x) = x \neq 0 \).

We remind the reader that any continuous injective function with a compact domain onto a Hausdorff space is a homeomorphism.

**Theorem 1.3.** Let \( S \) be a dense subgroup of \((\mathbb{R}, +)\). Then \( O^+(S) \) and \( O^-(S) \) are topologically isomorphic.

**Proof.** Let \( \psi : S \to S \) be defined by \( \psi(x) = -x \) and let \( \tilde{\psi} : \beta S_d \to \beta S_d \) be the continuous extension of \( \psi \) which is injective because \( \psi \) is injective. 

By \[4\] Corollary 4.22, \(\tilde{\psi}\) is a homomorphism, hence an isomorphism and ahomeomorphism. It is routine to verify that \(\tilde{\psi}[O^+(S)] = O^-(S).\)

We remark that if \(S\) is any countable regular topological space withoutisolated points, which is first countable, then the set of ultrafilters in \(\beta S_d\)which converge to any given point of \(S\), is homeomorphic to \(\mathbb{H}\). We leavethe easy proof as an exercise for the reader.

We close the introduction with the remark that it is well known, andreasonably easy to see, that any compact nonempty \(G_\delta\) subset of \(\mathbb{N}^* = \beta \mathbb{N} \setminus \mathbb{N}\)is either homeomorphic to \(\mathbb{N}^*\) or to \(\mathbb{H}\). Depending on set-theoreticassumptions, \(\mathbb{H}\) may or may not be homeomorphic to \(\mathbb{N}^*\). (Under CH, it is aconsequence of \([4]\) Corollary 1.2.4] that \(\mathbb{H}\) and \(\mathbb{N}^*\) are homeomorphic, whileit is shown in \([9]\) that it is consistent that \(\mathbb{H}\) and \(\mathbb{N}^*\) are not homeomorphic.)It is a consequence of \([4]\) Exercises 6.1.1 and 6.1.3] that \(\mathbb{N}^*\) and \(\mathbb{H}\) are nottopologically isomorphic.

2. Elementary results. The results of this section utilize representationsof numbers to various bases. We summarize some basic facts abouttheses expansions now, leaving verification of the assertions to the reader.

First, of course, every \(x \in \mathbb{N}\) has a unique expansion in the form\(x = \sum_{t \in F} 2^t\) for some \(F \in \mathcal{P}_f(\omega)\), and we define \(\text{supp}_2(x) = F\). Given \(n \in \mathbb{N}\),\(x \in 2^n\mathbb{N}\) if and only if \(\text{min supp}_2(x) \geq n\).

Every \(x \in \mathbb{N}\) has a unique expression in the form \(\sum_{t \in F} c_x(t) t!\) where\(F \in \mathcal{P}_f(\mathbb{N})\) and \(c_x \in \times_{t \in F} \{1, \ldots , t\}\), and we define \(\text{supp}_f(x) = F\). Given \(x\)and \(n \in \mathbb{N}\), \(x \in n!\mathbb{N}\) if and only if \(\text{min supp}_f(x) \geq n\).

Every \(x \in \mathbb{Z} \setminus \{0\}\) has a unique expansion in the form \(x = \sum_{t \in F} (-2)^t\) forsome \(F \in \mathcal{P}_f(\omega)\); we define \(\text{supp}_{-2}(x) = F\). We also define \(\text{supp}_{-2}(0) = \emptyset\).Given \(n \in \mathbb{N}\), \(x \in 2^n\mathbb{Z} \setminus \{0\}\) if and only if \(x \neq 0\) and \(\text{min supp}_{-2}(x) \geq n\).Also, \(x \in \mathbb{N}\) if and only if \(x \neq 0\) and \(\text{max supp}_{-2}(x)\) is even.

Let \(\mathbb{D} = \{n/2^k : n \in \mathbb{Z}\text{ and } k \in \omega\}\), the set of dyadic rationals. Every\(x \in \mathbb{D} \cap (0, 1)\) has a unique expansion in the form \(x = \sum_{t \in F} 1/2^t\) where\(F \in \mathcal{P}_f(\mathbb{N})\), and we define \(\text{supp}_{1/2}(x) = F\). Further, every such expansionis in \(\mathbb{D} \cap (0, 1)\). We note that for \(n \in \mathbb{N}\), \(x \in \mathbb{D} \cap (0, 1/2^n)\) if and only if\(\text{min supp}_{1/2}(x) > n\).

Every \(x \in \mathbb{D} \cap ((-2/3, 0) \cup (0, 1/3))\) has a unique expression of the form\(x = \sum_{t \in F} 1/(-2)^t\) where \(F \in \mathcal{P}_f(\mathbb{N})\), and we define \(\text{supp}_{-1/2}(x) = F\). Further, every such expression is in \(\mathbb{D} \cap ((-2/3, 0) \cup (0, 1/3))\).

Every \(x \in \mathbb{Q} \cap (0, 1)\) has a unique expression in the form \(\sum_{t \in F} a_x(t)/t!\)where \(F \in \mathcal{P}_f(\mathbb{N})\) with \(\text{min } F > 1\) and \(a_x \in \times_{t \in F} \{1, \ldots , t-1\}\), and we define \(\text{supp}_f(x) = F\). Further, every such expression is in \(\mathbb{Q} \cap (0, 1)\).

Let \(c = \sum_{t=1}^{\infty} 2t/(2t+1)\) and let \(d = \sum_{t=1}^{\infty} 2t/(2t)!\). Every \(x \in (\mathbb{Q} \setminus \{0\}) \cap (-c, d)\)has a unique expression in the form \(\sum_{t \in F} (-1)^t b_x(t)/t!\) where \(F \in \mathcal{P}_f(\mathbb{N})\)
with $\min F > 1$ and $b_x \in \bigwedge_{t \in F} \{1, \ldots, t - 1\}$, and we define $\text{supp}_-(x) = F$. Further, every such expression is in $(\mathbb{Q} \setminus \{0\}) \cap (c, d)$.

One property that all of these expansions have in common is that if $x$ and $y$ have disjoint supports, then the support of $x + y$ is the union of the supports of $x$ and $y$.

The fact in the following theorem that $\bigcap_{n=1}^{\infty} 2^n \mathbb{Z} \setminus \{0\}$ is a copy of $\mathbb{H}$ is [4, Exercise 7.2.1], except that we erroneously failed to exclude 0 in the statement of the exercise. But the intent of the exercise was to use [4, Theorem 7.24], which in turn cited [4, Lemma 7.4], whose proof is quite complicated. Here we present a simple description of the function which is simultaneously an isomorphism and a homeomorphism.

**Theorem 2.1.** Define $\varphi : \mathbb{Z} \setminus \{0\} \to \mathbb{N}$ by $\varphi(x) = \sum_{t \in \text{supp}_-(x)} 2^t$ and let $\tilde{\varphi} : \beta \mathbb{Z} \to \beta \mathbb{N}$ be the continuous extension of $\varphi$. Then the restriction of $\tilde{\varphi}$ to $\bigcap_{n=1}^{\infty} 2^n \mathbb{Z} \setminus \{0\}$ is an isomorphism and a homeomorphism onto $\mathbb{H}$.

**Proof.** Since $\varphi$ is injective, so is $\tilde{\varphi}$. Given $n \in \mathbb{N}$, we see that $2^n \mathbb{Z} \setminus \{0\} = \{x \in \mathbb{Z} \setminus \{0\} : \min \text{supp}_-(x) \geq n\}$ and $2^n \mathbb{N} = \{x \in \mathbb{N} : \min \text{supp}_2(x) \geq n\}$. Therefore, $\tilde{\varphi}(\bigcap_{n=1}^{\infty} 2^n \mathbb{Z} \setminus \{0\}) = \bigcap_{n=1}^{\infty} 2^n \mathbb{N} = \mathbb{H}$.

Thus it suffices to show that the restriction of $\tilde{\varphi}$ to $\bigcap_{n=1}^{\infty} 2^n \mathbb{Z} \setminus \{0\}$ is a homomorphism. To do this, we use [4, Theorem 4.21]. Let $n \in \mathbb{N}$ and let $x \in 2^n \mathbb{Z} \setminus \{0\}$. Let $m = \max \text{supp}_-(x) + 1$ and let $y \in 2^m \mathbb{Z} \setminus \{0\}$. Then

$$\text{supp}_-(x + y) = \text{supp}_-(x) \cup \text{supp}_-(y)$$

so $\varphi(x + y) = \varphi(x) + \varphi(y)$ as required. ■

In the following corollary we get quite explicit descriptions of disjoint copies of $\mathbb{H}$ that are each left ideals of $\mathbb{H}$. Corollary 3.6 has a much stronger result (but with nonelementary proof).

**Corollary 2.2.** Let $E = \{x \in \mathbb{N} : \max \text{supp}_2(x) \text{ is even}\}$. Let $\mathbb{H}_1 = \mathbb{H} \cap \overline{E}$ and let $\mathbb{H}_2 = \mathbb{H} \cap \overline{\mathbb{N} \setminus E}$. Then $\mathbb{H}_1$ and $\mathbb{H}_2$ partition $\mathbb{H}$ into clopen sets, each of which is topologically isomorphic to $\mathbb{H}$ and is a left ideal of $\mathbb{H}$.

**Proof.** Let $T = \bigcap_{n=1}^{\infty} 2^n \mathbb{Z} \setminus \{0\}$ and let $\varphi$ be the function of Theorem 2.1. Now $T \cap \mathbb{N}^* = \mathbb{H}$ so $\tilde{\varphi}[T \cap \mathbb{N}^*]$ is topologically isomorphic to $\mathbb{H}$. If $\psi : \mathbb{Z} \to \mathbb{Z}$ is defined by $\psi(x) = -x$, then $\psi$ is a homomorphism, so by [4, Corollary 4.22] its continuous extension $\tilde{\psi} : \beta \mathbb{Z} \to \beta \mathbb{Z}$ is an isomorphism and a homeomorphism. Since $T \cap (-\mathbb{N}^*) = \tilde{\psi}[T \cap \mathbb{N}^*]$, we find that $T \cap (-\mathbb{N}^*)$ is also topologically isomorphic to $\mathbb{H}$, and hence so is $\tilde{\varphi}[T \cap (-\mathbb{N}^*)]$. Using the fact that $\mathbb{N} = \{x \in \mathbb{Z} \setminus \{0\} : \max \text{supp}_2(x) \text{ is even}\}$, we see that $\tilde{\varphi}[T \cap \mathbb{N}^*] = \mathbb{H}_1$ and $\tilde{\varphi}[T \cap (-\mathbb{N}^*)] = \mathbb{H}_2$.

There are at least two ways to show that $\mathbb{H}_1$ and $\mathbb{H}_2$ are left ideals of $\mathbb{H}$. One is to use the fact that $\mathbb{N}^*$ and $-\mathbb{N}^*$ are both left ideals of $\beta \mathbb{Z}$. We will use a more direct route by letting $p \in \mathbb{H}_1$ and $q \in \mathbb{H}$ and showing that
q + p ∈ \mathbb{H}_1. (The proof for \mathbb{H}_2 is essentially the same.) We observe that 
q + p ∈ \mathbb{H} so we only need to show that \( E \in q + p \). We will in fact show that 
for all \( x ∈ \mathbb{N}, -x + E \in p \). So let \( x ∈ \mathbb{N} \) and let \( m = \max \supp_2(x) + 1 \). Then 
\( 2^n \mathbb{N} \cap E ⊆ -x + E \). ■

In the next two theorems we get a simple description of an isomorphism 
and a homeomorphism from \( O^+(\mathbb{D}) \) to \( \mathbb{H} \) and from \( O(\mathbb{D}) \setminus \{0\} \) to \( \mathbb{H} \).

**Theorem 2.3.** \( O^+(\mathbb{D}) \) and \( O^-(\mathbb{D}) \) are topologically isomorphic to \( \mathbb{H} \).

**Proof.** By Theorem 1.3 it suffices to show that \( O^+(\mathbb{D}) \) is a copy of \( \mathbb{H} \). Define \( ϕ : \mathbb{D} ∩ (0, 1) → \mathbb{N} \) by \( ϕ(x) = \sum_{t ∈ \supp_1(x)} 2^{t-1} \). Define \( ϕ \) at will for 
\( x ∈ \mathbb{D} \setminus (0, 1) \). Let \( \tilde{ϕ} : β\mathbb{D}_d → β\mathbb{N} \) be the continuous extension of \( ϕ \).

We claim that the restriction of \( \tilde{ϕ} \) to \( O^+(\mathbb{D}) \) is an isomorphism. It is 
injective because \( ϕ \) is injective on \( \mathbb{D}∩(0, 1) \). To see that it is a homomorphism, we use [4, Theorem 4.21]. Let \( x ∈ \mathbb{D} ∩ (0, 1) \) and let \( m = \max \supp_1(x) \). Let \( y ∈ \mathbb{D} ∩ (0, 1/2^m) \). Then \( \min \supp_1(y) > m \) so \( ϕ(x + y) = ϕ(x) + ϕ(y) \).

Also, given \( n ∈ \mathbb{N} \), we have \( ϕ[\mathbb{D} ∩ (0, 1/2^n)] = 2^n \mathbb{N} \) so \( \tilde{ϕ}[O^+(\mathbb{D})] = \bigcap_{n=1}^{∞} 2^n \mathbb{N} = \mathbb{H} \). ■

**Theorem 2.4.** \( O(\mathbb{D}) \setminus \{0\} \) is topologically isomorphic to \( \mathbb{H} \).

**Proof.** Define \( ϕ : \mathbb{N} → \mathbb{D} ∩ (-2/3, 1/3) \) by \( ϕ(x) = \sum_{t ∈ \supp_2(x)} 1/((-2)^{t+1}) \) and let \( \tilde{ϕ} : β\mathbb{N} → β\mathbb{D}_d \) be its continuous extension.

Let \( c_0 = 2/3 \). For \( n ∈ \omega \) let \( d_{2n} = d_{2n+1} = 1/(3 · 4^n) \), and if \( n > 0 \), let \( c_{2n-1} = c_{2n} = 2/(3 · 4^n) \). Then for each \( n ∈ \omega \), \( ϕ[2^n \mathbb{N}] = (\mathbb{D} \setminus \{0\}) ∩ (-c_n, d_n) \). (With apologies, we leave the tedious verification of this assertion to the reader.)

As a consequence, we find that \( \tilde{ϕ}[\mathbb{H}] = \bigcap_{n=1}^{∞} \cl_{β\mathbb{D}_d}((-c_n, 0) ∪ (0, d_n)) = \bigcap_{n=1}^{∞} \cl_{β\mathbb{D}_d}(-c_n, 0) ∪ \bigcap_{n=1}^{∞} \cl_{β\mathbb{D}_d}(0, d_n) = O^-(\mathbb{D}) ∪ O^+(\mathbb{D}) = O(\mathbb{D}) \setminus \{0\} \).

As before, one easily establishes that \( \tilde{ϕ} \) is an isomorphism and a homeo-
morphism. ■

We will obtain in Corollary 3.6 a much stronger result than the following.

**Corollary 2.5.** There exist \( \mathbb{H}_1 \) and \( \mathbb{H}_2 \) that partition \( \mathbb{H} \) into clopen sets, each of which is topologically isomorphic to \( \mathbb{H} \) and is a right ideal of \( \mathbb{H} \).

**Proof.** By Theorems 2.3 and 2.4, \( O^-(\mathbb{D}), O^+(\mathbb{D}) \), and \( O(\mathbb{D}) \setminus \{0\} \) are all 
topologically isomorphic to \( \mathbb{H} \). By [3, Lemma 2.5(d)], \( O^-(\mathbb{D}) \) and \( O^+(\mathbb{D}) \) are 
right ideals of \( O(\mathbb{D}) \setminus \{0\} \). ■

We conclude this section by showing that Theorems 2.3 and 2.4 hold 
if \( \mathbb{D} \) is replaced by \( \mathbb{Q} \). The proofs are not quite as constructive because we 
produce explicit functions that are isomorphisms and homeomorphisms onto 
\( \bigcap_{n=1}^{∞} n\mathbb{N} \) and then rely on the previously established fact that \( \bigcap_{n=1}^{∞} n\mathbb{N} \) is a copy of \( \mathbb{H} \).
Theorem 2.6. $O^+(\mathbb{Q})$ and $O^-(\mathbb{Q})$ are topologically isomorphic to $\mathbb{H}$.

Proof. It suffices to establish that $O^+(\mathbb{Q})$ is a copy of $\mathbb{H}$. Define $\varphi : \mathbb{Q} \cap (0, 1) \to \mathbb{N}$ by $\varphi(x) = \sum_{t \in \text{supp}_f(x)} a_x(t)(t-1)!$, and define $\varphi$ at will on $\mathbb{Q} \setminus (0, 1)$. Let $\widetilde{\varphi} : \beta \mathbb{Q}_d \to \beta \mathbb{N}$ be its continuous extension. As in previous proofs, one shows that $\widetilde{\varphi}[O^+(\mathbb{Q})] = \bigcap_{n=1}^{\infty} \overline{n\mathbb{N}} = \bigcap_{n=1}^{\infty} \overline{\mathbb{N}}$ and the restriction of $\widetilde{\varphi}$ to $O^+(\mathbb{Q})$ is an isomorphism. One then invokes [4, Corollary 7.26], which says that $\bigcap_{n=1}^{\infty} \overline{n\mathbb{N}}$ is topologically isomorphic to $\mathbb{H}$. ■

Theorem 2.7. $O(\mathbb{Q}) \setminus \{0\}$ is topologically isomorphic to $\mathbb{H}$.

Proof. Define $\varphi : \mathbb{N} \to \mathbb{Q} \setminus \{0\}$ by $\varphi(x) = \sum_{t \in \text{supp}_f(x)} (-1)^{t+1} c_x(t) / (t+1)!$ and let $\widetilde{\varphi} : \beta \mathbb{N} \to \beta \mathbb{Q}_d$ be its continuous extension. Using $\widetilde{\varphi}$ one shows in a now familiar fashion that $O(\mathbb{Q}) \setminus \{0\}$ is topologically isomorphic to $\bigcap_{n=1}^{\infty} \overline{n\mathbb{N}}$. ■

3. Copies of $\mathbb{H}$ in $\mathbb{H}$. In this section we obtain results which are applications of theorems whose main ideas are due to Yevhen Zelenyuk in his proof that $\beta \mathbb{N}$ does not contain any nontrivial finite groups. A topology $\mathcal{T}$ on a group $G$ is left invariant provided that for every $U \in \mathcal{T}$ and every $a \in G$, $aU \in \mathcal{T}$. A space is zero-dimensional provided it has a base of clopen sets.

Theorem 3.1 ([4, Theorem 7.24]). Let $G$ be a group with a left invariant zero-dimensional Hausdorff topology, and let $X$ be a countable subspace of $G$ which contains the identity $e$ of $G$ and has no isolated points. Suppose also that, for each $a \in X$, $aX \cap X$ is a neighborhood of $a$ in $X$ and that there is a neighborhood $V(a)$ of $e$ in $X$, with $V(e) = X$, for which $aV(a) \subseteq X$. If the filter of neighborhoods of $e$ in $X$ has a countable base, then $\bigcap\{c \ell_{\beta X_d} W : W$ is a neighborhood of $e$ in $X \} \setminus \{e\}$ is topologically isomorphic to $\mathbb{H}$. ■

Corollary 3.2. Let $G$ be a countable metrizable topological group, with identity $e$, which has no isolated points. Then $O(G) \setminus \{e\}$ is topologically isomorphic to $\mathbb{H}$.

Proof. To see that $G$ is zero-dimensional, let $d$ denote a metric for $G$. For every $a \in G$ and every $r > 0$, there exists $r' \in (0, r)$ for which $\{x \in G : d(a, x) = r'\} = \emptyset$. This implies that $\{x \in G : d(a, x) < r'\}$ is clopen. Let $X = G$ and for $a \in G$, let $V(a) = G$. Then apply Theorem 3.1. ■

In Section 5 we will derive a generalization of Theorem 3.1 which replaces the assumption that $G$ is a group with the assumption that $G$ is a cancellative semigroup with identity. The proof of the generalization is quite complicated, but still significantly simpler than the proof of [4, Lemma 7.4] on which the proof of [4, Theorem 7.24] (and thus of Theorem 3.1) relies.
The reader who is primarily interested in the results about copies of \( \mathbb{H} \) and has been through the proof of [4, Theorem 7.24], or is willing to take Theorem 3.1 on faith, is advised to ignore Section 5. On the other hand, if she has not been through the proof of [4, Theorem 7.24] and wants to understand the proofs of Theorems 3.4 and 3.5, she is advised to read the proof of Theorem 5.3.

**Definition 3.3.** Let \( E \) be an infinite subset of \( \mathbb{N} \). Then
\[
Y_E = \mathbb{H} \cap c\ell_{\beta\mathbb{N}}\{x \in \mathbb{N} : \text{max supp}_2(x) \in E\},
\]
\[
Z_E = \mathbb{H} \cap c\ell_{\beta\mathbb{N}}\{x \in \mathbb{N} : \text{min supp}_2(x) \in E\}.
\]

It has been known for some time that for any infinite \( E \subseteq \mathbb{N} \), \( Y_E \) is a left ideal of \( \mathbb{H} \) and \( Z_E \) is a right ideal of \( \mathbb{H} \). The new information in the following two theorems is that they are copies of \( \mathbb{H} \).

**Theorem 3.4.** Let \( E \) be an infinite subset of \( \mathbb{N} \). Then \( Y_E \) is a clopen subset of \( \mathbb{H} \) which is a left ideal of \( \mathbb{H} \) and is topologically isomorphic to \( \mathbb{H} \).

**Proof.** Let \( X = \{0\} \cup \{x \in \mathbb{N} : \text{max supp}_2(x) \in E\} \). Then \( Y_E = \mathbb{H} \cap X \). Since \( E \) is infinite, we have \( Y_E \neq \emptyset \), and \( Y_E \) is trivially clopen in \( \mathbb{H} \). To verify that \( Y_E \) is a left ideal of \( \mathbb{H} \), let \( p \in Y_E \) and let \( q \in \mathbb{H} \). We claim that for all \( x \in \mathbb{N} \), \( -x + X \in p \) so that \( X \in q + p \). (Then, since \( q + p \in \mathbb{H} \), we have \( q + p \in Y_E \).) So let \( x \in \mathbb{N} \) and let \( n = 1 + \text{max supp}_2(x) \). Then \( X \cap 2^n\mathbb{N} \subseteq p \) and \( X \cap 2^n\mathbb{N} \subseteq -x + X \).

We show that \( Y_E \) is topologically isomorphic to \( \mathbb{H} \) using Theorem 3.1 with \( G = \mathbb{Z} \). Let \( S = \{x + 2^n\mathbb{Z} : x \in \mathbb{Z} \text{ and } n \in \mathbb{N}\} \). It is routine to verify that \( S \) is a base for a Hausdorff zero-dimensional left invariant topology on \( \mathbb{Z} \).

To see that \( X \) has no isolated points in the relative topology, let \( a \in X \) and \( n \in \mathbb{N} \). Pick \( t \in E \) such that \( t > n \) and \( t > \text{max supp}_2(a) \) if \( a \neq 0 \). Then \( a + 2^t \in (a + 2^n\mathbb{Z}) \cap X \).

We now let \( a \in X \) and show that \( (a + X) \cap X \) is a neighborhood of \( a \) in \( X \). If \( a = 0 \) this is trivial. So assume that \( a \neq 0 \) and pick \( n \in \mathbb{N} \) such that \( n > \text{max supp}_2(a) \). We claim that \( (a + 2^n\mathbb{Z}) \cap X \subseteq (a + X) \cap X \), so let \( y \in (a + 2^n\mathbb{Z}) \cap X \). Pick \( w \in \mathbb{Z} \) such that \( y = a + 2^n w \). Since \( 2^n > a \), \( w \geq 0 \). If \( w = 0 \), then \( y = a + 0 \in (a + X) \), so assume \( w > 0 \). Then \( \text{max supp}_2(2^n w) = \text{max supp}_2(y) \in E \) so \( 2^n w \in X \).

Let \( V(0) = X \). For \( a \in X \setminus \{0\} \), pick \( n \in \mathbb{N} \) with \( n > \text{max supp}_2(a) \) and let \( V(a) = 2^n\mathbb{Z} \cap X \). Then \( a + V(a) \subseteq X \).

For \( n \in \mathbb{N} \), let \( W_n = X \cap 2^n\mathbb{Z} = (X \cap 2^n\mathbb{N}) \cup \{0\} \). Then \( \langle W_n \rangle_{n=1}^{\infty} \) is a neighborhood base for \( 0 \) in \( X \).

We have established that the hypotheses of Theorem 3.1 hold. Therefore
\[
\bigcap_{n=1}^{\infty} W_n \setminus \{0\} = \bigcap\{c\ell_{\beta X}, W : W \text{ is a neighborhood of } 0 \text{ in } X \} \setminus \{0\} \text{ is topologically isomorphic to } \mathbb{H}, \text{ and } \bigcap_{n=1}^{\infty} W_n \setminus \{0\} = Y_E.
\]
One is tempted to modify slightly the proof of Theorem 3.4 to prove Theorem [3.3]: However, if $\mathbb{N} \setminus E$ is infinite and $X = \{0\} \cup \{x \in \mathbb{N} : \min \text{supp}_2(x) \in E\}$, it is not true that for each $a \in X$, $(a + X) \cap X$ is a neighborhood of $a$ in $X$ for the topology on $\mathbb{Z}$ generated by $\{x + 2^n\mathbb{Z} : x \in \mathbb{Z} \text{ and } n \in \mathbb{N}\}$. For example, let $t \in E$ and let $a = 2^t$. Suppose we have $n \in \mathbb{N}$ such that $(a + 2^n\mathbb{Z}) \cap X \subseteq (a + X) \cap X$. Pick $j \in \mathbb{N} \setminus E$ such that $j > \max\{t, n\}$. Then $a + 2^t \in ((a + 2^n\mathbb{Z}) \cap X) \setminus (a + X)$.

**Theorem 3.5.** Let $E$ be an infinite subset of $\mathbb{N}$. Then $Z_E$ is a clopen subset of $\mathbb{H}$ which is a right ideal of $\mathbb{H}$ and is topologically isomorphic to $\mathbb{H}$.

**Proof.** Let $Y = \{x \in \mathbb{N} : \min \text{supp}_2(x) \in E\}$. Then $Z_E = \mathbb{H} \cap \overline{Y}$. Since $E$ is infinite, we have $Z_E \neq \emptyset$, and $Z_E$ is trivially clopen in $\mathbb{H}$. To verify that $Z_E$ is a right ideal of $\mathbb{H}$, let $p \in Z_E$ and let $q \in \mathbb{H}$. We claim that $Y \setminus \{0\} \subseteq \{x \in \mathbb{N} : -x + Y \subseteq q\}$ so that $Y \subseteq p + q$. (Then, since $p + q \in \mathbb{H}$, we have $p + q \in Z_E$.) So let $x \in Y \setminus \{0\}$ and let $n = 1 + \max \text{supp}_2(x)$. Then $2^nN \subseteq q$ and $2^nN \subseteq -x + Y$.

We show that $Z_E$ is is topologically isomorphic to $\mathbb{H}$ using Theorem 3.1 with $G = \bigoplus_{n=0}^{\infty} \mathbb{Z}$. For $\vec{x} \in G$, let $\text{supp}(\vec{x}) = \{n \in \omega : x_n \neq 0\}$. For each $n \in \mathbb{N}$, let

$$T_n = \{\vec{0}\} \cup \{\vec{x} \in G \setminus \{\vec{0}\} : \min \text{supp}(\vec{x}) \in E \text{ and } \min \text{supp}(\vec{x}) \geq n\}.$$ 

Let $B = \{\vec{a} + T_n : \vec{a} \in G, n \in \mathbb{N}, \text{ and if } \vec{a} \neq \vec{0}, \text{ then } n > \max \text{supp}(\vec{a})\}$. It is routine to verify that $B$ is a base for a left invariant Hausdorff topology on $G$.

We show now that this topology is zero-dimensional by verifying that each member of $B$ is closed. To this end let $\vec{a} + T_n \in B$. Assume first that $\vec{a} = \vec{0}$. Let $\vec{x} \in G \setminus T_n$. Then $\vec{x} \neq \vec{0}$. Let $m = \min \text{supp}(\vec{x})$ and let $k = 1 + \max \text{supp}(\vec{x})$. Then $m \notin E$ or $m < n$. We claim that $(\vec{x} + T_k) \cap T_n = \emptyset$. Let $\vec{z} \in T_k \setminus \{\vec{0}\}$. Then $m = \min \text{supp}(\vec{x} + \vec{z})$ so $\vec{x} + \vec{z} \notin T_n$.

Now assume that $\vec{a} \neq \vec{0}$ and $n > \max \text{supp}(\vec{a})$. Let $\vec{x} \in G \setminus (\vec{a} + T_n)$. Then $\vec{x} \neq \vec{a}$. Assume first that $\max \text{supp}(\vec{x}) < n$. Then there is some $i \in \{0, 1, \ldots, n-1\}$ such that $x_i \neq a_i$ so $(\vec{x} + T_n) \cap (\vec{a} + T_n) = \emptyset$. Thus we may assume that $k = 1 + \max \text{supp}(\vec{x}) \geq n$. We claim that $(\vec{x} + T_k) \cap (\vec{a} + T_n) = \emptyset$, so suppose instead that we have $\vec{u} \in T_k$ and $\vec{v} \in T_n$ such that $\vec{x} + \vec{u} = \vec{a} + \vec{v}$. Then $\text{supp}(\vec{x} + \vec{v}) = \text{supp}(\vec{x}) \cup \text{supp}(\vec{u})$ and $\text{supp}(\vec{a} + \vec{v}) = \text{supp}(\vec{a}) \cup \text{supp}(\vec{v})$. As $\max \text{supp}(\vec{x}) \geq n > \max \text{supp}(\vec{a})$, we see that $\text{supp}(\vec{a})$ is an initial segment of $\text{supp}(\vec{x})$ and $\text{supp}(\vec{v})$ is a final segment of $\text{supp}(\vec{v})$, so $\text{supp}(\vec{v} - \vec{u}) = \text{supp}(\vec{v}) \setminus \text{supp}(\vec{u})$ and thus $\min \text{supp}(\vec{v} - \vec{u}) = \min \text{supp}(\vec{v}) \in E$ and $\min \text{supp}(\vec{u}) \geq n$ so $\vec{v} - \vec{u} \in T_n$. Therefore $\vec{x} = \vec{a} + (\vec{v} - \vec{u}) \in \vec{a} + T_n$, a contradiction.

Let $X = \{\vec{0}\} \cup \{\vec{x} \in G \setminus \{\vec{0}\} : \min \text{supp}(\vec{x}) \in E\}$. Trivially $X$ has no isolated points and $(0 + X) \cap X$ is a neighborhood of $0$ in $X$. Let $V(\vec{0}) = X$. 
Let $a \in X \setminus \{0\}$ and let $n = 1 + \max \supp(a)$. Then $(a + T_n) \cap X \subseteq (a + X) \cap X$ so $(a + X) \cap X$ is a neighborhood of $a$ in $X$. Let $V(a) = T_n$. Then $a + V(a) \subseteq X$.

We have established that the hypotheses of Theorem 3.1 hold. Since \( \{T_n : n \in \mathbb{N}\} \) is a neighborhood base for $0$ in $X$, \( \bigcap_{n=1}^{\infty} c\ell_{\beta}X_d T_n \setminus \{0\} \) is a topological and algebraic copy of $\mathbb{H}$.

Define $\psi : \mathbb{N} \to G$ by, for $x \in \mathbb{N}$, $\psi(x) = \tilde{y}$ where $\supp(\tilde{y}) = \supp_2(x)$; let $\tilde{\psi} : \beta\mathbb{N} \to \beta G_d$ be its continuous extension. For each $n \in \mathbb{N}$, $\psi[2^nN \cap Y] = T_n \setminus \{0\}$, so the restriction of $\tilde{\psi}$ to $Z_E$ is an isomorphism and a homeomorphism onto $\bigcap_{n=1}^{\infty} c\ell_{\beta}X_d T_n \setminus \{0\}$. ■

**Corollary 3.6.** Let $n \in \mathbb{N}$.

1. There is a partition of $\mathbb{H}$ into $n$ clopen subsets, each of which is topologically isomorphic to $\mathbb{H}$ and is a left ideal of $\mathbb{H}$.
2. There is a partition of $\mathbb{H}$ into $n$ clopen subsets, each of which is a topological and algebraic copy of $\mathbb{H}$ and is a right ideal of $\mathbb{H}$.

**Proof.** Partition $\mathbb{N}$ into $n$ infinite sets $E(1), \ldots, E(n)$. Then we see that $Y_{E(1)}, \ldots, Y_{E(n)}$ are as required for conclusion (1) and $Z_{E(1)}, \ldots, Z_{E(n)}$ are as required for (2). ■

Recall that a collection $\mathcal{A}$ of sets is almost disjoint if the intersection of any two members of $\mathcal{A}$ is finite. Recall also that there exists a collection of $c$ almost disjoint subsets of $\mathbb{N}$ where $c = |\mathbb{R}|$. (An easy way of seeing this is to pick a sequence of rationals converging to $\alpha$ for every $\alpha \in \mathbb{R}$.) Then by Zorn’s Lemma, one may pick a maximal almost disjoint family of cardinality $c$.

**Corollary 3.7.** Let $\langle E(\sigma) \rangle_{\sigma < \varsigma}$ enumerate a maximal almost disjoint family of infinite subsets of $\mathbb{N}$.

1. For each $\sigma < \varsigma$, $Y_{E(\sigma)}$ is a clopen topological and algebraic copy of $\mathbb{H}$ which is a left ideal of $\mathbb{H}$, if $\sigma < \tau < \varsigma$, then $Y_{E(\sigma)} \cap Y_{E(\tau)} = \emptyset$, and $\bigcup_{\sigma < \varsigma} Y_{E(\sigma)}$ is dense in $\mathbb{H}$.
2. For each $\sigma < \varsigma$, $Z_{E(\sigma)}$ is a cloopen topological and algebraic copy of $\mathbb{H}$ which is a right ideal of $\mathbb{H}$, if $\sigma < \tau < \varsigma$, then $Z_{E(\sigma)} \cap Z_{E(\tau)} = \emptyset$, and $\bigcup_{\sigma < \varsigma} Z_{E(\sigma)}$ is dense in $\mathbb{H}$.

**Proof.** (1) If $\sigma < \tau < \varsigma$, pick $n \in \mathbb{N}$ such that $E(\sigma) \cap E(\tau) \subseteq \{1, \ldots, n\}$. Then $2^{n+1}\mathbb{N} \cap \{x \in \mathbb{N} : \max \supp_2(x) \in E(\sigma)\} \cap \{x \in \mathbb{N} : \max \supp_2(x) \in E(\tau)\} = \emptyset$ so $Y_{E(\sigma)} \cap Y_{E(\tau)} = \emptyset$.

To see that $\bigcup_{\sigma < \varsigma} Y_{E(\sigma)}$ is dense in $\mathbb{H}$, suppose we have $A \subseteq \mathbb{N}$ such that $\emptyset \neq A \cap \mathbb{H}$ and $(A \cap \mathbb{H}) \cap \bigcup_{\sigma < \varsigma} Y_{E(\sigma)} = \emptyset$. Choose $x_1 \in A$. Inductively, having chosen $(x_t)_{t=1}^{n}$, choose $x_{n+1} \in A$ such that $\min \supp_2(x_{n+1}) > \max \supp_2(x_n)$. For each $n \in \mathbb{N}$ let $y_n = \max \supp_2(x_n)$. By the maximality of $\{E(\sigma) : \sigma < \varsigma\}$ pick $\sigma < \varsigma$ such that $E(\sigma) \cap \{y_n : n \in \mathbb{N}\}$ is infinite. Let
$D = \{n \in \mathbb{N} : y_n \in E(\sigma)\}$. Pick $p \in \mathbb{N}^*$ such that $\{x_n : n \in D\} \subseteq p$. Then $p \in \mathcal{A} \cap \mathcal{H}$. Let $X = \{x \in \mathbb{N} : \max \text{supp}_2(x) \in E(\sigma)\}$. Then $\{x_n : n \in D\} \subseteq X$ so $p \in \mathcal{X} \cap \mathcal{H} = Y_{E(\sigma)}$, a contradiction.

The proof of (2) is essentially the same, with $y_n = \min \text{supp}_2(x_n)$.

We saw in Theorems 2.3, 2.4, 2.6, and 2.7 by elementary arguments that if $S = \mathbb{D}$ or $S = \mathbb{Q}$, then $O^+(S), O^-(S)$, and $O(S) \setminus \{0\}$ are copies of $\mathbb{H}$. We see now in the next two theorems, as a consequence of Theorem 3.1, that if $S$ is any countable dense subgroup of $(\mathbb{R}, +)$, the same conclusions hold.

**Theorem 3.8.** Let $S$ be a countable dense subsemigroup of $([0, \infty), +)$ with the property that $b - a \in S$ whenever $a, b \in S$ and $a \leq b$. Then $O^+(S)$ is topologically isomorphic to $\mathbb{H}$.

**Proof.** We apply Theorem 3.1 with $G = \mathbb{R}$ and $X = S$. We give $G$ the zero-dimensional first countable topology for which the sets of the form $[a, b)$, where $a, b \in \mathbb{R}$ and $a < b$, are a base. It is easy to check that the hypotheses of Theorem 3.1 are satisfied. \[\blacksquare\]

One may also derive Theorem 3.8 using only Corollary 5.4 by letting $T_n = [0, 1/n) \cap S$ and $\mathcal{B} = \{a + T_n : a \in S$ and $n \in \mathbb{N}\}$.

When we add the assumption that $S$ is dense in $\mathbb{R}$, we need to add the assumption that $b - a \in S$ when $b < a$. But once we have done that, we in fact have a subgroup.

**Theorem 3.9.** Let $S$ be a countable dense subgroup of $(\mathbb{R}, +)$. Then $O(S) \setminus \{0\}$ is topologically isomorphic to $\mathbb{H}$.

**Proof.** This is immediate from Corollary 3.2. \[\blacksquare\]

4. **Direct sums.** Many of the results of this section depend on the notions of $\text{id}$ and $\text{oid}$ introduced by John Pym [7]. Before introducing these objects, we recall the notions of partial semigroup and adequate partial semigroup.

**Definition 4.1.**

(a) A partial multiplication on a set $S$ is a function $*$ taking a nonempty subset $D$ of $S \times S$ to $S$. If $(x, y) \in D$, we write $x * y$ for $*(x, y)$ and say that $x * y$ is defined.

(b) A partial semigroup is a pair $(S, *)$ where $*$ is a partial multiplication on $S$ such that $(x * y) * z = x * (y * z)$ for all $x, y, z \in S$ in the sense that if either side is defined, so is the other and they are equal.

(c) Let $(S, *)$ be a partial semigroup and let $F \in \mathcal{P}_f(S)$. Then $\sigma(F) = \{y \in S : x * y$ is defined for all $x \in F\}$.

(d) A partial semigroup $(S, *)$ is adequate if $\sigma(F) \neq \emptyset$ for every $F \in \mathcal{P}_f(S)$.
DEFINITION 4.2.

(a) An id is a partial semigroup \((S_i, \cdot)\) with an identity 1. That is, for each \(x \in S_i\), \(1 \cdot x = x \cdot 1 = x\).

(b) If for each \(i\) in some index set \(I\), \(S_i\) is an id, then \(\bigoplus_{i \in I} S_i = \{ \bar{x} \in \bigtimes_{i \in I} S_i \mid \{i \in I : x_i \neq 1\} \text{ is finite}\} \) is an oid.

(c) For \(\bar{x} \in \bigoplus_{i \in I} S_i\), \(\text{supp}(\bar{x}) = \{i \in I : x_i \neq 1\}\).

(d) If \(S = \bigoplus_{i \in I} S_i\) is an oid and \(\bar{x}, \bar{y} \in S\), then the oid operation \(\bar{x} \cdot \bar{y}\) is defined if and only if \(\text{supp}(\bar{x}) \cap \text{supp}(\bar{y}) = \emptyset\) in which case \((\bar{x} \cdot \bar{y})_i = x_i \cdot y_i\) for each \(i \in I\).

If each \(S_i\) is a semigroup, Definition 4.2(d) conflicts with the usual definition of the operation on \(\bigoplus_{i \in I} S_i\) which is defined for all \(\bar{x}\) and \(\bar{y}\), but the values agree wherever they are defined for both operations.

By [1] Lemma 6.14.1, if \(I\) is infinite and for each \(i \in I\), \(S_i\) is an id with at least two elements, then the oid \(\bigoplus_{i \in I} S_i\) is an adequate partial semigroup.

**Lemma 4.3.** Assume that \(I\) is infinite and for each \(i \in I\), \((S_i, \cdot)\) is an id with at least two elements. Let \(\delta S = \bigcap_{F \in \mathcal{P}_f(S)} c\ell_{\beta_S} \sigma(F)\). For \(p, q \in \delta S\) and \(A \subseteq S\), agree that \(A \in p \cdot q\) if and only if \(\{\bar{x} \in S \mid \{y \in S : \text{supp}(\bar{x}) \cap \text{supp}(\bar{y}) = \emptyset\} \text{ and } \bar{x} \cdot \bar{y} \in A\} \in q\) \(\in p\). Then \((\delta S, \cdot)\) is a semigroup.

**Proof.** This is an immediate consequence of [1] Theorem 4.22.2.

In the case that \(|S_n| = 2\) for each \(n \in \mathbb{N}\), the following result was established by Pym [7].

**Theorem 4.4.** For each \(n \in \mathbb{N}\) let \(S_n\) be a countable id with at least two members and identity 1, and let \(S = \bigoplus_{n=1}^{\infty} S_n\). Let \(\bar{T} = \{1\}_{n \in \mathbb{N}} \in S\). For \(n \in \mathbb{N}\), let \(U_n = \{\bar{T}\} \cup \{\bar{x} \in S \mid \{\bar{T}\} : \min \text{supp}(\bar{x}) > n\}\). Then \(\delta S = \bigcap_{n=1}^{\infty} c\ell_{\beta_S} U_n\) and \(\delta S \setminus \{\bar{T}\}\) is topologically isomorphic to \(\mathbb{H}\).

**Proof.** To see that \(\delta S = \bigcap_{n=1}^{\infty} c\ell_{\beta_S} U_n\), assume first that \(p \in \delta S\) and let \(n \in \mathbb{N}\). For \(i \in \{1, \ldots, n\}\) pick \(a_i \in S_i \setminus \{1_i\}\) and define \(\bar{x} \in S\) by \(x_i = a_i\) if \(i \leq n\) and \(x_i = 1_i\) otherwise. Then \(\sigma(\{\bar{x}\}) = U_n\) so \(U_n \subseteq p\). Now assume that \(p \in \bigcap_{n=1}^{\infty} c\ell_{\beta_S} U_n\) and let \(F \in \mathcal{P}_f(S)\). Let \(n = \max \bigcup \{\text{supp}(\bar{x}) : \bar{x} \in F\}\). Then \(U_n \subseteq \sigma(F)\) so \(\sigma(F) \subseteq p\).

For \(n \in \mathbb{N}\), if \(S_n\) is infinite, let \(G_n = \mathbb{Z}\) if \(|S_n| = k \in \mathbb{N} \setminus \{1\}\), let \(G_n = \mathbb{Z}_k\). Let \(G = \bigoplus_{n=1}^{\infty} G_n\) with the topology inherited from \(\bigtimes_{n=1}^{\infty} G_n\) with the product topology where each \(G_n\) is discrete. Given \(\bar{x} \in G\), let \(\text{supp}(\bar{x}) = \{n \in \mathbb{N} : x_n \neq 0\}\). For \(n \in \mathbb{N}\), let \(T_n = \{0\} \cup \{\bar{x} \in G \mid \{0\} : \min \text{supp}(\bar{x}) > n\}\). It is well known, and easy to see, that \(G\) is a Hausdorff zero-dimensional topological group with no isolated points. And \(\{T_n : n \in \mathbb{N}\}\) is a neighborhood base for \(0\) in \(G\). Thus by Corollary 3.2 \(O(G) \setminus \{0\}\) is topologically isomorphic to \(\mathbb{H}\).
For each $n \in \mathbb{N}$, pick a bijection $\varphi_n : S_n \to G_n$ with $\varphi_n(1) = 0$, and define $\psi : S \to G$ by $\psi(x) = \varphi_n(x_n)$ for $n \in \mathbb{N}$ and $x \in S$, noting that $\psi$ is a bijection. Let $\tilde{\psi} : \beta S_d \to \beta G_d$ be the continuous extension of $\psi$ and note that $\tilde{\psi}$ is also a bijection.

If $\bar{x}, \bar{y} \in S$ and $\text{supp}(\bar{x}) \cap \text{supp}(\bar{y}) = \emptyset$, then $\psi(\bar{x} \cdot \bar{y}) = \psi(\bar{x}) + \psi(\bar{y})$. That is, $\psi$ is a surjective partial semigroup homomorphism from $S$ to $G$, so by [4, Theorem 4.22.3], the restriction of $\tilde{\psi}$ to $\delta S$ is a homomorphism into $\delta G = \beta G_d$.

Given $n \in \mathbb{N}$, $\psi[U_n] = T_n$ so $\tilde{\psi}[\delta S \setminus \{\bar{1}\}] = \tilde{\psi}[(\bigcap_{n=1}^\infty c\ell_{\beta S_d}[U_n]) \setminus \{\bar{1}\}] = (\bigcap_{n=1}^\infty c\ell_{\beta G_d} T_n) \setminus \{0\} = O(G) \setminus \{0\}$. □

**Corollary 4.5.** For $n \in \mathbb{N}$, let $S_n$ be a countable id with identity 1 and at least two members and let $S = \bigoplus_{n=1}^\infty S_n$. Let $\bar{1} = \{1\}_{n \in \mathbb{N}} \in S$. Give $S$ the topology it inherits from $\bigtimes_{n=1}^\infty S_n$ where each $S_n$ is discrete. For $n \in \mathbb{N}$, let $U_n = \{\bar{1}\} \cup \{\bar{x} \in S \setminus \{\bar{1}\} : \min \text{supp}(\bar{x}) > n\}$. Then $O(S) = \bigcap_{n=1}^\infty c\ell_{\beta S_d} U_n$ and $O(S) \setminus \{\bar{1}\}$ is topologically isomorphic to $\mathbb{H}$.

**Proof.** This follows immediately from Theorem 4.4 and the fact that the sets $U_n$ are a base for the neighborhoods of $\bar{1}$ in $S$. ■

**Corollary 4.6.** Enumerate the primes as $\langle p_n \rangle_{n=1}^\infty$. For $n \in \mathbb{N}$, let $T_n = \{x \in \mathbb{N} : \text{for all } i \in \{1, \ldots, n\}, \text{ } p_i \text{ } \text{does not divide } x\}$. Then the semigroup $((\bigcap_{n=1}^\infty c\ell_{\beta \mathbb{N}} T_n) \setminus \{1\}, \cdot)$ is topologically isomorphic to $\mathbb{H}$.

**Proof.** $(\mathbb{N}, \cdot)$ is isomorphic to $((\bigoplus_{n=1}^\infty \mathbb{Z}, \cdot))$ so this follows from Corollary 4.5 with $S_n = \omega$ for each $n$. ■

We write $\mathbb{Q}^+ = \mathbb{Q} \cap (0, \infty)$. Given $x \in \mathbb{Q}^+$ and a prime $p$ we say that $p$ does not occur in $x$ provided that when $x$ is written in lowest terms, $p$ does not divide either the numerator or denominator of $x$.

**Corollary 4.7.** Enumerate the primes as $\langle p_n \rangle_{n=1}^\infty$. For $n \in \mathbb{N}$, let $M_n = \{x \in \mathbb{Q}^+ : \text{for all } i \in \{1, \ldots, n\}, \text{ } p_i \text{ } \text{does not occur in } x\}$. Then the semigroup $((\bigcap_{n=1}^\infty c\ell_{\beta \mathbb{Q}_d} M_n) \setminus \{1\}, \cdot)$ is topologically isomorphic to $\mathbb{H}$.

**Proof.** $(\mathbb{Q}^+, \cdot)$ is isomorphic to $((\bigoplus_{n=1}^\infty \mathbb{Z}, \cdot))$ so the assertion follows from Corollary 4.5 with each $S_n = \mathbb{Z}$. ■

This last corollary raises the question of whether, letting

$$R_n = \{x \in \mathbb{Q} \setminus \{0\} : \text{for all } i \in \{1, \ldots, n\}, \text{ } p_i \text{ } \text{does not occur in } x\},$$

one has $((\bigcap_{n=1}^\infty c\ell_{\beta \mathbb{Q}_d} R_n) \setminus \{1\}, \cdot)$ topologically isomorphic to $\mathbb{H}$. That is easily seen to be false because $-1 \in (\bigcap_{n=1}^\infty c\ell_{\beta \mathbb{Q}_d} R_n) \setminus \{1\}$ and $\mathbb{H}$ has no isolated points. Naturally, we ask whether $((\bigcap_{n=1}^\infty c\ell_{\beta \mathbb{Q}_d} R_n) \setminus \{1, -1\}, \cdot)$ is topologically isomorphic to $\mathbb{H}$. We see in fact it is not even isomorphic to any subset of $\beta \mathbb{N}$.■
THEOREM 4.8. Enumerate the primes as \( \langle p_n \rangle_{n=1}^\infty \). For \( n \in \mathbb{N} \), let \( R_n \) be as above. Then \( ((\bigcap_{n=1}^\infty \text{cl}_{\beta\mathbb{Q}_d} R_n) \setminus \{1, -1\}, \cdot) \) is not isomorphic to any subset of \( \beta\mathbb{N} \).

Proof. Let \( X = (\bigcap_{n=1}^\infty \text{cl}_{\beta\mathbb{Q}_d} R_n) \setminus \{1, -1\} \). For \( p \in \beta\mathbb{Q}_d \) let \( -p = (-1) \cdot p \). By \cite[Theorem 4.23]{4}, \( \mathbb{Q} \) is contained in the center of \( (\beta\mathbb{Q}_d, \cdot) \) so for \( p, q \in \beta\mathbb{Q}_d \), \((-p) \cdot (-q) = p \cdot q \) and \((-p) \cdot q = p \cdot (-q) = -(p \cdot q) \). For \( n \in \mathbb{N} \), \((-1) \cdot R_n = R_n \) so if \( p \in X \), then \(-p \in X \). By \cite[Theorem 4.20]{4}, \( (X, \cdot) \) is a compact right topological semigroup, so pick an idempotent \( p \in X \). Then \( \{p, -p\} \) is a two-element topological subgroup of \( X \). By Zelenyuk’s Theorem \cite[Theorem 7.17]{4}, \( (\mathbb{N}^*, +) \) has no nontrivial finite groups, so neither does \( (\beta\mathbb{N}, +) \).

We say that a commutative semigroup \( S \) with identity \( 0 \) has a base if there exist sequences \( \langle b_n \rangle_{n=1}^\infty \) in \( S \) and \( \langle s_n \rangle_{n=1}^\infty \) in \( \mathbb{N} \setminus \{1\} \) such that each \( x \in S \setminus \{0\} \) has a unique expression in the form \( \sum_{n\in F} m_x(n)b_n \), where \( F \in \mathcal{P}_f(\mathbb{N}) \) and \( m_x \in \bigwedge_{n\in F} \{1, \ldots, k_n - 1\} \). We let \( \text{supp}(x) = F \). We have seen many examples of semigroups that have a base.

THEOREM 4.9. Let \( S \) be a semigroup with a base and let \( \langle b_n \rangle_{n=1}^\infty \) and \( \langle s_n \rangle_{n=1}^\infty \) be as in the definition. For \( n \in \mathbb{N} \), let \( T_n = \{0\} \cup \{x \in S \setminus \{0\} : \text{min supp}(x) \geq n\} \). Then \( (\bigcap_{n=1}^\infty \text{cl}_{\beta S_d} T_n) \setminus \{0\} \) is topologically isomorphic to \( \mathbb{H} \).

Proof. Let \( G = \bigoplus_{n=1}^\infty \mathbb{Z}_{s_n} \) and let \( G \) have the topology inherited from \( \bigwedge_{n=1}^\infty \mathbb{Z}_{s_n} \) with the product topology. As in the proof of Theorem 4.4, we find that \( \text{O}(G) \setminus \{0\} \) is topologically isomorphic to \( \mathbb{H} \).

For \( n \in \mathbb{N} \), define \( \overline{c}_n \in G \) by \( \overline{c}_n(n) = 1 \) and \( \overline{c}_n(j) = 0 \) if \( j \neq n \). Define \( \varphi : S \to G \) by \( \varphi(0) = 0 \) and if \( x \in S \setminus \{0\} \), \( \varphi(x) = \sum_{n\in \text{supp}(x)} m_x(n) \overline{c}_n \). Let \( \overline{\varphi} : \beta S_d \to \beta G_d \) be the continuous extension of \( \varphi \). Then \( \overline{\varphi}[\bigcap_{n=1}^\infty \text{cl}_{\beta S_d} T_n] = \text{O}(G) \) and by \cite[Theorem 4.21]{4} the restriction of \( \overline{\varphi} \) to \( \bigcap_{n=1}^\infty \text{cl}_{\beta S_d} T_n \) is a homomorphism, and since \( \varphi \) is bijective, this restriction is an isomorphism and a homeomorphism.

As a consequence of Theorem 4.9, for any \( k \in \mathbb{N} \setminus \{1\} \), using base \( k + 1 \), one sees that \( \bigcap_{n=1}^\infty \text{cl}_{\beta N} (k^n N) \) is topologically isomorphic to \( \mathbb{H} \); using base \(- (k + 1) \), one sees that \( \bigcap_{n=1}^\infty \text{cl}_{\beta \mathbb{Z}} (k^n) \setminus \{0\} \) is topologically isomorphic to \( \mathbb{H} \). Using the factorial base, one can also conclude that \( \bigcap_{n=1}^\infty \text{cl}_{\beta \mathbb{N}} (n N) \) is topologically isomorphic to \( \mathbb{H} \), a fact that we knew already because \( \bigcap_{n=1}^\infty \text{cl}_{\beta \mathbb{N}} (n N) = \bigcap_{n=1}^\infty \text{cl}_{\beta \mathbb{N}} (n N) \).

If for each \( n \in \mathbb{N} \), \( S_n \) is a countable dense subsemigroup of \( ([0, \infty), +) \) or of \((\mathbb{R}, +)\), then by Corollary 4.5, if each \( S_n \) is given the discrete topology and \( S = \bigoplus_{n=1}^\infty S_n \) is given the product topology, then \( \text{O}(S) \setminus \{0\} \) is topologically isomorphic to \( \mathbb{H} \). We see now that with additional assumptions the same conclusion follows when \( S_n \) has the relative topology inherited from \( \mathbb{R} \):
Theorem 4.10. For each \( i \in \mathbb{N} \), let \( S_i \) be a countable dense subsemigroup of \(([0, \infty), +)\) such that whenever \( a \leq b \) in \( S_i \), one has \( b - a \in S_i \), and let \( S_i \) have the relative topology from \( \mathbb{R} \) with its usual topology. Let \( S = \bigoplus_{i=1}^{\infty} S_i \) with the topology inherited from \( \bigtimes_{i=1}^{\infty} S_i \) with the product topology. Then \( O(S) \setminus \{ \vec{0} \} \) is topologically isomorphic to \( \mathbb{H} \).

Proof. As in the proof of Theorem 3.8, we give \( \mathbb{R} \) the zero-dimensional first countable topology which has the sets of the form \([a, b)\), where \( a, b \in \mathbb{R} \) and \( a \leq b \), as a base. For each \( n \in \mathbb{N} \), let \( \mathbb{R}_n \) denote a copy of \( \mathbb{R} \) with this topology, and let \( T_n \) denote \( S_n \) with the relative topology that it has as a subspace of \( \mathbb{R}_n \). It is easy to check that the hypotheses of Theorem 3.1 are satisfied with \( G = \bigoplus_{n=1}^{\infty} \mathbb{R}_n \) and \( X = \bigoplus_{n=1}^{\infty} T_n \). So \( O(T) \) is a copy of \( \mathbb{H} \). Our claim now follows from the fact that the neighborhoods of 0 in \( S \) and in \( \bigoplus_{n=1}^{\infty} T_n \) coincide. \( \blacksquare \)

Alternatively, one may prove Theorem 4.10 using Corollary 5.4 instead of Theorem 3.1 by letting \( T_n = \{ \vec{x} \in S : (\forall i \in \{1, \ldots, n\})(x_i \in [0, 1/n) \cap S_i) \} \).

Theorem 4.11. For each \( i \in \mathbb{N} \), let \( S_i \) be a countable dense subgroup of \((\mathbb{R}, +)\) and let \( S_i \) have the relative topology from \( \mathbb{R} \) with its usual topology. Let \( S = \bigoplus_{i=1}^{\infty} S_i \) with the topology inherited from \( \bigtimes_{i=1}^{\infty} S_i \) with the product topology. Then \( O(S) \setminus \{ \vec{0} \} \) is topologically isomorphic to \( \mathbb{H} \).

Proof. This is an immediate consequence of Corollary 3.2. \( \blacksquare \)

5. Generalization of Theorem 3.1. In this section we derive Theorem 5.3, which is a generalization of Theorem 3.1. It replaces the assumption that \( G \) is a group by the assumption that \( G \) is a cancellative semigroup with identity.

The proof of Theorem 5.3 is a consequence of Theorem 5.2 which is adapted from the proof of [4, Lemma 7.4] whose main ideas are due to Yevhen Zelenyuk. The proof of Theorem 5.2 is simpler than the proof of [4, Lemma 7.4] because it proves less. (Here we only include the conclusions needed to prove Theorem 5.3 in [4], additional conclusions were needed for other applications of Lemma 7.4.)

We now incorporate some of the notation from [4, Section 7.1].

Definition 5.1.

(a) \( F \) will denote the free semigroup on the letters 0 and 1 with identity \( \emptyset \).
(b) If \( m \in \omega \) and \( i \in \{0, 1, \ldots, m\} \), then \( s_i^m \) will denote the element of \( F \) consisting of \( i \) 0’s followed by \( m - i \) 1’s. We also write \( u_m = s_m^m \) (so \( u_0 = \emptyset \)).
(c) If \( s \in F \), then \( l(s) \) will denote the length of \( s \) and \( \text{supp}_F(s) = \{ i \in \{1, \ldots, l(s)\} : s_i = 1 \} \) where \( s_i \) is the \( i \)th letter of \( s \).

(d) If \( s, t \in F \), then we shall write \( s \ll t \) if \( \max \text{supp}_F(s) + 1 < \min \text{supp}_F(t) \).

(e) If \( s, t \in F \), then we define \( s + t \) to be the element of \( F \) for which \( l(s + t) = \max\{l(s), l(t)\} \) and \( (s + t)_i = 1 \) if and only if \( s_i = 1 \) or \( t_i = 1 \).

(f) Given any \( t \in F \), \( \text{t} \) has a unique representation in the form \( t = s_{i_0}^{m_0} + s_{i_1}^{m_1} + \cdots + s_{i_k}^{m_k} \) where \( 0 \leq i_0 < m_0 < i_1 < m_1 < \cdots < i_k \leq m_k \) (except that, if \( k = 0 \), the requirement is \( 0 \leq i_0 \leq m_0 \)). We shall call this the canonical representation of \( t \). When we write \( t = s_{i_0}^{m_0} + s_{i_1}^{m_1} + \cdots + s_{i_k}^{m_k} \), we shall assume that this is the canonical representation.

(g) Given \( t \in F \), if \( t = s_i^m \) for some \( i, m \in \omega \), then \( t' = \emptyset \) and \( t^* = t \). Otherwise, if \( t = s_{i_0}^{m_0} + s_{i_1}^{m_1} + \cdots + s_{i_k}^{m_k} \), then \( t' = s_{i_0}^{m_0} + s_{i_1}^{m_1} + \cdots + s_{i_k}^{m_k} \) and \( t^* = s_{i_k}^{m_k+1} \).

For example \( 011001110001 = s_1^3 + s_5^8 + s_{12}^2 \) and \( 101110 = s_0^1 + s_2^5 + s_6^6 \).

**Theorem 5.2.** Assume that \( G \) is a cancellative semigroup with identity \( e \) and that \( X \) is a countably infinite subset of \( G \) with \( e \in X \). Well order \( X \) in order type \( \omega \). Assume that \( G \) has a Hausdorff topology and there is a decreasing sequence \( \langle T_n \rangle_{n=1}^\infty \) of subsets of \( G \) such that \( \{ T_n : n \in \mathbb{N} \} \) is a neighborhood base for \( e \) consisting of sets clopen in \( G \), and for each \( a \in X \) there exists \( m \in \mathbb{N} \) such that \( \{ aT_n : n \geq m \} \) is a neighborhood base for a consisting of sets clopen in \( G \). Assume that \( X \) has no isolated points, for each \( a \in X \), \( aX \cap X \) is a neighborhood of \( a \) in \( X \), and for each \( a \in X \), we have a neighborhood \( V(a) \) of \( e \) in \( X \) such that \( aV(a) \subseteq X \) and \( V(e) = X \). Assume that \( \langle W_n \rangle_{n=1}^\infty \) is a sequence of neighborhoods of \( e \) in \( X \).

Then we can define \( x(t) \in X \) and \( X(t) \subseteq X \) for each \( t \in F \) such that

(i) \( a_k = \min X \setminus \{ x(t) : t \in F \text{ and } l(t) \leq k \} \);

(ii) \( a_k \in X(t_k \setminus 1) \);

(iii) \( a_k = x(t_k \setminus 1) \); and

(iv) \( l(t_k) = k \).

Moreover, for each \( t \in F \),

(1) \( X(t) \) is clopen in \( X \);

(2) \( x(t) \in X(t) \);

(3) \( X(t) = X(t \setminus 0) \cup X(t \setminus 1) \) and \( X(t \setminus 0) \cap X(t \setminus 1) = \emptyset \);

(4) \( x(t \setminus 0) = x(t) \);

(5) \( x(t) = x(t')x(t^*) \);

(6) \( X(t) = x(t')X(t^*) \); and

(7) \( X(t^*) \subseteq V(x(t')) \).
Further,

(8) for \( s, t \in F \), \( x(s) = x(t) \) if and only if \( s = t, s = t^{\sim}0 \ldots 0 \) or \( t = s^{\sim}0 \ldots 0 \);

(9) for each \( n \in \mathbb{N} \setminus \{1\} \), \( X(u_n) \subseteq W_{n-1} \);

(10) if \( t, r \in F \) and \( t \ll r \), then \( x(t + r) = x(t)x(r) \); and

(11) for every \( n \in \omega \), \( x[u_nF] = X(u_n) \) and in particular \( x[F] = X \).

Proof. Let \( U = \{ T_n : n \in \mathbb{N} \} \). The proof proceeds by induction on \( l(t) \) establishing that induction hypotheses (i) through (iv) and (1) through (9) hold.

When the induction is complete, conclusions (10) and (11) will be established.

We let \( x(\emptyset) = e \) and \( X(\emptyset) = X \). Let \( a_0 = \min (X \setminus \{ e \}) \), let \( t_0 = \emptyset \), pick \( U_0 \in U \) such that \( a_0 \notin U_0 \). Let \( X(0) = U_0 \cap X \), \( X(1) = X \setminus U_0 \), \( x(0) = e \), and \( x(1) = a_0 \).

Now let \( n \in \mathbb{N} \), and assume we have chosen \( \langle a_k \rangle_{k=0}^{n-1}, \langle t_k \rangle_{k=0}^{n-1}, \) and \( x(t) \) and \( X(t) \) for all \( t \in F \) with \( l(t) \leq n \) satisfying (i)–(iv) and (1)–(9), where as an induction hypothesis (8) includes the assertion that \( l(s) \leq n \) and \( l(t) \leq n \) and (3) and (4) include the assertion that \( l(t) < n \).

We first verify the conclusions for \( n = 1 \) and \( l(t) \leq 1 \). One can simply check (i)–(iv) and (1)–(4). For (5)–(7), note that \( 0' = \emptyset = 0 \) and \( 0' = 1' = 0 \), \( 0* = 0 \), and \( 1* = 1 \). For (8), if \( l(s) \leq 1, l(t) \leq 1, t \neq s \) and \( x(t) = x(s) \), then \( \{ t, s \} = \{ 0, 0 \} \). Conclusion (9) is vacuous. Thus the conclusions are satisfied for \( n = 1 \).

By (3), \( \{ X(t) : l(t) = n \} \) is a partition of \( X \). Let

\[
a_n = \min (X \setminus \{ x(t) : l(t) \leq n \})
\]

and pick the unique \( t_n \) with \( l(t) = n \) such that \( a_n \in X(t_n) \). Then (i) and (iv) are satisfied. (We have to wait until \( X(t_n^{\sim}1) \) and \( x(t_n^{\sim}1) \) have been defined to verify (ii) and (iii).)

By (6), \( X(t_n) = x(t_n^*)X(t_n^*) \) so pick \( c_n \in X(t_n^*) \) such that \( a_n = x(t_n^*)c_n \).

For \( s \in \{ s_i^n : i \in \{ 0, 1, \ldots, n \} \} \), choose \( b_s \in X(s) \) such that \( b_s \neq x(s) \). (Recall that \( X \) has no isolated points so \( X(s) \) is not finite.) If \( s = t_n^* \), let \( b_s = c_n \). (By (5), \( x(t_n) = x(t_n^*)x(t_n^*) = x(t_n^*)x(s) \) and \( a_n = x(t_n^*)c_n \). Since \( a_n \neq x(t_n) \), \( c_n \neq x(s) \).) Define \( x(s^{\sim}0) = x(s) \) and \( x(s^{\sim}1) = b_s \).

Now assume that \( t \in F \), \( l(t) = n \), and \( t \notin \{ s_i^n : i \in \{ 0, 1, \ldots, n \} \} \). Then \( t^* \in \{ s_i^n : i \in \{ 0, 1, \ldots, n \} \} \) so \( x(t^{\sim}1) \) has been defined. We define \( x(t^{\sim}0) = x(t) \) and \( x(t^{\sim}1) = x(t')x(t^{\sim}1) \). Note that if \( s \in \{ s_i^n : i \in \{ 0, 1, \ldots, n \} \} \), then \( s' = \emptyset \) and \( s^* = s \) so \( x(s^{\sim}1) = ex(s^{\sim}1) = x(s')x(s^{\sim}1) \).

We choose \( U_n \in U \) such that

(a) \( U_n \cap X \subseteq W_n \),
(b) \( U_n \cap X \subseteq V(x(v)) \) for all \( v \in F \) with \( l(v) \leq n \),
(c) \(x(v)U_n \cap X \subseteq x(v)X\) and \(x(v)U_n \cap X\) is clopen in \(X\) for all \(v \in F\) with \(l(v) \leq n\),
(d) \(a_n \notin x(t_n)U_n\),
(e) if \(t \in F\) and \(l(t) = n\), then \(x(t)U_n \cap X \subseteq X(t)\), and
(f) if \(t \in F\) and \(l(t) = n\), then \(x(t^{-1}) \notin x(t)U_n\).

It is enough to show that for each of the items, a member of \(\mathcal{U}\) can be found satisfying that item. It is trivially possible to satisfy (a) and (b). For (c), let \(v \in F\) with \(l(v) \leq n\). Pick \(m \in \mathbb{N}\) such that \(\{x(v)T_n : n \geq m\}\) is a neighborhood base for \(x(v)\) in \(G\) consisting of sets clopen in \(G\). Also \(x(v)X \cap X\) is a neighborhood of \(x(v)\) in \(X\). So we may pick \(k \geq m\) such that \(x(v)T_k \cap X \subseteq x(v)X \cap X\). For (d) we use the fact that the topology is Hausdorff and \(a_n \notin \{x(t) : l(t) \leq n\}\). For (e), given \(t \in F\) with \(l(t) = n\) we see that \(x(t)\) is in \(X(t)\), which is clopen in \(X\), and for sufficiently large \(k\), \(x(t)T_k\) is clopen in \(G\). To see that we can satisfy (f), assume \(t \in F\) with \(l(t) = n\). By (5), \(x(t) = x(t')x(t^*)\), \(x(t^*-1) = b_{t^*} \neq x(t^*)\), and \(x(t^{-1}) = x(t')x(t^*-1)\), so \(x(t^{-1}) \neq x(t)\).

Let \(X(t^{-0}) = x(t)U_n \cap X\) and \(X(t^{-1}) = X(t) \setminus X(t^{-0})\). This completes the inductive definitions for \(n\).

We claim that for any \(v \in F\) with \(l(v) \leq n\), \(x(v)U_n \cap X = x(v)(U_n \cap X)\), so let such \(v\) be given. By (c), \(x(v)U_n \cap X \subseteq x(v)X\) so
\[x(v)U_n \cap X \subseteq x(v)U_n \cap x(v)X = x(v)(U_n \cap X)\]
Also, by (b), \(U_n \cap X \subseteq V(x(v))\) and, by the hypotheses of the theorem, \(x(v)V(x(v)) \subseteq X\), so \(x(v)(U_n \cap X) \subseteq x(v)V(x(v)) \subseteq X\), and consequently \(x(v)(U_n \cap X) \subseteq x(v)U_n \cap X\).

We need to verify that (ii), (iii), and (1)–(9) are satisfied. For (ii), using (d) we have \(a_n \notin x(t_n)U_n\) so \(a_n \notin X(t_n^{-0})\). Also, \(a_n \in X(t_n)\), so \(a_n \in X(t_n)^{-1} = X(t_n^{-1})\).

For (iii) we have \(a_n = x(t_n')c_n = x(t_n')b_{t_n^*} = x(t_n')x(t_n^*^{-1})\). If \(t_n \notin \{s_i^n : i \in \{0, 1, \ldots, n\}\}\), then we defined \(x(t_n^{-1}) = x(t_n')x(t_n^*^{-1})\) so \(a_n = x(t_n^{-1})\). If \(t_n \in \{s_i^n : i \in \{0, 1, \ldots, n\}\}\), then \(t_n' = \emptyset\) and \(t_n^* = t_n\) so \(x(t_n^{-1}) = ex(t_n^*^{-1}) = x(t_n')x(t_n^*^{-1})\) and thus \(a_n = x(t_n^{-1})\).

To verify (1)–(7), let \(t \in F\) with \(l(t) = n\). Since \(x(t)U_n \cap X\) is clopen in \(X\), conclusion (1) holds immediately for \(t^{-0}\) and \(t^{-1}\). For (2), \(x(t^{-0}) = x(t) \in x(t)U_n \cap X = X(t^{-0})\). Also, \(x(t^{-1}) = x(t')x(t^*^{-1}) \in x(t')X(t^*) = X(t)\) by (6), and \(x(t^{-1}) \notin x(t)U_n\) by (f), so \(x(t^{-1}) \in X(t) \setminus x(t)U_n \subseteq X(t) \setminus X(t^{-0}) = X(t^{-1})\).

Since \(x(t)U_n \cap X \subseteq X(t)\), conclusion (3) holds immediately, as does (4).

Now let \(v = t^{-1}\) and \(w = t^{-0}\). We show that (5)–(7) hold for \(v\) and \(w\). There are four possibilities for \(t\), namely \(t = s_i^n\), \(t = s_i^0\) for some \(i < n\), \(t = s_i^m + s_i^{m_1} + \cdots + s_i^{m_k}\), or \(t = s_i^m + s_i^{m_1} + \cdots + s_i^{m_k}\) for some \(i_k\) with
Now \( x(v) = x(t')x(t^* \sim 1) = x(v')x(v^*) \) and \( x(w) = x(t) = x(t)e = x(w')x(w^*) \) so (5) holds.

Next observe that \( X(u_{n+1}) = X(u_n \sim 0) = x(u_n)U_n \cap X = U_n \cap X \) so \( X(w) = x(t)U_n \cap X = x(t)(U_n \cap X) = x(w')X(u_{n+1}) = x(w')X(w^*) \) so (6) holds for \( w \). To verify (6) for \( v \), we use the fact that \( l(t^*) = n \):

\[
X(v) = X(t) \setminus X(t \sim 0) = X(t) \setminus (x(t)U_n \cap X) = x(t')(X(t^*) \setminus (x(t')x(t^*)(U_n \cap X))) = x(t')(X(t^*) \setminus (x(t^*)U_n \cap X)) = x(t')(X(t^* \sim 1) = x(v')X(v^*).
\]

To verify (7), note that \( X(v^*) = X(t^* \sim 1) \subseteq X(t^*) \subseteq V(x(t')) = V(x(v')) \) and \( X(w^*) = X(u_{n+1}) = x(u_n)U_n \cap X = U_n \cap X \subseteq V(x(t)) = V(x(w')) \).

Conclusion (8) follows immediately from the fact that for any \( t \) we have \( x(t \sim 0) \in X(t \sim 0), x(t^* \sim 1) \in X(t^* \sim 1), \) and \( X(t \sim 0) \cap X(t^* \sim 1) = \emptyset \).

For (9), we have \( X(u_{n+1}) = X(u_n \sim 0) = x(u_n)U_n \cap X = U_n \cap X \subseteq W_n \).

This completes the inductive construction. We still need to verify that conclusions (10) and (11) hold.

If \( t = s_{i_0}^{m_0} + s_{i_1}^{m_1} + \cdots + s_{i_k}^{m_k} \), one easily shows by induction on \( k \) using (5) that \( x(t) = x(s_{i_0}^{m_0})x(s_{i_1}^{m_1}) \cdots x(s_{i_k}^{m_k}) \). If \( t \ll r, t = s_{i_0}^{m_0} + s_{i_1}^{m_1} + \cdots + s_{i_k}^{m_k} \), and \( r = s_{j_0}^{n_0} + s_{j_1}^{n_1} + \cdots + s_{j_l}^{n_l} \), then the canonical representation of \( t + r \) is \( s_{i_0}^{m_0} + s_{i_1}^{m_1} + \cdots + s_{i_k}^{m_k} + s_{j_0}^{n_0} + s_{j_1}^{n_1} + \cdots + s_{j_l}^{n_l} \), unless \( i_k = m_k \) in which case it is \( s_{i_0}^{m_0} + s_{i_1}^{m_1} + \cdots + s_{i_{k-1}}^{m_{k-1}} + s_{j_0}^{n_0} + s_{j_1}^{n_1} + \cdots + s_{j_l}^{n_l} \). Since \( x(s_{i_k}^{m_k}) = e \), one sees in either case that \( x(t + r) = x(t)x(r) \) so (10) holds.

To verify (11), we note first that each \( a \in X \) occurs eventually as a value of \( x \). For otherwise, let \( a = \min(X \setminus x[F]) \). There are only finitely many predecessors so \( a = a_n \) for some \( n \), and then by (iii), \( a_n = x(t_n \sim 1) \).

Now for \( n \in \omega \) and \( t \in F, x(u_nt) \in X(u_nt) \subseteq X(u_n) \) so \( x[u_nF] \subseteq X(u_n) \).

Suppose that \( a \in X(u_n) \setminus x[u_nF] \). We have seen that \( a = x(v) \) for some \( v \) so \( v \in F \setminus u_nF \). Since \( c = x(u_n) \in x[u_nF], a \neq e \) so \( v \notin \{ u_m : m < \omega \} \) so \( \text{supp}_F(v) \neq \emptyset \). Let \( k = \min \text{supp}_F(v) \). Then \( k \leq n \) since otherwise \( v \notin u_nF \). Therefore \( x(v) \in X(s_{k-1}^k) \) and \( X(u_n) \subseteq X(u_k) \). Since \( X(s_{k-1}^k) \cap X(u_k) = \emptyset \), we have \( a = x(v) \notin X(u_n) \), a contradiction. ■
Theorem 5.3. Assume that $G$ is a cancellative semigroup with identity $e$ and that $X$ is a countably infinite subset of $G$ with $e \in X$. Assume that $G$ has a Hausdorff topology and there is a decreasing sequence $(T_n)_{n=1}^{\infty}$ of subsets of $G$ such that for each $a \in X$ there exists $m \in \mathbb{N}$ such that $\{aT_n : n \geq m\}$ is a neighborhood base for a consisting of sets clopen in $G$. Assume that $X$ has no isolated points, for each $a \in X$, $aX \cap X$ is a neighborhood of $a$ in $X$, and for each $a \in X$, we have a neighborhood $V(a)$ of $e$ in $X$ such that $aV(a) \subseteq X$ and $V(e) = X$. Assume that $(W_n)_{n=1}^{\infty}$ is a neighborhood base for $e$ in $X$ and let $Y = (\bigcap_{n=1}^{\infty} \mathcal{cl}_{\beta X_d} W_n) \setminus \{e\}$. Then $Y$ is is topologically isomorphic to $\mathbb{H}$.

Proof. Pick $x(t)$ and $X(t)$ for $t \in F$ satisfying conclusions (1)–(11) of Theorem 5.2. Given $n \in \mathbb{N}$, $X(u_n)$ is a neighborhood of $x(u_n) = e$ in $X$ so there is $m \in \mathbb{N}$ such that $W_m \subseteq X(u_n)$. By (9), for each $n$, $X(u_{n+1}) \subseteq W_n$. Therefore $V = \bigcap_{n=1}^{\infty} X(u_n) \setminus \{e\}$.

Define $\psi : X \to \omega$ by $\psi(e) = 0$ and for $t \in F \setminus \{e\}$,

$$\psi(x(t)) = \sum_{i \in \text{supp}_F(t)} 2^{i-1}.$$

By (8) we see that $\psi$ is well defined and injective. It is trivially surjective. Let $\tilde{\psi} : \beta X_d \to \beta \omega$ be the continuous extension of $\psi$.

Given $n \in \mathbb{N}$, we have $X(u_n) \setminus \{e\} = x[u_nF] \setminus \{e\}$ so $X(u_n) \setminus \{e\} = \{x(t) : t \in F, \text{supp}_F(t) \neq \emptyset, \text{min}\text{supp}_F(t) > n\}$. Hence, $\psi[X(u_n) \setminus \{e\}] = 2^n \mathbb{N}$ so that $\tilde{\psi}[Y] = \mathbb{H}$.

We show that $Y$ is a semigroup using [4, Theorem 4.20], and that $\tilde{\psi}$ is a homomorphism using [4, Theorem 4.21]. Since $\tilde{\psi}$ is injective, this will suffice.

So let $n \in \mathbb{N}$ and let $v \in X(u_n) \setminus \{e\}$. Pick $t \in F$ such that $\text{supp}_F(t) \neq \emptyset$, $\text{min}\text{supp}_F(t) > n$, and $x(t) = v$. Let $m = \max\text{supp}_F(t) + 1$ and let $w \in X(u_m) \setminus \{e\}$. To complete the proof we will show that $vw \in X(u_n) \setminus \{e\}$ and that $\psi(vw) = \psi(v) + \psi(w)$. Pick $r \in F$ such that $\text{supp}_F(r) \neq \emptyset$, $\text{min}\text{supp}_F(r) > m$, and $x(r) = w$. Then $vw = x(t)x(r) = x(t+r)$ by conclusion (10) and $\text{supp}_F(t+r) = \text{supp}_F(t) \cup \text{supp}_F(r)$. So $x(t+r) \in X(u_n) \setminus \{e\}$ and finally $\psi(vw) = \sum_{i \in \text{supp}_F(t+r)} 2^{i-1} = \sum_{i \in \text{supp}_F(t)} 2^{i-1} + \sum_{i \in \text{supp}_F(r)} 2^{i-1} = \psi(v) + \psi(w)$.

Most of our results involve semigroups written additively, and some are consequences of the following corollary.

Corollary 5.4. Let $(S,+)$ be a countable cancellative semigroup with identity 0. Assume that there exists a decreasing sequence $(T_n)_{n=1}^{\infty}$ of subsets of $S$ and $m : S \to \mathbb{N}$ such that $m(0) = 1$, $\{a + T_n : a \in S$ and $n \geq m(a)\}$ is a base for a Hausdorff topology on $S$ consisting of clopen sets, and $S$ has no isolated points. Assume also that for each $a \in S$, $\{a + T_n : n \geq m(a)\}$ is
a neighborhood base for \( a \). Then \( O(S) = (\bigcap_{n=1}^{\infty} c\ell_{\beta S_d} T_n) \) and \( O(S) \setminus \{0\} \) is topologically isomorphic to \( \mathbb{H} \).

**Proof.** It is immediate that \( O(S) = (\bigcap_{n=1}^{\infty} c\ell_{\beta S_d} T_n) \). We apply Theorem 5.3 with \( G = S = X \). For \( a \in X \), we have \( a + T_{m(a)} \subseteq (a + X) \cap X \) so \( (a + X) \cap X \) is a neighborhood of \( a \) in \( X \). For each \( a \in X \), let \( V(a) = X \).

The proof of Theorem 3.4 using Theorem 5.3 is verbatim the same as its proof in Section 3. The proof of Theorem 3.5 using Theorem 5.3 is slightly shorter than its proof in Section 3 since one may let \( G = \omega \) and dispense with the last paragraph of the proof.

**References**


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