A construction of Nöbeling manifolds of arbitrary weight

by

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Abstract. For each cardinal $\kappa$, each natural number $n$, and each at most $n$-dimensional simplicial complex $K$ we construct a space $\nu^n_\kappa(K)$ and a map $\pi: \nu^n_\kappa(K) \to K$ such that the following conditions are satisfied:

1. $\nu^n_\kappa(K)$ is a complete $n$-dimensional metric space of weight $\kappa$;
2. $\nu^n_\kappa(K)$ is an absolute neighborhood extensor in dimension $n$;
3. $\nu^n_\kappa(K)$ is strongly universal in the class of complete $n$-dimensional metric spaces of weight $\kappa$; and
4. $\pi$ is an $n$-homotopy equivalence.

For $\kappa = \omega$ these spaces are $n$-dimensional separable Nöbeling manifolds. They have very interesting fractal-like internal structure that allows easy construction, subdivision, and surgery on brick partitions.

1. Introduction. The $n$-dimensional Nöbeling space is the subset $\nu^n$ of $\mathbb{R}^{2n+1}$ consisting of points with at most $n$ rational coordinates. Such spaces were introduced in 1931 by Nöbeling [19]. The $n$-dimensional Nöbeling space is a universal space in the class of $n$-dimensional separable metric spaces, i.e. it contains a topological copy of every $n$-dimensional separable metric space as a subspace.

In the 1980s it was conjectured that $\nu^n$ is an $n$-dimensional analog of the Hilbert space $\ell^2$; see for example [9, Question 7.2], [7, Chapter 5], [14, Question 6.1], [21, TC 10]. The conjecture was inspired by the observation that $\nu^n$ satisfies $n$-dimensional analogs of properties that topologically characterize Hilbert space. The analogy is formalized in the following theorem.

Theorem 1.1. A space $X$ is homeomorphic to $\nu^n$ if and only if the following conditions are satisfied:

1. $X$ is a separable complete $n$-dimensional metric space;

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(2) $X$ is an absolute extensor in dimension $n$; and

(3) $X$ is strongly universal in the class of $n$-dimensional complete separable metric spaces (i.e., every continuous map from a separable $n$-dimensional metric space into $X$ can be arbitrarily closely approximated by closed embeddings).

Observe that for $n = \infty$ we have $\nu^\infty = \mathbb{R}^\infty$, which is homeomorphic to $\ell^2$. Hence for $n = \infty$ this theorem is a reformulation of a famous characterization theorem of Toruńczyk [20]. For $n < \infty$ this was a long-standing open conjecture that was proved independently in 2006 in [1, 2, 3, 15, 16, 18].

Toruńczyk [20] gave a topological characterization of non-separable Banach spaces of weight $\kappa$ that completed the program of topological characterization of Banach spaces begun by Fréchet. The analogous result for Nöbeling spaces is not known. We state it in the form of the following conjecture, which is also a rigidity statement for $n$-dimensional Nöbeling manifolds of weight $\kappa$: A space is a Nöbeling manifold if it is locally homeomorphic to a Nöbeling space. There is an analogous rigidity result for separable Nöbeling manifolds proved in [18]. In particular that result implies that a separable Nöbeling manifold is homeomorphic to $\nu^n$ if and only if it has vanishing homotopy groups in dimensions less than $n$.

**Definition.** A space $X$ is an abstract $n$-dimensional Nöbeling manifold of weight $\kappa$ if it satisfies the following conditions:

(1) $X$ is a complete $n$-dimensional metric space of weight $\kappa$;

(2) $X$ is an absolute neighborhood extensor in dimension $n$; and

(3) $X$ is strongly universal in the class of complete $n$-dimensional metric spaces of weight $\kappa$.

An abstract $n$-dimensional Nöbeling space of weight $\kappa$ is an abstract $n$-dimensional Nöbeling manifold of weight $\kappa$ that has vanishing homotopy groups in dimensions less than $n$.

**Conjecture 1.2.** Two abstract $n$-dimensional Nöbeling manifolds of weight $\kappa$ are homeomorphic if and only if they are weakly $n$-homotopy equivalent.

Note that in the separable case, by the Open Embedding Theorem [18], every separable $n$-dimensional Nöbeling manifold is homeomorphic to an open subset of $\nu^n$.

In the present paper we construct abstract Nöbeling spaces of arbitrary weight $\kappa$ and abstract Nöbeling manifolds that are $n$-homotopy equivalent to an arbitrary simplicial complex $K$.

**Theorem 1.3.** For each cardinal $\kappa$ and each simplicial complex $K$ there exists an abstract $n$-dimensional Nöbeling manifold $\nu^n_\kappa(K^{(n)})$ of weight $\kappa$ and a map $\pi: \nu^n_\kappa(K^{(n)}) \to K$ that is a weak $n$-homotopy equivalence.
In [8, Theorem 2.7] a construction of an abstract $n$-dimensional Nöbeling space of weight $\kappa$ along with an $n$-soft projection onto the Hilbert space of weight $\kappa$ was given. That space is a limit of an inverse sequence of completely metrizable spaces with $n$-soft and $n$-full bonding maps. The space $\nu^n_\kappa(K^{(n)})$ constructed in the present paper is a limit of an inverse sequence of simplicial complexes with quasi-simplicial bonding maps that satisfy a much weaker condition that is a combinatorial analog of $n$-softness and $n$-fullness. This comes at a price: we could not claim $n$-softness of the long projection.

It is acknowledged that the difficulty of proving the characterization theorem for the separable $n$-dimensional Nöbeling space $\nu^n$ for $1 \leq n < \infty$ lay in the fact that both $\nu^0$ and $\nu^\infty$ possess a natural product structure, while $\nu^n$ does not [7]. The spaces that we construct have a very nice internal fractal-like structure, as they are constructed as inverse limits of sequences whose bonding maps possess a high degree of symmetry. This provides an easy construction of brick partitions [13] for these spaces, easy subdivision of these partitions, and allows for easy surgery on these brick partitions. The construction is new and interesting even in the separable case ($\kappa = \omega$), where the characterization theorem is known.

Note that the constructed spaces in the separable case are similar to the universal space $\mathbb{U}_n$ constructed in [4], although the present construction is much simpler. Similar arguments to the ones given here show that $\mathbb{U}_n$ is homeomorphic to $\nu^n$, which answers a question stated in [4].

The spaces $\nu^n_\kappa(K)$ that we construct are Markov spaces in the sense of [5]. Theorem 5.4 gives a sufficient condition for a Markov space to be strongly universal for the class of complete $n$-dimensional metric spaces of weight $\kappa$.

2. Preliminaries. In this section we set the basic definitions and collect some known results that will be used in the later sections.

2.1. Absolute extensors in dimension $n$

**Definition.** We say that a space $X$ is $k$-**connected** if each map $\varphi : S^k \to X$ from a $k$-dimensional sphere into $X$ is null-homotopic in $X$. We let $\mathcal{C}^{n-1}$ denote the class of all spaces that are $k$-connected for each $k < n$.

**Definition.** We say that a space $X$ is **locally $k$-connected** if for each point $x \in X$ and each open neighborhood $U \subset X$ of $x$ there exists an open neighborhood $V$ of $x$ such that each map $\varphi : S^k \to V$ from a $k$-dimensional sphere into $V$ is null-homotopic in $U$. We let $\mathcal{LC}^{n-1}$ denote the class of all spaces that are locally $k$-connected for each $k < n$.

**Definition.** We say that a metric space $X$ is an absolute neighborhood extensor in dimension $n$ if every map from a closed subset $A$ of an $n$-dimensional metric space into $X$ extends over an open neighborhood of $A$. 

The class of absolute neighborhood extensors in dimension \( n \) is denoted by \( \text{ANE}(n) \) and its elements are called \( \text{ANE}(n) \)-spaces.

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**Definition.** We say that a metric space \( X \) is an absolute extensor in dimension \( n \) if every map from a closed subset of an \( n \)-dimensional metric space \( Y \) into \( X \) extends over the entire space \( Y \). The class of absolute extensors in dimension \( n \) is denoted by \( \text{AE}(n) \) and its elements are called \( \text{AE}(n) \)-spaces.

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Absolute extensors and absolute neighborhood extensors in dimension \( n \) \((n < \infty)\) were characterized by Dugundji in the following theorem.

**Theorem 2.1 ([10]).** Let \( X \) be a metric space and let \( n < \infty \). Then

- \( X \in \text{ANE}(n) \Leftrightarrow X \) is locally \( k \)-connected for all \( k < n \), i.e. \( X \in \mathcal{LC}^{n-1} \); and
- \( X \in \text{AE}(n) \Leftrightarrow X \in \text{ANE}(n) \) and if \( X \) is \( k \)-connected for all \( k < n \), i.e. \( X \in \mathcal{C}^{n-1} \).

**2.2. Simplicial complexes.** For the reasons given in [6], we always endow simplicial complexes with the metric topology. We recall the definition presently.

**Definition ([12, p. 100]).** Let \( V \) denote the set of vertices of a simplicial complex \( K \). Let \( \ell_2 \) be a Hilbert space with an orthonormal basis \( \{ \xi_v \}_{v \in V} \), endowed with the \( \| \cdot \|_2 \) norm. Let \( r > 0 \). We embed \( K \) into \( \ell_2 \) by mapping each vertex \( v \) of \( K \) to a vector \( \frac{r}{\sqrt{2}} \xi_v \) in \( \ell_2 \) and by extending the embedding to an affine mapping on every simplex of \( K \) (the proof of Lemma 2.3 explains the choice of the scaling factor). The metric on \( K \) of scale \( r \) is the metric induced on \( K \) by the above embedding; a metric topology on \( K \) is a topology on \( K \) induced by a metric on \( K \) of any scale (they are all equivalent).

**Definition.** Let \( \kappa \) be a cardinal number. A simplicial complex \( \Delta_n(\kappa) \) is a full \( n \)-complex on \( \kappa \) vertices if it has \( \kappa \) vertices and if every set of at most \( n + 1 \) vertices of \( \Delta_n(\kappa) \) spans a simplex in \( \Delta_n(\kappa) \). We allow \( n = \infty \), denote \( \Delta_{\infty}(\kappa) \) by \( \Delta(\kappa) \) and call it a full complex on \( \kappa \) vertices. If \( \kappa \) is countably infinite, then we call \( \Delta(\kappa) \) an infinite full complex.
**Lemma 2.2.** Let $K$ be a simplicial complex endowed with the metric topology.

1. If $K$ is locally finite-dimensional, then $K$ is a complete ANE-space.
2. If $K$ is a full $n$-complex, then $K$ is a complete $AE(n)$-space.

**Proof.** By [12, Lemma 11.5] a simplicial complex with the metric topology is complete if and only if it does not contain an infinite full complex. This shows completeness of both simplicial complexes.

By [12, Theorem 11.3], every simplicial complex with the metric topology is an ANE-space.

By Theorem 2.1, the full $n$-complex is an $AE(n)$ space, as it has vanishing homotopy groups in dimensions less than $n$. ■

**Lemma 2.3.** If a simplicial complex $K$ is endowed with a metric of scale $r$, then the diameter of every simplex of $K$ is at most $r$.

**Proof.** Let $v, w$ be two vertices of $K$. Since $\xi_v$ and $\xi_w$ are orthogonal, we have

$$\left\| \frac{r}{\sqrt{2}} \xi_v - \frac{r}{\sqrt{2}} \xi_w \right\|^2 = \left( \frac{r}{\sqrt{2}} \right)^2 (\|\xi_v\|^2 + \|\xi_w\|^2) = r^2.$$ 

Hence the distance between any two vertices is equal to $r$. The diameter of any simplex of positive dimension is equal to its edge length, hence the diameters do not exceed $r$. ■

### 2.3. $n$-Homotopy equivalence

**Definition ([22]).** We say that a map is a weak $n$-homotopy equivalence if it induces isomorphisms on homotopy groups of dimensions less than $n$, regardless of the choice of basepoint.

**Definition ([22]).** We say that two maps $f, g : X \to Y$ are $n$-homotopic if for every map $\Phi$ from a complex of dimension less than $n$ into $X$, the compositions $f \circ \Phi$ and $g \circ \Phi$ are homotopic in the usual sense.

**Theorem 2.4 ([18]).** A map of two $n$-dimensional ANE($n$)-spaces is a weak $n$-homotopy equivalence if and only if it is an $n$-homotopy equivalence.

### 2.4. Carrier Theorem

**Definition.** Let $\mathcal{C}$ be a class of topological spaces. We let $AE(\mathcal{C})$ denote the class of absolute extensors for all spaces from the class $\mathcal{C}$. We write $AE(X)$ for $AE(\{X\})$.

**Definition.** We say that a cover $\mathcal{U}$ of a space $X$ is locally finite if every point $x$ of $X$ has an open neighborhood that has non-empty intersection with only finitely many elements of $\mathcal{U}$. We say that a cover $\mathcal{U}$ of a space is locally finite-dimensional if the nerve of $\mathcal{U}$ is locally finite-dimensional. We
say that a cover \( \mathcal{U} \) is \textit{closed} if every element of \( \mathcal{U} \) is closed. Let \( \mathcal{C} \) be a class of topological spaces. We say that a cover is a \textit{\( \mathcal{C} \)-cover} if the intersection of each non-empty collection of its elements is either empty or belongs to \( \mathcal{C} \).

**Definition.** Let \( \mathcal{C} \) be a class of topological spaces. A cover of a topological space \( X \) is said to be \textit{regular for the class} \( \mathcal{C} \) if it is a closed, locally finite, locally finite-dimensional \( AE(\mathcal{C}) \)-cover of \( X \).

We write \( f(A) \) for \( f(A \cap \text{dom } f) \) for each map \( f \) and each set \( A \), i.e. we do not require \( A \) to be a subset of the domain of \( f \).

**Definition.** A \textit{carrier} is a function \( C: \mathcal{F} \to \mathcal{G} \) from a cover \( \mathcal{F} \) of a space \( X \) into a collection \( \mathcal{G} \) of subsets of a topological space such that for each \( \mathcal{A} \subset \mathcal{F} \) if \( \bigcap \mathcal{A} \neq \emptyset \), then \( \bigcap_{A \in \mathcal{A}} C(A) \neq \emptyset \). We say that a map \( f \) is \textit{carried by} \( C \) if it is defined on a closed subset of \( X \) and \( f(F) \subset C(F) \) for each \( F \in \mathcal{F} \).

**Carrier Theorem.** Assume that \( C: \mathcal{F} \to \mathcal{G} \) is a carrier such that \( \mathcal{F} \) is a closed cover of a space \( X \) and \( \mathcal{G} \) is an \( AE(X) \)-cover of another space. If \( \mathcal{F} \) is locally finite and locally finite-dimensional, then each map carried by \( C \) extends to a map of the entire space \( X \), also carried by \( C \).

The Carrier Theorem is proved in [17].

**2.5. Local \( k \)-connectedness of inverse limits.** Theorem [2.5] below is proved in [6]. We cite it here for completeness.

**Definition.** Let \( K \) and \( L \) be simplicial complexes. We say that a map \( p: K \to L \) is \textit{n-regular} if it is quasi-simplicial (i.e. it is simplicial into the first barycentric subdivision \( \beta^1 K \) of \( K \)) and if for each simplex \( \delta \) of \( \beta^1 K \) the inverse image \( p^{-1}(\delta) \) has vanishing homotopy groups in dimensions less than \( n \) (regardless of the choice of basepoint).

**Theorem 2.5 ([6]).** Let
\[
X = \varprojlim\left(K_1 \xleftarrow{p_1} K_2 \xleftarrow{p_2} \cdots \right).
\]
Assume that for each \( i \) the following conditions are satisfied:

(I) \( K_i \) is a simplicial complex with the metric topology; and

(II) \( p_i \) is surjective and n-regular.

Then

1. \( X \) is an \( ANE(n) \);
2. each short projection \( \pi_i^k: K_k \to K_i \) and each long projection \( \pi_i: X \to K_i \) is a weak n-homotopy equivalence; and
3. for each \( i \), the covers \( O_i \) and \( B_i \) are \( AE(n) \)-covers of \( X \).
3. A lifting property. In this section we define a lifting property \((qts_{\kappa,n})\) of quasi-simplicial maps. The assumption that bonding maps of an inverse sequence satisfy \((qts_{\kappa,n})\) is sufficient for strong universality of its inverse limit. However, we immediately show that if we work with simplicial complexes of weight \(\kappa\), then any map that satisfies \((qts_{\kappa,n})\) is unique up to a simplicial isomorphism (Lemma 3.3).

**Definition 3.1.** Let \(p: K \to L\) be a quasi-simplicial map. We say that \(p\) has the quasi-simplicial-to-simplicial lifting property with respect to \(n\)-dimensional complexes of weight \(\kappa\) if the following condition is satisfied:

- For each pair \(A \supset B\) of at most \(n\)-dimensional simplicial complexes of weight at most \(\kappa\) and each commutative diagram

\[
\begin{array}{ccc}
K & \xrightarrow{p} & L \\
\uparrow{g} & & \uparrow{\tilde{g}} \\
B & \xrightarrow{\subset} & A
\end{array}
\]

\((qts_{\kappa,n})\)

in which \(g\) is a simplicial map and \(G\) is a quasi-simplicial map, there exists a simplicial map \(\tilde{g}: A \to K\) such that \(\tilde{g}\) is an embedding on \(A \setminus B\), \(p \circ \tilde{g} = G\) and \(\tilde{g}\) is equal to \(g\) on \(B\).

**Lemma 3.2.** If \(p: K \to L\) is a quasi-simplicial map that satisfies \((qts_{\kappa,n})\), then for each simplex \(\delta \in \beta^1L\) the inverse image \(p^{-1}(\delta)\) is a full \(n\)-complex with at least \(\kappa\) vertices. In particular, \(p\) is \(n\)-regular and surjective.

**Proof.** Let \(D\) denote a simplicial complex with \(\kappa\) vertices and no higher dimensional simplices. Let \(v\) be a vertex of \(\beta^1L\). Let \(f: D \to \{v\}\) be the constant map. Since \(f\) is simplicial, by \((qts_{\kappa,n})\) it lifts to a simplicial embedding into \(p^{-1}(v)\), hence \(p^{-1}(v)\) has at least \(\kappa\) vertices. The same holds for \(p^{-1}(\delta)\), where \(\delta\) is any simplex in \(\beta^1L\).

Fix \(\delta\) in the triangulation of \(\beta^1L\) (the triangulation of the first barycentric subdivision of \(L\)). Let \(A\) be a set of vertices in \(p^{-1}(\delta)\) with \(#A \leq n+1\). Let \(\Delta\) be the simplex spanned by \(A\). Let \(G: \Delta \to L\) be a simplicial map such that \(G|_{A} = p|_{A}\). Such a \(G\) exists since \(p(A) \subset \delta\). The dimension of \(\Delta\) is at most \(n\), hence by \((qts_{\kappa,n})\), \(G\) lifts to a map \(\tilde{g}\) into \(K\) such that \(\tilde{g}|_{A} = \text{id}_{A}\). Hence the vertices of \(A\) span a simplex in \(K\), therefore \(p^{-1}(\delta)\) is a full \(n\)-complex with at least \(\kappa\) vertices.

By Lemma 2.2 every full \(n\)-complex is \(AE(n)\), hence \(p\) is \(n\)-regular. \(\blacksquare\)
Lemma 3.3. Let \( p_1: K_1 \to L \) and \( p_2: K_2 \to L \) be a pair of quasi-simplicial maps that satisfy \((\text{QTS}_{K,n})\). If \( K_1 \) and \( K_2 \) have weight at most \( \kappa \), then there is a simplicial isomorphism \( h: K_1 \to K_2 \) such that \( p_2 \circ h = p_1 \).

Proof. Since the cardinality of the vertex sets of \( K_1 \) and \( K_2 \) is bounded by \( \kappa \), Lemma 3.2 implies that for each simplex \( \delta \) in the triangulation of \( \beta^1 L \), the inverse images \( p_1^{-1}(\delta) \) and \( p_2^{-1}(\delta) \) are full \( n \)-complexes. Hence any bijection \( h_0 \) that maps vertices of \( K_1 \) onto vertices of \( K_2 \) and with the property that \( p_2 \circ h_0 = p_1 \) extends to a simplicial isomorphism. \( \blacksquare \)

4. Regular covers of inverse limits of polyhedra

Definition. Let \( X = \lim_{\leftarrow} (K_1 \xleftarrow{p_1} K_2 \xleftarrow{p_2} \cdots) \). We let \( \pi_i^k: K_k \to K_i \) denote the short projections and \( \pi_i: X \to K_i \) denote the long projections.

Definition. Let \( K \) be a simplicial complex. Let \( L \subset K \) be a subcomplex of \( K \). The open star \( \text{ost}_K L \) of \( L \) in \( K \) is the complement of the union of closed simplices of \( K \) that have empty intersection with \( L \):

\[
\text{ost}_K L = K \setminus \bigcup \{ \delta \in \tau(K): \delta \cap L = \emptyset \}.
\]

Definition 4.1. Let \( X = \lim_{\leftarrow} (K_1 \xleftarrow{p_1} K_2 \xleftarrow{p_2} \cdots) \). Let \( v(K_i) \) denote the set of vertices of \( K_i \). We let

\[
\mathcal{O}_{K_i} = \{ O_v = \text{ost}_{K_i} v \}_{v \in v(K_i)}
\]

be the cover of \( K_i \) by the open stars of vertices of \( K_i \) and

\[
\mathcal{O}_i = \{ \pi_i^{-1}(\text{ost}_{K_i} v) \}_{v \in v(K_i)}
\]

be the cover of \( X \) by the sets of threads that pass through elements of \( \mathcal{O}_{K_i} \).

Definition. Let \( \mathcal{F} \) and \( \mathcal{G} \) be families of subsets of a given space. We write \( \mathcal{F} < \mathcal{G} \) if the following condition is satisfied:

- Whenever \( F_1, F_2 \in \mathcal{F} \) and \( F_1 \cap F_2 \neq \emptyset \), there exists a \( G \in \mathcal{G} \) such that \( F_1 \cup F_2 \subset G \).

Lemma 4.2. Let \( X = \lim_{\leftarrow} (K_1 \xleftarrow{p_1} K_2 \xleftarrow{p_2} \cdots) \), where each \( p_i \) is a surjective \( n \)-regular quasi-simplicial map. Then

1. \( \mathcal{O}_i \) is an open \( \text{AE}(n) \)-cover of \( X \);
2. \( \mathcal{O}_{i+1} < \mathcal{O}_i \); and
3. if we endow \( K_i \) with a metric of scale \( 2^{-i/2} \), then

\[
\sum_i \text{mesh} \mathcal{O}_i < \infty.
\]
Proof. Condition (1) follows from Theorem 2.5.

To show (2), observe that if \( \text{ost}_{K_{i+1}} v \cap \text{ost}_{K_{i+1}} w \neq \emptyset \), where \( v, w \) are vertices in \( K_{i+1} \), then both \( p_i(v) \) and \( p_i(w) \) are adjacent to a single vertex \( z \) of \( K_i \) (since \( p_i \) is quasi-simplicial). Then \( \pi^{-1}_{i+1}(\text{ost}_{K_{i+1}} v) \cup \pi^{-1}_{i+1}(\text{ost}_{K_{i+1}} w) \subset \pi^{-1}_i(\text{ost}_{K_i} z) \).

Condition (3) follows from Lemma 2.3. 

**Lemma 4.3.** Let \( U_i \) be a sequence of covers with the property that \( U_{i+1} \subset U_i \). Let \( x_i \) be a sequence of points such that \( x_{i+1} \in \text{st}_{U_i} x_i \). Then for each \( k \) and each \( i \geq k \) we have \( x_i \in \text{st}_{U_k} x_k \).

**Proof.** Fix \( i \geq k \). Since \( x_i \in \text{st}_{U_{i-1}} x_{i-1} \), there exists an element \( U_{i-1} \in U_{i-1} \) such that \( x_i, x_{i-1} \in U_{i-1} \). Likewise, there exists \( U_{i-2} \in U_{i-2} \) such that \( x_{i-1}, x_{i-2} \in U_{i-2} \). Since \( U_{i-1} \subset U_{i-2} \), there exists \( U'_{i-2} \subset U_{i-2} \) such that \( U_{i-1} \subset U'_{i-2} \). Since \( U_{i-2} \subset U_{i-3} \), there exists \( U'_{i-3} \subset U_{i-3} \) such that \( U'_{i-2} \cup U_{i-2} \subset U'_{i-3} \). Therefore \( x_i \in \text{st}_{U_{i-3}} x_{i-3} \). A recursive application of the above argument finishes the proof. 

**5. Strong universality of inverse limits of polyhedra**

**Lemma 5.1.** Let \( X = \varprojlim (K_1 \xleftarrow{p_1} K_2 \xleftarrow{p_2} \cdots) \), where each \( p_i \) is a surjective \( n \)-regular quasi-simplicial map.

Fix \( i \) and let \( \pi_i : X \to K_i \) denote the long projection. Let \( Y \) be a metric space of dimension at most \( n \). Let \( W \) be an open locally finite cover of \( Y \) with multiplicity at most \( n + 1 \). Let \( C : W \to O_i \) be a carrier. For each \( W \), let \( v_W \) denote a vertex of \( K_i \) such that \( C(W) = \pi^{-1}_i(\text{ost}_{K_i} v_W) \). For \( y \in Y \) let \( \Delta_y \) denote a simplex in \( K_i \) that is spanned by \( \{v_W : W \in W, y \in Y\} \).

Then there exists a map \( g : Y \to X \) such that for each \( y \in Y \) we have

\[ g(y) \in \pi^{-1}_i(\Delta_y). \]

In particular, if \( C \) is one-to-one, then

\[ \text{mesh } g^{-1}(O_i) \leq \text{mesh } \text{st } W. \]

**Proof.** For each \( A \subset W \) let

\[ F_A = \left( \text{Cl}_Y \bigcap A \right) \setminus \bigcup (W \setminus A). \]

Let \( F = \{F_A\}_{A \subset W} \). Clearly \( F \) is a closed cover of \( Y \). Since \( W \) is locally finite and has finite multiplicity, \( F \) is locally finite and is locally finite-dimensional.

For each \( A_i \subset W \) we have

\[ \bigcap_i F_{A_i} \neq \emptyset \implies \bigcap_i A_i \neq \emptyset. \]

Let \( \Delta_A \) denote the simplex in \( K_i \) that is spanned by the vertices \( \{v_A : A \in A\} \).
We have
\[ \bigcap_i A_i \neq \emptyset \implies \bigcap_i \Delta A_i \neq \emptyset \]
(see Figure 1).

Fig. 1. Defining the map \( g \) in Lemma 5.1

Let
\[ \mathcal{D} = \{ \pi_i^{-1}(\Delta A) \}_{A \subset \mathcal{W}}. \]

By the above considerations, the map \( D: \mathcal{F} \rightarrow \mathcal{D} \) defined by the formula
\[ D(F_A) = \pi_i^{-1}(\Delta A) \]
is a carrier.

Since each \( p_i \) is surjective and \( n \)-regular, Theorem 2.5 shows that \( \pi^{-1}(\Delta A) \) is an \( ANE(n) \) and is weakly \( n \)-homotopic to the simplex \( \Delta A \), therefore the cover \( \mathcal{D} \) is an \( AE(n) \)-cover. Therefore by the Carrier Theorem there exists a map \( g: Y \rightarrow X \) that is carried by \( D \). This map satisfies the condition
\[ g(F_A) \in \pi_i^{-1}(\Delta A) \]
by the definition of the carrier \( D \).

Let \( y \in Y \) and let \( \mathcal{A} = \{ W \in \mathcal{W} : y \in W \} \). Then we have
\[ g(y) \in g(F_A) \subset \pi_i^{-1}(\Delta A) = \pi_i^{-1}(\Delta y). \]

**Lemma 5.2.** Let \( X = \lim_{\leftarrow} (K_1 \xrightarrow{p_1} K_2 \xrightarrow{p_2} \cdots) \). Assume that each bonding map \( p_i \) satisfies \( \text{QTS}_{\kappa,n} \). Let \( Y \) be a metric space of dimension at most \( n \) and weight at most \( \kappa \). Let \( A \) be a closed subset of \( Y \). Then for each map \( f: Y \rightarrow X \), each \( \varepsilon > 0 \), each open neighborhood \( U \) of \( A \) in \( Y \) and each \( i \) there exists a map \( g: Y \rightarrow X \) that satisfies the following conditions:

1. \( g|_A = f|_A \);
2. \( g \) is \( \mathcal{O}_i \)-close to \( f \); and
3. \( \text{mesh}(g|_{Y \setminus U})^{-1}(\mathcal{O}_{i+1}) \leq \varepsilon \).
Proof. Let $W$ be a cover of $Y$ that satisfies the following conditions:

1. $W \prec f^{-1}(O_{i+1})$;
2. $W$ is open, has multiplicity at most $n + 1$, is locally finite, and has cardinality at most $\kappa$;
3. mesh $st W \leq \varepsilon$; and
4. $st_W A \subset U$.

Let $C : f^{-1}(O_i) \rightarrow O_i$ be defined by $C(f^{-1}(O)) = O$ for each $O \in O_i$. It is a carrier and $f$ is carried by $C$. Let $\tilde{C} : N(f^{-1}(O_i)) \rightarrow N(O_i) = K_i$ be the map between nerves that is induced by $C$.

Let $R = \{F \in W : F \cap A \neq \emptyset\}$. Let $J : N(R) \rightarrow N(W)$ be the identity map ($N(R)$ is a subcomplex of $N(W)$).

Let $D : f^{-1}(O_{i+1}) \rightarrow O_{i+1}$ be defined by the formula $D(f^{-1}(O)) = O$ for each $O \in O_{i+1}$, and define $\tilde{D}$ as above to be the induced map.

Let $K : N(W) \rightarrow N(f^{-1}(O_{i+1}))$ be a map such that if $K(v(F)) = v(U)$, then $F \subset U$. Such a map exists because $W$ refines $f^{-1}(O_{i+1})$; moreover, it is a simplicial map.

Let $L : N(f^{-1}(O_{i+1})) \rightarrow N(f^{-1}(O_i))$ be a map induced by inclusions. The following diagram is commutative:

$$
\begin{array}{ccc}
N(f^{-1}(O_i)) & \xrightarrow{\tilde{C}} & N(O_i) = K_i \\
L & & \downarrow{p_i} \\
N(R) & \xrightarrow{J} & N(W) \xrightarrow{K} N(f^{-1}(O_{i+1})) \xrightarrow{\tilde{D}} N(O_{i+1}) = K_{i+1}
\end{array}
$$

It follows that the following diagram is commutative:

$$
\begin{array}{ccc}
N(W) & \xrightarrow{\tilde{C} \circ L \circ K} & N(O_i) = K_i \\
J & & \downarrow{p_i} \\
N(R) & \xrightarrow{\tilde{D} \circ K \circ J} & N(O_{i+1}) = K_{i+1}
\end{array}
$$

Since $p_i$ satisfies $(qts_{\kappa,n})$ we can lift $\tilde{C} \circ L \circ K$ to a simplicial map $\tilde{E} : N(W) \rightarrow K_{i+1}$ such that

1. $p_i \circ \tilde{E} = \tilde{C} \circ L \circ K$;
2. $\tilde{E} \circ J = \tilde{D} \circ K \circ J$; and
3. $\tilde{E}$ is an embedding on $N(W) \setminus N(R)$.

Let $E : W \rightarrow O_{i+1}$ be a map such that $\tilde{E}(v(F)) = v(E(F))$. It is a carrier and $f|_A$ is carried by $E$. 
Let $Z = Y \setminus \bigcup \mathcal{R}$. By Lemma 5.1 there exists a map $h: Z \to X$ that is carried by $E|_{\mathcal{W}\mathcal{R}}$ and such that $\text{mesh } h^{-1}(\mathcal{O}_{i+1}) \leq \text{mesh } \mathcal{W} \leq \varepsilon$. It is $\mathcal{O}_i$-close to $f$ as both maps are carried by $C$. By the Carrier Theorem, there exists a map $g: Y \to X$ that extends $f|_A \cup h$ and is carried by $C$; this map satisfies the desired conditions. ■

**Lemma 5.3.** Let $X$ be a Polish space and let $Y$ be a metric space. Let $f: X \to Y$ be a continuous map. If for each $y \in Y$,

$$\lim_{n \to \infty} \text{diam } f^{-1}(B(y, 1/n)) = 0,$$

then $f$ is a closed embedding of $X$ into $Y$.

**Proof.** If there are distinct points $x_0, x_1$ in $X$ with $f(x_0) = f(x_1) = y$, then there exists $\eta > 0$ such that $\text{diam } f^{-1}(B(y, \delta)) \geq \eta$ for all $\delta > 0$, which contradicts the assumption. Hence $f$ is one-to-one. We will prove that $f$ is a closed map. Let $A$ be a closed subset of $X$. Let $y_0 \in \text{Cl } f(A)$. Let

$$A_n = A \cap \text{Cl}_X f^{-1}(B(y_0, 1/n) \cap f(A)).$$

By the assumption $\lim_{n \to \infty} \text{diam } A_n = 0$. Each $A_n$ is non-empty and closed, and $X$ is complete. By Cantor’s Intersection Theorem [11] there exists an $x_0 \in \bigcap_{n \in \mathbb{N}} A_n$. We have $x_0 \in A$ and $f(x_0) = y_0$ by the definition of $A_n$. Therefore $y_0 \in f(A)$ and $f(A)$ is closed. ■

**Theorem 5.4.** Let $X = \lim_{\leftarrow n} (K_1 \xleftarrow{p_1} K_2 \xleftarrow{p_2} \cdots)$. If each bonding map $p_i$ satisfies (QTS$_{\kappa, n}$), then $X$ is strongly universal for the class of complete $n$-dimensional metric spaces of weight $\kappa$.

**Proof.** Let $Y$ be a complete, at most $n$-dimensional metric space of weight at most $\kappa$. Let $f: Y \to X$. Let $\mathcal{U}$ be an open cover of $X$. Let $\mathcal{O}_i$ be a cover of $X$ as in Definition 4.1. Let

$$R_i = \text{Cl}_X \{y \in Y : \forall U \in \mathcal{U} \text{ Cl}_X \text{ st}_i^2 f(y) \not\subset U\}$$

for $i > 1$ and let $R_1 = Y$.

Let $g_1 = f$. By recursive application of Lemma 5.2 we construct a sequence of maps $g_i: X \to Y$ satisfying the following conditions:

1. $g_i|_{R_i} = f|_{R_i}$;
2. $g_i$ is $\mathcal{O}_{i-1}$-close to $g_{i-1}$; and
3. $\text{mesh } (g_i|_{Y \setminus R_i})^{-1}(\mathcal{O}_{i-1}) \leq 1/i$.

By (2) and by Lemma 4.2 the sequence $g_i$ is uniformly convergent. As an inverse limit of complete spaces, $X$ is complete. Therefore the limit

$$g = \lim_{i \to \infty} g_i$$

exists and is continuous. Let $y \in Y$ and let $i$ be such that $y \in Y \setminus R_i$ and $y \in R_{i-1}$. We have $g_{i-1}(y) = f(y)$. In view of (2) and Lemma 4.3...
Construction of Nöbeling manifolds

6. Construction of Nöbeling manifolds of weight \( \kappa \)

Construction 6.1. Let \( K \) be an at most \( n \)-dimensional simplicial complex. Let \( \kappa \) be a cardinal number. We construct a simplicial complex \( N^n_\kappa(K) \) and a simplicial map \( \pi : N^n_\kappa(K) \to \beta^1 K \). The complex \( N^n_\kappa(K) \) has \( \kappa \) vertices corresponding to each vertex of \( \beta^1 K \). The map \( \pi \) maps each vertex of \( N^n_\kappa(K) \) to a corresponding vertex of \( \beta^1 K \). The simplicial structure on \( N^n_\kappa(K) \) is the maximal \( n \)-dimensional simplicial structure such that \( \pi \) is a simplicial map into \( \beta^1 K \), i.e. vertices \( v_1, v_2, \ldots, v_k \) span a vertex in \( N^n_\kappa(K) \) if and only if \( k \leq n + 1 \) and vertices \( \pi(v_1), \pi(v_2), \ldots, \pi(v_k) \) span a vertex in \( \beta^1 K \).

![Diagram](image1.png)

Fig. 2. The sequence \( K \leftarrow N^l_3(K) \leftarrow N^l_3(N^l_3(K)) \leftarrow \cdots \) for a single-edge starting graph \( K \). Here \( \kappa = 3 \); to get a Nöbeling manifold take infinite \( \kappa \).

Lemma 6.2. Let \( K \) be an at most \( n \)-dimensional simplicial complex. The map \( \pi : N^n_\kappa(K) \to K \) is a quasi-simplicial map satisfying \((Q\text{TS}_{\kappa,n})\).

Proof. By the definition, \( \pi \) is constructed to be a simplicial map into \( \beta^1 K \), hence it is quasi-simplicial into \( K \). Let

\[
\begin{array}{c}
N^n_\kappa(K) \xrightarrow{\pi} \beta^1 K \\
g \downarrow \quad \downarrow \hat{g} \\
B \subset A
\end{array}
\]
be a commutative diagram from Definition 3.1. Let \( v \) be a vertex of \( A \setminus B \). We define \( \tilde{g}(v) \) to be a vertex of \( \pi^{-1}(G(v)) \). Since the simplicial complex \( \pi^{-1}(G(v)) \) has \( \kappa \) vertices and \( A \) has weight at most \( \kappa \), we can make \( \tilde{g} \) be one-to-one on vertices of \( A \setminus B \). Since the simplicial structure on \( \mathcal{N}_\kappa^n(K) \) was maximal with respect to \( \pi \), the map \( \tilde{g} \) extends from vertices of \( A \setminus B \) to a simplicial map from \( A \) into \( \mathcal{N}_\kappa^n(K) \).

**Construction 6.3.** Let \( K \) be an at most \( n \)-dimensional simplicial complex. Let \( \kappa \) be a cardinal number. Let \( n \) be a natural number. Let \( K_0 = K \). Let \( K_{i+1} = \mathcal{N}_\kappa^n(K_i) \) and let \( p_{i+1} \) be the canonical map from \( \mathcal{N}_\kappa^n(K_i) \) to \( \beta^1 K_i \). Let

\[
\nu^n_{\kappa}(K) = \lim_{\leftarrow}(K_0 \xrightarrow{p_1} K_1 \xrightarrow{p_2} K_2 \xrightarrow{p_3} \cdots)
\]

and let \( \pi_K : \nu^n_{\kappa}(K) \to K \) denote the long projection.

**Lemma 6.4.** Let \( K \) be an at most \( n \)-dimensional simplicial complex. Then \( \nu^n_{\kappa}(K) \) is a complete \( n \)-dimensional metric space of weight \( \kappa \).

**Proof.** We have \( \nu^n_{\kappa}(K) = \nu^n_{\kappa}(K^{(n)}) \) (where \( K^{(n)} \) denotes the \( n \)-dimensional skeleton of \( K \)) so without loss of generality we may assume that \( \dim K \leq n \). Then \( \nu^n_{\kappa}(K) \) is an inverse limit of a sequence of \( n \)-dimensional spaces, hence it is at most \( n \)-dimensional. It contains an \( n \)-dimensional simplex as a subspace, hence it is \( n \)-dimensional.

Since each \( K_i \) is finite-dimensional, it is complete. Thus \( \nu^n_{\kappa}(K) \) is complete.

**Lemma 6.5.** Let \( K \) be an at most \( n \)-dimensional simplicial complex. Then \( \nu^n_{\kappa}(K) \) is an absolute neighborhood extensor in dimension \( n \).

**Proof.** Each \( p_i \) is \( n \)-regular by Lemma 3.2. By Theorem 2.5(1), \( \nu^n_{\kappa}(K) \) is an ANE(\( n \)).

**Lemma 6.6.** Let \( K \) be an at most \( n \)-dimensional simplicial complex. Then \( \nu^n_{\kappa}(K) \) is strongly universal in the class of complete \( n \)-dimensional metric spaces of weight \( \kappa \).

**Proof.** This follows from Theorem 5.4.

**Lemma 6.7.** Let \( K \) be an at most \( n \)-dimensional simplicial complex. Then the projection \( \pi_K : \nu^n_{\kappa}(K) \to K \) is an \( n \)-homotopy equivalence.

**Proof.** Each \( p_i \) is \( n \)-regular by Lemma 3.2. By Theorem 2.5(2) and Theorem 2.4, \( \nu^n_{\kappa}(K) \) is an ANE(\( n \)).

**Theorem 6.8.** If \( K \) is at most \( n \)-dimensional simplicial complex, then \( \nu^n_{\kappa}(K) \) is an \( n \)-dimensional abstract Nöbeling manifold of weight \( \kappa \) and the projection \( \pi : \nu^n_{\kappa}(K) \to K \) is an \( n \)-homotopy equivalence.

**Proof.** Apply Lemmas 6.4–6.7.
 Remark 6.9. If \( L \) is a simplicial complex of arbitrary dimension, then the inclusion \( L^{(n)} \subset L \) of the \( n \)-dimensional skeleton of \( L \) into \( L \) is a weak \( n \)-homotopy equivalence, \( \nu_\kappa^n(L^{(n)}) \) is an \( n \)-dimensional abstract Nöbeling manifold of weight \( \kappa \), and the projection \( \pi : \nu_\kappa^n(L^{(n)}) \to L^{(n)} \) composed with the inclusion \( L^{(n)} \subset L \) is a weak \( n \)-homotopy equivalence.

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