MULTILINEAR OPERATORS BETWEEN ASYMMETRIC NORMED SPACES

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Abstract. We introduce and characterize the continuity of multilinear mappings between asymmetric normed spaces. In particular, we study the completeness properties of the asymmetric normed semilinear space of these mappings. As an application, we prove multilinear versions of the Banach–Steinhaus and closed graph theorems in the framework of asymmetric normed spaces.

1. Introduction and preliminaries. The paper is divided into three sections. After the introductory one, in Section 2 we extend to multilinear mappings the concept of continuity in asymmetric normed spaces. In Section 3 we establish some fundamental theorems: separate continuity of multilinear mappings in asymmetric normed spaces, the multilinear asymmetric Banach–Steinhaus theorem and the closed graph theorem for continuous multilinear operators between asymmetric normed spaces.

The notation used in the paper is in general standard. Let \( m \in \mathbb{N} \) and let \( X_j \ (j = 1, \ldots, m) \), \( Y \) be normed spaces over \( \mathbb{K} \) (either \( \mathbb{R} \) or \( \mathbb{C} \)). A mapping \( T : X_1 \times \cdots \times X_m \to Y \) is called multilinear (or \( m \)-linear) if it is linear separately in each coordinate, i.e. the mappings

\[
T_j : X_j \to Y, \quad x^j \mapsto T(x^1, \ldots, x^j, \ldots, x^m),
\]

are linear for any fixed \( x^k \in X_k, k \neq j \). The linear space of such mappings is denoted by \( L(X_1, \ldots, X_m; Y) \).

An \( m \)-linear mapping \( T : X_1 \times \cdots \times X_m \to Y \) is continuous if it is continuous as a function between two normed spaces. As a consequence, \( T \) is continuous if and only if there is a constant \( C \geq 0 \) such that

\[
\|T(x^1, \ldots, x^m)\| \leq C\|x^1\| \cdots \|x^m\|
\]

for all \( x^j \in X_j \ (j = 1, \ldots, m) \).

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We will denote by $L(X_1,\ldots,X_m;Y)$ the linear space of all continuous $m$-linear mappings from $X_1 \times \cdots \times X_m$ into $Y$. If $Y = \mathbb{K}$, we write $L(X_1,\ldots,X_m)$. It is easy to see that

$$\|T\| = \inf\{C \geq 0 \text{ satisfying (1.1)}\}$$

defines a norm on $L(X_1,\ldots,X_m;Y)$ which is a complete norm when $\|\cdot\|_Y$ is complete. For the general theory of continuous multilinear mappings we refer to [M86], [S10] or [D99].

Regarding the asymmetric normed spaces, a function $p : X \rightarrow \mathbb{R}^+$ is an asymmetric norm on a real linear space $X$ if for any $x,y \in X$ and $\alpha \in \mathbb{R}^+$,

(a) $p(x) = p(-x) = 0$ if and only if $x = 0$,
(b) $p(\alpha x) = \alpha p(x)$,
(c) $p(x + y) \leq p(x) + p(y)$.

We say that the pair $(X,p)$ is an asymmetric normed space.

The asymmetric norm conjugate to $p$ is the function $\overline{p} : X \rightarrow \mathbb{R}^+$ defined by $\overline{p}(x) = p(-x)$. As a consequence, the asymmetric norm $p$ induces a norm $p^s$ on $X$ defined by the formula $p^s(x) = \max\{p(x),p(-x)\}$; this norm is referred to as the symmetrization of $p$.

The asymmetric norm $p$ induces a $T_0$ topology $\tau_p$ on $X$ that is generated by the asymmetric open balls $B_p(x,\varepsilon) = \{y \in X : p(y-x) < \varepsilon\}$, where $\varepsilon > 0$. Moreover the collection $\{B_p(x,\varepsilon) : \varepsilon > 0\}$ forms a fundamental system of neighborhoods for the topology $\tau_p$. However, in general this topology is not Hausdorff.

A sequence $(x_n)_n$ in an asymmetric normed space $(X,p)$ is convergent to $x \in X$ with respect to $\tau_p$ if and only if $\lim_{n \to +\infty} p(x_n - x) = 0$. From this we obtain the following result (see [MCT15, Remark 1.1]).

**Proposition 1.1.** Let $Z$ be a linear subspace of $X$. Then $Z$ is closed in $(X,p)$ if and only if it is closed in $(X,\overline{p})$.

**Completeness in asymmetric normed spaces.** There are several notions of Cauchy sequence and several notions of completeness in asymmetric normed spaces (see [CT13] and [GRS02]). We present only the following notions:

- A sequence $(x_n)_n$ of elements of $X$ is said to be **left (right) $K$-Cauchy** if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $p(x_n - x_k) < \varepsilon$ (resp. $p(x_k - x_n) < \varepsilon$) whenever $n \geq k \geq n_0$.
- $(X,p)$ is called **left (right) $K$-complete** if every left (right) $K$-Cauchy sequence in $X$ is convergent with respect to $\tau_p$.
- $(X,p)$ is called **bicomplete** (or bi-Banach) if the normed space $(X,p^s)$ is complete.
Example 1.2. As an important example, let $u$ be the asymmetric norm on the usual linear space $\mathbb{R}$ defined by

$$u(x) := x^+ = \max\{x, 0\}.$$  

In this case $u = \max\{-x, 0\} = x^{-}$ and $u^* = \max\{-x, x\} = |x|$. Obviously $(\mathbb{R}, u)$ is a bi-Banach space.

Continuous linear operators. Let $(X, p)$ and $(Y, q)$ be asymmetric normed spaces. We denote by $LC(X, Y)$ the set of all continuous linear mappings from $(X, p)$ into $(Y, q)$ and by $LC^s(X, Y)$ the set of all continuous linear mappings from $(X, p^s)$ into $(Y, q^s)$. The set $LC(X, Y)$ is not necessarily a linear space but it is a cone (or normed semilinear space) with $LC(X, Y) \subset LC^s(X, Y)$ (see [GRS03]).

The next result and its consequences can be found in [FGA93] and [GRS03] and will be used later.

Proposition 1.3. A linear mapping $T$ belongs to $LC(X, Y)$ if and only if there is a constant $M > 0$ such that $q(T(x)) \leq Mp(x)$ for all $x \in X$.

Following [GRS03, Theorem 1], we can consider an asymmetric norm on the cone $LC(X, Y)$ of all linear continuous mappings $T$ from $(X, p)$ into $(Y, q)$ defined by the formula

$$p^*_q(T) := \sup\{q(T(x)) : p(x) \leq 1\}$$

and also

$$p^*_q(T) = \inf\{M > 0 : q(T(x)) \leq Mp(x)\}.$$ 

A relevant special case arises when we take $(Y, q) = (\mathbb{R}, u)$. In this case $p^*_u$ will be simply denoted by $p^*$ and we put

$$X^* = \{T : (X, p) \to (\mathbb{R}, u) : T \text{ is linear and continuous}\}.$$ 

This cone is referred to as the dual space of $(X, p)$. Observe that $(X^*, p^*)$ is a bi-Banach cone (see [GRS03, Theorem 1]). For the general theory of asymmetric normed spaces we refer the reader to the monograph [C13].

2. Continuous $m$-linear mappings between asymmetric normed spaces. We give a characterization of continuous multilinear mappings in a way analogous to that used to characterize linear mappings between asymmetric normed spaces. To study the continuity of multilinear mappings between asymmetric normed spaces, we use $N$-asymmetric norms instead of asymmetric norms. We will see the reason for this in Remark 2.4 after characterizing the continuity by means of an inequality.

Definition 2.1. An $N$-asymmetric norm is an asymmetric norm $p$ on the real linear space $X$ for which $p(x) = 0$ implies $x = 0$. We say that the pair $(X, p)$ is an $N$-asymmetric normed space.
For example, on the linear space $\mathbb{R}$ we define the usual $N$-asymmetric norm $p$ by
\begin{equation}
(2.1) \quad p(x) = |x| + \max\{x, 0\}.
\end{equation}
More generally, we can define an $N$-asymmetric norm $p$ on a Banach lattice $X$ by the formula
\[ p(x) = \|x\| + \|\max\{x, 0\}\|. \]
Throughout the paper, $(X_1, p_1), \ldots, (X_m, p_m)$ will be real $N$-asymmetric normed spaces and $(Y, q)$ an asymmetric normed space. Let us consider the space $X_1 \times \cdots \times X_m$ endowed with the $N$-asymmetric norms $p_\infty$ and $s$ defined by
\[ p_\infty(x) = \max_{1 \leq j \leq m} p_j(x^j) \quad \text{and} \quad s(x) = \sum_{j=1}^m p_j(x^j) \]
for all $x = (x^1, \ldots, x^m) \in X_1 \times \cdots \times X_m$. We know that $s$ and $p_\infty$ are equivalent asymmetric norms on $X_1 \times \cdots \times X_m$ for which the induced topology coincides with the product topology (see [A09, Lemma 6]).

**Definition 2.2.** An $m$-linear mapping $T : X_1 \times \cdots \times X_m \to Y$ is continuous if it is continuous as a function between asymmetric normed spaces.

We denote by $LC_{(p_1, \ldots, p_m; q)}(X_1, \ldots, X_m; Y)$ the set of all continuous multilinear mappings between the $N$-asymmetric normed space $X_1 \times \cdots \times X_m$ and the asymmetric normed space $Y$, and by $LC^s_{(p_1^s, \ldots, p_m^s; q^s)}(X_1, \ldots, X_m; Y)$ the normed linear space of all continuous multilinear operators between the normed linear spaces $(X_1, p_1^s) \times \cdots \times (X_m, p_m^s)$ and $(Y, q^s)$.

**Theorem 2.3.** Let $(X_1, p_1), \ldots, (X_m, p_m)$ be $N$-asymmetric normed spaces, $(Y, q)$ an asymmetric normed space and $T : X_1 \times \cdots \times X_m \to Y$ a multilinear mapping. The following statements are equivalent:

(i) $T$ is continuous.
(ii) $T$ is continuous at $(0, \ldots, 0)$.
(iii) There is a constant $M \geq 0$ such that
\begin{equation}
(2.2) \quad q(T(x^1, \ldots, x^m)) \leq Mp_1(x^1) \cdots p_m(x^m) \quad \text{for all } x^j \in X_j, \ j = 1, \ldots, m.
\end{equation}
(iv) We have
\begin{equation}
(2.3) \quad \|T\|_{(p_1, \ldots, p_m; q)} := \sup_{p_j(x^j) \leq 1, j = 1, \ldots, m} q(T(x^1, \ldots, x^m)) < \infty.
\end{equation}

**Proof.** It is easy to see that (iii)$\iff$(iv), and the fact that (i) implies (ii) is obvious.
which yields

\[ p_\infty\left(\frac{rx^1}{2p_1(x^1)}, \ldots, \frac{rx^m}{2p_m(x^m)}\right) = \frac{r}{2} < r \]

for all \((x^1, \ldots, x^m) \in X_1 \times \cdots \times X_m\) with \(x^j \neq 0, j = 1, \ldots, m\), one has

\[ q\left(T\left(\frac{rx^1}{2p_1(x^1)}, \ldots, \frac{rx^m}{2p_m(x^m)}\right)\right) < 1. \]

By multilinearity of \(T\) we obtain (2.2) with \(M = 2^m/r^m\). If \(x^j = 0\) for some \(j = 1, \ldots, m\) we have \(T(x^1, \ldots, x^m) = 0\) and (2.2) remains valid.

(ii)\(\Rightarrow\)(iii). Assume that \(T\) is continuous at \((0, \ldots, 0)\). Then we can choose \(r > 0\) such that \(T(B_{p_\infty}(0, r)) \subset B_q(0, 1)\). Since

\[ p_\infty\left(\frac{rx^1}{2p_1(x^1)}, \ldots, \frac{rx^m}{2p_m(x^m)}\right) = \frac{r}{2} < r \]

we obtain (2.2) with \(M = 2^m/r^m\). If \(x^j = 0\) for some \(j = 1, \ldots, m\) we have \(T(x^1, \ldots, x^m) = 0\) and (2.2) remains valid.

(iii)\(\Rightarrow\)(i). Let us consider the space \(X_1 \times \cdots \times X_m\) endowed with the asymmetric norm \(s\). Let us fix \(a = (a^1, \ldots, a^m) \in X_1 \times \cdots \times X_m\); we prove that \(T(B_s(a, r)) \subset B_q(T(a), \epsilon)\) for every \(\epsilon > 0\), where \(r < \min\{1, \epsilon/(kM)\}\) and \(k = \max_{1 \leq j \leq m}\{p_\infty(a)^{j-1}(1 + p_\infty(a))^{m-j}\}\). Let \(y = T(z) \in T(B_s(a, r))\) with \(z = (z^1, \ldots, z^m)\). Then, using (2.2) and taking into account

\[ T(z) - T(a) = \sum_{j=1}^m T(a^1, \ldots, a^{j-1}, z^j - a^j, z^{j+1}, \ldots, z^m), \]

we obtain

\[ q(y - T(a)) \leq \sum_{j=1}^m M p_1(a^1) \cdots p_{j-1}(a^{j-1}) p_j(z^j - a^j) p_{j+1}(z^{j+1}) \cdots p_m(z^m) \]

\[ \leq \sum_{j=1}^m M p_j(z^j - a^j) p_\infty(a)^{j-1} p_\infty(z)^{m-j}. \]

On the other hand, since \(p_\infty \leq s\), we get

\[ p_\infty(z) \leq s(z - a) + p_\infty(a) < 1 + p_\infty(a), \]

which yields

\[ q(y - T(a)) < \sum_{j=1}^m M p_j(z^j - a^j) k = k M s(z - a) < \epsilon. \]

**Remark 2.4.** For many asymmetric norms, there are no continuous multilinear mappings. Those are the ones that may have non-zero elements \(x^1 \in X_1\) such that \(p_1(x^1) = 0\) (i.e. \(p_1\) is not an \(N\)-asymmetric norm). Indeed, if the \(m\)-linear mapping \(T\) is continuous, then \(T(x^1, \ldots, x^m) = 0\) for all \(x^j \in X_j\) with \(j > 1\). This is obvious, since if \(T(x^1, \ldots, x^m) \neq 0\) we have

\[ q(T(x^1, x^2, \ldots, x^m)) > 0 \quad \text{or} \quad q(T(x^1, -x^2, \ldots, x^m)) > 0, \]

which contradicts the fact that

\[ 0 < q(T(x^1, x^2, \ldots, x^m)) \leq M p_1(x^1)p_2(x^2) \cdots p_m(x^m) = 0 \]
or
\[ 0 < q(T(x^1, -x^2, \ldots, x^m)) \leq Mp_1(x^1)p_2(-x^2) \cdots p_m(x^m) = 0. \]

Now we give an easy example of a continuous bilinear mapping.

**Example 2.5.** We can define the bilinear map \( T : (\mathbb{R}, p) \times (\mathbb{R}, p) \to (\mathbb{R}, u) \) by \( T(x, y) = xy \), where \( u \) is the (usual) asymmetric norm on \( \mathbb{R} \) defined by (1.2) and \( p \) is the usual \( N \)-asymmetric norm defined by (2.1). It is easy to see that
\[ u(T(x, y)) = \max\{xy, 0\} \leq p(x)p(y) \text{ for all } x, y \in \mathbb{R}. \]

**Proposition 2.6.** Let \( T \in LC_{(p_1, \ldots, p_m; q)}(X_1, \ldots, X_m; Y) \). Then
\[ q(T(x^1, \ldots, x^m)) \leq \|T\|_{(p_1, \ldots, p_m; q)}p_1(x^1) \cdots p_m(x^m) \]
for all \( x^j \in X_j, j = 1, \ldots, m \). Moreover \( \|T\|_{(p_1, \ldots, p_m; q)} \) can be calculated also by the formula
\[ \|T\|_{(p_1, \ldots, p_m; q)} = \inf\{M > 0 : M \text{ satisfies } (2.2)\}. \]

**Proof.** For every \( x^j \in X_j \setminus \{0\}, j = 1, \ldots, m \), we get, from (2.3),
\[ q\left(T\left(\frac{x^1}{p_1(x^1)}, \ldots, \frac{x^m}{p_m(x^m)}\right)\right) \leq \|T\|_{(p_1, \ldots, p_m; q)} \]
and we obtain (2.4). If \( x^j = 0 \) for some \( j = 1, \ldots, m \), the inequality is obvious. On the other hand, if \( \lambda \) is the right side of (2.5) then it is clear that \( \lambda \leq \|T\|_{(p_1, \ldots, p_m; q)} \). For the reverse inequality, if \( M > 0 \) satisfies (2.2), it follows that
\[ \|T\|_{(p_1, \ldots, p_m; q)} = \sup_{p_j(x^j) \leq 1, j = 1, \ldots, m} q(T(x^1, \ldots, x^m)) \leq M \]
and so \( \|T\|_{(p_1, \ldots, p_m; q)} \leq \lambda \). □

An immediate consequence of Theorem 2.3 is the following corollary.

**Corollary 2.7.** Let \( m \in \mathbb{N} \). The following are equivalent for a multilinear mapping \( T : X_1 \times \cdots \times X_{2m+1} \to Y \):

(i) \( T \) is continuous from \( (X_1, p_1) \times \cdots \times (X_{2m+1}, p_{2m+1}) \) to \( (Y, q) \).

(ii) \( T \) is continuous from \( (X_1, \overline{p}_1) \times \cdots \times (X_{2m+1}, \overline{p}_{2m+1}) \) to \( (Y, \overline{q}) \).

Consequently,
\[ LC_{(p_1, \ldots, p_{2m+1}; q)}(X_1, \ldots, X_{2m+1}; Y) \subset LC^s_{(p_1, \ldots, p_{2m+1}; q^s)}(X_1, \ldots, X_{2m+1}; Y), \]
and
\[ \|T\| := \|T\|_{(p_1, \ldots, p_{2m+1}; q^s)} \leq \|T\|_{(p_1, \ldots, p_{2m+1}; q)} \]
for all \( T \in LC_{(p_1, \ldots, p_{2m+1}; q)}(X_1, \ldots, X_{2m+1}; Y) \).
Proof. The equivalence (i)⇔(ii) follows from

\[
\sup_{\mathcal{P}_j(x^j) \leq 1} \tilde{q}(T(x^1, \ldots, x^{2m+1})) = \sup_{\mathcal{P}_j(-x^j) \leq 1} q(T(-x^1, \ldots, -x^{2m+1})) = \sup_{\mathcal{P}_j(x^j) \leq 1} q(T(x^1, \ldots, x^{2m+1})).
\]

Thus \(\|T\|_{(p_1, \ldots, p_{2m+1}; q)} = \|T\|_{(p_1, \ldots, \tilde{p}_{2m+1}; q)}\).

Now let \(T \in LC_{(p_1, \ldots, p_{2m+1}; q)}(X_1, \ldots, X_{2m+1}; Y)\). For all \(x^j \in X_j\) such that \(p_j^s(x^j) \leq 1, j = 1, \ldots, 2m+1\), we have

\[
q(T(x^1, \ldots, x^{2m+1})) \leq \|T\|_{(p_1, \ldots, p_{2m+1}; q)}
\]

and

\[
\tilde{q}(T(x^1, \ldots, x^{2m+1})) \leq \|T\|_{(p_1, \ldots, \tilde{p}_{2m+1}; q)}.
\]

This implies that

\[
\|T\| = \sup_{p_j^s(x^j) \leq 1, j=1,\ldots,2m+1} q^s(T(x^1, \ldots, x^{2m+1})) \leq \|T\|_{(p_1, \ldots, p_{2m+1}; q)} < \infty,
\]

and the assertion follows.  

Remark 2.8. As in the linear case, \(LC := LC_{(p_1, \ldots, p_m; q)}(X_1, \ldots, X_m; Y)\) is a cone (or a normed semilinear subspace) of \(LC^{s}_{(p_1, \ldots, p_m; q)}(X_1, \ldots, X_m; Y)\), that is, \(T + S, \alpha T \in LC\) for all \(T, S \in LC\) and all \(\alpha > 0\). We do not know whether this set is a linear space or not.

Now we introduce an asymmetric norm on \(LC_{(p_1, \ldots, p_m; q)}(X_1, \ldots, X_m; Y)\).

Proposition 2.9. \(\|\cdot\|_{(p_1, \ldots, p_m; q)}\) is an asymmetric norm on the cone of all continuous multilinear mappings between \(N\)-asymmetric normed spaces \(X_1 \times \cdots \times X_m\) and \(Y\).

Proof. Let \(T \in LC_{(p_1, \ldots, p_m; q)}(X_1, \ldots, X_m; Y)\) be such that

\[
\|T\|_{(p_1, \ldots, p_m; q)} = 0.
\]

Then for all \(x^j, j = 1, \ldots, m\), we have

\[
0 \leq q(T(x^1, \ldots, x^m)) \leq \|T\|_{(p_1, \ldots, p_m; q)}p_1(x^1) \cdots p_m(x^m) = 0
\]

and

\[
0 \leq q(-T(x^1, \ldots, x^m)) \leq \|T\|_{(p_1, \ldots, p_m; q)}p_1(-x^1) \cdots p_m(x^m) = 0.
\]

Hence \(T(x^1, \ldots, x^m) = 0\) for every \(x^j \in X_j\) (\(j = 1, \ldots, m\)), and therefore \(T = 0\). The remaining properties of the asymmetric norm are easy to verify. 

Corollary 2.10. For all $T \in LC_{s_{p_1,\ldots,p_m;qs}}(X_1,\ldots,X_m;Y)$ we have

\begin{equation}
\|T\| \leq \|T\|_s^{(p_1,\ldots,p_m;qs)}.
\end{equation}

Proof. In order to establish (2.6), we may suppose $\|T\|_s^{(p_1,\ldots,p_m;qs)} < \infty$. Then we obtain $T,-T \in LC_{(p_1,\ldots,p_m;qs)}(X_1,\ldots,X_m;Y)$. For every $x_j \in X_j$, $j = 1,\ldots,m$, we get

$q(T(x^1,\ldots,x^m)) \leq \|T\|_s^{(p_1,\ldots,p_m;qs)} p_1(x^1) \cdots p_m(x^m)$

and

$q^s(T(x^1,\ldots,x^m)) \leq \|T\|_s^{(p_1,\ldots,p_m;qs)} p_1^s(x^1) \cdots p_m^s(x^m).$

Consequently,

$q^s(T(x^1,\ldots,x^m)) \leq \|T\|_s^{(p_1,\ldots,p_m;qs)} p_1^s(x^1) \cdots p_m^s(x^m)$

and thus $\|T\| \leq \|T\|_s^{(p_1,\ldots,p_m;qs)}.$

3. Some fundamental theorems

3.1. Adjoint of multilinear mapping. The definition of the adjoint of an $m$-linear mapping between normed spaces is due to Ramanujan and Schoc \[RS85\]. We now present a similar definition for multilinear mappings between asymmetric normed spaces; the adjoint operator obtained in this way is additive, positively homogeneous and acts from a bi-Banach cone to another bi-Banach cone.

Definition 3.1. Let $(X_1,p_1),\ldots,(X_m,p_m)$ be $N$-asymmetric normed spaces and $(Y,q)$ be an asymmetric normed space.

If $T \in LC_{(p_1,\ldots,p_m;qs)}(X_1,\ldots,X_m;Y)$ we define the adjoint of $T$ by

$T^* : Y^* \to LC_{(p_1,\ldots,p_m;qs)}(X_1,\ldots,X_m;\mathbb{R}), \quad f \mapsto T^*(f) : X_1 \times \cdots \times X_m \to \mathbb{R},$ with

$T^*(f)(x^1,\ldots,x^m) = f(T(x^1,\ldots,x^m)).$

Theorem 3.2. The mapping $T^*$ is additive, positively homogeneous, and bounded, and $\|T^*\| = \|T\|_s^{(p_1,\ldots,p_m;qs)}$, where $\|T^*\|$ is the smallest bounded constant for $T^*$. 
Proof. For all $f \in Y^*$ we can write
\[
\|T^*(f)\|_{(p_1,\ldots,p_m;u)} = \sup_{p_j(x^j) \leq 1} \|u(T(x^1,\ldots,x^m))\|_{p_j(x^j) \leq 1, j=1,\ldots,m} 
\leq q^*(f) \sup_{p_j(x^j) \leq 1} q(T(x^1,\ldots,x^m)) = q^*(f)\|T\|_{(p_1,\ldots,p_m;q)},
\]
which means that $T^*$ is bounded and $\|T^*\| \leq \|T\|_{(p_1,\ldots,p_m;q)}$. In order to establish the reverse inequality, let $x^j \in X_j$ be such that $p_j(x^j) \leq 1$ for all $j = 1,\ldots,m$ and $T(x^1,\ldots,x^m) \neq 0$. By the Hahn–Banach theorem for asymmetric normed spaces (see [CT3, Theorem 2.2.2]), there exists $f \in Y^*$ such that $q^*(f) = 1$ and $f(T(x^1,\ldots,x^m)) = q(T(x^1,\ldots,x^m))$. Hence
\[
\|T^*(f)\|_{(p_1,\ldots,p_m;u)} \geq (f(T(x^1,\ldots,x^m)))^+ = q(T(x^1,\ldots,x^m)),
\]
from which it follows that
\[
\|T^*\| \geq \|T^*(f)\|_{(p_1,\ldots,p_m;u)} 
\geq \sup_{p_j(x^j) \leq 1, j=1,\ldots,m} q(T(x^1,\ldots,x^m)) = \|T\|_{(p_1,\ldots,p_m;q)}.
\]

3.2. Completeness properties. To study the completeness properties of $LC_{(p_1,\ldots,p_m;\alpha)}(X_1,\ldots,X_m;Y)$ we need the following.

Lemma 3.3. The norm $\|\cdot\|_{(p_1,\ldots,p_m;\alpha)}$ can be calculated by
\[
(3.1) \quad \|T\|_{(p_1,\ldots,p_m;\alpha)} = \sup_{p_j(x^j) \leq 1, j=1,\ldots,m} \|q^s(T(x^1,\ldots,x^m))\|
\]
for all $T \in LC^s_{(p_1,\ldots,p_m;\alpha)}(X_1,\ldots,X_m;Y)$.

Proof. Let $\alpha$ be the right side member of (3.1). Then
\[
\|-T\|_{(p_1,\ldots,p_m;\alpha)} = \sup_{p_j(x^j) \leq 1, j=1,\ldots,m} q(-T(x^1,\ldots,x^m)) 
\leq \sup_{p_j(x^j) \leq 1, j=1,\ldots,m} q^s(T(x^1,\ldots,x^m)) = \alpha
\]
and
\[
\|T\|_{(p_1,\ldots,p_m;\alpha)} = \sup_{p_j(x^j) \leq 1, j=1,\ldots,m} q(T(x^1,\ldots,x^m)) 
\leq \sup_{p_j(x^j) \leq 1, j=1,\ldots,m} q^s(T(x^1,\ldots,x^m)) = \alpha.
\]
This implies $\|T\|_{(p_1,\ldots,p_m;\alpha)} \leq \alpha$. Also, for all $x^j \in X_j$ such that $p_j(x^j) \leq 1$, $j = 1,\ldots,m$, we have
\[
q(T(x^1,\ldots,x^m)) \leq \|T\|_{(p_1,\ldots,p_m;\alpha)} \leq \|T\|_{(p_1,\ldots,p_m;\alpha)}^s.
\]
and
\[ q(-T(x^1, \ldots, x^m)) \leq \| -T \|_{(p_1, \ldots, p_m; q)} \leq \| T \|_{(p_1, \ldots, p_m; q)}^{s}. \]

Then \( \alpha \leq \| T \|_{(p_1, \ldots, p_m; q)}^{s} \). ■

**Proposition 3.4.** Let \( m \in \mathbb{N} \). The set \( LC_{(p_1, \ldots, p_m; q)}(X_1, \ldots, X_m; Y) \) is closed in \( (LC_{(p_1, \ldots, p_m; q^{s})}(X_1, \ldots, X_m; Y), \| \cdot \|_{(p_1, \ldots, p_m; q^{s})}) \).

**Proof.** Let \((T_n)_n\) be a sequence in \( LC_{(p_1, \ldots, p_m; q)}(X_1, \ldots, X_m; Y) \) that converges to \( T \in LC_{(p_1, \ldots, p_m; q^{s})}(X_1, \ldots, X_m; Y) \) with respect to \( \| \cdot \|_{(p_1, \ldots, p_m; q^{s})} \). We will show that \( T \in LC_{(p_1, \ldots, p_m; q)}(X_1, \ldots, X_m; Y) \). Indeed, there exists \( n_0 \in \mathbb{N} \) such that
\[
\| T \|_{(p_1, \ldots, p_m; q)} \leq \| T \|_{(p_1, \ldots, p_m; q^{s})} \leq \| T_{n_0} - T \|_{(p_1, \ldots, p_m; q)} + \| T_{n_0} \|_{(p_1, \ldots, p_m; q^{s})} < 1 + \| T_{n_0} \|_{(p_1, \ldots, p_m; q^{s})} < \infty. \]

In the following theorem we show that \( LC_{(p_1, \ldots, p_m; q)}(X_1, \ldots, X_m; Y) \) is bi-Banach if \( Y \) is a bi-Banach asymmetric space.

**Theorem 3.5.** Let \((X_1, p_1), \ldots, (X_m, p_m)\) be \( N \)-asymmetric normed spaces, \((Y, q)\) be an asymmetric normed space and assume that \((Y, q)\) is bi-Banach. Then \( LC_{(p_1, \ldots, p_m; q^{s})}(X_1, \ldots, X_m; Y) \) is a bi-Banach space with respect to the asymmetric norm \( \| \cdot \|_{(p_1, \ldots, p_m; q^{s})} \). Consequently, the space \( (LC_{(p_1, \ldots, p_m; q)}(X_1, \ldots, X_m; Y), \| \cdot \|_{(p_1, \ldots, p_m; q^{s})}) \) is a bi-Banach cone.

**Proof.** Let \((T_n)_n \subset LC_{(p_1, \ldots, p_m; q^{s})}(X_1, \ldots, X_m; Y)\) be a Cauchy sequence with respect to the norm \( \| \cdot \|_{(p_1, \ldots, p_m; q^{s})} \). Hence for all \( \varepsilon > 0 \), there exists \( n_0 \in \mathbb{N} \) such that \( \| T_n - T_k \| \leq \| T_n - T_k \|_{(p_1, \ldots, p_m; q^{s})} < \varepsilon \), for all \( n, k \geq n_0 \). This means that \((T_n)_n\) is a Cauchy sequence in the Banach space
\[ (LC_{(p_1, \ldots, p_m; q^{s})}(X_1, \ldots, X_m; Y), \| \cdot \|_{(p_1, \ldots, p_m; q^{s})}). \]

Thus, there exists \( T \in LC_{(p_1, \ldots, p_m; q^{s})}(X_1, \ldots, X_m; Y) \) such that \( \| T_n - T \| \to 0 \). As \( T_n - T \) is continuous we get
\[ q^s(T_n(x^1, \ldots, x^m) - T(x^1, \ldots, x^m)) \leq \| T_n - T \| p_1^s(x^1) \cdots p_m^s(x^m) \]
and \((T_n(x^1, \ldots, x^m))_n\) is convergent to \( T(x^1, \ldots, x^m) \) in the Banach space \((Y, q^{s})\) for all \( x^j \in X_j, j = 1, \ldots, m \). Then there is \( k \geq n_0 \) such that \( q^s((T - T_k)(x^1, \ldots, x^m)) \leq \varepsilon \). By (3.1), for any \( n \geq n_0 \) and \( x^j \in X_j \) with \( p_j(x^j) \leq 1, j = 1, \ldots, m \), we have
The second claim is a consequence of the first one and of Proposition 3.4.

By taking the supremum over all \(x^j \in X_j\) with \(p_j(x^j) \leq 1, j = 1, \ldots, m\), we obtain

\[
\|T - T_n\|_{(p_1, \ldots, p_m; q)}^s < 2\varepsilon \quad \text{for every } n \geq n_0.
\]

The second claim is a consequence of the first one and of Proposition 3.4.

### 3.3. Separate continuity of multilinear mappings.

A multilinear mapping \(T : X_1 \times \cdots \times X_m \to Y\) between normed spaces is *separately continuous* if it is continuous with respect to each variable while the other variables are fixed. Obviously, continuity implies separate continuity, but the reverse implication is true if \(X_1, \ldots, X_m\) are Banach spaces (see [PM14, p. 4] or [DF92, p. 8]). In the following we see that this result is true in the asymmetric framework under some requirements.

Recall that an asymmetric normed space \((X, p)\) is said to be *of the half second category* if the condition \(X = \bigcup_{n \geq 1} E_n\) implies \(\text{int}_p(\text{cl}_p(E_m)) \neq \emptyset\) for some \(m \in \mathbb{N}\), where \(\text{int}_p(A)\) is the interior of a set \(A\) in \((X, \tau_p)\) and \(\text{cl}_p(A)\) is the closure of \(A\) in \((X, \tau_p)\). Note that if \(p\) is a norm on \(X\), the notion of space of the half second category coincides with the classical notion of space of the second category (see [ARV12] or [MC15]).

The next result, an asymmetric version of the Banach–Steinhaus theorem for linear operators, can be found in [ARV12] and will be used later.

**Theorem 3.6 ([ARV12, Theorem 2.6]).** Let \((X, p)\) and \((Y, q)\) be asymmetric normed spaces. Suppose that \((X, p)\) is of the half second category. If \(\mathcal{F}\) is a family of continuous linear operators such that \(\sup_{T \in \mathcal{F}} q(T(x)) < \infty\) for every \(x \in X\), then

\[
\sup_{T \in \mathcal{F}} \sup_{p(x) \leq 1} \{q(T(x)) : p(x) \leq 1\} < \infty.
\]

**Theorem 3.7.** Let \((X_1, p_1), \ldots, (X_m, p_m)\) be \(N\)-asymmetric normed spaces and \((Y, q)\) an asymmetric normed space. Suppose \((X_j, p_j)\) is of the half second category for all \(j = 1, \ldots, m\). An \(m\)-linear mapping \(T : X_1 \times \cdots \times X_m \to Y\) is separately continuous if and only if \(T\) is continuous.

**Proof.** The “if” part is obvious. We prove the “only if” part by induction on \(m \in \mathbb{N}\). For \(m = 1\) there is nothing to prove. Suppose that the statement is true for \(m - 1\). Given \((x^1, \ldots, x^m) \in X_1 \times \cdots \times X_m\), define the mappings \(T_{x^1, \ldots, x^{m-1}} : (X_m, p_m) \to (Y, q)\) and \(T_{x^m} : (X_1, p_1) \times \cdots \times (X_{m-1}, p_{m-1}) \to (X_m, p_m)\)
By the assumption it is clear that the linear mapping $T_{x_1,\ldots,x_{m-1}}$ is continuous and the $(m-1)$-linear mapping $T_{x_m}$ is separately continuous. Then $T_{x_m}$ is continuous by the inductive hypothesis. Now consider the family

$$F = \{T_{x_1,\ldots,x_{m-1}} : p_j(x^j) \leq 1, j = 1, \ldots, m-1\}.$$ 

The continuity of $T_{x_m}$ implies that

$$\sup_{T_{x_1,\ldots,x_{m-1}} \in F} q(T_{x_1,\ldots,x_{m-1}}(x^m)) < \infty$$

for every $x^m \in X_m$. So by the asymmetric version of the Banach–Steinhaus theorem (Theorem 3.6), we have

$$\sup_{T_{x_1,\ldots,x_{m-1}} \in F} \left( \sup_{p_m(x^m) \leq 1} q(T_{x_1,\ldots,x_{m-1}}(x^m)) \right) < \infty.$$ 

It follows that

$$\|T\|_{(p_1,\ldots,p_m;q)} = \sup_{p_j(x^j) \leq 1, j=1,\ldots,m} q(T(x^1,\ldots,x^m))$$

$$= \sup_{p_j(x^j) \leq 1, j=1,\ldots,m-1} \left( \sup_{p_m(x^m) \leq 1} q(T_{x_1,\ldots,x_{m-1}}(x^m)) \right)$$

$$= \sup_{T_{x_1,\ldots,x_{m-1}} \in F} \left( \sup_{p_m(x^m) \leq 1} q(T_{x_1,\ldots,x_{m-1}}(x^m)) \right) < \infty,$$

which proves that $T$ is continuous from $(X_1,p_1) \times \cdots \times (X_m,p_m)$ to $(Y,q)$. □

3.4. Multilinear asymmetric Banach–Steinhaus theorem. Let $(X_1,p_1),\ldots,(X_m,p_m)$ be $N$-asymmetric normed spaces and $(Y,q)$ be an asymmetric normed space. As in the linear case, we say that a family $F \subset LC_{(p_1,\ldots,p_m;q)}(X_1,\ldots,X_m;Y)$ is called pointwise bounded if

$$\sup_{T \in F} q(T(x^1,\ldots,x^m)) < \infty$$

for every $x^j \in X_j, j = 1,\ldots,m$.

In this case (3.2) is equivalent to

$$\sup_{T \in F} \bar{q}(T(x^1,\ldots,x^m)) < \infty,$$

for every $x^j \in X_j, j = 1,\ldots,m$.

Mimicking the proof of [B09, Theorem 1] we present a Banach–Steinhaus theorem for multilinear mappings between asymmetric normed spaces.

**Theorem 3.8.** Let $(X_1,p_1),\ldots,(X_m,p_m)$ be right $K$-complete $N$-asymmetric normed spaces, $(Y,q)$ an asymmetric normed space and $F$ a pointwise
bounded family in $LC_{(p_1,\ldots,p_m,q)}(X_1,\ldots,X_m; Y)$. Then

$$\sup_{T \in \mathcal{F}} \|T\|_{(p_1,\ldots,p_m,q)} < \infty.$$  

Proof. First, note that the space $X_1 \times \cdots \times X_m$, endowed with the $N$-asymmetric norm $p_{\infty} := \max\{p_1,\ldots,p_m\}$, is right K-complete (see [A09 Lemma 6]). It follows that $X_1 \times \cdots \times X_m$ is also right K-complete with $\overline{p}_{\infty}$ (see [A09 Lemma 4]). For each integer $n \in \mathbb{N}$ consider the subset $F_n$ of $X_1 \times \cdots \times X_m$ consisting of all $(x_1,\ldots,x_m)$ such that

$$\sup_{T \in \mathcal{F}} q(T(x^1,\ldots,x^m)) \leq n \quad \text{and} \quad \sup_{T \in \mathcal{F}} \overline{q}(T(x^1,\ldots,x^m)) \leq n.$$

Each $F_n$ is closed. Perhaps the simplest way to see this is to note that the maps $\sup_{T \in \mathcal{F}} q\circ T$ and $\sup_{T \in \mathcal{F}} \overline{q}\circ T$ are lower semicontinuous on $X_1 \times \cdots \times X_m$ with respect to the $N$-asymmetric norm $p_{\infty}$, since $q,\overline{q} : Y \to \mathbb{R}$ are semicontinuous with respect to $q$ and $\overline{q}$ respectively (see [C13 Proposition 1.18]) and all $T \in \mathcal{F}$ are continuous. In addition, $X_1 \times \cdots \times X_m = \bigcup_{n \geq 1} F_n$, because if $(x_1,\ldots,x_m) \in X_1 \times \cdots \times X_m$, there exist $n_1, n_2 \in \mathbb{N}$ such that

$$\sup_{T \in \mathcal{F}} q(T(x^1,\ldots,x^m)) \leq n_1 \quad \text{and} \quad \sup_{T \in \mathcal{F}} \overline{q}(T(x^1,\ldots,x^m)) \leq n_2.$$

Then $(x^1,\ldots,x^m) \in F_n$ with $n = \max\{n_1,n_2\}$. By the asymmetric Baire category theorem (see [MC15 Theorem 1.11]), there must be some $n_0 \in \mathbb{N}$ such that $F_{n_0}$ contains an open ball $B_{p_{\infty}}(a,r)$ with $(a^1,\ldots,a^m) \in X_1 \times \cdots \times X_m$. It is clear that, for every $(t^1,\ldots,t^m) \in B_{p_{\infty}}(0,r)$ we have $(0,t^2,t^3,\ldots,t^m) \in B_{p_{\infty}}(0,r)$. We can compute, for $T \in \mathcal{F}$,

$$q(T(t^1,a^2+t^2,\ldots,a^m+t^m)) \leq q(T(a^1+t^1,a^2+t^2,\ldots,a^m+t^m)) + \overline{q}(T(a^1,a^2+t^2,\ldots,a^m+t^m)) \leq 2n_0.$$

Using the same argument and taking into account the fact that $(t^1,0,t^3,\ldots,t^m)$ and $(0,t^3,\ldots,t^m)$ belong to $B_{p_{\infty}}(0,r)$, we get $\overline{q}(T(t^1,a^2,a^3+t^3,\ldots,a^m+t^m)) \leq 2n_0$ and then

$$q(T(t^1,t^2,a^3+t^3,\ldots,a^m+t^m)) \leq q(T(t^1,a^2+t^2,\ldots,a^m+t^m)) + \overline{q}(T(t^1,a^2,a^3+t^3,\ldots,a^m+t^m)) \leq 4n_0.$$

By repeating this argument $m$ times, we obtain

$$q(T(t^1,\ldots,t^m)) \leq 2^m n_0 \quad \text{for every } (t^1,\ldots,t^m) \in B_{p_{\infty}}(0,r).$$

On the other hand,
∥T∥_{\text{p}_1,\ldots,\text{p}_m, \text{q}} = \sup\{q(T(x^1, \ldots, x^m)) : p_j(x^j) \leq 1, j = 1, \ldots, m\}
= \sup\{q(T(t^1/r, \ldots, t^m/r)) : p_j(t^j) \leq r, j = 1, \ldots, m\}
\leq \frac{2m n_0}{r^m}.

Since this holds for every \(T \in F\), the result is proved.

3.5. Closed graph theorem. The closed graph theorem for the continuous multilinear operators between asymmetric normed spaces can be easily derived from the asymmetric closed graph theorem in the linear case [MC15, Theorem 4.2] and Theorem 3.7. The proof of this result is similar to the proof of the main result in [F96]. We include the detailed proof for completeness.

Let \(T : X_1 \times \cdots \times X_m \to Y\) be a multilinear mapping between asymmetric spaces \((X_1, \text{p}_1), \ldots, (X_m, \text{p}_m)\) and \((Y, \text{q})\). The graph of \(T\), denoted \(G(T)\), is the set of pairs \(((x^1, \ldots, x^m), y) \in (X_1 \times \cdots \times X_m) \times Y\) such that \(y = T(x^1, \ldots, x^m)\). Consider the space \((X_1 \times \cdots \times X_m) \times Y\) endowed with the asymmetric norm \(r\) defined by
\[r(z) = \text{q}(y) + \sum_{j=1}^{m} \text{p}_j(x^j)\]
for all \(z = ((x^1, \ldots, x^m), y) \in (X_1 \times \cdots \times X_m) \times Y\).

Theorem 3.9. Assume that \((X_j, \text{p}_j), j = 1, \ldots, m,\) are Hausdorff \(N\)-asymmetric normed spaces, right \(K\)-complete and of the half second category, \((Y, \text{q})\) is right \(K\)-complete and \(G(T)\) is closed in \((X_1, \ldots, X_m) \times Y\). Then \(T\) is continuous.

Proof. For every \(j = 1, \ldots, m\), define
\[T_j : X_j \to Y, \quad T_j(x) := T(x^1, \ldots, x^{j-1}, x, x^{j+1}, \ldots, x^m)\]
and set
\[Z_j = G(T) \cap (\{x^1\} \times \cdots \times \{x^{j-1}\} \times X_j \times \{x^{j+1}\} \times \cdots \times \{x^m\}) \times Y.\]
It is easy to check that \(Z_j\) is closed in \((X_1 \times \cdots \times X_m) \times Y\) and \(G(T_j) = \psi(Z_j)\), for every \(j = 1, \ldots, m\), where \(\psi\) is the homeomorphism
\[\psi : (\{x^1\} \times \cdots \times \{x^{j-1}\} \times X_j \times \{x^{j+1}\} \times \cdots \times \{x^m\}) \times Y \to X_j \times Y\]
defined by
\[\psi((x^1, \ldots, x^{j-1}, x, x^{j+1}, \ldots, x^m), y) := (x, y).\]
Then \(G(T_j)\) is closed in \(X_j \times Y\) and by the closed graph theorem (see [MC15, Theorem 4.2]) each \(T_j\) is continuous. It follows that \(T\) is separately continuous. Therefore, by Theorem 3.7 \(T\) is continuous.

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