

Transformation laws for generalized Dedekind sums associated to Fuchsian groups

by

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1. Introduction

1.1. The reciprocity law for the Dedekind sum. The *Dedekind eta function* is defined by the infinite product

$$(1.1) \quad \eta(z) := e^{\pi iz/12} \prod_{n=1}^{\infty} (1 - e^{2\pi inz})$$

with z in the upper half-plane \mathbb{H} . When $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$ it can be shown that

$$(1.2) \quad \log \eta\left(\frac{az + b}{cz + d}\right) = \log \eta(z) + \frac{1}{2} \log(cz + d) + \pi i S(a, b, c, d),$$

where the real numbers $S(a, b, c, d)$ are independent of z . Indeed, Dedekind obtained the evaluation

$$(1.3) \quad S(a, b, c, d) = \frac{a + d}{12c} - \frac{c}{|c|} \left(\frac{1}{4} + s(d, |c|) \right) \quad \text{for } c \neq 0,$$

and

$$(1.4) \quad S(a, b, 0, d) = \frac{b}{12d} + \frac{d/|d| - 1}{4}.$$

The term $s(d, |c|)$ in (1.3) is a *Dedekind sum* which is defined as

$$(1.5) \quad s(d, c) := \sum_{m=0}^{c-1} \left(\left(\frac{dm}{c} \right) \right) \left(\left(\frac{m}{c} \right) \right)$$

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with

$$((x)) := \begin{cases} x - [x] - 1/2 & \text{if } x \in \mathbb{R}, x \notin \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}, \end{cases}$$

for d, c coprime and $c \geq 1$; see [18, Chap. IX]. The periodicity relation

$$(1.6) \quad s(d+c, c) = s(d, c)$$

is immediate. Dedekind showed that (1.2) implies the reciprocity law

$$(1.7) \quad s(c, d) + s(d, c) = \frac{1}{12} \left(\frac{c}{d} + \frac{d}{c} + \frac{1}{cd} \right) - \frac{1}{4}$$

for c and d relatively prime positive integers. Several other proofs of the identity (1.7) are given in [23].

1.2. Generalizations of Dedekind sums. There are various directions in which authors have obtained generalizations of Dedekind sums, including sums of Bernoulli polynomials, cotangent sums, and rational functions evaluated at roots of unity. Many of these investigations take the evaluation (1.5) as a starting point. Alternatively, one can begin with (1.2), meaning that one defines S as coming from the root of unity that appears when considering the transformation property of the logarithm of a modular form, or more generally, the holomorphic section of a power of the canonical bundle twisted by a flat line bundle.

The Dedekind eta function (1.1) arises naturally in the constant term of a Laurent expansion for the non-holomorphic Eisenstein series for $\mathrm{SL}(2, \mathbb{Z})$. In other words, $\eta(z)$ appears in terms of a Kronecker limit function; see, for example, [6] for a discussion of several intricate connections between Kronecker limit functions and number theory. Goldstein in [8, 9, 10] takes this point of view and studies Kronecker limit functions of Eisenstein series associated to general Fuchsian groups Γ . In particular, Goldstein develops the notion of generalized Dedekind sums associated to Γ . A deeper study in this direction is given in [2] and [3] where Burrin also studies distribution results for general Dedekind sums, which were previously only known in a few instances.

We will use the notation S_Γ for the generalization of S from (1.2) to other groups Γ , and call this object a *modular Dedekind symbol*. We will use H_Γ for the corresponding *generalized Dedekind sum* extending (1.5). The periodicity (1.6) may be traced back to the parabolic generator $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ of $\mathrm{SL}(2, \mathbb{Z})$. Similarly, the reciprocity (1.7) comes from the elliptic generator $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The subgroups Γ of $\mathrm{SL}(2, \mathbb{R})$ which we consider will be finitely generated by parabolic, elliptic and hyperbolic elements. For each parabolic generator we will obtain a *periodicity relation* which generalizes (1.6), while each elliptic generator will provide a *reciprocity relation* as in (1.7). There will also be

hyperbolic periodicity relations from the hyperbolic generators, and our focus of interest is to describe all of these transformations.

The above ideas also extend naturally to a type of non-holomorphic Eisenstein series that includes modular symbols in its definition; see (2.11) for the definition of the modular symbol pairing. The simplest such series, which contains one modular symbol, was studied in [20], for example, and its Kronecker limit was obtained in [14]. The authors of [15] defined and studied a new modular Dedekind symbol, denoted by S_Γ^* , obtained from the Kronecker limit of Eisenstein series constructed with the square of the absolute value of a modular symbol. These types of series are not automorphic in the usual sense but rather transform with a higher-order condition as described in (2.13). Recently [22] these higher-order Eisenstein series played a key role in the proof of conjectures by Mazur, Rubin and Stein on the statistics of modular symbols. These series have been generalized further in [4].

1.3. Definitions and main results. Much of the following background material is found in the early chapters of [13, 24], for example, but is included here for the convenience of the reader.

The group $\mathrm{SL}(2, \mathbb{R})$ acts by linear fractional transformations on \mathbb{H} , and this action may be extended to $\mathbb{H} \cup \mathbb{R} \cup \{\infty\}$. Let Γ be a discrete subgroup of $\mathrm{SL}(2, \mathbb{R})$ where the quotient $\Gamma \backslash \mathbb{H}$ has finite hyperbolic volume V_Γ . This means that Γ is a Fuchsian group of the first kind. We assume that Γ contains at least one parabolic element fixing a cusp in $\mathbb{R} \cup \{\infty\}$. If \mathfrak{a} is a cusp of Γ then we label the subgroup of parabolic elements fixing \mathfrak{a} as $\Gamma_{\mathfrak{a}}$. The image of $\Gamma_{\mathfrak{a}}$ under the map $\mathrm{SL}(2, \mathbb{R}) \rightarrow \mathrm{SL}(2, \mathbb{R})/\pm I$ is isomorphic to \mathbb{Z} . If we denote this image by $\overline{\Gamma}_{\mathfrak{a}}$, there exists a scaling matrix $\sigma_{\mathfrak{a}} \in \mathrm{SL}(2, \mathbb{R})$ such that

$$(1.8) \quad \sigma_{\mathfrak{a}}\infty = \mathfrak{a} \quad \text{and} \quad \sigma_{\mathfrak{a}}^{-1}\overline{\Gamma}_{\mathfrak{a}}\sigma_{\mathfrak{a}} = \left\{ \pm \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} \mid \ell \in \mathbb{Z} \right\}.$$

The matrix $\sigma_{\mathfrak{a}}$ is unique up to multiplication on the right by $\pm \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ for any $t \in \mathbb{R}$. By conjugating Γ , if necessary, we may assume that Γ has a cusp at infinity, which is denoted by ∞ , with scaling matrix σ_{∞} equal to the identity matrix I . Therefore $\overline{\Gamma}_{\infty}$ is generated by $\pm P_{\infty}$ for $P_{\infty} := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. We also assume for simplicity that P_{∞} is in Γ . It follows that if $-I \in \Gamma$ then Γ_{∞} is generated by P_{∞} and $-I$. If $-I \notin \Gamma$ then Γ_{∞} is just generated by P_{∞} . Note that we are excluding the case where $-I$ and P_{∞} are both not in Γ and $-P_{\infty}$ generates Γ_{∞} . On the other hand, if $\gamma \in \Gamma$ has bottom left entry equal to 0, then $\gamma \in \Gamma_{\infty}$. This is clear for $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ and true more generally for any discrete subgroup of $\mathrm{SL}(2, \mathbb{R})$; see [24, Proposition 1.17].

The modular Dedekind symbols S_Γ and S_Γ^* , associated to Γ , are defined in [15, Sect. 1.3]. The definitions and relevant details are reviewed in Section 2 below. For now, recall that S_Γ^* requires modular symbols for its construction,

and these depend on the choice of a non-zero weight 2 holomorphic cusp form for Γ . The space of such forms, $S_2(\Gamma)$, has dimension equal to the genus g of $\Gamma \backslash \mathbb{H}$. Therefore, $S_\Gamma^* = S_{\Gamma, f}^*$ depends on a group Γ of positive genus and a non-zero $f \in S_2(\Gamma)$.

Let $\theta_\Gamma := S_\Gamma^* - S_\Gamma$ and set $A_\Gamma := V_\Gamma / (4\pi)$. It follows from the Gauss–Bonnet formula (3.2) that A_Γ is rational. For $r \in \mathbb{R}$, define $\text{sgn}(r) := r/|r|$ if $r \neq 0$ and $\text{sgn}(0) := 0$. We fix Γ and will usually omit it from the notation in what follows.

DEFINITION 1.1 (Generalized Dedekind sums). Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. If $c \neq 0$ then define

$$H(\gamma) := -S(\gamma) + A \frac{a+d}{c} - \frac{1}{4} \text{sgn}(c) \quad \text{and} \quad H^*(\gamma) := \theta(\gamma) + A \frac{a+d}{c}.$$

If $c = 0$, set

$$H(\gamma) := -S(\gamma) + A \frac{b}{d} + \frac{1}{4}(\text{sgn}(d) - 1) \quad \text{and} \quad H^*(\gamma) := \theta(\gamma) + A \frac{b}{d}.$$

When Γ is the modular group $\text{SL}(2, \mathbb{Z})$, the constant A is $1/12$. In this case, the above definition coincides with (1.3) and one sees that $H(\gamma) = \frac{c}{|c|} s(d, |c|)$ for $c \neq 0$ and $H(\gamma) = 0$ for $c = 0$. Therefore, $H(\gamma)$ may be viewed as a *signed* generalization of the Dedekind sum, as introduced in [2].

PROPOSITION 1.2. *The functions H and H^* are left Γ_∞ -invariant. Therefore, H and H^* depend only on the bottom rows of their arguments, thus allowing us to write*

$$(1.9) \quad H(d, c) := H(\gamma) \quad \text{and} \quad H^*(d, c) := H^*(\gamma)$$

for $\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma$. In this form one has the periodicity relations

$$(1.10) \quad H(d+c, c) = H(d, c) \quad \text{and} \quad H^*(d+c, c) = H^*(d, c).$$

The values taken by H for certain groups Γ have been studied by many authors. On the other hand, the values of H^* are just beginning to be understood. It follows from [15, Theorem 4] that, in the case of genus 1 Hecke congruence groups, H^* is always rational. These are exactly the groups $\Gamma_0(N)$ for which $N \in \{11, 14, 15, 17, 19, 20, 21, 24, 27, 32, 36, 49\}$. See Section 5.3 for a detailed description of H^* when $\Gamma = \Gamma_0(11)$, for example.

The next theorem gives a density result for the genus 1 Hecke congruence groups mentioned above. For $r \in \mathbb{R}$, let $\{r\}$ denote its fractional part.

THEOREM 1.3. *Let Γ be a genus 1 Hecke congruence group. Then there exists $t = t(\Gamma) \in \mathbb{N}$ with the following property: for all d, c such that $0 < d \leq c \leq X$ and $\begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma$, the points*

$$(\{d/c\}, \{tH^*(d, c)\})$$

become equidistributed in the square $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ as $X \rightarrow \infty$.

For $\Gamma = \mathrm{SL}(2, \mathbb{Z})$, Myerson [19] proved a more general version of this result for H , with $t \in \mathbb{N}$ replaced by any non-zero $t \in \mathbb{R}$. Also, for any cofinite Fuchsian group Γ and any non-zero $t \in \mathbb{R}$, it is shown in [2] that $\{tH(d, c)\}$ becomes equidistributed in \mathbb{R}/\mathbb{Z} .

We now turn our attention to the transformation laws of the generalized Dedekind sums H and H^* . In what follows, we use the notation $\begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix}$ for the entries of a matrix γ .

THEOREM 1.4. *Let $\gamma, \tau \in \Gamma$ with $c_\gamma c_\tau c_{\gamma\tau} \neq 0$. Then*

$$(1.11) \quad H(\gamma\tau) - H(\gamma) - H(\tau) \\ = -A \left(\frac{c_\gamma}{c_\tau c_{\gamma\tau}} + \frac{c_\tau}{c_\gamma c_{\gamma\tau}} + \frac{c_{\gamma\tau}}{c_\gamma c_\tau} \right) + \frac{1}{4} \mathrm{sgn}(c_\gamma c_\tau c_{\gamma\tau}),$$

$$(1.12) \quad H^*(\gamma\tau) - H^*(\gamma) - H^*(\tau) \\ = -A \left(\frac{c_\gamma}{c_\tau c_{\gamma\tau}} + \frac{c_\tau}{c_\gamma c_{\gamma\tau}} + \frac{c_{\gamma\tau}}{c_\gamma c_\tau} \right) + \frac{V_f}{2\pi} \mathrm{Im}(\langle \gamma, f \rangle \overline{\langle \tau, f \rangle})$$

where

$$V_f := V_\Gamma / (16\pi^2 \|f\|^2),$$

$\|f\|$ is the Petersson norm of f , and $\langle \cdot, f \rangle$ is the modular symbol pairing as defined in (2.11).

The identity (1.11) first appeared in [3, Sect. 2.4]. By writing $\gamma_1 = \gamma$, $\gamma_2 = \tau$ and $\gamma_3 = (\gamma\tau)^{-1}$, the above transformation laws may be given the following elegant formulation, first observed by Dieter [5] for the classical Dedekind sums (1.5).

COROLLARY 1.5. *Let*

$$\gamma_1 = \begin{pmatrix} * & * \\ c_1 & * \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} * & * \\ c_2 & * \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} * & * \\ c_3 & * \end{pmatrix} \in \Gamma$$

be such that $c_1 c_2 c_3 \neq 0$ and $\gamma_1 \gamma_2 \gamma_3 = I$. Then

$$(1.13) \quad H(\gamma_1) + H(\gamma_2) + H(\gamma_3) \\ = -A \left(\frac{c_1}{c_2 c_3} + \frac{c_2}{c_1 c_3} + \frac{c_3}{c_1 c_2} \right) + \frac{1}{4} \mathrm{sgn}(c_1 c_2 c_3),$$

$$(1.14) \quad H^*(\gamma_1) + H^*(\gamma_2) + H^*(\gamma_3) \\ = -A \left(\frac{c_1}{c_2 c_3} + \frac{c_2}{c_1 c_3} + \frac{c_3}{c_1 c_2} \right) - \frac{V_f}{2\pi} \mathrm{Im}(\langle \gamma_1, f \rangle \overline{\langle \gamma_2, f \rangle}).$$

Equation (1.13) is [3, Theorem 2], with the signs on the right corrected. These three-term relations also imply succinct two-term relations, though it is simpler to prove the following directly.

THEOREM 1.6. *Let $\gamma, \tau \in \Gamma$ with $c_\gamma c_\tau c_{\gamma\tau} \neq 0$. Then, in the notation of (1.9),*

$$(1.15) \quad H(d_\gamma, c_\gamma) - H(d_{\gamma\tau}, c_{\gamma\tau}) \\ = A \left(\frac{d_\gamma}{c_\gamma} - \frac{d_{\gamma\tau}}{c_{\gamma\tau}} + \frac{c_\tau}{c_\gamma c_{\gamma\tau}} \right) + S(\tau) + \frac{1}{4} (\text{sgn}(c_\tau) - \text{sgn}(c_\gamma c_\tau c_{\gamma\tau})),$$

$$(1.16) \quad H^*(d_\gamma, c_\gamma) - H^*(d_{\gamma\tau}, c_{\gamma\tau}) \\ = A \left(\frac{d_\gamma}{c_\gamma} - \frac{d_{\gamma\tau}}{c_{\gamma\tau}} + \frac{c_\tau}{c_\gamma c_{\gamma\tau}} \right) - \theta(\tau) - \frac{V_f}{2\pi} \text{Im}(\langle \gamma, f \rangle \overline{\langle \tau, f \rangle}).$$

We will develop some consequences of Theorem 1.6. For example, if τ is elliptic or parabolic then $S(\tau)$ in (1.15) is not difficult to evaluate, from which one obtains reciprocity and periodicity relations for H . If τ in (1.16) is elliptic or parabolic, and not fixing a cusp equivalent to $\infty \bmod \Gamma$, then we will see that only the first term on the right can be non-zero, thus giving the simple law

$$(1.17) \quad H^*(d_\gamma, c_\gamma) - H^*(d_{\gamma\tau}, c_{\gamma\tau}) = A \left(\frac{d_\gamma}{c_\gamma} - \frac{d_{\gamma\tau}}{c_{\gamma\tau}} + \frac{c_\tau}{c_\gamma c_{\gamma\tau}} \right).$$

Among the various relations obtained, we highlight here an interesting identity, involving the classical Dedekind sum (1.5) and the Möbius function μ , which we derive from generalized Dedekind sums associated to the group $\Gamma_0(N)$ for any integer $N \geq 2$. Let c, d and k be positive integers with c and d relatively prime and $N \mid c$. Then, as seen in (5.5),

$$\sum_{v|N} \frac{\mu(v)}{v} \left(s \left(d, \frac{vc}{N} \right) - s \left(d, \frac{vc}{N} + vkd \right) \right) = \frac{N^2 k (d^2 + 1)}{12c(c + kNd)} \prod_{p|N} (1 - p^{-2}).$$

The material that follows is divided into four sections. Section 2 presents initial results needed for our analysis. The proofs of the main results are given in Section 3. Section 4 provides additional transformation formulas, and in Section 5 we give some examples for specific groups.

Finally, we note that our definitions of S, S^*, H and H^* above are for the cusp at infinity. There are analogous definitions for arbitrary cusps and our formulas may be extended to cover these cases. For details on the general parametrization of the modular Dedekind symbols S and S^* , see [15, Sect. 5]. For the general parametrization of H , see [2].

2. Preliminaries. In this section, for the sake of completeness, we recall results from [3], [8] and [15] that will be required. Throughout, as described in Section 1.3, Γ is a discrete subgroup of $\text{SL}(2, \mathbb{R})$ with finite volume and at least one cusp, at ∞ . It is assumed that the parabolic subgroup fixing ∞ is generated by $P_\infty := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and also $-I$ if $-I \in \Gamma$.

2.1. The phase factor. For any two matrices M, N in $\mathrm{SL}(2, \mathbb{R})$ define the *phase factor* $\omega(M, N)$ by

$$(2.1) \quad \omega(M, N) := (-\log j(MN, z) + \log j(M, Nz) + \log j(N, z))/(2\pi i).$$

We are using the principal branch of the logarithm with argument in $(-\pi, \pi]$. The right side of (2.1) is independent of $z \in \mathbb{H}$ and may only take values in the set $\{-1, 0, 1\}$. This phase factor is treated carefully by Petersson in [21] where an explicit formula for evaluation of $\omega(M, N)$ is given in terms of the bottom rows of M, N and MN . Here, we will recall a slightly different formulation from [15, Proposition 3.2] which will be used throughout this paper.

For all M, N in $\mathrm{SL}(2, \mathbb{R})$ set $\mathbf{c} = (\mathrm{sgn}(c_M), \mathrm{sgn}(c_N), \mathrm{sgn}(c_{MN}))$. Then

$$(2.2) \quad \omega(M, N) = \begin{cases} 1 & \text{if } \mathbf{c} = (1, 1, -1) \text{ or } (0, 1, -1) \text{ or } (1, 0, -1), \\ 1 & \text{if } \mathbf{c} = (0, 0, 0) \text{ and } \mathrm{sgn}(d_M) = \mathrm{sgn}(d_N) = -1, \\ -1 & \text{if } \mathbf{c} = (-1, -1, 1) \text{ or } (-1, -1, 0), \\ 0 & \text{otherwise.} \end{cases}$$

Applying the above relations, it is easy to see that for $M \in \mathrm{SL}(2, \mathbb{R})$ and $k \in \mathbb{Z}$ one has

$$(2.3) \quad \omega(P_\infty^k, M) = \omega(M, P_\infty^k) = 0,$$

$$(2.4) \quad \omega(-P_\infty^k, M) = \omega(M, -P_\infty^k) = \begin{cases} (1 + \mathrm{sgn}(c_M))/2 & \text{if } c_M \neq 0, \\ (1 - \mathrm{sgn}(d_M))/2 & \text{if } c_M = 0. \end{cases}$$

The next formula of Petersson may also be verified easily: for $c_M c_N c_{MN} \neq 0$,

$$(2.5) \quad \omega(M, N) = (\mathrm{sgn}(c_M) + \mathrm{sgn}(c_N) - \mathrm{sgn}(c_{MN}) - \mathrm{sgn}(c_M c_N c_{MN}))/4.$$

2.2. The modular Dedekind symbol S . The non-holomorphic Eisenstein series associated to Γ and the cusp at ∞ is defined for $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ by

$$E_\infty(z, s) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathrm{Im}(\gamma z)^s.$$

This series admits a meromorphic continuation to all $s \in \mathbb{C}$; at $s = 1$, the Laurent expansion has a first-order pole, and the Kronecker limit function is defined to be the next order term in the expansion. The Kronecker limit function can be shown to have the form $\log |\mathrm{Im}(z)\eta_{\Gamma, \infty}(z)|$ for some function $\eta_{\Gamma, \infty}(z)$ which is non-vanishing on \mathbb{H} , locally holomorphic, and transforms as a weight $1/2$ modular form with respect to Γ . This modular transformation includes the multiplier system $e^{\pi i S_{\Gamma, \infty}(\gamma)}$. In other words, we have

$$(2.6) \quad \log \eta_{\Gamma, \infty}(\gamma z) = \log \eta_{\Gamma, \infty}(z) + \frac{1}{2} \log j(\gamma, z) + \pi i S_{\Gamma, \infty}(\gamma) \quad \text{for all } \gamma \in \Gamma$$

where $j(\gamma, z) := c_\gamma z + d_\gamma$. In the case when $\Gamma = \mathrm{SL}(2, \mathbb{Z})$, the function $\eta_{\Gamma, \infty}(z)$ is the classical Dedekind eta function (1.1), and (2.6) coincides with (1.2). In what follows, we consider the group Γ and the cusp at ∞ as fixed, so we will write $S(\gamma) = S_{\Gamma, \infty}(\gamma)$ and call this function the *modular Dedekind symbol* for Γ . This construction was initiated in [8] and [9].

Define an automorphism ι of $\mathrm{SL}(2, \mathbb{R})$ as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{\iota} \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}.$$

Also set

$$(2.7) \quad \rho(\gamma) := \begin{cases} 1 & \text{if } c_\gamma = 0 \text{ and } d_\gamma < 0, \\ 0 & \text{otherwise.} \end{cases}$$

The properties we will need of the symbol S are assembled in the next theorem.

THEOREM 2.1. *For all γ and τ in Γ the following assertions are true:*

- (i) *We have $S(\gamma\tau) = S(\gamma) + S(\tau) + \omega(\gamma, \tau)$.*
- (ii) *We have $S(I) = 0$, and if $-I \in \Gamma$, then $S(-I) = -1/2$.*
- (iii) *If $\iota(\Gamma) = \Gamma$, then $S(\iota(\gamma)) = -S(\gamma) - \rho(\gamma)$, where ρ is defined in (2.7).*
- (iv) *Suppose γ is parabolic, fixing the cusp \mathfrak{a} . Choose a scaling matrix $\sigma_{\mathfrak{a}}$ satisfying (1.8). Then $\gamma_{\mathfrak{a}} := \sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{a}} = \pm \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$ for a well-defined $h \in \mathbb{Z}$ and*

$$(2.8) \quad S(\gamma) = \delta(\infty, \mathfrak{a}) \cdot hA - \log(j(\gamma_{\mathfrak{a}}, i))/(2\pi i) - \omega(\sigma_{\mathfrak{a}}^{-1}, \gamma) + \omega(\gamma_{\mathfrak{a}}, \sigma_{\mathfrak{a}}^{-1}).$$

Here $\delta(\infty, \mathfrak{a}) = 1$ in the case $\mathfrak{a} \equiv \infty \pmod{\Gamma}$ and $\delta(\infty, \mathfrak{a}) = 0$ otherwise.

- (v) *Let γ be elliptic, fixing $z_0 \in \mathbb{H}$ and with $\gamma^r = I$ for $r > 0$. Then*

$$S(\gamma) = -\frac{1}{2\pi i} \log j(\gamma, z_0) = -\frac{1}{r} \sum_{k=1}^{r-1} \omega(\gamma^k, \gamma).$$

Proof. All these results are proved in [15]. Part (i) is shown there in Theorem 1. The first assertion of part (ii) follows immediately from part (i), and the second assertion is [15, (3.9)]. Propositions 6.3, 5.6 and 5.8 of [15] give parts (iii), (iv) and (v) respectively. ■

Note that (2.8) above simplifies to

$$(2.9) \quad S(\gamma) = \delta(\infty, \mathfrak{a}) \cdot hA$$

when $\gamma_{\mathfrak{a}} = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$; see [15, Proposition 5.6].

2.3. The higher-order modular Dedekind symbol S^* . Assume now that the group Γ has positive genus. This implies that the space $S_2(\Gamma)$ of weight 2 holomorphic cusp forms with respect to Γ has positive dimension.

Let $f \in S_2(\Gamma)$ be a non-zero cusp form which remains fixed throughout the discussion.

We define a non-holomorphic Eisenstein series at the cusp at infinity twisted by powers of modular symbols by

$$(2.10) \quad E_\infty^{m,n}(z, s; f) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \langle \gamma, f \rangle^m \overline{\langle \gamma, f \rangle}^n \operatorname{Im}(\gamma z)^s$$

for $\operatorname{Re}(s) > 1$. The *modular symbol pairing* is defined by

$$(2.11) \quad \langle \gamma, f \rangle := 2\pi i \int_{z_0}^{\gamma z_0} f(w) dw.$$

Since $f \in S_2(\Gamma)$, the modular symbol pairing is independent of $z_0 \in \mathbb{H} \cup \{\text{the cusps of } \Gamma\}$, and the pairing satisfies the identity

$$(2.12) \quad \langle \gamma\tau, f \rangle = \langle \gamma, f \rangle + \langle \tau, f \rangle \quad \text{for all } \gamma, \tau \in \Gamma.$$

If γ is parabolic or elliptic, then letting z_0 be the point fixed by γ shows that $\langle \gamma, f \rangle = 0$. In particular, (2.10) is well-defined.

Let $\gamma \in \Gamma$ act on a map ψ by $(\psi|\gamma)(z) := \psi(\gamma z)$ and extend this action to all $\mathbb{C}[\Gamma]$ by linearity. We may call ψ an *n th-order automorphic form* if

$$(2.13) \quad \psi|(\gamma_1 - I) \cdots (\gamma_n - I) = 0 \quad \text{for all } \gamma_1, \dots, \gamma_n \in \Gamma.$$

With this notation, Γ -invariant functions such as $E_\infty(z, s)$ are first-order. The series $E_\infty^{m,n}(z, s; f)$ has order $m + n + 1$. See for example [4, Sect. 5.1] for more details.

For $m = n$ the series (2.10) has a pole at $s = 1$ of order $m + 1$. An examination of the Kronecker limit formula for $E_\infty^{m,m}(z, s; f)$ as $s \rightarrow 1$, as developed in [15, Sect. 4], proves the following: there exists a function $\eta_{\Gamma, \infty, f}^*$ which, up to a factor of modulus 1, transforms with weight $1/2$ with respect to Γ so that

$$(2.14) \quad \log \eta_{\Gamma, \infty, f}^*(\gamma z) = \log \eta_{\Gamma, \infty, f}^*(z) + \frac{1}{2} \log j(\gamma, z) + \pi i S_{\Gamma, \infty, f}^*(\gamma)$$

for all $\gamma \in \Gamma$.

This $\eta_{\Gamma, \infty, f}^*$ is a product of a holomorphic part and an explicit, real-valued non-holomorphic part; see [15, Def. 4.5]. The terms $S_{\Gamma, \infty, f}^*(\gamma)$ turn out to be independent of m for all $m \geq 1$, and so the function $S^* = S_{\Gamma, \infty, f}^*$ is called a *higher-order modular Dedekind symbol* in [15]. It has the following properties.

THEOREM 2.2. *For all γ and τ in Γ the following assertions are true:*

- (i) *We have $S^*(\gamma\tau) = S^*(\gamma) + S^*(\tau) + \frac{V_f}{2\pi} \operatorname{Im}(\langle \gamma, f \rangle \overline{\langle \tau, f \rangle}) + \omega(\gamma, \tau)$ where $V_f := V_\Gamma / (16\pi^2 \|f\|^2)$ and $\|f\|$ denotes the Petersson norm of f .*
- (ii) *If $-I \in \Gamma$, then $S^*(-I) = -1/2$.*
- (iii) *If $\iota(\Gamma) = \Gamma$, then $S^*(\iota(\gamma)) = -S^*(\gamma) - \rho(\gamma)$.*

(iv) Suppose γ is parabolic, fixing the cusp \mathfrak{a} . Choose a scaling matrix $\sigma_{\mathfrak{a}}$ satisfying (1.8) and set $\gamma_{\mathfrak{a}} := \sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{a}}$. Then

$$(2.15) \quad S^*(\gamma) = -\log(j(\gamma_{\mathfrak{a}}, i))/(2\pi i) - \omega(\sigma_{\mathfrak{a}}^{-1}, \gamma) + \omega(\gamma_{\mathfrak{a}}, \sigma_{\mathfrak{a}}^{-1}).$$

(v) If γ is elliptic then $S^*(\gamma) = S(\gamma)$.

Proof. Part (i) is in Theorem 2 of [15]. Part (ii) is contained in Corollary 5.4 there. Propositions 6.2, 5.7 and 5.8 of [15] show parts (iii), (iv) and (v) respectively. ■

2.4. The difference $\theta := S^* - S$. We now consider the difference $\theta := S^* - S$ from which the generalized Dedekind sum H^* is constructed. We could have used just S^* to define H^* , but the advantage of using θ is that all the phase factors disappear, as we see now.

COROLLARY 2.3. *For all γ and τ in Γ the following assertions are true:*

- (i) We have $\theta(\gamma\tau) = \theta(\gamma) + \theta(\tau) + \frac{V_f}{2\pi} \operatorname{Im}(\langle \gamma, f \rangle \overline{\langle \tau, f \rangle})$.
- (ii) If $-I \in \Gamma$, then $\theta(-I) = 0$, and $\theta(-\gamma) = \theta(\gamma)$.
- (iii) If $\iota(\Gamma) = \Gamma$, then $\theta(\iota(\gamma)) = -\theta(\gamma)$.
- (iv) Assume γ is a parabolic element fixing a cusp \mathfrak{a} . Choose a scaling matrix $\sigma_{\mathfrak{a}}$ satisfying (1.8). Then $\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{a}} = \pm \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$ for a well-defined $h \in \mathbb{Z}$. We have $\theta(\gamma) = -\delta(\infty, \mathfrak{a}) \cdot hA$ where $\delta(\infty, \mathfrak{a}) = 1$ in the case $\mathfrak{a} \equiv \infty \pmod{\Gamma}$ and $\delta(\infty, \mathfrak{a}) = 0$ otherwise.
- (v) If γ is an elliptic element, then $\theta(\gamma) = 0$.
- (vi) If γ or τ is either elliptic or parabolic, then $\theta(\gamma\tau) = \theta(\gamma) + \theta(\tau)$ since modular symbols are zero for elliptic or parabolic elements.
- (vii) For each $n \in \mathbb{Z}$, $\theta(\gamma^n) = n\theta(\gamma)$.

Before continuing, we describe some further properties of this map. The following result states that, in effect, θ detects the difference between Γ and its abelianization.

LEMMA 2.4. *For all γ and τ in Γ ,*

$$\theta(\gamma\tau) - \theta(\gamma) - \theta(\tau) = \frac{1}{2}\theta([\gamma, \tau]),$$

where the commutator $[\gamma, \tau]$ is defined as $[\gamma, \tau] := \gamma\tau\gamma^{-1}\tau^{-1}$.

Proof. We first claim that

$$(2.16) \quad \theta([\gamma, \tau]) = \frac{V_f}{\pi} \operatorname{Im}(\langle \gamma, f \rangle \overline{\langle \tau, f \rangle}).$$

Indeed,

$$\begin{aligned} \theta([\gamma, \tau]) &= \theta(\gamma\tau\gamma^{-1}) - \theta(\tau) = \theta(\gamma\tau) - \theta(\gamma) - \theta(\tau) - \frac{V_f}{2\pi} \operatorname{Im}(\langle \tau, f \rangle \overline{\langle \gamma, f \rangle}) \\ &= \frac{V_f}{2\pi} \operatorname{Im}(\langle \gamma, f \rangle \overline{\langle \tau, f \rangle}) - \frac{V_f}{2\pi} \operatorname{Im}(\langle \tau, f \rangle \overline{\langle \gamma, f \rangle}) = \frac{V_f}{\pi} \operatorname{Im}(\langle \gamma, f \rangle \overline{\langle \tau, f \rangle}). \end{aligned}$$

Using (2.16) in Corollary 2.3(i) completes the proof. ■

The following corollary justifies the use of the term *sum* when referring to H^* .

COROLLARY 2.5. For any word $\gamma_1^{k_1} \cdots \gamma_m^{k_m}$ in Γ ,

$$\theta(\gamma_1^{k_1} \cdots \gamma_m^{k_m}) = k_1\theta(\gamma_1) + \cdots + k_m\theta(\gamma_m) + \frac{1}{2} \sum_{1 \leq i < j \leq m} k_i k_j \theta([\gamma_i, \gamma_j]).$$

Proof. The assertion follows by iterating Lemma 2.4 and simplifying with (2.16) and (2.12). ■

We note here that θ can be viewed as the *homogenization* of the generalized Dedekind sum H^* .

PROPOSITION 2.6. For all $\tau \in \Gamma$ with $\tau \notin \Gamma_\infty$ we have

$$(2.17) \quad \lim_{n \rightarrow \infty} \frac{H^*(\tau^n)}{n} = \theta(\tau).$$

Proof. We will use the notation $\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$ for the entries of τ^n . If $\tau = \pm I$ then (2.17) is true since $H^*(\tau^n) = \theta(\tau) = 0$. Now assume that τ is parabolic, so that $|\text{tr}(\tau)| = 2$. We may choose $\sigma \in \text{SL}(2, \mathbb{R})$ such that $\sigma^{-1}\tau\sigma = \pm \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$ for non-zero $h \in \mathbb{Z}$. If we write $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ then

$$\tau^n = \pm \sigma \begin{pmatrix} 1 & nh \\ 0 & 1 \end{pmatrix} \sigma^{-1} = \pm \begin{pmatrix} 1 - nh\alpha\gamma & nh\alpha^2 \\ -nh\gamma^2 & 1 + nh\alpha\gamma \end{pmatrix}.$$

Since $\tau \notin \Gamma_\infty$, we have $c_1 \neq 0$ and therefore $c_n \neq 0$ for all n . Hence

$$H^*(\tau^n) = \theta(\tau^n) + A \frac{a_n + d_n}{c_n} = n\theta(\tau) + A \frac{2}{-nh\gamma^2}$$

and (2.17) follows.

In the remaining case we have $|\text{tr}(\tau)| \neq 2$ and so $c_1 \neq 0$ (cf. [24, Proposition 1.17]), τ has two distinct eigenvalues λ_1, λ_2 and is diagonalizable. We find

$$(2.18) \quad \begin{aligned} \tau^n &= \pm \sigma \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} \sigma^{-1} \\ &= \pm \begin{pmatrix} \lambda_2^n + \alpha\delta(\lambda_1^n - \lambda_2^n) & -\alpha\beta(\lambda_1^n - \lambda_2^n) \\ \gamma\delta(\lambda_1^n - \lambda_2^n) & \lambda_1^n - \alpha\delta(\lambda_1^n - \lambda_2^n) \end{pmatrix}. \end{aligned}$$

This implies the formulas (for the positive sign in (2.18), otherwise multiply by -1)

$$\begin{aligned} a_n &= \lambda_2^n + (a_1 - \lambda_2) \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}, & b_n &= b_1 \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}, \\ c_n &= c_1 \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}, & d_n &= \lambda_1^n + (d_1 - \lambda_1) \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}. \end{aligned}$$

Then

$$(2.19) \quad H^*(\tau^n) = \theta(\tau^n) + A \frac{a_n + d_n}{c_n} = n\theta(\tau) + A \frac{\lambda_1^n + \lambda_2^n}{c_1(\lambda_1^n - \lambda_2^n)}(\lambda_1 - \lambda_2).$$

The last term in (2.19) goes to $\pm A(\lambda_1 - \lambda_2)/c_1$ as $n \rightarrow \infty$, and (2.17) follows. ■

Equation (2.17) is not true when $\tau = \pm P_\infty^k$: the left side is 0 and the right side is $-kA$. For a further discussion of these points, we refer to the proof of Lemma 3.3 below.

3. Proofs of the main results. The following initial lemmas establish basic properties of H and H^* which are required in our analysis.

LEMMA 3.1. *Assume that $-I$ and γ are in Γ . Then $H(-\gamma) = H(\gamma)$ and $H^*(-\gamma) = H^*(\gamma)$.*

Proof. Write $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. For $c \neq 0$ we have

$$\begin{aligned} H(-\gamma) &= -S(-\gamma) + A \frac{a+d}{c} - \frac{\text{sgn}(-c)}{4} \\ &= -S(-I) - S(\gamma) - \omega(-I, \gamma) + A \frac{a+d}{c} + \frac{\text{sgn}(c)}{4} \\ &= \frac{1}{2} - S(\gamma) - \frac{1 + \text{sgn}(c)}{2} + A \frac{a+d}{c} + \frac{\text{sgn}(c)}{4} = H(\gamma). \end{aligned}$$

We have used Theorem 2.1(i) and (2.4) for $k = 0$. The case when $c = 0$ is similar. The proof that $H^*(-\gamma) = H^*(\gamma)$ is simpler, using only the fact that $\theta(-\gamma) = \theta(\gamma)$, which comes from Corollary 2.3(ii). ■

LEMMA 3.2. *If $\iota(\Gamma) = \Gamma$ and $\gamma \in \Gamma$ then $H(\iota(\gamma)) = -H(\gamma)$ and $H^*(\iota(\gamma)) = -H^*(\gamma)$.*

Proof. Use Definition 1.1, Theorem 2.1(iii) and Corollary 2.3(iii). ■

LEMMA 3.3. *The functions $H : \Gamma \rightarrow \mathbb{R}$ and $H^* : \Gamma \rightarrow \mathbb{R}$ are left and right Γ_∞ -invariant.*

Proof. Recall that Γ_∞ is generated by P_∞ and $-I$ if $-I \in \Gamma$. If $-I \notin \Gamma$ then Γ_∞ is generated by P_∞ alone. For γ in Γ , Theorem 2.1(i, iv), along with (2.9), (2.3), shows that

$$S(P_\infty^k \gamma) = S(P_\infty^k) + S(\gamma) + \omega(P_\infty^k, \gamma) = kA + S(\gamma).$$

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $P_\infty^k \gamma$ equals $\begin{pmatrix} a+kc & b+kd \\ c & d \end{pmatrix}$. By definition, if $c \neq 0$,

$$\begin{aligned} H(P_\infty^k \gamma) &= -S(P_\infty^k \gamma) + A \frac{a+kc+d}{c} - \frac{\text{sgn}(c)}{4} \\ &= -S(\gamma) + A \frac{a+d}{c} - \frac{\text{sgn}(c)}{4} = H(\gamma). \end{aligned}$$

If $c = 0$ then

$$\begin{aligned} H(P_\infty^k \gamma) &= -S(P_\infty^k \gamma) + A \frac{b + kd}{d} + \frac{\operatorname{sgn}(d) - 1}{4} \\ &= -S(\gamma) + A \frac{b}{d} + \frac{\operatorname{sgn}(d) - 1}{4} = H(\gamma). \end{aligned}$$

Combining this with Lemma 3.1 shows that H is left-invariant under Γ_∞ . The left-invariance of H^* follows similarly by employing

$$\theta(P_\infty^k \gamma) = \theta(P_\infty^k) + \theta(\gamma) = -kA + \theta(\gamma)$$

from Corollary 2.3(i, iv). On the right, γP_∞^k equals $\begin{pmatrix} a & b+ka \\ c & d+kc \end{pmatrix}$. By definition, if $c \neq 0$, then

$$\begin{aligned} H(\gamma P_\infty^k) &= -S(\gamma P_\infty^k) + A \frac{a + d + kc}{c} - \frac{\operatorname{sgn}(c)}{4} \\ &= -S(\gamma) + A \frac{a + d}{c} - \frac{\operatorname{sgn}(c)}{4} = H(\gamma). \end{aligned}$$

If $c = 0$ then $a = d = \pm 1$, as we have noted, and

$$\begin{aligned} H(\gamma P_\infty^k) &= -S(\gamma P_\infty^k) + A \frac{b + ka}{d} + \frac{\operatorname{sgn}(d) - 1}{4} \\ &= -S(\gamma) + A \frac{b}{d} + \frac{\operatorname{sgn}(d) - 1}{4} = H(\gamma). \end{aligned}$$

As previously, combining this with Lemma 3.1 shows that H is right-invariant under Γ_∞ . Similar calculations imply that H^* is right-invariant as well. ■

Analogous results to the above for H are contained in [2, Theorem 2].

LEMMA 3.4. *For all $\gamma \in \Gamma$,*

$$(3.1) \quad H(\gamma^{-1}) = -H(\gamma) \quad \text{and} \quad H^*(\gamma^{-1}) = -H^*(\gamma).$$

Proof. This is again a straightforward computation, using $\omega(\gamma, \gamma^{-1}) = \rho(\gamma)$ in the notation of (2.7). In fact, for $c_\gamma \neq 0$,

$$H(\gamma) + H(\gamma^{-1}) = \omega(\gamma, \gamma^{-1}) = \rho(\gamma) = 0.$$

For $c_\gamma = 0$, we know $a_\gamma = d_\gamma$ and hence

$$H(\gamma) + H(\gamma^{-1}) = \rho(\gamma) + \frac{\operatorname{sgn}(d_\gamma) - 1}{4} + \frac{\operatorname{sgn}(a_\gamma) - 1}{4} = 0.$$

The result for H^* is shown similarly. ■

LEMMA 3.5. *Let us write the product $\gamma\tau$ of any two matrices γ and τ from Γ as*

$$\begin{pmatrix} a_{\gamma\tau} & b_{\gamma\tau} \\ c_{\gamma\tau} & d_{\gamma\tau} \end{pmatrix} = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \begin{pmatrix} a_\tau & b_\tau \\ c_\tau & d_\tau \end{pmatrix}.$$

Then, assuming that $c_\gamma c_\tau c_{\gamma\tau} \neq 0$, we have

$$\begin{aligned} \frac{a_\gamma + d_\gamma}{c_\gamma} + \frac{a_\tau + d_\tau}{c_\tau} - \frac{a_{\gamma\tau} + d_{\gamma\tau}}{c_{\gamma\tau}} &= \frac{c_\gamma}{c_\tau c_{\gamma\tau}} + \frac{c_\tau}{c_\gamma c_{\gamma\tau}} + \frac{c_{\gamma\tau}}{c_\gamma c_\tau}, \\ \frac{a_\gamma + d_\gamma}{c_\gamma} - \frac{a_{\gamma\tau} + d_{\gamma\tau}}{c_{\gamma\tau}} &= \frac{c_\tau}{c_\gamma c_{\gamma\tau}} - \frac{d_{\gamma\tau}}{c_{\gamma\tau}} + \frac{d_\gamma}{c_\gamma}. \end{aligned}$$

Proof. The claim is proved by straightforward computation. ■

Proof of the periodicity relations (Proposition 1.2). We already showed in Lemma 3.3 that H and H^* are left Γ_∞ -invariant.

Assume that both $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\eta = \begin{pmatrix} \alpha & \beta \\ c & d \end{pmatrix}$ are in Γ . Then

$$\gamma\eta^{-1} = \begin{pmatrix} ad - bc & \alpha b - a\beta \\ 0 & \alpha d - \beta c \end{pmatrix} = \begin{pmatrix} 1 & \alpha b - a\beta \\ 0 & 1 \end{pmatrix}.$$

This parabolic element fixes ∞ and necessarily equals P_∞^k . Hence $\gamma = P_\infty^k \eta$ and by Lemma 3.3, $H(\gamma) = H(\eta)$, and $H^*(\gamma) = H^*(\eta)$. This shows that $H(\gamma)$ and $H^*(\gamma)$ depend solely on the lower row of γ , and therefore we may use the notation (1.9).

We have $\gamma P_\infty = \begin{pmatrix} a & b+a \\ c & d+c \end{pmatrix}$. Then by the right Γ_∞ -invariance of H ,

$$H(d+c, c) = H(\gamma P_\infty) = H(\gamma) = H(d, c).$$

The same is true for H^* .

Proof of the three-term relations (Theorem 1.4, Corollary 1.5). By Definition 1.1, one has

$$\begin{aligned} H^*(\gamma\tau) - H^*(\gamma) - H^*(\tau) \\ = \theta(\gamma\tau) - \theta(\gamma) - \theta(\tau) - A \left(\frac{a_\gamma + d_\gamma}{c_\gamma} + \frac{a_\tau + d_\tau}{c_\tau} - \frac{a_{\gamma\tau} + d_{\gamma\tau}}{c_{\gamma\tau}} \right). \end{aligned}$$

Applying Corollary 2.3(i) and Lemma 3.5 on the right gives (1.12). The analogous relation for H in (1.11) is proved in the same way, using Theorem 2.1(i) and (2.5). This completes the proof of Theorem 1.4.

Let $\gamma = \gamma_1$, $\tau = \gamma_2$, and $(\gamma\tau)^{-1} = \gamma_3$ in Theorem 1.4. By Lemma 3.4, we have $H(\gamma_3) = -H(\gamma\tau)$ and $H^*(\gamma_3) = -H^*(\gamma\tau)$. Corollary 1.5 follows upon noting that $c_3 = -c_{\gamma\tau}$.

Proof of the two-term relations (Theorem 1.6). By Definition 1.1, one has

$$\begin{aligned} H(\gamma) - H(\gamma\tau) \\ = -S(\gamma) + S(\gamma\tau) + A \left(\frac{a_\gamma + d_\gamma}{c_\gamma} - \frac{a_{\gamma\tau} + d_{\gamma\tau}}{c_{\gamma\tau}} \right) - \frac{1}{4} \operatorname{sgn}(c_\gamma) + \frac{1}{4} \operatorname{sgn}(c_{\gamma\tau}). \end{aligned}$$

The relation (1.15) is then established with Theorem 2.1(i), (2.5), Lemma 3.5 and using the notation (1.9). Similarly,

$$H^*(\gamma) - H^*(\gamma\tau) = \theta(\gamma) - \theta(\gamma\tau) + A \left(\frac{a_\gamma + d_\gamma}{c_\gamma} - \frac{a_{\gamma\tau} + d_{\gamma\tau}}{c_{\gamma\tau}} \right)$$

and then (1.16) follows from Corollary 2.3(i) and Lemma 3.5.

Suppose that τ is parabolic or elliptic. We have seen that this means $\langle \tau, f \rangle = 0$. Also, if τ does not fix a cusp equivalent to $\infty \bmod \Gamma$, then $\theta(\tau) = 0$ by Corollary 2.3(iv, v). Therefore (1.16) reduces to (1.17) in this case.

Proof of the density result (Theorem 1.3). By Corollary 2.5, θ depends only on its values on a fixed set of generators for Γ and their commutators. More precisely, $\theta(\gamma)$ is a $\frac{1}{2}\mathbb{Z}$ -linear combination of this finite set of values. Recall that it is shown in [15, Sect. 6.4] that θ is rational on the set of generators and their commutators. Let L be the least common multiple of all the denominators. Then $2L \cdot \theta(\gamma) \in \mathbb{Z}$ for all $\gamma \in \Gamma$.

On the other hand, by the Gauss–Bonnet formula,

$$(3.2) \quad A = g - 1 + \frac{n}{2} + \frac{1}{2} \sum_{j=1}^{\ell} \left(1 - \frac{1}{m_j} \right)$$

where g is the genus of Γ , n is the number of inequivalent cusps, and the numbers m_j give the orders of the generating elliptic elements. There can only be finitely many elliptic generators. Let L' be the least common multiple of m_1, \dots, m_ℓ . It then follows that $A \in \frac{1}{2L'}\mathbb{Z}$. Set t to be the positive integer $4LL'$. Let $e(x) := e^{2\pi i x}$. Then for any $m, n \in \mathbb{N}$,

$$\begin{aligned} \sum_{c \leq X} \sum_{0 \leq d < c} e \left(m \cdot \frac{d}{c} + n \cdot t H^*(d, c) \right) &= \sum_{c \leq X} \sum_{0 \leq d < c} e \left(m \cdot \frac{d}{c} + n \cdot t A \frac{a + d}{c} \right) \\ &= \sum_{c \leq X} \sum_{0 \leq d < c} e \left(ntA \frac{a}{c} + (m + ntA) \frac{d}{c} \right). \end{aligned}$$

By the preceding discussion, both $M := ntA$ and $N := m + M$ are positive integers. Therefore, the series

$$S(M, N, c) = \sum_{0 \leq d < c} e \left(\frac{Ma + Nd}{c} \right)$$

defines a Selberg–Kloosterman sum. By Weyl’s equidistribution mod 1 criterion [26, 17], Theorem 1.3 follows by showing that

$$(3.3) \quad \sum_{c \leq X} S(M, N, c) = o \left(\sum_{c \leq X} S(0, 0, c) \right)$$

for each pair M, N of positive integers.

We may use the following results to establish (3.3). Theorem 2 in [7] shows the existence of a real number $\beta \in (0, 1]$ such that

$$(3.4) \quad \sum_{c \leq X} \frac{S(M, N, c)}{c} = O(X^{\beta/3+\varepsilon}).$$

In fact, this result of Goldfeld and Sarnak is more complicated, taking into account the possible presence of exceptional eigenvalues in the interval $(0, 1/4)$ for the Laplacian on $\Gamma \backslash \mathbb{H}$. There are none in the case of Hecke congruence groups, as seen in [27] for example. Then (3.4) follows from relating sums of Kloosterman sums to certain zeta functions appearing in the Fourier coefficients of non-holomorphic Poincaré series. Similarly, the sum $\sum_{c \leq X} S(0, 0, c)$ relates to the zeta function

$$Z(s) := \sum_{c > 0} \frac{S(0, 0, c)}{c^{2s}} = \frac{\pi^{-s/2} \Gamma(s)}{\Gamma(s - 1/2)} \varphi(s).$$

Here $\varphi(s)$ is the scattering determinant appearing in the Fourier expansion of the non-holomorphic Eisenstein series. It is absolutely convergent for $\operatorname{Re}(s) > 1$ and admits a meromorphic continuation to $s \in \mathbb{C}$, with no poles in $\operatorname{Re}(s) \geq 1/2$ save for a simple pole at $s = 1$ of residue $(\pi V_\Gamma)^{-1}$. Ikehara's Tauberian Theorem [12] now implies that

$$(3.5) \quad \sum_{c \leq X} S(0, 0, c) \sim \frac{X^2}{\pi V_\Gamma}.$$

Hence (3.4) and (3.5) imply (3.3) and this concludes the proof.

We note that (3.3) may also be deduced directly from [11, Theorem 4, p. 116] which, when specialized to our case, establishes

$$\sum_{c \leq X} S(M, N, c) = \delta_0 \frac{X^2}{\pi V_\Gamma} + O(X^{4/3})$$

where $\delta_0 = 1$ if $M = N = 0$ and is zero otherwise. Good's results in the monograph [11] are much more general, applying to various double cosets sums, though the difficult notation makes them hard to decipher.

4. Further transformation relations. The two-term relations in Theorem 1.6 yield additional arithmetic properties of the generalized Dedekind sums H and H^* , some of which we now state.

COROLLARY 4.1. *Let $w := \begin{pmatrix} 0 & -1/\lambda \\ \lambda & 0 \end{pmatrix} \in \Gamma$ for $\lambda > 0$. Let $\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma$ be such that $cd \neq 0$. Then*

$$(4.1) \quad H(d, c) - H(-c/\lambda, d\lambda) = A\left(\frac{d}{c} + \frac{c}{d\lambda^2} + \frac{1}{cd}\right) - \frac{1}{4} \operatorname{sgn}(cd),$$

$$(4.2) \quad H^*(d, c) - H^*(-c/\lambda, d\lambda) = A\left(\frac{d}{c} + \frac{c}{d\lambda^2} + \frac{1}{cd}\right).$$

Moreover, if $\iota(\Gamma) = \Gamma$ then $-H(-c/\lambda, d\lambda) = H(c/\lambda, d\lambda)$ and $-H^*(-c/\lambda, d\lambda) = H^*(c/\lambda, d\lambda)$.

Proof. The relation (1.17) for $\tau = w$ implies (4.2) directly. Also, (1.15) implies

$$H(d, c) - H(-c/\lambda, d\lambda) = A\left(\frac{d}{c} + \frac{c}{d\lambda^2} + \frac{1}{cd}\right) + S(w) + \frac{1}{4}(1 - \operatorname{sgn}(cd)).$$

Since $S(w^2) = S(-I) = -1/2$, and $S(w^2) = 2S(w) + \omega(w, w) = 2S(w)$, we can conclude that $S(w) = -1/4$. For the last statement, Lemmas 3.1 and 3.2 combine to show $H(\gamma) = -H(\iota(\gamma)) = -H(-\iota(\gamma))$. Note that $-I = w^2 \in \Gamma$. The same properties hold for H^* . ■

When zero is a cusp for the group Γ there is a particularly elegant form of Theorem 1.6.

COROLLARY 4.2. *Suppose that $P_0 := \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \in \Gamma$ for $s > 0$, and assume that the cusps 0 and ∞ are not Γ -equivalent. Then for $k \in \mathbb{Z}$ and $\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma$ with $c(c + ksd) \neq 0$, we have*

$$(4.3) \quad H^*(d, c) - H^*(d, c + ksd) = A \frac{ks(d^2 + 1)}{c(c + ksd)}.$$

Furthermore, if $c(c + ksd) > 0$, then

$$(4.4) \quad H(d, c) - H(d, c + ksd) = A \frac{ks(d^2 + 1)}{c(c + ksd)}.$$

Proof. Both (4.3) and (4.4) are true for $k = 0$ so we may assume $k \neq 0$. The identity (4.3) follows by taking $\tau = P_0^k$ in (1.17). By (1.15),

$$H(d, c) - H(d, c + ksd) = A \frac{ks(d^2 + 1)}{c(c + ksd)} + S(P_0^k) + \frac{1}{4} \operatorname{sgn}(k)(1 - \operatorname{sgn}(c(c + ksd))).$$

By Theorem 2.1(iv) and (2.9) we have $S(P_0^k) = 0$ and the identity (4.4) follows. ■

Numerous further relations and identities for H and H^* are possible. For example, suppose $\iota(\Gamma) = \Gamma$ and $\gamma = \begin{pmatrix} a & * \\ c & d \end{pmatrix} \in \Gamma$ with $c \neq 0$. Then by Lemmas 3.2 and 3.4 one finds that $H(d, c) = H(a, c)$ and $H^*(d, c) = H^*(a, c)$.

5. Examples. In this section we present transformation laws for the generalized Dedekind sums H and H^* associated to Hecke triangle groups,

Hecke congruence groups and Helling/moonshine type groups. We include the underlying group in the notation for clarity.

5.1. Hecke triangle groups. Let $\lambda_q := 2 \cos(\pi/q)$. The *Hecke triangle group* G_q is generated by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & \lambda_q \\ 0 & 1 \end{pmatrix}$ for integers $q \geq 3$. For each group G_q , the corresponding Riemann surface has genus 0 with one cusp, two elliptic points and volume $\pi(q-2)/q$. Except for $q = 3, 4, 6$ and ∞ , G_q is non-arithmetic.

We conjugate G_q into our desired form to obtain

$$\hat{G}_q := \sigma^{-1} G_q \sigma = \left\langle \left(\begin{pmatrix} 0 & -1/\lambda_q \\ \lambda_q & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \right\rangle \quad \text{for } \sigma = \begin{pmatrix} \lambda_q^{1/2} & 0 \\ 0 & \lambda_q^{-1/2} \end{pmatrix}.$$

The generalized Dedekind sums H^* do not exist for these groups as there are no weight 2 cusp forms. For $H_{\hat{G}_q}$ associated to \hat{G}_q , Corollary 4.1 implies that, with any $\begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \hat{G}_q$ and $cd > 0$,

$$(5.1) \quad H_{\hat{G}_q}(d, c) + H_{\hat{G}_q}(c/\lambda_q, d\lambda_q) = \frac{q-2}{4q} \left(\frac{d}{c} + \frac{c}{d\lambda_q^2} + \frac{1}{cd} \right) - \frac{1}{4}.$$

This is equivalent to the reciprocity from [1, p. 12]. We have used $\iota(\hat{G}_q) = \hat{G}_q$, valid as ι maps the generators to their inverses and has $\iota(\gamma\tau) = \iota(\gamma)\iota(\tau)$.

When $q = 3$ we have $\lambda_3 = 1$ and $G_3 = \hat{G}_3 = \text{SL}(2, \mathbb{Z})$. Then (5.1) becomes

$$H_{\text{SL}(2, \mathbb{Z})}(c, d) + H_{\text{SL}(2, \mathbb{Z})}(d, c) = \frac{1}{12} \left(\frac{d}{c} + \frac{c}{d} + \frac{1}{cd} \right) - \frac{1}{4},$$

which is the reciprocity law for classical Dedekind sums.

5.2. Hecke congruence groups. Define $\Gamma_0(N)$ as the subgroup of $\text{SL}(2, \mathbb{Z})$ with bottom left matrix elements divisible by the level N . In this subsection we take Γ to be one of these Hecke congruence groups, with subscript N in our notation indicating the level.

With the work of Vassileva [25], it can be seen that H_N may be expressed by a finite sum, as in the classical case. Define the products $\alpha_N := \prod_{p|N} 1/(1-p^{-1})$ and $\beta_N := \prod_{p|N} (1-p^{-2})/(1-p^{-1})$ where p is prime, and let μ denote the Möbius function. In [25, Thm. 4.1.3, Prop. 4.1.1] it is shown that for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$,

$$(5.2) \quad S_N(\gamma) = \beta_N \frac{N(a+d)}{12c} - \frac{c}{4|c|} - \alpha_N \frac{c}{|c|} \sum_{v|N} \frac{\mu(v)}{v} s \left(d, \frac{v|c|}{N} \right) \quad \text{if } c \neq 0$$

and

$$S_N(\gamma) = \beta_N \frac{Nb}{12d} + \frac{d/|d| - 1}{4} \quad \text{if } c = 0.$$

Since $N\beta_N$ is the index of the group $\Gamma_0(N)$ in $\mathrm{SL}(2, \mathbb{Z})$, one has $A_N = N\beta_N/12$. Hence

$$(5.3) \quad H_N(d, c) = \alpha_N \frac{c}{|c|} \sum_{v|N} \frac{\mu(v)}{v} s\left(d, \frac{v|c|}{N}\right) \quad \text{if } c \neq 0$$

and $H_N(d, c) = 0$ if $c = 0$.

For each parabolic generator of $\Gamma_0(N)$ we obtain a periodicity relation for H_N . The generator P_∞ gives the usual periodicity $H_N(d+c, c) = H_N(d, c)$ from Proposition 1.2. This also follows from (5.3) and (1.6). Next, if we assume that $N \geq 2$, then the cusps 0 and ∞ are not $\Gamma_0(N)$ -equivalent. Let c, d and k be positive integers with c and d relatively prime and $N|c$. Corollary 4.2 may be applied with $s = N$ to obtain the periodicity

$$(5.4) \quad H_N(d, c) - H_N(d, c + kNd) = A_N \frac{kN(d^2 + 1)}{c(c + kNd)}.$$

Inserting (5.3) into (5.4) then yields the summation formula

$$(5.5) \quad \sum_{v|N} \frac{\mu(v)}{v} \left(s\left(d, \frac{vc}{N}\right) - s\left(d, \frac{vc}{N} + vkd\right) \right) = \frac{N^2 k(d^2 + 1)}{12c(c + kNd)} \prod_{p|N} (1 - p^{-2}).$$

We are unaware of this identity involving the classical Dedekind sum (1.5) and the Möbius function appearing elsewhere in the literature. However, a direct proof of (5.5) may be given by applying the reciprocity law (1.7) and then the periodicity relation (1.6) to the second Dedekind sum. With another application of reciprocity the Dedekind sums disappear, and we are left with an identity that follows from the properties of μ .

The groups $\Gamma_0(N)$ are genus 0 for finitely many N . Outside these levels, H_N^* may be constructed as there are weight 2 cusp forms. It is not known if H_N^* may be expressed as a finite sum, similar to (5.3). Indirect evidence that such sums exist is given by the fact that the values of H_N^* are rational in all the cases examined so far. In particular, it follows from [15, Theorem 4] that H_N^* is always rational for $N \in \{11, 14, 15, 17, 19, 20, 21, 24, 27, 32, 36, 49\}$. These are the levels where $\Gamma_0(N)$ has genus 1 and so H_N^* is independent of the choice of non-zero cusp form.

5.3. The group $\Gamma_0(11)$. Let us focus on the Hecke congruence group of level $N = 11$, the simplest case where both $H_N = H_{11}$ and $H_N^* = H_{11}^*$ are not trivial. The group $\Gamma = \Gamma_0(11)$ is generated by the parabolic and hyperbolic elements

$$(5.6) \quad P_\infty := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad P_0 := \begin{pmatrix} 1 & 0 \\ 11 & 1 \end{pmatrix}, \quad Q := \begin{pmatrix} -7 & -1 \\ 22 & 3 \end{pmatrix}, \quad R := \begin{pmatrix} 4 & 1 \\ -33 & -8 \end{pmatrix},$$

along with $-I$, and they satisfy the relation $QRQ^{-1}R^{-1}P_0^{-1}P_\infty = I$. Thus, H_{11} and H_{11}^* have the periodicity relations corresponding to P_∞ , P_0 from Proposition 1.2 and Corollary 4.2. There are also hyperbolic periodicity relations corresponding to Q and R , from Theorem 1.6, which we describe next. In the notation of Section 5.2 above, $A_{11} = 11\beta_{11}/12 = 1$.

By the computations in [15, Sect. 7.1], with A and B replaced notionally by Q and R , we have

$$S_{11}(P_\infty) = 1, \quad S_{11}(P_0) = 0, \quad S_{11}(Q) = -\frac{2}{5}, \quad S_{11}(R) = \frac{2}{5},$$

and, by extending that work slightly, the evaluations

$$(5.7) \quad \theta_{11}(P_\infty) = -1, \quad \theta_{11}(P_0) = 0, \quad \theta_{11}(Q) = \frac{3}{10}, \quad \theta_{11}(R) = -\frac{3}{10}.$$

We may take $f(z)$ to be $\eta(z)^2\eta(11z)^2$. If $\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma$ then

$$\langle \gamma, f \rangle = -\langle \gamma^{-1}, f \rangle = -2\pi i \int_{\infty}^{\gamma^{-1}\infty} f(w) dw = 2\pi i \int_{-d/c}^{\infty} f(w) dw.$$

For all $c, d \in \mathbb{Z}$ that are relatively prime, with $11 \mid c$ and $c(22d - 7c) \neq 0$, the hyperbolic periodicity relations associated to Q are

$$\begin{aligned} & H_{11}(d, c) + H_{11}(c - 3d, -7c + 22d) \\ &= \left(\frac{d}{c} + \frac{c - 3d}{22d - 7c} + \frac{22}{c(22d - 7c)} \right) - \frac{2}{5} + \frac{1}{4}(1 - \operatorname{sgn}(c(22d - 7c))), \end{aligned}$$

and

$$\begin{aligned} (5.8) \quad & H_{11}^*(d, c) + H_{11}^*(c - 3d, -7c + 22d) \\ &= \left(\frac{d}{c} + \frac{c - 3d}{22d - 7c} + \frac{22}{c(22d - 7c)} \right) - \frac{3}{10} \\ &\quad - \frac{1}{2\|f\|^2} \operatorname{Im} \left(\int_{-d/c}^{\infty} f(w) dw \cdot \overline{\int_{-3/22}^{\infty} f(w) dw} \right). \end{aligned}$$

The relations for R are similar, and we may also give the transforms corresponding to the inverses of Q and R .

Using the above transformations, $H_{11}(d, c)$ and $H_{11}^*(d, c)$ can be computed for any pair d and c corresponding to $\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma_0(11)$ as follows. Write γ as a word in the generators (5.6). At each step apply the transform corresponding to the inverse of the rightmost generator appearing in the word. This eventually reduces γ to I or $-I$. For H_{11}^* this procedure will involve the modular symbols in (5.8).

A more efficient method to evaluate $H_{11}^*(\gamma)$ is to apply Corollary 2.5 to the word representing γ . This gives $\theta_{11}(\gamma)$ in terms of θ_{11} evaluated on generators and commutators of pairs of generators. It follows easily from

Lemma 2.4 that

$$\theta_{11}([\gamma, \tau]) = \theta_{11}(\gamma\tau) - \theta_{11}(\tau\gamma)$$

and so the problem reduces further. It is shown in [15, Sect. 7.1] how to compute θ_{11} of products of pairs of generators. Using the values (5.7) then completes the calculation.

5.4. The groups $\Gamma_0(N)^+$. Assume that N is a square-free positive integer with r factors. In this subsection we let Γ be the *Helling* or *moonshine type group* of level N , which is

$$\Gamma_0(N)^+ := \left\{ \frac{1}{\sqrt{e}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}) : a, b, c, d, e \in \mathbb{Z}, e \mid N, e \mid a, e \mid d, N \mid c \right\}.$$

Then this Γ has only one cusp, at ∞ , with Γ_∞ generated by P_∞ and $-I$. The constant $A = A_{N,+}$ in this case is $\sigma(N)/(12 \cdot 2^r)$ where $\sigma(N)$ indicates the sum of all divisors of N . See [16] and its included references for further information on $\Gamma_0(N)^+$.

The generalized Dedekind sum $H_{N,+}$ associated to $\Gamma_0(N)^+$ satisfies the periodicity relation of Proposition 1.2. Moreover, if $p \mid N$ then $\begin{pmatrix} 0 & -1/\sqrt{p} \\ \sqrt{p} & 0 \end{pmatrix} \in \Gamma_0(N)^+$. Let $\begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma_0(N)^+$ with $cd > 0$. By Corollary 4.1 we obtain the reciprocity relation

$$(5.9) \quad H_{N,+}(d, c) + H_{N,+}(c/\sqrt{p}, d\sqrt{p}) = \frac{\sigma(N)}{12 \cdot 2^r} \left(\frac{d}{c} + \frac{c}{pd} + \frac{1}{cd} \right) - \frac{1}{4}.$$

Further elliptic elements will result in additional reciprocity laws. For example, when $N = 37$, there are three more elliptic elements of $\Gamma_0(37)^+$:

$$\epsilon_1 := \begin{pmatrix} 6 & -1 \\ 37 & -6 \end{pmatrix}, \quad \epsilon_2 := \frac{1}{\sqrt{37}} \begin{pmatrix} 37 & -19 \\ 74 & -37 \end{pmatrix}, \quad \epsilon_3 := \begin{pmatrix} 11 & -3 \\ 37 & -10 \end{pmatrix}.$$

The first two are of order 4 and the third one is of order 6 (see [15, Sect. 7.2]). Computing with Theorem 2.1(v) yields $S_{37,+}(\epsilon_1) = S_{37,+}(\epsilon_2) = -1/4$ and $S_{37,+}(\epsilon_3) = -1/6$. Then, for $c, d \in \mathbb{Z}$ or $c, d \in \mathbb{Z}\sqrt{37}$, $cd > 0$ such that $\begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma_0(37)^+$, applying Theorem 1.6 with $\tau = \epsilon_1, \epsilon_2, \epsilon_3$ (using Lemmas 3.1 and 3.2 to adjust the signs) gives the following formulas:

$$(5.10) \quad H_{37,+}(d, c) + H_{37,+}(c + 6d, 6c + 37d) \\ = \frac{19}{12} \left(\frac{d}{c} + \frac{c + 6d}{6c + 37d} + \frac{37}{c(6c + 37d)} \right) - \frac{1}{4},$$

$$(5.11) \quad H_{37,+}(d, c) + H_{37,+}((19c + 37d)/\sqrt{37}, \sqrt{37}(c + 2d)) \\ = \frac{19}{12} \left(\frac{d}{c} + \frac{19c + 37d}{37(c + 2d)} + \frac{2}{c(c + 2d)} \right) - \frac{1}{4},$$

$$(5.12) \quad H_{37,+}(d, c) + H_{37,+}(3c + 10d, 11c + 37d) \\ = \frac{19}{12} \left(\frac{d}{c} + \frac{3c + 10d}{11c + 37d} + \frac{37}{c(11c + 37d)} \right) - \frac{1}{6}.$$

Recall that $\Gamma_0(37)^+$ has genus 1, and hence the Dedekind sum $H_{37,+}^*$ exists for this group. It is uniquely defined as $\dim S_2(\Gamma) = 1$. Using (1.17), one obtains reciprocity formulas for $H_{37,+}^*$ that are almost the same as those for $H_{37,+}$ above; replace $H_{37,+}$ by $H_{37,+}^*$ in (5.9)–(5.12) and remove the fractions $1/4$ and $1/6$ on the right.

The group $\Gamma_0^+(37)$ is generated by P_∞ , the elliptic elements $\begin{pmatrix} 0 & -1/\sqrt{37} \\ \sqrt{37} & 0 \end{pmatrix}$, $\epsilon_1, \epsilon_2, \epsilon_3$ and the hyperbolic elements

$$L := \frac{1}{\sqrt{37}} \begin{pmatrix} 148 & -89 \\ 185 & -111 \end{pmatrix}, \quad M := \begin{pmatrix} 20 & -13 \\ 37 & -24 \end{pmatrix}.$$

Hyperbolic periodicity relations associated to L , as an example, can be easily computed by combining Theorem 1.6 with computations from [15, Sect. 7.2].

The compactified quotient $X_0(37)^+ := \overline{\Gamma_0(37)^+} \backslash \mathbb{H}$ is isomorphic to the genus 1 curve $y^2 = 4x^3 - 4x + 1$ over \mathbb{C} , where $\overline{\Gamma_0(37)^+}$ is the projection of $\Gamma_0(37^+)$ onto $\mathrm{PSL}(2, \mathbb{R})$. Hence, we can take $f(z)$ to be the weight 2 cusp form on $\Gamma_0(37)^+$ related to the isomorphism between $\Gamma \backslash \mathbb{H} \cup \{\infty\}$ and the algebraic curve $y^2 = 4x^3 - 4x + 1$ sending ∞ to 0 in such a way that the pull-back of the canonical differential dx/y is $-2\pi i f(z) dz$. The Petersson norm of f is then $\|f\| = \omega_1 \omega_2 / (4\pi^2 i)$, where ω_1 and ω_2 are the real and complex periods of the curve $y^2 = 4x^3 - 4x + 1$.

By the computations in [15, Sect. 7.2], one has $\theta_{37,+}(L) = -19/24$. Hence, for all $\begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma_0(37)^+$ such that $c(4c + 5d) \neq 0$ we find

$$H_{37,+}^*(d, c) + H_{37,+}^*((89c + 111d)/\sqrt{37}, (4c + 5d)\sqrt{37}) \\ = \frac{19}{12} \left(\frac{d}{c} + \frac{89c + 111d}{37(4c + 5d)} + \frac{5}{c(4c + 5d)} \right) + \frac{19}{24} \\ - \frac{19}{24\|f\|^2} \mathrm{Im} \left(\int_{-d/c}^{\infty} f(w) dw \cdot \overline{\int_{3/5}^{\infty} f(w) dw} \right).$$

A similar expression can be easily deduced for the sum $H_{37,+}$.

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