

Fixed points of the Ruelle–Thurston operator and the Cauchy transform

by

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Abstract. We give necessary and sufficient conditions for a function in a naturally appearing function space to be a fixed point of the Ruelle–Thurston operator associated to a rational function (see Lemma 2.1). The proof uses essentially a 2020 paper of the author. As an immediate consequence, in Theorem 1 and Lemma 2.2 we revisit Theorem 1 and Lemma 5.2 of Levin (2014).

1. Introduction. Let f be a rational function of degree at least 2. The Ruelle–Thurston (pushforward) operator T_f associated to f acts on a function g as follows:

$$T_f g(x) = \sum_{w: f(w)=x} \frac{g(w)}{f'(w)^2}$$

whenever the right-hand side is well-defined. (See [28], [8], [22] and references therein for some background.) We say that $H : \mathbb{C} \rightarrow \mathbb{C}$ which is defined Lebesgue almost everywhere is a *fixed point* of T_f if $T_f H(z) = H(z)$ for almost every $z \in \mathbb{C}$.

In [15] (see also [22]) we calculate the action of the resolvent $(1 - \rho T_f)^{-1}$ on the Cauchy kernel to study Padé approximations to the function $\int d\mu(u)/(z - u)$ in the basin of ∞ of a polynomial f where μ is the equilibrium measure of f . On the other hand, T_f appears [8] in a duality relation $\int_{\mathbb{C}} H(z)(f^*\nu)(z) d\sigma_z = \int_{\mathbb{C}} (T_f H)(z)\nu(z) d\sigma_z$. Here and below, σ_z is the Lebesgue measure on the z -plane and $f^* : \nu \mapsto \nu \circ f(\bar{f}'/f')$ is the pullback operator that acts on the Beltrami coefficients ν . Absence of non-trivial fixed points of f^* supported on $J(f)$ (unless f is a so-called flexible Lattès map)

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is equivalent to the ‘no invariant line fields’ conjecture, which in turn would imply the fundamental ‘density of hyperbolicity’, or the Fatou conjecture (see, e.g. [24]). The operator T_f has been used by Douady and Hubbard in their proof of Thurston’s topological realisation of rational maps [8], and then applied to transversality, ‘no invariant line field’ and other problems; see e.g., [2], [5], [9], [10], [16]–[19], [21], [23], [29], [30]. We give an example of application after Theorem 1 of this paper. For an alternative (local) approach and discussions, see [20].

A key point in those applications is that 1 is not an eigenvalue of the operator T_f , i.e., T_f has no non-trivial fixed points in a relevant space (cf. [20]). The space of functions (quadratic differentials) which are meromorphic and integrable on the Riemann sphere is covered in [8]; other forms of fixed points are considered in [9], [16], [19], [23], [29].

Our main result, Lemma 2.1, describes the set of fixed points of T_f as well as gives conditions for the triviality of this set in a rather general and natural space of functions. It includes, for example, the Cauchy transforms of finite discrete measures supported on critical orbits of f . Fixed points of T_f of this form, for specific classes of maps f , appear in the above mentioned works. In the present paper we consider the general situation allowing the postcritical set to intersect the boundaries of Herman rings of f , the only case that has not been covered before (cf. [23], [19], [2], [6]). This case needs separate special considerations.

Lemma 2.1, more precisely its corollary, Lemma 2.2, allow us to revisit Theorem 1 of [19] (see Theorem 1 below), to cover also maps with Herman rings. Theorem 1 of [19] and its revision, Theorem 1 of the present note, include or imply many of the previous results in this direction, e.g. in [29], [16], [23], [30], [5], and have found new applications in [12], [1], [3].

2. Statements and comments. Given a finite complex measure μ on \mathbb{C} , consider its *Cauchy transform*

$$(1) \quad \hat{\mu}(z) = \int \frac{d\mu(w)}{w - z}.$$

The integral converges absolutely Lebesgue almost everywhere and μ is holomorphic in $\bar{\mathbb{C}} \setminus \text{supp}(\mu)$ (see Sect. 3 for more details). We will assume additionally that μ satisfies the following condition at ∞ :

$$(2) \quad \int_{|z| > 10} |z| \log |z| d|\mu|(z) < \infty.$$

In particular, (2) holds if μ has compact support on \mathbb{C} . Denote

$$A = \int d\mu(z) = \mu(\mathbb{C}), \quad B = \int z d\mu(z),$$

existing by (2). Note that $\hat{\mu}$ is integrable at ∞ if and only if $A = B = 0$. We use the following terminology and notations.

DEFINITION 2.1 (cf. [19]). A compact $E \subset \mathbb{C}$ is an *A-compact* if $A(E) = R(E)$ where $A(E)$ is the algebra of all continuous functions on E which are analytic in the interior of E , and $R(E)$ is the algebra of uniform limits on E of rational functions with poles outside E (=uniform limits on E of functions holomorphic on E). A set $E \subset \bar{\mathbb{C}}$ is an *A-compact* if $M(E)$ is an A-compact for some (hence, any) Möbius transformation M such that $M(E) \subset \mathbb{C}$. If an A-compact E is nowhere dense, it is called a *C-compact* ($C =$ continuous since in this case $A(E) = C(E)$, the set of all continuous functions on E).

Necessary and sufficient conditions for a compact in the plane to be A- or C-compact are given by Vitushkin [31]. In particular [11], a compact E is a C-compact if the area of E is zero. A set E is C-compact (respectively, A-compact) if every point of E (respectively, every point of ∂E) belongs to the boundary of a component of the complement $\mathbb{C} \setminus E$, in particular, if the complement to E consists of finitely many components or if E is (a closed subset of) the boundary of an open set (then E is a C-compact).

Recall that a *Herman ring* A is a periodic component of the Fatou set $F(f)$ of f which is homeomorphic to an annulus. The boundary ∂A of A consists of two connected components.

DEFINITION 2.2. Given a closed subset K of the Julia set $J(f)$ of f , denote by $\mathcal{H}(K)$ the collection of all Herman rings A of f such that $\partial A \subset K$ and let

$$\mathcal{H}_K = \bigcup \{\bar{A} : A \in \mathcal{H}(K)\}.$$

Our main result is the following. Note that the conditions on f at ∞ as well as (1.1)–(1.2) in the next lemma serve for the proof of Theorem 1, as in [19].

LEMMA 2.1. *Let f be a rational function which is not a flexible Lattès map.*

1. *Suppose f is normalized so that $f(z) = \sigma z + b + O(1/z)$ for some $\sigma \neq 0, \infty$. Let μ be a measure that satisfies (2) such that the function $H(z) := \hat{\mu}(z)$ is a fixed point of the operator T_f . Assume that either (1.1) or (1.2) holds, where*

$$(1.1) \quad A = B = 0,$$

$$(1.2) \quad \text{either } |\sigma| > 1 \text{ or } \sigma^q = 1 \text{ for some } q \in \mathbb{N} \text{ and } b = 0 \text{ or } \sigma = 1 \text{ and } A = 0.$$

Assume that $K := \text{supp}(\mu) \subset J(f)$ and moreover

(CL) *K is a C-compact.*

Then $\mu = 0$ outside \mathcal{H}_K , i.e., $K \subset \partial\mathcal{H}_K = \bigcup_{A \in \mathcal{H}(K)} \partial A$. In particular, $\mu = 0$ if $\mathcal{H}_K = \emptyset$. If $\mu \neq 0$ and, additionally to (CL),

(AL) \mathcal{H}_K is an A -compact,

then the following representation holds:

$$(3) \quad \mu = \sum_{A \in \mathcal{H}(K)} \mu_A$$

where μ_A is a measure supported on ∂A , μ_A is absolutely continuous with respect to harmonic measure of A , at least one of μ_A is not trivial, and $\mu_A, \mu_{A'}$ are mutually singular for $A \neq A'$. In particular, μ is non-atomic. Moreover, if $\mu_A \neq 0$, then $A \in \mathcal{H}(K)$ must satisfy the following: if $\psi_A : \Delta_A \rightarrow A$ is a holomorphic homeomorphism from a round annulus Δ_A onto A , then $1/\psi'_A$ is in the H^1 -Hardy space, i.e.,

$$(4) \quad \limsup_{\epsilon \rightarrow 0} \int_{\{z \in \Delta_A : \text{dist}(z, \partial\Delta_A) = \epsilon\}} \frac{|dw|}{|\psi'_A(w)|} < \infty$$

if $\mathcal{H}_K \subset \mathbb{C}$ is bounded, and otherwise (4) holds for $M(A)$ instead of A where M is a Möbius transformation such that $M(\mathcal{H}_K)$ is a bounded subset of \mathbb{C} .

2. Conversely, let $\mathcal{H} = \{A, f(A), \dots, f^{q-1}(A)\}$ be the cycle of a Herman ring A . Assume that $1/\psi'_A \in H^1$. Then $1/\psi'_B \in H^1$ for every $B \in \mathcal{H}$ and there exists a finite complex measure $\mu \neq 0$ supported on $\bigcup_{B \in \mathcal{H}} \partial B$ such that:

$$(2.1) \quad T_f \hat{\mu} = \hat{\mu} \text{ in } \mathbb{C} \setminus \text{supp}(\mu) \text{ and } \hat{\mu} = 0 \text{ in } \bar{\mathbb{C}} \setminus \bigcup_{B \in \mathcal{H}} \bar{B},$$

$$(2.2) \quad \text{the representation (3) holds (with } \mathcal{H} \text{ instead of } \mathcal{H}(K)\text{)}.$$

The measure μ is unique in the following sense: if ν is another measure with the same support $\text{supp}(\nu) = \bigcup_{B \in \mathcal{H}} \partial B$ for which (2.1) holds with μ replaced by ν , then $\nu = k\mu$ for some constant $k \in \mathbb{C}$.

Notice that if the boundary curves of a Herman ring A happen to be smooth enough (say C^2) then $1/\psi'_A \in H^1$.

COMMENT 1. Presumably, the closures of Herman rings are mutually disjoint and the complement to the closure of any Herman ring consists of finitely many components. That would imply that the condition (AL) always holds. Note that (AL) holds if, for example, the boundaries of Herman rings of f are locally connected.

COMMENT 2 (cf. [2]). Condition (CL) can be replaced by the following one:

$$(\widetilde{\text{CL}}) \quad f \text{ has no invariant line field on } K.$$

This follows at once from Step III of the proof of Lemma 2.1 (see Sect. 4). In the case when \mathcal{H}_K is empty, i.e., K contains no boundaries of Herman rings, the condition (CL) was, in fact, observed in [2].

The proof of Lemma 2.1 goes along the following lines (see Section 4). First, using the contraction property of the operator T_f , it is shown that $H = 0$ outside $K \cup \mathcal{H}_K$. If K contains no boundaries of Herman rings and K is a C-compact, then $\mu = 0$. In fact, in this case the proof is not original and is more or less a minor variation of arguments scattered through [8], [9], [16], [23], [18], [19]. The case that K does contain boundaries of Herman rings is the main content of the present note. In this case, we use some recent results about the Cauchy transform from [14]: the claim involving (CL) will follow from [14, Lemma 2.1], and that involving also (AL), from [14, Corollary 2.1]; we state them in Sect. 3 for the reader's convenience.

Let us draw a corollary which is suitable for the main application, Theorem 1. Suppose that $V := \{v_1, \dots, v_\ell\} \subset J(f)$ is a collection of critical values of f . Let

$$K = \overline{\bigcup_{j=1}^{\ell} O^+(v_j)}$$

where $O^+(x) = \{f^i(x)\}_{i \geq 0}$ denotes the forward orbit of a point x . Note that $K \subset J(f)$.

DEFINITION 2.3. Let $\mathcal{H}_{\text{crit}}(V)$ be a subcollection of $\mathcal{H}(K)$ of those Herman rings A such that there is a pair of different indices $1 \leq i < i' \leq \ell$ with the property that $O^+(v_i)$ is a dense subset of $L := \bigcup_{k=0}^{q-1} f^k(L_A)$ and $O^+(v_{i'})$ is a dense subset of $L' := \bigcup_{k=0}^{q-1} f^k(L'_A)$ where L_A, L'_A are two components of ∂A and q is the period of A . Denote

$$\mathcal{H}_{\text{crit},V} := \bigcup \{\bar{A} : A \in \mathcal{H}_{\text{crit}}(V)\}.$$

Now, suppose that, as in Lemma 2.1, f is not a flexible Lattès map with the normalization $f(z) = \sigma z + b + O(1/z)$ at ∞ .

LEMMA 2.2 (cf. [19, Lemma 5.2]). *For each $j \in \{1, \dots, \ell\}$, let m_j be a discrete, finite, complex measure supported in $O^+(v_j) \cap \mathbb{C}$, and m is a linear combination of m_1, \dots, m_ℓ . Let*

$$(5) \quad H(x) := \hat{m}(x) = \sum_{k=0}^{\infty} \frac{\alpha_k}{b_k - x} \quad \text{with} \quad \sum_{k \geq 0} |\alpha_k| (1 + |b_k|^2) < \infty.$$

In particular, $A = \sum_{k \geq 0} \alpha_k$, $B = \sum_{k \geq 0} \alpha_k b_k$ exist. Assume that $m_i(\{v_i\}) \neq 0$ whenever v_i is neither periodic nor in the forward orbit of any other $v_{i'}$, $i' \neq i$. Assume additionally that the following conditions of part 1 of Lemma 2.1 are

satisfied: either (1.1) or (1.2), as well as (CL). Finally, assume

(AL_{cr}) $\mathcal{H}_{\text{crit},V}$ is an A-compact.

Then $T_f H = H$ implies $m = 0$ (i.e., $H = 0$).

Indeed, suppose for contradiction that $m \neq 0$ so that after perhaps throwing away some indices and renumbering the rest, one can assume that $m = \sum_{1 \leq j \leq \ell} a_j m_j$ where $a_j \neq 0$ for all j . By (CL), K is a C-compact, so by Lemma 2.1, part 1, $\text{supp}(m) \subset \partial \mathcal{H}_K$, in particular one can assume from the beginning that each $O^+(v_j)$ falls into the boundary of some $A \in \mathcal{H}(K)$. If v_j is periodic, then this obviously implies that it is in the boundary of some Herman ring. If v_j is not periodic, then it is in the forward orbit of some v_i which is neither periodic nor in the forward orbit of any other $v_{i'}$. Hence, $m(\{v_i\}) = a_i m_i(\{v_i\}) \neq 0$, which implies that again v_i , and hence v_j , is in the boundary of some Herman ring. This proves that any v_j is in the boundary of some Herman ring. Therefore, $K = \bigcup_{j=1}^{\ell} O^+(v_j)$ is a subset of boundaries of Herman rings, so by the definition of $\mathcal{H}(K)$, the union $\bigcup_{1 \leq j \leq \ell} O^+(v_j)$ is a dense subset of $\partial \mathcal{H}_K$. Then the following claim shows that $\mathcal{H}_{\text{crit},V} = \mathcal{H}_K$, and hence, by (AL_{cr}), that \mathcal{H}_K is an A-compact. Therefore, by Lemma 2.1, part 1, m has no atom, a contradiction.

CLAIM. Let x be in a component L_A of the boundary of a Herman ring A . Then $O^+(x)$ is either nowhere dense or (everywhere) dense in $L = \bigcup_{k=0}^{q-1} f^k(L_A)$, where q is the period of A .

Indeed, assume without loss of generality that $q = 1$ and that there is a ball B centered at a point of $L (= L_A)$ such $B \cap L = B \cap \omega(x)$. As the harmonic measure of $B \cap L$ is positive, there is a subset X_B of positive Lebesgue length in one of the boundary circles S of the annulus Δ_A which uniformizes A such that for each $w \in X_B$ the radial limit $\psi_A(w)$ of $\psi_A : \Delta_A \rightarrow A$ exists and $\psi_A(w) \in B$. Hence, the set $X = \bigcup_{i \geq 0} \lambda^i X_B$, for the corresponding $\lambda \in S^1$ where $\psi_A^{-1} \circ f \circ \psi_A : w \mapsto \lambda w$, has full length in S and, for each $w \in X$, $\psi_A(w)$ exists and is in $\omega(x)$. Therefore, $\omega(x)$ is a closed and dense subset of L , i.e., $\omega(x) = L$. Thus $\omega(x)$ is either nowhere dense in L or equal to L .

The proof of the following theorem is (literally) identical to the proof of Theorem 1 of [19] (see Sect. 5.3 there), after replacing Lemmas 5.2–5.3 of [19] by Lemma 2.2. Recall that a critical point c of f with the forward orbit in \mathbb{C} is called *summable* if, for $v = f(c)$, $\sum_{n=0}^{\infty} \frac{1+|f^n(v)|^2}{1+|v|^2} \frac{1}{|(f^n)'(v)|} < \infty$.

THEOREM 1 (cf. [19, Theorem 1]). Let f be an arbitrary rational function of degree $d \geq 2$ which is not a flexible Lattès example. Suppose that

$\{c_1, \dots, c_r\}$ is a collection of r summable critical points of f , and the union

$$\mathcal{C} := \bigcup_{j=1}^r \omega(c_j)$$

of their ω -limit sets satisfies the following conditions:

(C) \mathcal{C} is a C -compact,

(A_{cr}) $\mathcal{H}_{\text{crit}, \mathbf{v}}$ is an A -compact where $\mathbf{v} := \{f(c_1), \dots, f(c_r)\}$.

Replacing if necessary f by its equivalent (i.e., Möbius conjugate), one can assume the forward orbits of c_1, \dots, c_r avoid infinity. Consider the set X_f of all rational functions of degree d which are close enough to f and have the same number p' of different critical points with the same corresponding multiplicities. Then there is a p' -dimensional manifold Λ_f and its r -dimensional submanifold Λ , $f \in \Lambda \subset \Lambda_f \subset X_f$, with the following properties:

(a) every $g \in X_f$ is equivalent to some $\hat{g} \in \Lambda_f$,

(b) for every one-dimensional family $f_t \in \Lambda$ through f such that $f_t(z) = f(z) + tu(z) + O(|t|^2)$ as $t \rightarrow 0$, if $u \neq 0$, then, for some $1 \leq j \leq r$, the limit

$$(6) \quad \lim_{m \rightarrow \infty} \frac{\frac{d}{dt} \Big|_{t=0} f_t^m(c_j(t))}{(f^{m-1})'(f(c_j))} = \sum_{n=0}^{\infty} \frac{u(f^n(c_j))}{(f^n)'(f(c_j))} \neq 0$$

exists and is a non-zero number. Here $c_j(t)$ is the critical point of f_t such that $c_j(0) = c_j$.

Note the following particular case: all points c_1, \dots, c_r belong to the same grand orbit, i.e., $f^{n_i}(c_i) = f^{n_j}(c_j)$ for any i, j and some $n_i, n_j > 0$. Then by Definition 2.3, $\mathcal{H}_{\text{crit}, \mathbf{v}}$ is empty, hence, the condition (A_{cr}) is void:

COMPLEMENT TO THEOREM 1. *If all points c_1, \dots, c_r are in the same grand orbit (e.g., if either $r = 1$ or $r > 1$, but $f(c_1) = \dots = f(c_r)$), then the condition (A_{cr}) is unnecessary.*

In [19, Theorem 1], the condition (A_{cr}) of the present Theorem 1 was absent. Presumably, (A_{cr}) always holds—see Comment 1. The case $\mathcal{H}(\mathcal{C}) = \emptyset$ was covered in [19] (see also [2]). Here we treat the missing case, when the boundaries of some Herman rings are contained in \mathcal{C} .

Condition (6) is equivalent to the following: for a coordinate system $\{x_1, \dots, x_{p'}\}$ in Λ_f , the rank of the matrix $\{L(c_j, x_k)\}_{1 \leq k \leq p', 1 \leq j \leq r}$, where

$$L(c_j, x_k) = \lim_{m \rightarrow \infty} \frac{\frac{\partial g^m(c_j)}{\partial x_k} \Big|_{g=f}}{(f^{m-1})'(f(c_j))},$$

is maximal, i.e., equal to r . As mentioned in Section 1, Theorem 1 covers many previous results in this direction (as well as has new applications). For

example, consider the unicritical family $f_v(z) = z^d + v$. If 0 is a summable critical point of f_{v_0} , then, by Corollary 2.1(2) below, Theorem 1 applies, hence, (6) holds and it turns into

$$\lim_{m \rightarrow \infty} \frac{\frac{d}{dv}|_{v=v_0} f_v^m(0)}{(f^{m-1})'(v_0)} = \sum_{n=0}^{\infty} \frac{1}{(f^n)'(v_0)} \neq 0.$$

This implies in particular that f_{v_0} is unstable in the family $\{f_v\}_{v \in \mathbb{C}}$. Indeed, if f_{v_0} were structurally stable in $\{f_v\}$, then the sequence of functions $\{f_v^m(0)\}$ near $v = v_0$ (hence, the sequence of derivatives $\{\frac{d}{dv}|_{v=v_0} f_v^m(0)\}$, too) would be bounded, which along with $(f^{m-1})'(v_0) \rightarrow \infty$ would imply that $\frac{\frac{d}{dv}|_{v=v_0} f_v^m(0)}{(f^{m-1})'(v_0)} \rightarrow 0$, a contradiction (see [16] for details).

More generally, let f be a rational function (not a flexible Lattès map) with a summable critical point c such that $\omega(c)$ is a C-compact. Then, by the Complement to Theorem 1 (with $r = 1$), the inequality (6) must hold for some family $f_t(z) = f(z) + tu(z) + O(|t|^2) \in \Lambda$ with $u \neq 0$. On the other hand, if we assume that f is structurally stable in the (one-dimensional) space Λ then repeating the argument for the unicritical family, we see that the limit in (6) must be zero. This contradiction shows that such an f is unstable in Λ , and therefore also in the bigger space Λ_f ; see [19, Corollary 1.2] for more details. In particular, f is unstable in the space of all rational maps of the same degree [23].

COMMENT 3. • Since \mathcal{C} is closed and forward invariant, either \mathcal{C} is nowhere dense or $\mathcal{C} = \bar{\mathcal{C}} = J(f)$, and hence, under the condition (C), \mathcal{C} has to be nowhere dense.

• \mathcal{C} is a C-compact if and only if $\omega(c_j)$ is a C-compact for every $j \in \{1, \dots, r\}$. If $c_j \in \partial U$ where U is a component of the Fatou set of f (say, an iterate of c_j is in the boundary of a Herman ring), then $\omega(c_j) \subset \bigcup_{n \geq 0} f^n(\partial U)$, hence $\omega(c_j)$ is a C-compact (we use that U is (pre-)periodic, by Sullivan's no wandering domain theorem),

Let us list some classes of rational maps f and corresponding sets \mathcal{C} for which the conclusion of [19, Theorem 1] (=conclusion of Theorem 1 of this note) holds:

COROLLARY 2.1. *The conclusion of Theorem 1 holds whenever f is not a flexible Lattès map and one of (1)–(8) below holds:*

- (1) f has no Herman rings and \mathcal{C} is a C-compact,
- (2) f is a polynomial,
- (3) $J(f) = \bar{\mathcal{C}}$, and \mathcal{C} is a C-compact,
- (4) $\mathcal{C} \neq \bar{\mathcal{C}}$ and the complement to \mathcal{C} consists of finitely many components,
- (5) note two particular cases of (4): (i) \mathcal{C} is totally disconnected, for example, finite, (ii) \mathcal{C} lies in a finite union of disjoint Jordan curves in \mathbb{C} , for

- example, $\mathcal{C} \subset \mathbb{R}$,
- (6) $f : \mathcal{C} \rightarrow \mathcal{C}$ is expanding (e.g., the critical points c_1, \dots, c_r satisfy Misiurewicz’s condition),
- (7) the following two conditions hold:
- (7.1) the area of $\mathcal{C} \setminus \partial\mathcal{H}_{\text{crit},\mathbf{v}}$ is zero,
- (7.2) either $\mathcal{H}_{\text{crit},\mathbf{v}}$ is empty (e.g., all c_1, \dots, c_r are in a single grand orbit), or the boundary of every Herman ring of f is locally connected.
- (8) All critical values of f which are in $J(f)$ are summable. Here either \mathcal{C} is nowhere dense or $\mathcal{C} = \bar{\mathcal{C}} = J(f)$.

Corollary 2.1 follows immediately from Corollary 2.2 below, which is stated right after the following comment about applications of Theorem 1:

Corollary 2.1 implies that Corollaries 1.1–1.2 of [19] and their proofs remain untouched. Indeed, in [19, Corollary 1.1] case (6) of Corollary 2.1 above applies, and in [19, Corollary 1.2], Complement to Theorem 1 (with $r = 1$) applies (see discussion after Theorem 1). For similar reasons, applications of Theorem 1 of [19] in [12] and in [1] are unaffected as well: in [12], f is a polynomial (so case (2) of Corollary 2.1 applies); and in [1], f is expanding on \mathcal{C} , i.e., case (6) of Corollary 2.1 works (in fact, case (1) applies as well because f as in [1] cannot have Herman rings).

COROLLARY 2.2. *In the notations of Lemma 2.1, $T_f H = H$ (where $H = \hat{\mu}$ with $\text{supp}(\mu) \subset J(f)$) implies $\mu = 0$ if f is not a flexible Lattès map and one of the conditions (1)–(8) of Corollary 2.1 holds with the following obvious changes: \mathcal{C} should be replaced by $K \subset J(f)$; in cases (1)–(7), K is nowhere dense; and in case (7), $\mathcal{H}_{\text{crit},(\mathbf{v})}$ is replaced by \mathcal{H}_K . In case (8), K can have interior points.*

Proof. We handle cases (1)–(7); for (8), see the end of Section 4. In cases (1)–(7), $K \subset J(f)$ is a C-compact. Indeed, in cases (1), (3) it is assumed; in cases (2) and (4), every point of K belongs to a component of the complement; and in cases (6), (7), K is of measure zero. It remains to note the following. In cases (1)–(3), f has no Herman rings. In case (4), the complements to K as well as to \mathcal{H}_K consist of finitely many components, so K is a C-compact (being also nowhere dense) and so \mathcal{H}_K is A-compact.

It remains to consider cases (6)–(7). In case (7), if the boundary of every Herman ring is locally connected it is easy to see that the closures of two different Herman rings are disjoint and the complement to the closure of every Herman ring consists of finitely many components. Therefore, \mathcal{H}_K is an A-compact. Finally, for case (6), if $f : K \rightarrow K$ is expanding then K cannot contain the boundary of a Herman ring, as shown in the next lemma. ■

LEMMA 2.3. *Let Ω be a Herman ring of a rational function f which is invariant by f . Then, for either component L of $\partial\Omega$, $f : L \rightarrow L$ is not expanding.*

Proof. Assume the contrary. Then, passing to an iterate, one can find a neighborhood U of L such that $|Df(z)| > 2$ for any $z \in U$. Fix a conformal homeomorphism $\psi : \{1 < |w| < r\} \rightarrow \Omega$ such that $f = \psi R \psi^{-1}$ where $R(w) = \lambda w$ is an irrational rotation. We can suppose that $L = \bigcap_{1 < r' < r} \overline{\psi(\{1 < |w| < r'\})}$. Fix $1 < \rho < r$ such that $U_0 := \psi(\{1 < |w| \leq \rho\})$ is compactly contained in U . Let $\Gamma = \psi(\{|w| = \rho\})$ and let $h > 0$ be the distance between two disjoint compact sets L and Γ . Find $u \in L$ and $v \in \Gamma$ such that $h = |u - v|$. The interval $(u, v]$ must be a subset of U_0 because otherwise there would exist $u' \in L$ and $v' \in \Gamma$ with $|u' - v'| < h$. Let $g := \psi R^{-1} \psi^{-1}$ be the branch of f^{-1} leaving Ω invariant and let $\gamma = g((u, v])$. Then γ is a semi-open curve in U_0 which begins at $v_{-1} = g(v)$ and tends to L . Since $|Df(x)| > 2$ for all $x \in \gamma$, the length of γ satisfies $l(\gamma) < h/2$. As $v_{-1} \in \Gamma$ and γ joins v_{-1} and L , we arrive at a contradiction with the definition of h . ■

3. The Cauchy transform of measures. Given a finite complex measure ν with $\text{supp}(\nu) \subset \mathbb{C}$, let

$$\hat{\nu}(z) = \int \frac{d\nu(w)}{w - z}$$

be the Cauchy transform of ν . For the following facts, see e.g. [13]. As ν is finite, by Fubini's theorem, $\int \frac{d|\nu|(w)}{|w-z|}$ (hence $\hat{\nu}$) is locally in $L^1(dx dy)$. In particular, $\hat{\nu}$ exists almost everywhere on \mathbb{C} . Moreover, $\hat{\nu}$ is holomorphic outside $\text{supp}(\nu)$, and $\nu(\infty) = 0$ if ν has a compact support. Moreover, $\hat{\nu} \neq 0$ on a set of a positive area unless $\nu = 0$.

The following two propositions are the main auxiliary statements we use. They appear in [14] as Lemma 2.1 and Corollary 2.1, respectively.

PROPOSITION 1.

- (a) *Any closed subset of a C -compact is a C -compact.*
- (b) *Let K be a nowhere dense compact in \mathbb{C} and μ a measure on K . Suppose that for a neighborhood W of a point $x \in K$, $K \cap \overline{W}$ is a C -compact and $\hat{\mu} = 0$ on $W \setminus K$. Then μ vanishes on $K \cap W$, i.e., $|\mu|(W) = 0$.*

PROPOSITION 2. *Suppose \mathcal{H} is a non-empty collection of bounded rotation domains of a rational function f . Let $V = \bigcup\{A : A \in \mathcal{H}\}$, let $E \subset \mathbb{C} \setminus V$ be a nowhere dense compact subset such that $\partial V \subset E$, and let ν be a bounded complex measure supported on E such that $\hat{\nu} = 0$ off $E \cup V$. If E is a C -compact and \overline{V} is an A -compact, then ν is in fact supported on $\partial V = \bigcup_{A \in \mathcal{H}} \partial A$, $\nu|_{\partial A}$, $A \in \mathcal{H}$, are mutually singular and, for each A , $\nu|_{\partial A}$ is*

absolutely continuous with respect to harmonic measure of A . In particular, ν is non-atomic. Moreover, for each $A \in \mathcal{H}$,

$$\limsup_{\epsilon \rightarrow 0} \int_{\{z \in \Delta_A: \mathbf{dist}(z, \partial \Delta_A) = \epsilon\}} |\hat{\nu} \circ \psi_A(z)| |\psi'_A(z)| |dz| < \infty,$$

where $\psi_A: \Delta_A \rightarrow A$ is a holomorphic homeomorphism from a round annulus Δ_A onto A .

4. Proof of Lemma 2.1. Let us start with part **1**. We split the proof into Steps I–VIII. Note that most of the arguments are not original and are included for completeness. Namely, Step I is taken from [19, proof of Lemma 5.2] and along with Steps II–V and the first claim of Lemma 4.5 of Step VI are indeed minor modifications of result of [8], [9], [16], [23], [18]. Next, the proof when K contains no boundaries of Herman rings is straightforward: see Step VII. In the short Step VIII we deal with the general case applying Propositions 1–2.

We prove part **1** by contradiction. So let H be the Cauchy transform of a finite complex measure μ that satisfies (2). Assume that $\mu \neq 0$, $T_f H = H$, and the assumptions of Lemma 2.1, part **1**, hold.

I. The function

$$\tilde{H}(z) := H(z) + \frac{A}{z} + \frac{B}{z^2} = \int \left[\frac{1}{w-z} + \frac{1}{z} + \frac{w}{z^2} \right] d\mu(w) = \int \frac{w^2}{z^2(w-z)} d\mu(w)$$

is integrable at ∞ . Indeed, for every w , the function (of z) $w^2/[z^2(w-z)]$ is integrable at ∞ , and one can write

$$\begin{aligned} \int_{|z|>1} \left| \frac{w^2}{z^2(w-z)} \right| d\sigma_z &= |w| \int_{|u|>1/|w|} \left| \frac{1}{u^2(1-u)} \right| d\sigma_u \\ &\leq C_1 |w| (1 + \ln^+ |w|), \end{aligned}$$

where $\ln^+ |w| = \max\{0, \ln |w|\}$. Hence,

$$\begin{aligned} \int_{|z|>1} |\tilde{H}(z)| d\sigma_z &\leq \int d|\mu|(w) \int_{|z|>1} \left| \frac{w^2}{z^2(w-z)} \right| d\sigma_z \\ &\leq C_1 \int |w| (1 + \ln^+ |w|) d|\mu|(w) < \infty, \end{aligned}$$

by condition (2). Now, take R large enough and consider the disk $D(R) = \{|x| < R\}$. We claim that

$$(7) \quad \limsup_{R \rightarrow \infty} \left\{ \int_{f^{-1}(D(R))} |H(x)| d\sigma_x - \int_{D(R)} |H(x)| d\sigma_x \right\} \leq 0.$$

Indeed, in case (1.1), i.e., if $A = B = 0$, this follows at once from the integrability of H at ∞ . In case (1.2), the conditions on σ imply that there

is $a > 0$ such that

$$(8) \quad f^{-1}(D(R)) \subset D(R + |b| + a/R)$$

(actually, $f^{-1}(D(R)) \subset D(R)$ if $|\sigma| > 1$). On the other hand,

$$(9) \quad \lim_{R \rightarrow \infty} \int_{R < |x| < R + |b| + a/R} |H(x)| d\sigma_x = 0$$

since $\tilde{H}(x) = H(x) + A/x + B/x^2$ is integrable at ∞ , and an easy calculation shows that the assumptions in case (1.2) guarantee that

$$(10) \quad \lim_{R \rightarrow \infty} \int_{R < |x| < R + |b| + a/R} \left| \frac{A}{x} + \frac{B}{x^2} \right| d\sigma_x = 0.$$

(9) along with (8) imply (7) in case (1.2). As in [19, proof of Lemma 5.2], (7) implies that

$$(11) \quad |H(x)| = |T_f H(x)| = \left| \sum_{w: f(w)=x} \frac{H(w)}{f'(w)^2} \right| = \sum_{w: f(w)=x} \frac{|H(w)|}{|f'(w)|^2}$$

almost everywhere. Indeed, otherwise there is a set $A \subset D(R_0)$ of positive measure (for some R_0) and $\delta > 0$ such that

$$|T_f H(x)| < (1 - \delta) \sum_{w: f(w)=x} \frac{|H(w)|}{|f'(w)|^2}$$

for $x \in A$. Then, for all $R > R_0$,

$$\begin{aligned} \int_{D(R)} |H(x)| d\sigma_x &= \int_{D(R) \setminus A} |T_f H(x)| d\sigma_x + \int_A |T_f H(x)| d\sigma_x \\ &< \int_{f^{-1}(D(R) \setminus A)} |H(x)| d\sigma_x + (1 - \delta) \int_{f^{-1}(A)} |H(x)| d\sigma_x \\ &= \int_{f^{-1}(D(R))} |H(x)| d\sigma_x - \delta \int_{f^{-1}(A)} |H(x)| d\sigma_x, \end{aligned}$$

which contradicts (7).

II. By Sect. 3, H is well-defined on a set $Y \subset \mathbb{C}$ such that $\mathbb{C} \setminus Y$ has zero Lebesgue measure and $H(x) \neq 0$ on a set $Z \subset Y$ of positive Lebesgue measure. Replacing Y by $\bigcap_{n \in \mathbb{Z}} f^n(Y)$, one can assume that Y is completely invariant. (11) immediately implies

LEMMA 4.1. *For every measurable $A \subset \mathbb{C}$ such that*

$$\Lambda(A) := \int_A |H(z)| d\sigma_z < \infty,$$

we have

$$\Lambda(A) = \Lambda(f^{-1}(A)).$$

In other words, Λ is an f -invariant positive measure on \mathbb{C} (which is finite in case (1.1) though not necessarily so in case (1.2)).

III.

LEMMA 4.2. Assume $x \in Y$, $f'(x) \neq 0$ and $H(x) \neq 0$. Then there is a real constant $L_x \geq 1$ such that

$$(12) \quad H(f(x))(f'(x))^2 = L_x H(x).$$

In particular, $x \in Z$ implies $f(x) \in Z$. If $f(x), x$ are in $K^c := \mathbb{C} \setminus K$ then there is a neighborhood U of x such that (12) holds for all $x \in U$ and moreover L_x is a constant function on U .

Proof. (12) follows at once from (11). If, additionally, $f(x), x \in K^c$ then there is a connected neighborhood U of x such that $U \cup f(U) \subset K^c$ and H, f' are never 0 in U . It follows that $L_x = H(f(x))(f'(x))^2/H(x)$ is a real holomorphic function in $x \in U$, and therefore a constant. ■

Let

$$l(z) = \overline{H(z)}/|H(z)|$$

whenever $H(z)$ is well-defined and not zero. Since $L_x > 0$ in (12), it follows that

$$l(f(z))\overline{f'(z)}/f'(z) = l(z)$$

whenever $z \in Z$ and $f'(z) \neq 0$. This means that $l(z)\overline{dz}/dz$ is an invariant line field defined initially on the set of all $z \in Z$ with $f'(z) \neq 0$.

Consider the case $J(f) = \bar{\mathbb{C}}$. The condition (CL) that K is a \mathbb{C} -compact implies that K is nowhere dense, hence, $\mathbb{C} \setminus K$ is a non-empty open set. Assume for contradiction that $H(z) \neq 0$ on a non-empty open $U \subset \mathbb{C} \setminus K$. Then $l(z)\overline{dz}/dz$ is an invariant holomorphic line field on U . Therefore, by Lemma 3.16 of [24], f is a flexible Lattès map, which contradicts condition (i). Thus $H \equiv 0$ off a \mathbb{C} -compact K , hence, by the (classical) fact [19, Lemma 5.3], $H \equiv 0$, i.e., we are done in this case.

IV. From now on, $J(f) \neq \bar{\mathbb{C}}$. To deal with the (non-)integrability of H at ∞ we use the following

LEMMA 4.3. Let g be a local holomorphic map in a neighborhood of 0 such that $g(0) = 0$ and $g'(0) = 1$. Let $h(z) = \tilde{A}/z^3 + \tilde{B}/z^2 + \tilde{h}(z)$ where $\tilde{A}, \tilde{B} \in \mathbb{C}$ and \tilde{h} is an integrable function in a neighborhood of 0. Assume that either (a) $g(z) = z + O(z^3)$, or (b) $g(z) = z + O(z^2)$ and $\tilde{A} = 0$. Then:

- (1) every attracting petal P of g at 0 contains a domain U_P such that $g(U_P) \subset U_P$, $U_P \setminus g(U_P)$ contains a disk, $0 \in \partial U_P$, every forward orbit in P enters U_P and

$$\int_{U_P} |h(z)| d\sigma_z < \infty,$$

- (2) *there is an open set U_- such that $g^{-1}(U_-) \subset U_-$, the union of U_- with all attracting petals of g at 0 and the point $\{0\}$ constitutes a neighborhood of 0, and*

$$\int_{U_-} |h(z)| d\sigma_z < \infty.$$

Proof. (1) To begin, we remark that given a local holomorphic injection (coordinate change) ψ near 0 such that $\psi(0) = 0$ and $\psi'(0) = 1$ it is easy to see that it is enough to show the existence of \tilde{U} instead of U as in (1) for $\tilde{g} = \psi^{-1} \circ g \circ \psi$ and $h \circ \psi$ instead of g and h respectively and then let $U = \psi(\tilde{U})$.

Consider case (a). One can assume that the asymptotic attracting direction of \tilde{U} is the positive real axis. We use classical facts about local dynamics near a parabolic fixed point; see e.g. [7]. There exists a local coordinate change ψ such that $\tilde{g}(z) = z - z^{\nu+1} + \alpha z^{2\nu+1} + O(|z|^{2\nu+2})$ where $\nu \geq 2$ (since $g(z) = z + O(z^3)$). Moreover, making a local change of coordinate $w = l(z) := 1/(\nu z^\nu)$ in the attracting petal $\tilde{P} = \psi^{-1}(P)$ of \tilde{g} we get $F(w) = l \circ \tilde{g} \circ l^{-1}(w) = w + 1 + C/w + O(1/|w|^2)$ as $w \rightarrow \infty$ in $P_\infty = l(P)$, which contains a set of the form $\{w = w_1 + iw_2 : w_1 > M_0 - \kappa|w_2|\}$ for some $M_0 > 0$ and $\kappa > 0$. Finally, there is a Fatou coordinate $\Psi : U_\infty \rightarrow \mathbb{C}$ such that $\Psi(w) = w - C \log w + O(1)$ for the main branch of \log such that $\Psi \circ F(w) = \Psi(w) + 1$. Now, given $\epsilon > 0$ and $M > 0$ let $U_\infty(\epsilon, M) := \{w = w_1 + iw_2 : w_1 > M, |w_2| < w_1^\epsilon\}$. It is straightforward to check that given $\epsilon > 0$ there is M_ϵ such that $F(U_\infty(\epsilon, M_\epsilon)) \subset U_\infty(\epsilon, M_\epsilon)$. Enlarging M_ϵ if necessary and using the asymptotics for the Fatou coordinate we also check that every forward orbit of F in P_∞ enters $U_\infty(\epsilon, M_\epsilon)$. Fix $\epsilon \in (0, 1/\nu)$ and the corresponding $U_\infty := U_\infty(\epsilon, M_\epsilon)$. Let $\tilde{U} := l^{-1}(U_\infty) \subset \tilde{P}$. Now, if $w = w_1 \pm iw_1^\epsilon \in \partial U_\infty$, for a big $w_1 > 0$, then $z = l^{-1}(w) = 1/(\nu w)^{1/\nu}(w) = u + iv$ where $u = \nu^{-1/\nu} w_1^{-1/\nu}(1 + O(w_1^{-2+2\epsilon}))$ and $v = \pm \nu^{-1-1/\nu} w_1^{-1-1/\nu+\epsilon}(1 + O(w_1^{-1+\epsilon}))$. Therefore, as $v \rightarrow 0$ the following asymptotics holds: $|v| = Bu^\gamma + O(|u|^{\gamma'})$ where $\gamma = \nu + 1 - \epsilon\nu > 2$ (as $\nu \geq 2$ and $0 < \epsilon < 1/\nu$) and $\gamma' = \gamma + \nu(1 - \epsilon) > \gamma$. Since $h(\psi(z)) = O(|z|^{-3})$ we then get $\int_{\tilde{U}} |h(\psi(z))| dx dy < \infty$. Case (b) is very similar to (a) though simpler and is left to the reader.

Part (2) follows if we apply part (1) to attracting petals of the local inverse g^{-1} , finding for each such petal R_j , $1 \leq j \leq \nu$, the set U_{R_j} corresponding to g^{-1} . Let $U_- = \bigcup_{j=1}^\nu U_{R_j}$. Since each forward orbit in R_j under g^{-1} enters U_{R_j} , the union of U_- and the attracting petals of g is a punctured neighborhood of 0. ■

V. Let Ω be a component of $F(f)$. Recall that A is a measure introduced in Lemma 4.1, Step II.

LEMMA 4.4. *If Ω is not periodic then $H|_\Omega = 0$.*

Proof. Assume the contrary and choose $A \subset \Omega$ such that $\mu(A) > 0$. Since Ω is not periodic, $f^{-n}(A) \cap f^{-m}(A) = \emptyset$ for all non-negative $n \neq m$. Hence, by Step II, $\Lambda(\bigcup_{n \geq 0} f^{-n}(A)) = \int_{\bigcup_{n \geq 0} f^{-n}(A)} |H(z)| d\sigma_z = \infty$. In case (1.1), H is integrable on $\bar{\mathbb{C}}$, a contradiction. Consider case (1.2). If $|\sigma| > 1$ then ∞ is an attracting fixed point of f . Hence, all $f^{-n}(A)$ stay away from a neighborhood of ∞ , and therefore $\Lambda(\bigcup_{n \geq 0} f^{-n}(A)) = \int_{\bigcup_{n \geq 0} f^{-n}(A)} |H(z)| d\sigma_z < \infty$, a contradiction. In the remaining two possibilities of case (1.2), ∞ is a parabolic fixed point. Let Ω_∞ be its immediate basin. As $\bigcup_{n \geq 0} f^{-n}(A) \subset \mathbb{C} \setminus \Omega_\infty$, we get a contradiction if show that

$$(13) \quad \int_{\mathbb{C} \setminus \Omega_\infty} |H(z)| d\sigma_z < \infty.$$

To this end, passing to f^q we get: either $f^q(z) = z + O(1/z)$, or $f(z) = z + O(1)$ and $A = 0$. Now, making the change $w = 1/z$ we arrive at a map $g(w) = 1/f(1/w)$ and $|H(z)|d\sigma_z = |h(w)|d\sigma_w$ where $h(w) = H(1/w)w^{-4}$, precisely as in Lemma 4.3 of Step IV. Let U_- be the set appearing in (2) of that lemma. Since $\mathbb{C} \setminus \Omega_\infty \cap \{|z| > R\} \subset 1/U_-$ for R large enough, (13) follows. ■

VI. We are left with the case when Ω is a periodic component of $F(f)$.

LEMMA 4.5.

- (1) If Ω is not a Herman ring then $H|_\Omega = 0$.
- (2) (cf. [16, p. 190], [2]) If Ω is a Herman ring and $H|_\Omega \neq 0$ then there is $C_\Omega \neq 0$ such that $H|_\Omega = C_\Omega \left\{ \frac{\varphi'_\Omega}{\varphi_\Omega} \right\}^2$ where $\varphi_\Omega : \Omega \rightarrow \{1 < |w| < R_\Omega\}$ is a conformal isomorphism. Moreover, $C_\Omega = C_{f^i(\Omega)}$ for all $i > 0$, i.e. depends only on the cycle that contains Ω . Moreover, $\partial\Omega \subset K$.

Proof. Let q be such that $f^q(\Omega) = \Omega$ and $H|_\Omega \neq 0$, i.e., H is a non-zero holomorphic function on a connected open set Ω . First, let Ω be an immediate basin of attraction of either an attracting or a parabolic point $a \in \bar{\Omega}$. To prove that $H|_\Omega = 0$ it is enough to find an open set $X \subset \Omega$ such that $\Lambda(X) < \infty$ and either (i) $X \subset f^{-q}(X)$ and $f^{-q}(X) \setminus X$ contains a ball or (ii) $X \supset f^{-q}(X)$, $X \setminus f^{-q}(X)$ contains a ball. Indeed, then $\Lambda(f^{-q}(X) \setminus X) = \Lambda(f^{-q}(X)) - \Lambda(X) = 0$ in case (i) and $\Lambda(X \setminus f^{-q}(X)) = 0$ in case (ii), hence in either case $H = 0$ on a ball in Ω , hence everywhere in Ω .

Let us show that such a set X exists. If a is attracting or parabolic and $a \neq \infty$, X can be taken a neighborhood of a if a is attracting, and an attracting petal at a if a is parabolic. If $a = \infty$ is attracting (i.e., $|\sigma| > 1$), define $X = \Omega \setminus U$ where U is a neighborhood of ∞ . Finally, if $a = \infty$ is parabolic, by Lemma 4.3 define $X = 1/U_P$ where P is an attracting petal at 0 of $g(z) = 1/f(1/z)$.

Now, let Ω be either a Siegel disk or a Herman ring. Since $f^q : \Omega \rightarrow \Omega$ is a homeomorphism, by Step V, H is a holomorphic non-zero map on Ω such that $H \circ f^q [(f^q)']^2 = H$ on Ω . Let $\varphi_\Omega : \Omega \rightarrow \Delta$ be a conformal homeomorphism onto either a disk (if Ω is a Siegel disk) or an annulus Δ (if Ω is a Herman ring) which conjugates $f^q : \Omega \rightarrow \Omega$ to an irrational rotation $w \mapsto \lambda w$ on Δ . For $\tilde{H} = H \circ \varphi_\Omega^{-1} [(\varphi_\Omega^{-1})']^2$, the equation for H turns into $\tilde{H}(\lambda w)\lambda^2 = \tilde{H}(w)$, $w \in \Delta$. Writing this equation in terms of a series $\tilde{H}(w) = \sum_{n=-\infty}^{\infty} a_n w^n$, it is immediate to see that the only solution is $\tilde{H}(w) = a_{-2}/w^2$, that is, the case of a disk (when $a_n = 0$ for $n < 0$) is impossible, while if Δ is an annulus, H must be proportional to $\{\varphi'_\Omega/\varphi_\Omega\}^2$. Moreover, since $H|_\Omega \neq 0$, every point $z \in \partial\Omega$ must belong to K . Indeed, otherwise H is a holomorphic function in a neighborhood V of z such that $H \neq 0$ in $\Omega \cap V$. On the other hand $z = \lim_{n \rightarrow \infty} z_n$ for a sequence z_n of points of some non-periodic components of $F(f)$, so by Lemma 4.4, $H(z_n) = 0$ and by the Uniqueness Theorem, $H|_V = 0$, a contradiction. ■

By Steps V–VI, $H = 0$ outside $K \cup \mathcal{H}_K$.

VII. Assume $\mathcal{H}(K) = \emptyset$, i.e., K contains no boundaries of Herman rings. Then by Steps V–VI, $H = \hat{\mu} = 0$ off K . Assume (CL), i.e., K is a C-compact. Then, by (well-known) [19, Lemma 5.3], $\mu = 0$, a contradiction.

VIII. $\mathcal{H}(K) \neq \emptyset$. Assuming (CL), i.e., K is a C-compact, by Lemma 1, $\mu = 0$ in $\mathbb{C} \setminus \mathcal{H}_K$, i.e., $K = \text{supp}(\mu) \subset \partial\mathcal{H}_K$ (cf. [14, proof of Theorem 1]). By Lemma 4.5(2), for each $A \in \mathcal{H}_K$, $H|_A = \hat{\mu}|_A = C_A \{\varphi'_A/\varphi_A\}^2$, and therefore

$$(14) \quad \hat{\mu} \circ \psi_A(w) \psi'_A(w) = C_A \{w^2 \psi'_A(w)\}^{-1}.$$

Now we also assume (AL). There are two cases. If \mathcal{H}_K is bounded, we directly apply Proposition 2 to $E = K$, $\mathcal{H} = \mathcal{H}(K)$, and $\nu = \mu$, and get the desired conclusion. If $\infty \in \mathcal{H}_K$, let $M(z) = 1/(x_0 - z)$ for some $x_0 \in \mathbb{C} \setminus \mathcal{H}_K$ and let $E = M(K)$, $\mathcal{H} = \{M(A) | A \in \mathcal{H}_K\}$, and ν is defined by $d\nu(u) = u d\mu(M^{-1}(u))$ so that $\hat{\nu}(v) = v^{-1} \hat{\mu}(M^{-1}(v)) = 0$ off the bounded set $E \cup V \subset \mathbb{C}$. Now we can apply Proposition 2 to the rational map $M \circ f \circ M^{-1}$, its collection of bounded Herman rings $\mathcal{H} = \{B = M(A) : A \in \mathcal{H}_K\}$, the set E and the measure ν just defined. We obtain the representation $\nu = \sum_{\{B \in \mathcal{H}\}} \nu_B$ where ν_B is supported on ∂B and absolutely continuous with respect to the harmonic measure ω_B of B , i.e., $d\nu_B = h_B d\omega_B$ where $h_B \in L^1(\partial B, \omega_B)$. By the connection of μ and ν and since $d\omega_{M(A)}(M(z)) = M'(z) d\omega_A(z)$, we get the representation (3), as in the conclusion of Lemma 2.1, part 1. That $1/\psi'_A \in H^1$ provided $C_A \neq 0$ follows from (14) as well.

Part 1 has been proved.

Let us prove Part 2. After perhaps a Möbius change, one can assume that all Herman rings $B \in \mathcal{H}$ are bounded subsets of \mathbb{C} . Let us show that

$1/\psi'_B \in H^1$ for all $B \in \mathcal{H}$. By assumption this holds for $B = A$. Let now $B = f^j(A)$ for some $j \in \{1, \dots, q-1\}$. Then $\psi_B = f^j \circ \psi_A : \Delta_A \rightarrow B$ and, for an appropriate $\lambda \in S^1$ and all $w \in \Delta_A$,

$$\begin{aligned} |\psi'_B(w)| &= |(f^j)'(\psi_A(w))\psi'_A(w)| = \left| \frac{\psi'_A(w)(f^q)'(\psi_A(w))}{(f^{q-j})'(f^j(\psi_A(w)))} \right| \\ &= \frac{|\lambda\psi'_A(\lambda w)|}{|(f^{q-j})'(f^j(\psi_A(w)))|} \geq M^{-(q-j)}|\psi'_A(\lambda w)| \end{aligned}$$

where $M = \sup\{|(f'(z))| : z \in \bigcup_{B \in \mathcal{H}} B\} < \infty$. We see immediately that $1/\psi'_B \in H^1$ as well. Now the existence of the measure μ follows easily from (14) if we apply [14, Theorem 1, **P2**] where we take $\Omega_i = f^{i-1}(A)$, $i = 1, \dots, q-1$, and $\kappa_i = (\varphi'_i/\varphi_i)^2$ where $\varphi_i : f^{i-1}(A) \rightarrow \{1 < |w| < R\}$ is a conformal homeomorphism. As for uniqueness, if ν is another measure as in Part **2**, by (14) there is $C \in \mathbb{C}$ such that, for the measure $\tau := \nu - C\mu$, $\hat{\tau} = 0$ off $K := \bigcup\{\partial B : B \in \mathcal{H}\}$. On the other hand, K is a \mathbb{C} -compact because every $x \in K$ lies at the boundary of one of the components $B \in \mathcal{H}$ of its complement. Hence, $\tau = 0$.

Case (8) of Corollary 2.2. Assume that every critical point in $J(f)$ is summable. If $J(f) \neq \bar{\mathbb{C}}$, then by [26], $J(f)$ is of measure zero and f has no Herman rings; hence $K \subset J(f)$ is a \mathbb{C} -compact and \mathcal{H}_K is empty, and Lemma 2.1 applies. Let $J(f) = \bar{\mathbb{C}}$, in particular f has no Herman rings. Assume H is non-trivial. By Step III of the above proof of Lemma 2.1, f admits a non-trivial invariant line field on a forward invariant set of a positive Lebesgue measure (which is the set Z minus forward orbits of critical points). On the other hand, by [27] (see also [25]), Lebesgue almost every point of \mathbb{C} is conical for f , which leads (as in [24]) to the existence of a holomorphic line field for f ; hence f is a flexible Lattès example, a contradiction.

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