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Counter-examples in parametric geometry of numbers

by

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1. Introduction. The basic object of Diophantine approximation is rational approximation to points $\mathbf{u}$ in $\mathbb{R}^n$. This is generally measured by elements of the extended real line $[-\infty, \infty]$ called *exponents of approximation* to $\mathbf{u}$. The *spectrum* of a family of exponents $(\mu_1, \dots, \mu_m)$ is the subset of $[-\infty, \infty]^m$ consisting of all m-tuples $(\mu_1(\mathbf{u}), \dots, \mu_m(\mathbf{u}))$ as $\mathbf{u}$ varies among the points of $\mathbb{R}^n$ with linearly independent coordinates over $\mathbb{Q}$. In all cases where such a spectrum has been explicitly determined, its *trace* on $\mathbb{R}^m$ (the set of its finite points) can be expressed as the set of common solutions of a finite system of polynomial inequalities (called *transference inequalities*). In particular, this trace is a *semialgebraic* subset of $\mathbb{R}^m$, namely a finite union of such solution sets. It is natural to ask if this is always so.

A general study of spectra is proposed in [7]. It is based on parametric geometry of numbers and the observation, due to Schmidt and Summerer [8], that the standard exponents of approximation to a point $\mathbf{u} \in \mathbb{R}^n$ can be computed from the knowledge of the successive minima of a certain one-parameter family of convex bodies in $\mathbb{R}^n$. Using the equivalent formalism of [5], we choose the family

$$\mathcal{C}_{\mathbf{u}}(q) := \{\mathbf{x} \in \mathbb{R}^n \,;\, \|\mathbf{x}\| \le 1 \text{ and } |\mathbf{x} \cdot \mathbf{u}| \le e^{-q}\} \quad (q \ge 0),$$

where $\mathbf{x} \cdot \mathbf{u}$ is the usual scalar product of $\mathbf{x}$ and $\mathbf{u}$, and where $\|\mathbf{x}\| = |\mathbf{x} \cdot \mathbf{x}|^{1/2}$ is the Euclidean norm of $\mathbf{x}$. For each integer i with $1 \le i \le n$ and each $q \ge 0$, we define $L_i(q) = \log \lambda_i(q)$ where $\lambda_i(q)$ is the ith minimum of $\mathcal{C}_{\mathbf{u}}(q)$ with respect to $\mathbb{Z}^n$, that is, the smallest real number $\lambda \ge 1$ such that $\lambda \mathcal{C}_{\mathbf{u}}(q)$ contains at least i linearly independent points of $\mathbb{Z}^n$.

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Let $\mathbf{L_u}\colon [0,\infty) \to \mathbb{R}^n$ be the map given by

$$\mathbf{L_u}(q) = (L_1(q), \dots, L_n(q)) \quad (q \geq 0).$$

Then each standard exponent of approximation to $\mathbf{u}$ can be computed as a linear fractional function of the quantity

$$\mu_F(\mathbf{L_u}) := \liminf_{q\to\infty} \frac{1}{q} F(\mathbf{L_u}(q)) \tag{1.1}$$

for some non-zero linear form $F\colon \mathbb{R}^n \to \mathbb{R}$. For example, as explained in [6], the exponents $\omega_{d-1}(\mathbf{u})$ and $\hat{\omega}_{d-1}(\mathbf{u})$ introduced by Laurent [2] for each integer d with $1 \leq d \leq n-1$, which provide measures of approximation to $\mathbf{u}$ by subspaces of $\mathbb{R}^n$ of dimension d defined over $\mathbb{Q}$, can be computed as

$$\omega_{d-1}(\mathbf{u}) = \mu_F(\mathbf{L_u})^{-1} - 1 \quad \text{and} \quad \hat{\omega}_{d-1}(\mathbf{u}) = -\mu_{-F}(\mathbf{L_u})^{-1} - 1 \tag{1.2}$$

for the linear form $F = \psi_{n-d}$ given by $\psi_{n-d}(x_1, \dots, x_n) = \sum_{i=1}^{n-d} x_i$. This observation is used in [7] to attach an abstract spectrum to each linear map $T = (T_1, \dots, T_m)\colon \mathbb{R}^n \to \mathbb{R}^m$. It is denoted $\mathrm{Im}^*(\mu_T)$ and consists of all m-tuples

$$\mu_T(\mathbf{L_u}) := (\mu_{T_1}(\mathbf{L_u}), \dots, \mu_{T_m}(\mathbf{L_u})) \tag{1.3}$$

where $\mathbf{u}$ runs through the points of $\mathbb{R}^n$ with linearly independent coordinates over $\mathbb{Q}$.

We refer the reader to [7] for a short description of the known spectra prior to 2018. To this list, we should now add the recent breakthrough of Marnat and Moshchevitin [3] who determined the spectra of the pairs $(\omega_0, \hat{\omega}_0)$ and $(\omega_{n-2}, \hat{\omega}_{n-2})$ by a combination of classical arguments and of parametric geometry of numbers, thereby proving a conjecture proposed by Schmidt and Summerer [9, §3]. We also refer to [4, Chapter 2] for a short alternative proof of this result based only on parametric geometry of numbers together with a general conjecture about the spectra of the pairs $(\omega_d, \hat{\omega}_d)$ with $0 \leq d \leq n-2$ and a proof of that conjecture for $n = 4$.

In [7] it is shown that, for each linear map $T\colon \mathbb{R}^n \to \mathbb{R}^m$, the spectrum $\mathcal{S} = \mathrm{Im}^*(\mu_T)$ is a compact connected subset of $\mathbb{R}^m$ and that, when $n \leq 3$, it is semialgebraic and closed under coordinatewise minimum. The last property means that for any two points $\mathbf{x} = (x_1, \dots, x_m)$ and $\mathbf{y} = (y_1, \dots, y_m)$ in $\mathcal{S}$, the point

$$\min\{\mathbf{x}, \mathbf{y}\} = (\min\{x_1, y_1\}, \dots, \min\{x_m, y_m\}) \tag{1.4}$$

also belongs to $\mathcal{S}$.

In this paper we show that both of these properties fail for $n \geq 4$. Our counter-examples involve linear maps $T\colon \mathbb{R}^n \to \mathbb{R}^m$ with $m = n+1$. It would be interesting to know, for given $n \geq 4$, what is the smallest value of m for which there exists a linear map $T\colon \mathbb{R}^n \to \mathbb{R}^m$ whose corresponding spectrum is not a semialgebraic subset of $\mathbb{R}^m$ or is not closed under coordinatewise minimum. In particular, we wonder if such counter-examples exist with $m = 2$ and, more precisely, if one could take $T = (F, -F)\colon \mathbb{R}^n \to \mathbb{R}^2$ for some linear form F on $\mathbb{R}^n$. Note that for $F = \psi_{n-d}$ with integers $0 \le d \le n-2$, the formulas (1.2) provide an algebraic bijection between the spectrum of $(F, -F)$ and that of $(\omega_d, \hat{\omega}_d)$ in $\mathbb{R}^n$.

2. Parametric geometry of numbers. Fix an integer $n \ge 2$. The main theorem of parametric geometry of numbers [5, Theorem 1.3] asserts that, modulo bounded functions, the classes of maps $\mathbf{L_u}$ attached to points $\mathbf{u}$ in $\mathbb{R}^n$ are the same as the classes of rigid n-systems of mesh δ for any given $\delta \ge 0$. There are several equivalent ways of defining an n-system (also called an $(n, 0)$-system). One of them is [5, Definition 2.8] (with $\gamma = 0$). Here we choose the simpler Definition 2.1 of [7] where

$$\mathbf{e}_1 = (1, 0, \dots, 0), \ \dots, \ \mathbf{e}_n = (0, \dots, 0, 1)$$

denote the elements of the canonical basis of $\mathbb{R}^n$.

DEFINITION 2.1. Let I be a closed subinterval of $[0, \infty)$ with non-empty interior. An *n-system* on I is a map $\mathbf{P} = (P_1, \dots, P_n)\colon I \to \mathbb{R}^n$ with the property that, for any $q \in I$:

(S1) $0 \le P_1(q) \le \cdots \le P_n(q)$ and $P_1(q) + \cdots + P_n(q) = q$;
(S2) there exist $\epsilon > 0$ and integers $k, \ell \in \{1, \dots, n\}$ such that

$$\mathbf{P}(t) = \begin{cases} \mathbf{P}(q) + (t-q)\mathbf{e}_\ell & \text{for any } t \in I \cap [q-\epsilon, q], \\ \mathbf{P}(q) + (t-q)\mathbf{e}_k & \text{for any } t \in I \cap [q, q+\epsilon]; \end{cases}$$

(S3) if q is in the interior of I and if the integers k and ℓ from (S2) satisfy $k > \ell$, then $P_\ell(q) = \cdots = P_k(q)$.

Moreover, we say that $\mathbf{P}$ is *proper* if $I = [q_0, \infty)$ for some $q_0 \ge 0$ and if $\lim_{q\to\infty} P_1(q) = \infty$.

As suggested by Luca Ghidelli, one can view an n-system on $[0, \infty)$ as describing a ball game with n players $P_1, \dots, P_n$ moving on the real line as a function of the time according to the following rules:

(R1) At time $t = 0$, all players are at position 0.
(R2) No player can pass another one, so that, at any time $t \ge 0$, their order remains $P_1 \le \cdots \le P_n$.
(R3) The only player that can move is the one who carries the ball and that player moves with constant speed 1.
(R4) A player can only pass the ball to a player that is behind or next to him/her.

Indeed, for $I = [0, \infty)$, the rules (R1) to (R3) codify (S1) and (S2) while (R4) codifies (S3), assuming that the ball moves instantaneously. This interpretation is useful in many ways. For example, when $n \geq 3$, we obtain an $(n-1)$-system out of an n-system by considering only the positions of $P_1, \dots, P_{n-1}$ and by stopping the time counter when P_n has the ball. Another way is to consider only the positions of $P_2, \dots, P_n$ and to stop counting the time when P_1 has the ball.

Let $\mathbf{P} = (P_1, \dots, P_n)$ be an n-system on an interval I as in Definition 2.1. Following the terminology of Schmidt and Summerer [8], the *division numbers* of $\mathbf{P}$ are the boundary points of I and the interior points q of I at which $\mathbf{P}$ is not differentiable, namely those for which we have $k \neq \ell$ in (S2). The *switch numbers* of $\mathbf{P}$ are the boundary points of I and the interior points q of I for which we have $k < \ell$ in (S2). The *division points* of $\mathbf{P}$ (resp. the *switch points* of $\mathbf{P}$) are the values of $\mathbf{P}$ at its division numbers (resp. switch numbers). When $I = [0, \infty)$, the non-zero division points of $\mathbf{P}$ represent the positions of the players when the ball is passed from a player to another one, and the non-zero switch points of $\mathbf{P}$ represent their positions when the ball is passed from a player to another one behind.

Definition 2.2. Let $\delta > 0$ and let $q_0 \geq 0$. We say that an n-system $\mathbf{P}$ on $[q_0, \infty)$ is *rigid of mesh* δ if each non-zero switch point of $\mathbf{P}$ has n distinct coordinates and if these coordinates are integer multiples of δ.

Equivalently, an n-system $\mathbf{P}\colon [q_0, \infty) \to \mathbb{R}^n$ is rigid of mesh δ if $q_0 \in \delta\mathbb{Z}$, if $\mathbf{P}(q) \in \delta\mathbb{Z}^n$ for each $q \in \delta\mathbb{Z}$ with $q \geq q_0$, and if for $q = q_0$ and each $q \in (q_0, \infty) \setminus \delta\mathbb{Z}$ the point $\mathbf{P}(q)$ has n distinct coordinates. In particular, the division numbers of such a system belong to $\delta\mathbb{Z}$.

The present paper relies on the following consequence of [5, Theorems 8.1 and 8.2] to which we alluded at the beginning of the section.

Theorem 2.3. *For each non-zero point* $\mathbf{u}$ *in* $\mathbb{R}^n$ *and each* $\delta > 0$, *there exist* $q_0 \in \delta\mathbb{Z}$ *with* $q_0 \geq 0$ *and a rigid* n*-system* $\mathbf{P}$ *of mesh* δ *on* $[q_0, \infty)$ *such that* $\mathbf{P} - \mathbf{L_u}$ *is bounded on* $[q_0, \infty)$. *Conversely, for any* $q_0 \geq 0$ *and any* n*-system* $\mathbf{P}$ *on* $[q_0, \infty)$, *there exists a non-zero point* $\mathbf{u} \in \mathbb{R}^n$ *such that* $\mathbf{P} - \mathbf{L_u}$ *is bounded on* $[q_0, \infty)$. *The point* $\mathbf{u}$ *has* $\mathbb{Q}$*-linearly independent coordinates if and only if the map* $\mathbf{P}$ *is proper.*

The last assertion follows from the preceding ones based on the fact that a point $\mathbf{u}$ in $\mathbb{R}^n$ has $\mathbb{Q}$-linearly independent coordinates if and only if the first component of the map $\mathbf{L_u}$ is unbounded (i.e. if $\lim_{q\to\infty} \lambda_1(\mathcal{C}_\mathbf{u}(q)) = \infty$).

It is interesting to compare the above notion of an n-system to that of an $1 \times (n-1)$-template according to Das, Fishman, Simmons and Urbański in [1, Definition 2.1]. Adapted to our present context, it becomes exactly the notion of a generalized n-system as in [6, Definition 4.5]. The formulation given below follows the clever and concise definition of a template by the four authors.

Definition 2.4. Let I be a closed subinterval of $[0,\infty)$ with non-empty interior. A *generalized n-system* on I is a continuous piecewise linear map $\mathbf{P} = (P_1,\dots,P_n)\colon I \to \mathbb{R}^n$ with the following properties:

(G1) We have $0 \le P_1(q) \le \cdots \le P_n(q)$ and $P_1(q)+\cdots+P_n(q) = q$ for each $q \in I$.
(G2) For each $j = 1,\dots,n$, the component $P_j\colon I \to \mathbb{R}$ is both monotone increasing and 1-Lipschitz.
(G3) For each $j=1,\dots,n{-}1$ and each subinterval H of I on which $P_j < P_{j+1}$, the sum $P_1+\cdots+P_j$ is convex on H with slopes among $\{0,1\}$.

Moreover, we say that $\mathbf{P}$ is *proper* if $I = [q_0,\infty)$ for some $q_0 \ge 0$ and if $\lim_{q\to\infty} P_1(q) = \infty$.

Recall that a function $f\colon I \to \mathbb{R}$ is 1-*Lipschitz* if $f(b) - f(a) \le b - a$ for any $a, b \in I$ with $a \le b$. So (G2) amounts to asking that each P_j has slopes belonging to $[0,1]$.

To analyze this definition and compare it to [6, Definition 4.5], fix such a map $\mathbf{P}$. Set $M_0 = 0$ and $M_j = P_1 + \cdots + P_j$ for each $j = 1,\dots,n$. Then consider a non-empty open subinterval H of I on which $\mathbf{P}$ is affine. For each $j = 1,\dots,n-1$, we have either $P_j = P_{j+1}$ or $P_j < P_{j+1}$ on the whole of H. In the latter case, M_j has constant slope 0 or 1 on H by (G3). Let $\underline{k} \ge 1$ be the largest index for which $M_{\underline{k}-1}$ has slope 0 on H, and let $\overline{k} \le n$ be the smallest one for which $M_{\overline{k}}$ has slope 1 on H. Then, for each index j with $\underline{k} \le j < \overline{k}$, the function M_j has constant slope $M_j' \in (0,1)$ (because of (G2)), and so $P_j = P_{j+1}$ on H. Thus $P_{\underline{k}},\dots,P_{\overline{k}}$ coincide and have slope $1/(\overline{k}-\underline{k}+1)$ on H while all other components of $\mathbf{P}$ are constant on H.

Now, consider an interior point q of I at which $\mathbf{P}$ is not differentiable and choose $\epsilon > 0$ such that $\mathbf{P}$ is defined and differentiable on both $(q-\epsilon, q)$ and $(q, q+\epsilon)$. For each $j = 1,\dots,n-1$ such that $P_j(q) < P_{j+1}(q)$, we have $P_j < P_{j+1}$ on $(q-\epsilon, q+\epsilon)$ and so M_j is convex with slopes in $\{0,1\}$ on that interval: either it has constant slope on $(q-\epsilon, q+\epsilon)$ or else it has slope 0 on $(q-\epsilon, q)$ and slope 1 on $(q, q+\epsilon)$. Let $\underline{\ell} \le \overline{\ell}$ and $\underline{k} \le \overline{k}$ be the indices for which $P_{\underline{\ell}} = \cdots = P_{\overline{\ell}}$ have slope $1/(\overline{\ell}-\underline{\ell}+1)$ on $(q-\epsilon, q)$, and $P_{\underline{k}} = \cdots = P_{\overline{k}}$ have slope $1/(\overline{k}-\underline{k}+1)$ on $(q, q+\epsilon)$. Then

$$P_{\underline{\ell}}(q) = \cdots = P_{\overline{k}}(q) \quad \text{if } \underline{\ell} < \overline{k}$$

because for each j with $\underline{\ell} \le j < \overline{k}$, the function M_j has slope > 0 on $(q-\epsilon, q)$ and slope < 1 on $(q, q+\epsilon)$, and so $P_j(q) = P_{j+1}(q)$ by a previous observation.

A generalized n-system on $[0,\infty)$ can therefore be viewed as describing a ball game where several players may carry the ball together (like in rugby or like young children generally do). We keep the same rules (R1) and (R2) but replace (R3) and (R4) by the following weaker rules:

(R3′) Only the players that carry the ball can move, and they move together at speed $1/m$ where m is the size of their group.
(R4′) The group of players carrying the ball can only pass the ball to a group of players that are waiting behind them or are next to them.

It follows from this interpretation that each n-system is a generalized n-system and that any generalized n-system is a uniform limit of n-systems (see [6, Corollary 4.7]). Thus Theorem 2.3 admits the following complement.

THEOREM 2.5. *For any $q_0 \geq 0$ and any generalized n-system $\mathbf{P}$ on $[q_0, \infty)$, there exists a non-zero point $\mathbf{u} \in \mathbb{R}^n$ such that $\mathbf{P} - \mathbf{L_u}$ is bounded on $[q_0, \infty)$. The point $\mathbf{u}$ has $\mathbb{Q}$-linearly independent coordinates if and only if the map $\mathbf{P}$ is proper.*

The fact that an n-system has property (G3) is very useful and we will often use it in Sections 4 and 5. In terms of a team of players following the rules (R1)–(R4), it simply expresses the fact that, for a given integer j with $1 \leq j < n$, when one of $P_1, \dots, P_j$ gets the ball, the ball remains within that group until P_j meets P_{j+1} with the ball.

3. Computing spectra from n-systems. Let $n \geq 2$ be an integer. For any $q_0 \geq 0$ and any Lipschitz map $\mathbf{P}\colon [q_0, \infty) \to \mathbb{R}^n$, we denote by $\mathcal{F}(\mathbf{P})$ the set of accumulation points of the quotients $q^{-1}\mathbf{P}(q)$ as q goes to infinity, and define $\mathcal{K}(\mathbf{P})$ to be the convex hull of $\mathcal{F}(\mathbf{P})$, as in [7, §3]. When $\mathbf{P}$ is an n-system or a generalized n-system, the set $\mathcal{F}(\mathbf{P})$ is contained in

$$\bar{\Delta} := \{(x_1, \dots, x_n) \in \mathbb{R}^n\, ;\, 0 \leq x_1 \leq \cdots \leq x_n \text{ and } x_1 + \cdots + x_n = 1\}.$$

As this is a compact convex subset of $\mathbb{R}^n$, both sets $\mathcal{F}(\mathbf{P})$ and $\mathcal{K}(\mathbf{P})$ are then compact subsets of $\bar{\Delta}$.

For each integer $m \geq 1$, we equip $\mathbb{R}^m$ with the coordinatewise ordering where

$$(x_1, \dots, x_m) \leq (y_1, \dots, y_m) \iff x_1 \leq y_1, \dots, x_m \leq y_m.$$

For that partial order, the minimum of two points is their coordinatewise minimum as defined in the introduction, and every bounded subset of $\mathbb{R}^m$ has an infimum in $\mathbb{R}^m$. Then, for any linear map $T = (T_1, \dots, T_m)\colon \mathbb{R}^n \to \mathbb{R}^m$ and any Lipschitz map $\mathbf{P}\colon [q_0, \infty) \to \mathbb{R}^n$, we define

$$\begin{aligned}\mu_T(\mathbf{P}) &= \inf T(\mathcal{K}(P)) = \inf T(\mathcal{F}(\mathbf{P})) = \liminf_{q\to\infty} q^{-1}T(\mathbf{P}(q)) \\ &= \Big(\liminf_{q\to\infty} q^{-1}T_1(\mathbf{P}(q)), \dots, \liminf_{q\to\infty} q^{-1}T_m(\mathbf{P}(q))\Big)\end{aligned}$$

as in [7, §3]. In view of Theorems 2.3 and 2.5, the computation of a spectrum is reduced to a problem about maps of a combinatorial nature.

THEOREM 3.1. *Let $\delta > 0$. Then the spectrum $\mathrm{Im}^*(\mu_T)$ of any linear map $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is the set of all numbers $\mu_T(\mathbf{P})$ where $\mathbf{P}\colon [q_0,\infty) \to \mathbb{R}^n$ is a proper rigid n-system of mesh δ (resp. a proper n-system, resp. a proper generalized n-system).*

For the purpose of this paper, we will need the following facts.

LEMMA 3.2. *Let $q_0 \geq 0$ and let $\mathbf{P}\colon [q_0,\infty) \to \mathbb{R}^n$ be a proper generalized n-system.*

(i) *Let $w_1 < w_2 < \cdots$ denote the points of (q_0,∞) at which $\mathbf{P}$ is not differentiable, listed in increasing order, and let E be the set of limit points of the sequence $(w_i^{-1}\mathbf{P}(w_i))_{i\geq 1}$. Then $\mathcal{K}(\mathbf{P})$ is the convex hull of E.*

(ii) *For each $\delta > 0$ there exists $Q_\delta \in (q_0,\infty)$ such that for each $q \geq Q_\delta$ we have $q^{-1}\mathbf{P}(q) \in \mathcal{F}(\mathbf{P}) + [-\delta,\delta]^n$ where*

$$\mathcal{F}(\mathbf{P}) + [-\delta,\delta]^n := \{\mathbf{x} \in \mathbb{R}^n\,;\, \|\mathbf{x}-\mathbf{y}\| \leq \delta \text{ for some } \mathbf{y} \in \mathcal{F}(\mathbf{P})\}.$$

(iii) *If there exists $\rho > 1$ such that $\mathbf{P}(\rho q) = \rho\mathbf{P}(q)$ for each $q \geq q_0$, then*

$$\mathcal{F}(\mathbf{P}) = \{q^{-1}\mathbf{P}(q)\,;\, q_0 \leq q \leq \rho q_0\}.$$

The property (i) is proved for proper n-systems in [7, Proposition 3.2] but the proof extends to generalized n-systems as it relies simply on the fact that, for each $i \geq 1$, the restriction of $\mathbf{P}$ to $[w_i, w_{i+1}]$ is an affine map and thus $\{t^{-1}\mathbf{P}(t)\,;\, w_i \leq t \leq w_{i+1}\}$ is the line segment joining $w_i^{-1}\mathbf{P}(w_i)$ to $w_{i+1}^{-1}\mathbf{P}(w_{i+1})$ in $\bar{\Delta}$. Similarly, (ii) is proved for n-systems in [7, Lemma 4.1] but the proof, based on a compactness argument, applies in fact to any Lipschitz map. Finally, (iii) is clear from the definition. A generalized n-system $\mathbf{P}$ which satisfies the condition in (iii) for some $\rho > 1$ is called *self-similar*.

4. Examples of spectra which are not closed under the minimum. We restrict here to dimension $n = 4$, although we believe that our examples can be adapted to any dimension $n \geq 4$. We will construct proper generalized 4-systems $\mathbf{R}$ and $\mathbf{S}$, and linear maps $T\colon \mathbb{R}^4 \to \mathbb{R}^5$ such that $\min\{\mu_T(\mathbf{R}), \mu_T(\mathbf{S})\}$ is not in the spectrum of T.

Note that, in dimension 4, the set

$$\bar{\Delta} = \{(x_1,\dots,x_4) \in \mathbb{R}^4\,;\, 0 \leq x_1 \leq \cdots \leq x_4 \text{ and } x_1 + \cdots + x_4 = 1\}$$

is a tetrahedron with vertices

$$E_1 = \left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right),\ E_2 = \left(0,\frac{1}{3},\frac{1}{3},\frac{1}{3}\right),\ E_3 = \left(0,0,\frac{1}{2},\frac{1}{2}\right),\ E_4 = (0,0,0,1).$$

For each $i = 1,2,3$, we denote by $\bar{\Delta}_i$ the face of $\bar{\Delta}$ consisting of the points $(x_1,\dots,x_4)$ in $\bar{\Delta}$ with $x_i = x_{i+1}$. The fourth face of $\bar{\Delta}$ is the triangle $\bar{\Delta}_0 = E_2E_3E_4$ defined by $x_1 = 0$.

Let $\alpha,\beta\in\mathbb{R}$ with $1<\alpha<\beta$. We first observe that there is a unique generalized 4-system $\mathbf{R}=(R_1,\dots,R_4)$ on $[3+\alpha,\alpha(3+\alpha)]$ with $R_1=R_2$, whose division points are

$$A_1=(1,1,1,\alpha),\ \ A_2=(1,1,\alpha,\alpha),\ \ A_3=(1,1,\alpha,\alpha^2),\ \ \alpha A_1=(\alpha,\alpha,\alpha,\alpha^2).$$

Its combined graph is shown in Figure 1 (left). Moreover, this map extends uniquely to a self-similar generalized 4-system on $[3+\alpha,\infty)$ also denoted $\mathbf{R}$ such that $\mathbf{R}(\alpha q)=\alpha\mathbf{R}(q)$ for each $q\ge 3+\alpha$.

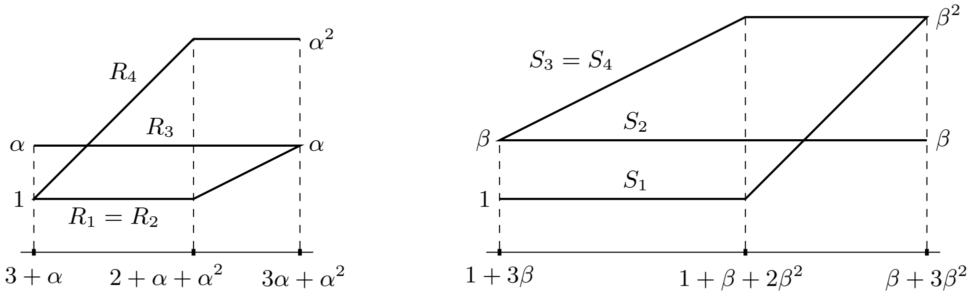


Fig. 1. The graphs of $\mathbf{R}$ and $\mathbf{S}$

Similarly, there is a unique generalized 4-system $\mathbf{S}=(S_1,\dots,S_4)$ on $[1+3\beta,\beta(1+3\beta)]$ with $S_3=S_4$, whose division points are

$$\begin{aligned} B_1&=(1,\beta,\beta,\beta), & B_3&=(\beta,\beta,\beta^2,\beta^2),\\ B_2&=(1,\beta,\beta^2,\beta^2), & \beta B_1&=(\beta,\beta^2,\beta^2,\beta^2).\end{aligned}$$

Its combined graph is shown in Figure 1 (right). The map extends uniquely to a self-similar generalized 4-system on $[1+3\beta,\infty)$ such that $\mathbf{S}(\beta q)=\beta\mathbf{S}(q)$ for each $q\ge 1+3\beta$.

For each $i=1,2,3$, let $\bar{A}_i$ (resp. $\bar{B}_i$) denote the quotient of A_i (resp. B_i) by the sum $|A_i|$ (resp. $|B_i|$) of its coordinates. Since $\mathbf{R}$ is self-similar and $R_1=R_2$, it follows from Lemma 3.2(i) that $\mathcal{K}(\mathbf{R})$ is the triangle $\bar{A}_1\bar{A}_2\bar{A}_3$ contained in the face $\bar{\Delta}_1=E_1E_3E_4$ of $\bar{\Delta}$. Similarly, since $S_3=S_4$, the convex set $\mathcal{K}(\mathbf{S})$ is the triangle $\bar{B}_1\bar{B}_2\bar{B}_3$ contained in $\bar{\Delta}_3=E_1E_2E_3$. These two triangles are shown in Figure 2 (left).

Let $\mathcal{K}$ denote the convex hull of the set $S:=\{\bar{B}_1,\bar{A}_1,\bar{A}_2,\bar{A}_3,E_3\}$. Since $\bar{B}_2$ and $\bar{B}_3$ belong respectively to the line segments $\bar{B}_1E_3$ and $\bar{A}_2E_3$, the convex $\mathcal{K}$ contains both $\mathcal{K}(\mathbf{R})$ and $\mathcal{K}(\mathbf{S})$. The right drawing in Figure 2 shows $\mathcal{K}$. Based on the relative positions of the points of S, we see that S is the set of vertices of $\mathcal{K}$ and that the boundary of $\mathcal{K}$ consists of four triangles $\bar{B}_1\bar{A}_1\bar{A}_2, \bar{B}_1\bar{A}_1\bar{A}_3, \bar{B}_1\bar{A}_3E_3, \bar{B}_1\bar{A}_2E_3\subset\bar{\Delta}_3$, and one quadrilateral $\bar{A}_1\bar{A}_2E_3\bar{A}_3\subset\bar{\Delta}_1$.

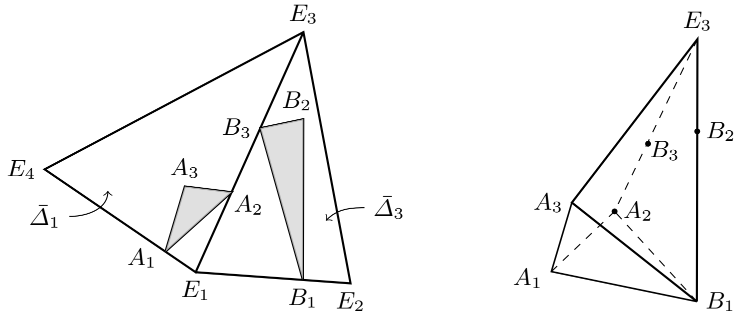


Fig. 2. Left: the triangles $\mathcal{K}(R)$ and $\mathcal{K}(S)$. Right: the convex $\mathcal{K}$.

Consider the linear map $T = (T_1, \dots, T_5)\colon \mathbb{R}^4 \to \mathbb{R}^5$ whose components are given by

$$\begin{aligned}
T_1(\mathbf{x}) &= -(\alpha-1)\beta x_1 - (\beta-\alpha)x_2 + (\beta-1)x_4,\\
T_2(\mathbf{x}) &= (\alpha-1)\beta x_1 - (\alpha-1)\beta x_2 + \alpha(\beta-1)x_3 - (\beta-1)x_4,\\
T_3(\mathbf{x}) &= \alpha\beta(\alpha-1)x_1 - \alpha(\alpha-1)x_2 + (\beta-1)x_3 - (\beta-1)x_4,\\
T_4(\mathbf{x}) &= x_3 - x_4,\\
T_5(\mathbf{x}) &= -x_2 + x_4.
\end{aligned}$$

The maps T_1, T_2 and T_3 are chosen so that they are non-negative on $\mathcal{K}$ and vanish respectively on the triangles $\bar{B}_1\bar{A}_1\bar{A}_2$, $\bar{B}_1\bar{A}_1\bar{A}_3$ and $\bar{B}_1\bar{A}_3E_3$. As the other two faces of $\mathcal{K}$ are contained in the faces $\bar{\Delta}_1$ and $\bar{\Delta}_3$ of $\bar{\Delta}$, we conclude that

$$\mathcal{K} = \{\mathbf{x} \in \bar{\Delta}\,;\, T_i(\mathbf{x}) \ge 0 \text{ for } i = 1,2,3\}. \tag{4.1}$$

We will prove the following result.

Theorem 4.1. *Suppose that a proper* 4*-system* $\mathbf{P}$ *satisfies* $\bar{B}_1 \in \mathcal{K}(\mathbf{P}) \subseteq \mathcal{K}$. *Then* $\mathcal{K}(\mathbf{P}) \subseteq \bar{\Delta}_3$.

In particular, this implies that there is no 4-system $\mathbf{P}$ for which $\mathcal{K}(\mathbf{P})$ is the convex hull of $\mathcal{K}(\mathbf{R}) \cup \mathcal{K}(\mathbf{S})$. This requires that the parameters α and β satisfy our current hypothesis $1 < \alpha < \beta$ because, for a choice of parameters satisfying $1 < \beta < \alpha$, the first author proved (unpublished work) that, on the contrary, such a 4-system $\mathbf{P}$ exists and so satisfies $\mu_L(\mathbf{P}) = \min\{\mu_L(\mathbf{R}), \mu_L(\mathbf{S})\}$ for any linear map $L\colon \mathbb{R}^4 \to \mathbb{R}^m$.

If we take Theorem 4.1 for granted, we deduce that the spectrum of T is not closed under the minimum.

Corollary 4.2. *There exists no proper* 4*-system* $\mathbf{P}$ *such that* $\mu_T(\mathbf{P}) = \min\{\mu_T(\mathbf{R}), \mu_T(\mathbf{S})\}$.

Proof. We find that

$$\mu_T(\mathbf{R}) = \min_{1\le i\le 3} T(\bar{A}_i) = (0,\, 0,\, 0,\, \alpha(1-\alpha)|A_3|^{-1},\, (\alpha-1)|A_2|^{-1}),$$
$$\mu_T(\mathbf{S}) = \min_{1\le i\le 3} T(\bar{B}_i) = (0,\, 0,\, 0,\, 0,\, 0),$$

and thus $\min\{\mu_T(\mathbf{R}), \mu_T(\mathbf{S})\} = (0,0,0,a,0)$ where $a = \alpha(1-\alpha)|A_3|^{-1}$ is negative. Suppose on the contrary that there exists a 4-system $\mathbf{P}$ such that $\mu_T(\mathbf{P}) = (0,0,0,a,0)$. Then we have $\inf T_i(\mathcal{K}(\mathbf{P})) = 0$ for $i = 1,2,3$ and so $\mathcal{K}(\mathbf{P}) \subseteq \mathcal{K}$ because of (4.1). We also have $\inf T_5(\mathcal{K}(\mathbf{P})) = 0$. However, T_5 is strictly positive at each vertex of $\mathcal{K}$ (the points of S) except at $\bar{B}_1$ where it vanishes. Thus $\bar{B}_1$ is the only point of $\mathcal{K}$ where T_5 vanishes and so $\mathcal{K}(\mathbf{P})$ must contain $\bar{B}_1$. Finally, we have $\inf T_4(\mathcal{K}(\mathbf{P})) = a < 0$ and so $\mathcal{K}(\mathbf{P}) \nsubseteq \bar{\Delta}_3$ because T_4 vanishes everywhere on $\bar{\Delta}_3$. This contradicts Theorem 4.1. ∎

Clearly, the corollary requires that there is no 4-system $\mathbf{P}$ for which $\mathcal{K}(\mathbf{P})$ is the convex hull $\widetilde{\mathcal{K}}$ of $\mathcal{K}(\mathbf{R}) \cup \mathcal{K}(\mathbf{S})$. Conversely, if we only assume this fact, then we can construct a linear map $T\colon \mathbb{R}^4 \to \mathbb{R}^{10}$ for which $\min\{\mu_T(\mathbf{R}), \mu_T(\mathbf{S})\}$ is not in the spectrum of T. It suffices to choose $T_1, \dots, T_4$ so that the conditions $T_i \ge 0$ $(1 \le i \le 4)$ define $\widetilde{\mathcal{K}}$ within $\bar{\Delta}$ and to choose the remaining six components $T_5, \dots, T_{10}$ so that they vanish at one of the six vertices $\bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{B}_1, \bar{B}_2, \bar{B}_3$ of $\widetilde{\mathcal{K}}$ and are strictly positive at the other vertices. The construction that we propose here is more economical as it uses a linear map T with only five components.

We now turn to the proof of Theorem 4.1. From now on we fix $q_0 > 0$ and a proper 4-system $\mathbf{P} = (P_1, \dots, P_4)\colon [q_0, \infty) \to \mathbb{R}^4$ satisfying $\bar{B}_1 \in \mathcal{K}(\mathbf{P}) \subset \mathcal{K}$, as in the statement of the theorem. For each $q \ge q_0$, we set

$$\kappa_1(q) = q^{-1}\big(\beta P_1(q) - P_2(q)\big),$$
$$\kappa_2(q) = q^{-1}\big(P_4(q) - P_2(q)\big), \tag{4.2}$$
$$\kappa_3(q) = q^{-1}\big(P_4(q) - P_3(q)\big).$$

We note that

$$\mathcal{K}(\mathbf{P}) \subseteq \bar{\Delta}_3 \iff \mathcal{F}(\mathbf{P}) \subseteq \bar{\Delta}_3 \iff \lim_{q\to\infty} \kappa_3(q) = 0. \tag{4.3}$$

So it remains to show that κ_3 vanishes at infinity.

We first rewrite the formula $q = P_1(q) + \cdots + P_4(q)$ $(q \ge q_0)$ as follows.

Lemma 4.3. *We have $q = (1+3\beta)P_1(q) + (2\kappa_2(q) - \kappa_3(q) - 3\kappa_1(q))q$ for each $q \ge q_0$.*

For each $\delta > 0$, we choose $Q_\delta > q_0$ as in Lemma 3.2(ii), such that $q^{-1}\mathbf{P}(q) \in \mathcal{F}(\mathbf{P}) + [-\delta, \delta]^4$ for each $q \ge Q_\delta$. We will need the following estimates.

LEMMA 4.4. *There exists a constant $c > 0$ with the following properties. For each $\delta > 0$ and each $q \geq Q_\delta$, we have*

(i) $-c\delta \leq \kappa_1(q) \leq \dfrac{\beta-1}{\alpha-1}\kappa_2(q) + c\delta$,

(ii) $\kappa_3(q) \leq \dfrac{\alpha^2-\alpha}{\beta-1}\kappa_1(q) + c\delta$.

Proof. Consider the linear forms f_1, f_2, f_3 on $\mathbb{R}^4$ given by

$$f_1(\mathbf{x}) = \beta x_1 - x_2, \qquad f_2(\mathbf{x}) = x_4 - x_2, \qquad f_3(\mathbf{x}) = x_4 - x_3. \tag{4.4}$$

A quick computation shows that

$$0 \leq f_1(\mathbf{x}) \leq \frac{\beta-1}{\alpha-1} f_2(\mathbf{x}) \quad \text{and} \quad f_3(\mathbf{x}) \leq \frac{\alpha^2-\alpha}{\beta-1} f_1(\mathbf{x})$$

for each $\mathbf{x} \in \{B_1, A_1, A_2, A_3, E_3\}$. Since f_1, f_2, f_3 are linear forms, these inequalities extend to the set of vertices $\{\bar{B}_1, \bar{A}_1, \bar{A}_2, \bar{A}_3, E_3\}$ of $\mathcal{K}$, and thus to their convex hull $\mathcal{K}$. We deduce that, for each $\delta > 0$ and each point $\mathbf{x}$ in $\mathcal{K} + [-\delta, \delta]^4$, we have

$$-c\delta \leq f_1(\mathbf{x}) \leq \frac{\beta-1}{\alpha-1} f_2(\mathbf{x}) + c\delta \quad \text{and} \quad f_3(\mathbf{x}) \leq \frac{\alpha^2-\alpha}{\beta-1} f_1(\mathbf{x}) + c\delta,$$

for a constant $c > 0$ that depends only on α and β. In particular, the latter inequalities hold at $\mathbf{x} = q^{-1}\mathbf{P}(q)$ for each $q \geq Q_\delta$; this yields (i) and (ii). ∎

LEMMA 4.5. *Let $q, r \in \mathbb{R}$ with $r > q \geq q_0$.*

(i) *If P_1 is constant on $[q, r]$, then $\kappa_1(t) \leq (q/t)\kappa_1(q)$ for each $t \in [q, r]$.*
(ii) *If P_4 is constant on $[q, r]$, then $\kappa_3(t) \leq (q/t)\kappa_3(q)$ for each $t \in [q, r]$.*
(iii) *We have $0 \leq \kappa_3(t) \leq \kappa_2(t)$ for each $t \geq q_0$.*

Proof. If P_1 is constant on $[q, r]$ and if $t \in [q, r]$, we find that

$$t\kappa_1(t) = \beta P_1(t) - P_2(t) = \beta P_1(q) - P_2(t) \leq \beta P_1(q) - P_2(q) = q\kappa_1(q),$$

which proves (i). The proof of (ii) is similar, and (iii) is clear. ∎

We conclude with the following lemma which, in view of (4.3), implies that $\mathcal{K}(\mathbf{P}) \subseteq \bar{\Delta}_3$ and thus proves the theorem.

LEMMA 4.6. *We have $\lim_{q\to\infty} \kappa_3(q) = 0$.*

Proof. Choose $\epsilon > 0$ small enough that

$$1 - 11\alpha\epsilon > 0 \quad \text{and} \quad \frac{\alpha}{\beta}\left(\frac{1+6\alpha\epsilon}{1-11\alpha\epsilon}\right) < 1. \tag{4.5}$$

This is possible since $1 < \alpha < \beta$. Then choose $\delta > 0$ such that

$$\frac{\alpha}{\beta}\left(\frac{1+6\alpha\epsilon}{1-11\alpha\epsilon}\right)\epsilon + \left(\frac{\beta-1}{\alpha-1} + 1\right)c\delta \leq \epsilon, \tag{4.6}$$

where c is as in Lemma 4.4. This requires in particular that $c\delta < \epsilon$. Since $\bar{B}_1 \in \mathcal{F}(\mathbf{P})$ and since each of the linear forms f_1, f_2, f_3 given by (4.4) vanish at $\bar{B}_1$, there exists $q \geq Q_\delta$ such that

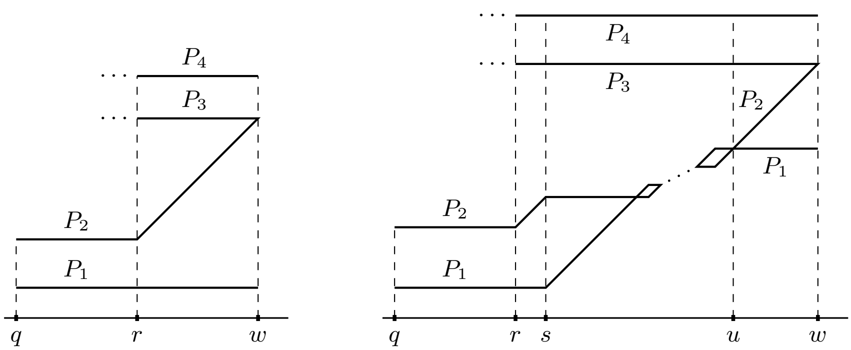
$$\kappa_1(q) \leq \epsilon \quad \text{and} \quad \kappa_3(q) \leq \kappa_2(q) \leq 2\alpha\epsilon. \tag{4.7}$$

These conditions allow us to perform the following induction argument. For q as above, let w be the smallest division point of $\mathbf{P}$ with $w > q$ for which $P_2(w) = P_3(w)$. We claim that

$$\kappa_1(w) \leq \epsilon, \quad \kappa_2(w) \leq 2\alpha\epsilon \quad \text{and} \quad \max_{q\leq t\leq w} \kappa_3(t) \leq 2\alpha\epsilon. \tag{4.8}$$

If we take this for granted, then (4.7) holds with q replaced by w and, since the division points of $\mathbf{P}$ form an infinite discrete sequence in $[q_0, \infty)$, we conclude, by induction, that $\kappa_3(t) \leq 2\alpha\epsilon$ for each $t \geq q_1$ where q_1 is the smallest solution of (4.7) with $q_1 \geq Q_\delta$. The lemma thus follows from this claim.

To prove (4.8), we first note that the combined graph of $\mathbf{P}$ over $[q, w]$ is as in Figure 3. Indeed, by the choice of w, we have $P_2 < P_3$ on (q, w), so $P_1 + P_2$ is convex on $[q, w]$: there exists $r \in [q, w]$ such that $P_1 + P_2$ is constant on $[q, r]$, and has slope 1 on $[r, w]$. Then P_1 and P_2 are constant on $[q, r]$ while P_3 and P_4 are constant on $[r, w]$. So, we must have $r < w$. Let s be the largest element of $[r, w]$ such that P_1 is constant on $[q, s]$. If $s = w$, the combined graph of $\mathbf{P}$ is as in Figure 3(a). Otherwise, it is as in Figure 3(b) where u denotes the largest element of $[s, w]$ at which $P_1(u) = P_2(u)$.



(a) Case where $P_1(q) = P_1(w)$ (b) Case where $P_1(q) < P_1(w)$

Fig. 3. The graph of $\mathbf{P}$ over $[q, w]$

Since $P_1(q) = P_1(s)$, Lemma 4.5(i) gives

$$\kappa_1(t) \leq \frac{q}{t}\kappa_1(q) \leq \epsilon \quad (q \leq t \leq s). \tag{4.9}$$

By Lemma 4.4(ii), this in turn implies that

$$\kappa_3(t) \le \frac{\alpha^2-\alpha}{\beta-1}\kappa_1(t) + c\delta < \alpha\epsilon + c\delta \le 2\alpha\epsilon \qquad (q \le t \le s), \tag{4.10}$$

since $c\delta \le \epsilon \le \alpha\epsilon$. If $s = w$, this proves (4.8) because then $\kappa_2(w) = \kappa_3(w) = \kappa_3(s) \le 2\alpha\epsilon$.

Suppose from now on that $s < w$. Since P_3 and P_4 are constant on $[r, w]$, we have

$$\kappa_3(t) = \frac{r}{t}\kappa_3(r) \le \kappa_3(r) \qquad (r \le t \le w), \tag{4.11}$$

and so (4.10) yields $\kappa_3(t) \le 2\alpha\epsilon$ for each $t \in [q, w]$. In particular, we obtain $\kappa_2(w) = \kappa_3(w) \le 2\alpha\epsilon$ because $P_2(w) = P_3(w)$. So, it only remains to prove that $\kappa_1(w) \le \epsilon$.

Applying Lemma 4.4(i) at the point w, and using $\kappa_2(w) = \kappa_3(w) = (r/w)\kappa_3(r)$ from above, we obtain

$$\kappa_1(w) \le \frac{\beta-1}{\alpha-1}\kappa_2(w) + c\delta = \frac{\beta-1}{\alpha-1}\cdot\frac{r}{w}\kappa_3(r) + c\delta.$$

Using the first parts of (4.9) and (4.10) with $t = r$, we also find that

$$\kappa_3(r) \le \frac{\alpha^2-\alpha}{\beta-1}\kappa_1(r) + c\delta \le \frac{\alpha^2-\alpha}{\beta-1}\cdot\frac{q}{r}\kappa_1(q) + c\delta.$$

Combining this inequality with the preceding one, we obtain

$$\kappa_1(w) \le \alpha\frac{q}{w}\kappa_1(q) + \left(\frac{\beta-1}{\alpha-1}+1\right)c\delta. \tag{4.12}$$

To estimate the ratio q/w from above, we use Lemma 4.3 at the points q and w together with the relations

$$P_1(w) = P_2(u) \ge P_2(q) = \beta P_1(q) - q\kappa_1(q) \ge \beta P_1(q) - \epsilon q \tag{4.13}$$

coming from the behavior of $\mathbf{P}$ on $[q, w]$ illustrated in Figure 3(b), the definition of κ_1, and the hypothesis (4.7). Since $\kappa_2(q) \ge \kappa_3(q) \ge 0$ and $\kappa_2(w) = \kappa_3(w) \ge 0$, Lemma 4.3 gives

$$\begin{aligned} q &\le (1+3\beta)P_1(q) + (2\kappa_2(q) - 3\kappa_1(q))q,\\ w &\ge (1+3\beta)P_1(w) - 3\kappa_1(w)w. \end{aligned}$$

By Lemma 4.4(i), we have $\kappa_1(q) \ge -c\delta \ge -\epsilon$. Using (4.7) this yields $|\kappa_1(q)| \le \epsilon$ and $2\kappa_2(q) - 3\kappa_1(q) \le 7\alpha\epsilon$. By (4.12), (4.6) and (4.7), we also have $\kappa_1(w) \le \alpha\kappa_1(q) + \epsilon \le 2\alpha\epsilon$. Together with (4.13), this gives

$$\begin{aligned} (1+6\alpha\epsilon)w \ge (1+3\beta)P_1(w) &\ge \beta(1+3\beta)P_1(q) - 4\beta\epsilon q\\ &\ge \beta(1-7\alpha\epsilon)q - 4\beta\epsilon q \ge \beta(1-11\alpha\epsilon)q. \end{aligned}$$

Substituting in (4.12), we conclude that

$$\kappa_1(w) \le \frac{\alpha}{\beta}\left(\frac{1+6\alpha\epsilon}{1-11\alpha\epsilon}\right)\kappa_1(q) + \left(\frac{\beta-1}{\alpha-1}+1\right)c\delta \le \epsilon,$$

using $\kappa_1(q) \le \epsilon$ and the hypothesis (4.6). ∎

Remark. Although the above shows that the spectrum $\mathrm{Im}^*(\mu_T)$ attached to a linear map $T\colon \mathbb{R}^n \to \mathbb{R}^m$ with $n \ge 4$ is not necessarily closed under the minimum, it is worth looking at conditions on T which ensure that this property holds. In his PhD thesis [4, Chapter 4], the first author shows that it holds for $n = 4$ when each component of T achieves its infimum on $\bar{\Delta}$ at the vertex $E_1 = (1/n, \dots, 1/n)$, but we do not know if this condition is sufficient for $n > 4$. In the notation of the introduction, an example of a linear map which fulfills this condition for any given $n \ge 2$ is $T = (\psi_1, \dots, \psi_{n-1})\colon \mathbb{R}^n \to \mathbb{R}^{n-1}$. Its spectrum is computed in [6] and yields via a simple algebraic transformation the spectrum of $(\omega_0, \omega_1, \dots, \omega_{n-2})$ in $\mathbb{R}^n$. In that case, Propositions 5.4 and 6.1 of [6] imply that the spectrum of T is closed under the minimum.

5. A family of non-semialgebraic spectra. Let $n \ge 4$ be an integer and let $\alpha > 1$ be a real number. Consider the linear map $T = (T_1, \dots, T_{n+1})\colon \mathbb{R}^n \to \mathbb{R}^{n+1}$ whose components are given by

$$T_1(\mathbf{x}) = x_1, \qquad T_n(\mathbf{x}) = x_n - \alpha^{n-3}x_2,$$
$$T_j(\mathbf{x}) = \alpha x_j - x_{j+1} \quad (2 \le j \le n-1), \qquad T_{n+1}(\mathbf{x}) = x_n - \alpha^{n-2}x_1,$$

for any $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$. The goal of this paragraph is to show that the spectrum $\mathrm{Im}^*(\mu_T)$ is not a semialgebraic subset of $\mathbb{R}^{n+1}$. More precisely, we will establish the following result, where $\mathbb{N}_+$ denotes the set of positive integers.

Theorem 5.1. *With the above notation, set $\beta = 1 + \alpha + \dots + \alpha^{n-2}$, and let E denote the set of all real numbers θ for which there exists a proper n-system $\mathbf{P}\colon [q_0, \infty) \to \mathbb{R}^n$ with $\mu_T(\mathbf{P}) = (\theta, 0, \dots, 0)$. Then we have*

$$E = \{0\} \cup \{(1 + \alpha^m\beta)^{-1}\,;\, m \in \mathbb{N}_+\}. \tag{5.1}$$

In particular, E contains infinitely many isolated points. So, E is not a semialgebraic subset of $\mathbb{R}$ and thus $\mathrm{Im}^*(\mu_T)$ is not a semialgebraic subset of $\mathbb{R}^{n+1}$.

As the proof will show, a proper n-system $\mathbf{P}$ with $\mu_T(\mathbf{P}) = (\theta, 0, \dots, 0)$ for some $\theta > 0$ has a very constrained behavior. Its graph decomposes into pieces which, after rescaling, converge to a graph of the type shown in Figure 5. We will need the following lemma.

LEMMA 5.2. *Let $a, b \in \mathbb{R}$ with $0 < a < b$, and let $\mathbf{P}_k\colon [a,b] \to \mathbb{R}^n$ be an n-system on $[a,b]$ for each $k \in \mathbb{N}_+$. Then there exists a subsequence of $(\mathbf{P}_k)_{k\geq 1}$ which converges uniformly on $[a,b]$. Its limit is a continuous function $\mathbf{f} = (f_1,\dots,f_n)\colon [a,b] \to \mathbb{R}^n$ with the following properties:*

(i) *for $j = 1,\dots,n$, its component $f_j\colon [a,b] \to \mathbb{R}$ is 1-Lipschitz and increasing;*
(ii) *we have $0 \leq f_1(t) \leq \cdots \leq f_n(t)$ and $f_1(t) + \cdots + f_n(t) = t$ for each $t \in [a,b]$;*
(iii) *if $f_j < f_{j+1}$ on (a,b) for some $j \in \{1,\dots,n-1\}$, then $f_1 + \cdots + f_j$ is convex on $[a,b]$ and piecewise-linear with slopes 0 then 1;*
(iv) *if $f_1(t) < f_2(t) < \cdots < f_n(t)$ for all but finitely many $t \in [a,b]$, then $\mathbf{f}$ is an n-system on $[a,b]$.*

Proof. The sequence $(\mathbf{P}_k)_{k\geq 1}$ is equicontinuous and uniformly bounded on $[a,b]$ because it consists of 1-Lipschitz maps whose maximum norm is bounded above by b. By the Arzelà–Ascoli theorem, it therefore admits a subsequence that converges uniformly on $[a,b]$. Let $\mathbf{f}=(f_1,\dots,f_n)\colon [a,b]\to\mathbb{R}^n$ be its limit. Then $\mathbf{f}$ is 1-Lipschitz on $[a,b]$. In particular, it is continuous and each of its components $f_1,\dots,f_n$ is 1-Lipschitz on $[a,b]$. The latter are also increasing on $[a,b]$ since the components of each $\mathbf{P}_k$ are such. This shows (i). Property (ii) is also immediate because for each $t \in [a,b]$, the coordinates of $\mathbf{P}_k(t)$ form an increasing sequence $P_{k,1}(t) \leq \cdots \leq P_{k,n}(t)$ with sum t. Now, suppose that $f_j < f_{j+1}$ on (a,b) for some $j \in \{1,\dots,n-1\}$, and let $[c,d]$ be any compact subinterval of (a,b). Then, for each large enough index k, we have $P_{k,j} < P_{k,j+1}$ on $[c,d]$. For those k, the sum $P_{k,1} + \cdots + P_{k,j}$ is convex on $[c,d]$ with slopes 0 then 1. We deduce that $f_1 + \cdots + f_j$ is convex on $[c,d]$ and piecewise linear with slopes 0 then 1. Property (iii) follows from this by letting c and d go to a and b respectively. Finally, (iv) follows from (i)–(iii). ∎

Proof of Theorem 5.1. It suffices to prove that E is given by (5.1). We start by proving that the non-zero points of E have the form $(1+\alpha^m\beta)^{-1}$ for some positive integer m.

Let $\mathbf{P} = (P_1,\dots,P_n)\colon [0,\infty) \to \mathbb{R}^n$ be a proper n-system such that $\mu_T(\mathbf{P}) = (\theta, 0, \dots, 0)$ for some $\theta > 0$. We denote by $q_1 < q_2 < \cdots$ the sequence of points $q \in [1,\infty)$ for which $P_1(q) = P_2(q)$, listed in increasing order. In each open interval (q_i, q_{i+1}), there is a point r_i such that P_1 is constant on $[q_i, r_i]$ while it has slope 1 on $[r_i, q_{i+1}]$. As the ratio $P_1(t)/t$ is bounded above by 1 for each $t \geq q_0$, it achieves its minimum on $[q_i, q_{i+1}]$ at the point r_i. By definition of $\mu_T(\mathbf{P})$, this means that

$$\theta = \liminf_{q\to\infty} \frac{P_1(q)}{q} = \liminf_{i\to\infty} \frac{P_1(r_i)}{r_i}. \tag{5.2}$$

So, for each sufficiently large i, we have $(\theta/2)r_i < P_1(r_i) = P_1(q_i) \leq q_i$, and therefore $1 \le r_i/q_i < 2/\theta$. It follows that there exists an infinite subset I of $\mathbb{N}_+$ and a real number $\rho \in [1, 2/\theta]$ such that $P_1(r_i)/r_i$ and r_i/q_i converge respectively to θ and ρ as i goes to infinity in I.

Set

$$a = 1 + \beta, \qquad c = \rho a, \qquad b = \frac{2}{\theta} a.$$

For each $i \in I$, we define an n-system $\mathbf{P}_i \colon [a,b] \to \mathbb{R}^n$ by

$$\mathbf{P}_i(t) = \frac{a}{q_i} \mathbf{P}\left(\frac{q_i t}{a}\right) \qquad (a \le t \le b).$$

By Lemma 5.2, there is an infinite subset I' of I such that $\mathbf{P}_i$ converges uniformly to a continuous map $\mathbf{f} = (f_1, \dots, f_n) \colon [a,b] \to \mathbb{R}^n$ as i goes to infinity in I'. We will show that $c = 1/\theta > a$ and that the restriction of $\mathbf{f}$ to $[a,c]$ is an n-system which is uniquely determined by θ. Then, from the explicit form of $\mathbf{f}$, we will deduce that $\theta = (1 + \alpha^m \beta)^{-1}$ for some integer $m \ge 1$. For the proof, we use freely the fact that the restriction of $\mathbf{f}$ to any closed subinterval of $[a,b]$ satisfies the properties (i) to (iv) in Lemma 5.2.

For each $i \in I$, we note that

$$P_{i,1}(a) = \frac{a}{q_i} P_1(q_i) = \frac{a}{q_i} P_2(q_i) = P_{i,2}(a).$$

Since P_1 is 1-Lipschitz, we also have

$$\begin{aligned} P_{i,1}(c) &= \frac{a}{q_i} P_1(\rho q_i) = \frac{a}{q_i} P_1(r_i + o(q_i)) = \frac{a}{q_i} P_1(r_i) + o(1) \\ &= \begin{cases} \dfrac{a}{q_i} P_1(q_i) + o(1) = P_{i,1}(a) + o(1), \\ a\dfrac{r_i}{q_i} \cdot \dfrac{P_1(r_i)}{r_i} + o(1) = c\theta + o(1). \end{cases} \end{aligned}$$

By passing to the limit as i goes to infinity in I', these estimates give

$$f_1(a) = f_2(a) \quad \text{and} \quad f_1(c) = f_1(a) = c\theta. \tag{5.3}$$

As f_1 is increasing, the second set of equalities implies that f_1 is constant on $[a,c]$. Moreover, for fixed $t \in [a,b]$, the ratio $q_i t/a$ tends to infinity with i. So, the hypothesis that $\mu_T(\mathbf{P}) = (\theta, 0, \dots, 0)$ yields

$$\liminf_{i\to\infty} T\left(\frac{1}{t}\mathbf{P}_i(t)\right) = \liminf_{i\to\infty} T\left(\frac{a}{q_i t}\mathbf{P}\left(\frac{q_i t}{a}\right)\right) \ge (\theta, 0, \dots, 0),$$

thus $T(t^{-1}\mathbf{f}(t)) \ge (\theta, 0, \dots, 0)$. Explicitly, this means that $t^{-1} f_1(t) \ge \theta$ and

$$\max\{\alpha^{n-2} f_1(t), \alpha^{n-3} f_2(t)\} \le f_n(t) \le \alpha f_{n-1}(t) \le \cdots \le \alpha^{n-2} f_2(t). \tag{5.4}$$

Using (5.3), we deduce that

$$\theta = \frac{f_1(c)}{c} = \min_{a \le t \le b} \frac{f_1(t)}{t}, \tag{5.5}$$

and so $f_1(t) > f_1(c)$ when $t > c$. When $f_1(t) = f_2(t)$, the inequalities (5.4) force $\mathbf{f}(t)$ to be a multiple of $(1, 1, \alpha, \dots, \alpha^{n-2})$. Since the coordinates of this point sum up to $1 + \beta = a$ and since those of $\mathbf{f}(t)$ sum up to t, we deduce that

$$\mathbf{f}(t) = \frac{t}{a}(1, 1, \alpha, \dots, \alpha^{n-2}) \quad \text{if } f_1(t) = f_2(t). \tag{5.6}$$

In particular, it follows from (5.3) that

$$\mathbf{f}(a) = (1, 1, \alpha, \dots, \alpha^{n-2}) \quad \text{and} \quad c = \theta^{-1}. \tag{5.7}$$

Similarly, if $f_j(t) = f_{j+1}(t)$ for some integer j with $2 \le j \le n-1$, these inequalities imply that

$$\mathbf{f}(t) = (r, s, s\alpha, \dots, s\alpha^{j-2}, s\alpha^{j-2}, \dots, s\alpha^{n-3}) \tag{5.8}$$

for some real numbers r, s with $0 \le r \le s$, and thus $0 \le r < s$ in view of (5.6). From this we infer that, for each $t \in [a, b]$, there is at most one index j with $1 \le j < n$ such that $f_j(t) = f_{j+1}(t)$.

The formula (5.6) implies that the points $t \in [a, b]$ for which $f_1(t) = f_2(t)$ are isolated. Indeed, if t is such a point, then $f_2 < f_3$ in some connected open neighborhood U of t in $[a, b]$. Hence, $f_1 + f_2$ is convex on U with slopes 0 and 1 (by Lemma 5.2(iii)). This implies that $f_1(u) + f_2(u) = 2u/a$ for at most two values of $u \in U$ (by comparing slopes since $0 < 2/a < 1$). By (5.6), these are the only possible $u \in U$ for which $f_1(u) = f_2(u)$.

By the above observation, since $f_1(a) = f_2(a)$, there exists a maximal d with $a < d \le b$ such that $f_1 < f_2$ on (a, d). Then f_1 is convex on $[a, d]$ with slopes 0 and 1. As f_1 is constant on $[a, c]$ and $f_1(t) > f_1(c)$ when $t > c$, we deduce that $c \in [a, d]$ and that f_1 has slope 1 on $[c, d]$. Since $f_1 + \dots + f_n$ has slope 1 on $[a, b]$, it follows that $f_2 + \dots + f_n$ is constant on $[c, d]$ and so each of $f_2, \dots, f_n$ is constant on $[c, d]$. In particular, we deduce that

$$\begin{aligned} f_2(d) &= f_2(c) \le f_2(a) + c - a = f_1(a) + c - a, \\ f_1(d) &= f_1(c) + d - c = f_1(a) + d - c. \end{aligned}$$

Since $f_1(d) \le f_2(d)$, this yields $d \le 2c - a$. We deduce that $c > a$ because $d > a$, and also that $d < b$ since $2c = 2/\theta < b$. By the choice of d, we conclude that $f_1(d) = f_2(d)$, and so $\mathbf{f}(d) = (d/a)\mathbf{f}(a)$.

For each $j = 1, \dots, n-1$, let S_j denote the closed subset of $[a, d]$ consisting of all points $t \in [a, d]$ with $f_j(t) = f_{j+1}(t)$. By an earlier remark, these sets are pairwise disjoint. Moreover, we have $S_1 = \{a, d\}$ and $S_j \cap [c, d] = \emptyset$ for $j = 2, \dots, n-1$. We claim that, like S_1, the sets $S_2, \dots, S_{n-1}$ are also finite. As the proof will show, this is where we need $n \ge 4$.

To prove this claim, fix $j \in \{2, \dots, n-1\}$. For each $t \in S_j$, we have $t \in (a, c)$, thus $f_1(t) = 1$, and so $\mathbf{f}(t)$ has the form (5.8) with $r = 1$ and

$s = (t-1)/\beta_j$ where

$$\beta_j = 1 + \cdots + \alpha^{j-2} + \alpha^{j-2} + \cdots + \alpha^{n-3}.$$

If $j > 2$, there is a neighborhood U of t in (a, c) on which $f_2 < f_3$. Then $f_1 + f_2$ is convex with slopes 0 and 1 on U, so $f_1(u) + f_2(u) = 1 + (u-1)/\beta_j$ has at most two solutions u in U (because $0 < 1/\beta_j < 1$), and therefore $U \cap S_j$ consists of at most two points. Similarly, if $j = 2$, there is a neighborhood U of t in (a, c) on which $f_3 < f_4$. Then $f_1 + f_2 + f_3$ is convex with slopes 0 and 1 on U, so $f_1(u) + f_2(u) + f_3(u) = 1 + 2(u-1)/\beta_2$ has at most two solutions u in U (because $0 < 2/\beta_2 < 1$). Hence $U \cap S_2$ consists again of at most two points. In both cases, this shows that S_j is a discrete subset of $[a, b]$ and so it is finite.

Since the set $S := S_1 \cup \cdots \cup S_{n-1}$ is finite, Lemma 5.2 shows that the restriction of $\mathbf{f}$ to $[a, d]$ is an n-system. Let $t_0 = a < t_1 < \cdots < t_N = d$ be the points of S listed in increasing order. Fix an integer i with $0 \le i < N$ and let k denote the index for which $t_i \in S_k$. Then the point $\mathbf{f}(t_i)$ is given by (5.8) with $r = 1$, $j = k$ and some value of s. We will show that the restriction of $\mathbf{f}$ to $H = [t_i, t_{i+1}]$ is entirely determined by $\mathbf{f}(t_i)$.

We first note that, for each $j = 1, \dots, n-1$, we have $f_j < f_{j+1}$ on (t_i, t_{i+1}), thus the sum $f_1 + \cdots + f_j$ is convex on H with slopes in $\{0, 1\}$.

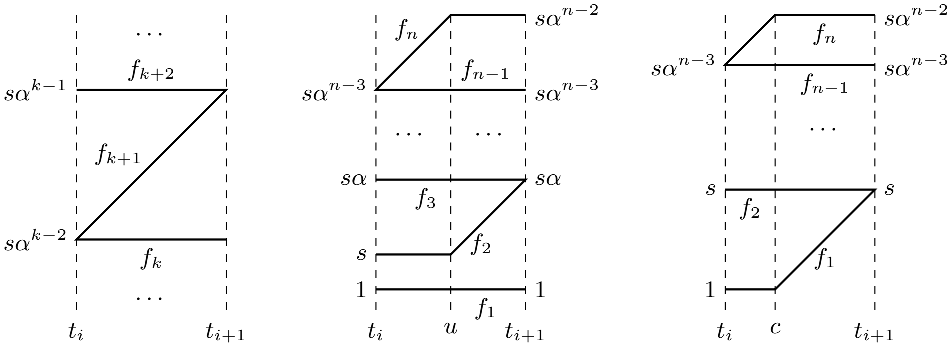
Suppose first that $k \le n-2$. Since f_{k+1} has slope 1 immediately to the right of t_i, the sum $f_1 + \cdots + f_{k+1}$ has constant slope 1 on H. So, f_{k+2} is constant on H and we have

$$\frac{f_{k+2}(t_{i+1})}{f_{k+1}(t_{i+1})} < \frac{f_{k+2}(t_i)}{f_{k+1}(t_i)} = \alpha.$$

Thus t_{i+1} belongs to S_{k+1} and $\mathbf{f}(t_{i+1})$ is given by (5.8) with $r = 1$, $j = k+1$ and the same value of s as for $\mathbf{f}(t_i)$. We conclude that $\mathbf{f}_{k+1}$ has slope 1 on H while all other components of $\mathbf{f}$ are constant on H. This situation is illustrated in Figure 4(a).

Suppose now that $k = n-1$. Let u be the largest point of $(t_i, t_{i+1}]$ such that f_n has slope 1 on $[t_i, u]$. Since $t_{i+1} \in S$ we must have $u < t_{i+1}$. Then $f_1 + \cdots + f_{n-1}$ changes slope from 0 to 1 at the point u and thus f_n is constant on $[u, t_{i+1}]$. Let ℓ be the smallest index with $1 \le \ell \le n-1$ such that $f_1 + \cdots + f_\ell$ changes slope from 0 to 1 at u. Then f_j is constant on H for each j with $\ell < j < n$. If $\ell = 1$, we must have $u = c$ and $t_{i+1} = d \in S_1$. This is illustrated in Figure 4(c). Suppose now that $\ell \ge 2$. Let v be the largest element of $(u, t_{i+1}]$ such that $f_1 + \cdots + f_{\ell-1}$ is constant on $[t_i, v]$. Then f_ℓ is constant on $[t_i, u]$, has slope 1 on $[u, v]$ and is constant on $[v, t_{i+1}]$, while $f_{\ell+1}$ is constant on $[u, t_{i+1}]$. Applying the main inequalities (5.4) with $t = u$, we deduce that

$$\alpha \ge \frac{f_{\ell+1}(u)}{f_\ell(u)} > \frac{f_{\ell+1}(u)}{f_\ell(v)} = \frac{f_{\ell+1}(t_{i+1})}{f_\ell(t_{i+1})},$$



(a) $k \le n-2$ (b) $k = n-1$ and $\ell = 2$ (c) $k = n-1$ and $\ell = 1$

Fig. 4. All possibilities for the graph of $\mathbf{f}$ over $[t_i, t_{i+1}]$

thus $v = t_{i+1} \in S_\ell$. In particular, $f_{\ell-1}$ is constant on $[t_i, v] = [t_i, t_{i+1}]$. If $\ell > 2$, this yields

$$\alpha = \frac{f_\ell(t_i)}{f_{\ell-1}(t_i)} < \frac{f_\ell(t_{i+1})}{f_{\ell-1}(t_{i+1})},$$

which is impossible. Hence, we must have $\ell = 2$ and all components of $\mathbf{f}$ other than f_2 and f_n are constant on $H = [t_i, t_{i+1}]$. This is illustrated in Figure 4(b).

By the above analysis, the points $\mathbf{f}(t_i)$ listed according to their index i are

$$(5.9)\qquad \begin{aligned} &\mathbf{f}(a) = (1, 1, \alpha, \dots, \alpha^{n-2}),\\ &(1, \alpha, \alpha, \dots, \alpha^{n-2}), (1, \alpha, \alpha^2, \alpha^2 \dots, \alpha^{n-2}), \dots, (1, \alpha, \dots, \alpha^{n-2}, \alpha^{n-2}),\\ &\qquad \cdots\\ &(1, \alpha^m, \alpha^m, \dots, \alpha^{m+n-3}), (1, \alpha^m, \alpha^{m+1}, \alpha^{m+1} \dots, \alpha^{m+n-3}),\\ &\qquad\qquad\qquad\qquad \dots, (1, \alpha^m, \dots, \alpha^{m+n-3}, \alpha^{m+n-3}),\\ &\mathbf{f}(d) = (\alpha^m, \alpha^m, \dots, \alpha^{m+n-2}), \end{aligned}$$

for some integer $m \ge 1$. The combined graph of $\mathbf{f}$ on $[a, d]$ (the union of the graphs of its components) is shown in Figure 5 for the case where $n = 4$ and $m = 3$. The switch points of $\mathbf{f}$ on (a, d) are

$$(5.10)\qquad \begin{aligned} &(1, \alpha, \alpha^2, \dots, \alpha^{n-1}),\ (1, \alpha^2, \alpha^3, \dots, \alpha^n),\\ &\qquad\qquad \dots,\ (1, \alpha^m, \alpha^{m+1}, \dots, \alpha^{m+n-2}). \end{aligned}$$

In particular, the last switch point is $\mathbf{f}(c)$, and so we get

$$\theta = c^{-1} = (1 + \alpha^m + \cdots + \alpha^{m+n-2})^{-1} = (1 + \alpha^m \beta)^{-1}.$$

This shows that $E \subseteq \{0\} \cup \{(1 + \alpha^m \beta)^{-1}\,;\, m \in \mathbb{N}_+\}$.

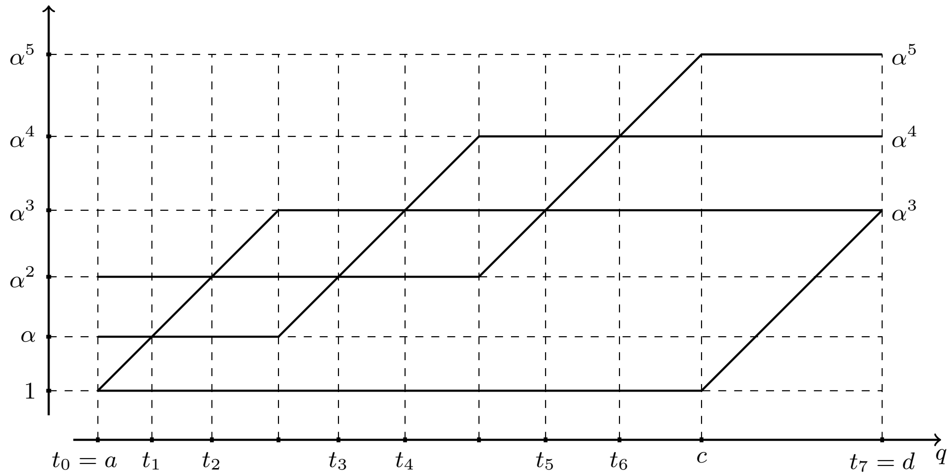


Fig. 5. The combined graph of $\mathbf{f}$ on $[a, d]$ for $n = 4$ and $m = 3$

Conversely, for each integer $m \geq 1$, there is a unique n-system $\mathbf{f}$ on $[a, d]$ with $a = 1 + \beta$ and $d = \alpha^m a$, whose division points are given by (5.9) and (5.10). The first component f_1 of that n-system is constant on $[a, c]$ where $c = 1 + \alpha^m \beta$, and it has slope 1 on $[c, d]$. Therefore the minimum of $f_1(t)/t$ on $[a, d]$ is $1/c$, achieved at $t = c$. Moreover, one verifies that $\mathbf{f}$ satisfies the main conditions (5.4) at each $t \in [a, d]$. More precisely, we find that

$$\min\{t^{-1}T(\mathbf{f}(t))\,;\, a \leq t \leq d\} = (c^{-1}, 0, \dots, 0).$$

Finally, we note that $\mathbf{f}$ extends uniquely to an n-system on $[a, \infty)$ such that $\mathbf{f}(\alpha^m t) = \alpha^m \mathbf{f}(t)$ for each $t \geq a$. This n-system is proper with $\mu_T(\mathbf{f}) = (c^{-1}, 0, \dots, 0)$. Thus the set E contains $c^{-1} = (1 + \alpha^m \beta)^{-1}$ for each $m \geq 1$. Since E is a closed subset of $\mathbb{R}$, it also contains 0. This completes the proof of (5.1) and so proves the theorem. ∎

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