FUNDAMENTA
MATHEMATICAE
Online First version

Nonregular ideals

by

Monroe Eskew (Wien)

Abstract. Generalizing Keisler's notion of regularity for ultrafilters, Taylor introduced degrees of regularity for ideals and showed that a countably complete nonregular ideal on ω_1 must be somewhere ω_1-dense. We prove a dichotomy about degrees of regularity for κ-complete ideals on successor cardinals κ and apply this to show that Taylor's Theorem does not generalize to higher cardinals. In particular, the existence of a nonregular ideal on ω_2 does not imply the existence of an ω_2-dense ideal on ω_2. We obtain similar results for normal ideals on $\mathcal{P}_\kappa(\lambda)$.

1. Introduction. An *ideal* on a set X is a collection of subsets of X closed under taking subsets and pairwise unions. If κ is a cardinal, an ideal I is called *κ-complete* if it is also closed under unions of size less than κ. An ideal I on X is called *nonprincipal* when for all $x \in X$, $\{x\} \in I$, and it is called *proper* when $X \notin I$. In this paper, we assume all our ideals are nonprincipal and proper. An ideal I on X gives a notion of a "negligible" subset of X, and members of I are called *I-measure-zero*. Subsets of X which are not in I are called *I-positive*, and the collection of these is typically denoted by I^+. The dual filter to I, the collection of all complements of members of I, constitutes the collection of *I-measure-one* sets and will be denoted by I^*. If an ideal I renders every subset of X either measure zero or measure one, then its dual filter is called an *ultrafilter*.

The notion of regularity of ultrafilters was introduced by Keisler [10] and has had many applications in set theory and model theory [3]. An ultrafilter $\mathcal{U}$ is called *(α, β)-regular* when there is a sequence of sets $\langle A_i : i < \beta\rangle \subseteq \mathcal{U}$ such that for all $z \subseteq \beta$ of ordertype α, $\bigcap_{i\in z} A_i = \emptyset$. Taylor [12] generalized this notion to arbitrary filters (or equivalently, ideals), defining an ideal I to be (α, β)-regular when for every sequence $\langle A_i : i < \beta\rangle \subseteq I^+$, there is a

2020 *Mathematics Subject Classification*: 03E02, 03E05, 03E35.
Key words and phrases: ideals, regularity, forcing.
Received 13 November 2018; revised 3 September 2020.
Published online 18 January 2021.

refinement $\langle B_i : i < \beta\rangle \subseteq I^+$, which means $B_i \subseteq A_i$ for each i, such that for all $z \subseteq \beta$ of ordertype α, $\bigcap_{i\in z} B_i = \emptyset$. An ideal on a cardinal κ is simply called *regular* when it is (ω,κ)-regular. Taylor showed some connections between regularity properties of ideals and the structure of their associated quotient Boolean algebras, most notably the following:

THEOREM 1.1 (Taylor). *A countably complete ideal I on ω_1 is nonregular if and only if there is a set $A \in I^+$ such that $\mathcal{P}(A)/I$ contains a dense set of size ω_1.*

Taylor also discussed degrees of regularity indexed by three ordinals. An ideal I is said to be (α,β,γ)*-regular* when for every sequence $\langle A_i : i < \gamma\rangle \subseteq I^+$, there is a refinement $\langle B_i : i < \gamma\rangle \subseteq I^+$ such that for every $x \subseteq \gamma$ of ordertype β, $|\bigcap_{i\in x} B_i| \le \alpha$. We note the following easy relations between the regularity properties:

(1) If $\alpha_0 < \alpha_1$, then (α_0,β,γ)-regularity implies (α_1,β,γ)-regularity.
(2) If $\beta_0 < \beta_1$, then (α,β_0,γ)-regularity implies (α,β_1,γ)-regularity.
(3) If $\gamma_0 < \gamma_1$, then (α,β,γ_1)-regularity implies (α,β,γ_0)-regularity.

Taylor [12] showed that if I is a κ-complete ideal on a regular uncountable cardinal κ, then I is (ω,κ)-regular if and only if it is $(2,\kappa)$-regular. The latter is known as the *disjoint refinement property* or *Fodor's property* [1]. In [5], the author showed that under GCH, many more degrees of regularity are equivalent for κ-complete ideals on κ, where κ is the successor of a regular cardinal, and this was used to examine the relationship between regularity and density of ideals on cardinals above ω_1.

In the present paper, as elsewhere, we only consider degrees of regularity of ideals on κ for which the last index in the degree is at most κ. Under this restriction, we show that without any assumptions, there are only two possible flavors of two-variable regularity at successor cardinals, and with GCH, only two possible flavors of three-variable regularity:

THEOREM 1.2. *Suppose μ is an infinite cardinal, $\kappa = \mu^+$, and I is a κ-complete ideal on κ. Then:*

(1) *I is $(\mathrm{cf}(\mu)+1,\kappa)$-regular, $(1,\mathrm{cf}(\mu),\kappa)$-regular, and $(2,\delta)$-regular for $\delta<\kappa$.*
(2) *If I is $(\mathrm{cf}(\mu),\kappa)$-regular, then I is $(2,\kappa)$-regular.*
(3) *If I is (α,β,κ)-regular for some $\alpha,\beta < \kappa$ such that $\mu^\beta = \mu$, then I is $(2,\kappa)$-regular.*

Part (1) already appeared in [12] for the case of μ regular, and though the extension to the general case does not require essentially new ideas, we include a proof here for completeness. We show similar results for κ-complete normal ideals on $\mathcal{P}_\kappa(\lambda)$.

We will simply say a normal ideal on $Z \subseteq \mathcal{P}(\lambda)$ is *regular* when it is $(2, \lambda)$-regular. Following Taylor [12], we show in Proposition 2.2 that $(2, \lambda)$-regularity is equivalent to (ω, λ)-regularity for such ideals, so this definition accords with the original terminology of Keisler.

It is easy to see that a λ-dense normal ideal on $\mathcal{P}(\lambda)$ is nonregular. Taylor's Theorem uses a result of Baumgarter–Hajnal–Máté [1], who showed that if a countably complete ideal on ω_1 is nowhere ω_1-dense, then it has the disjoint refinement property. This generalizes to normal ideals I on $Z \subseteq \mathcal{P}_\kappa(\lambda)$ for κ a successor cardinal, with an additional assumption about the quotient Boolean algebra $\mathcal{P}(Z)/I$ that is trivially satisfied for $\kappa = \omega_1$ (see [5]). However, it is possible to separate density and nonregularity above ω_1:

THEOREM 1.3. *Suppose $\kappa = \mu^+$, $\omega_1 \leq \mathrm{cf}(\mu)$, $\kappa \leq \lambda$, and there is a nonregular, κ-complete, normal ideal on $\mathcal{P}_\kappa(\lambda)$. There is a cardinal-preserving forcing extension that also has such an ideal, but in which there are no λ-dense, κ-complete, normal ideals on $\mathcal{P}_\kappa(\lambda)$.*

By results in [5], the existence of a λ-dense, κ-complete, normal ideal on $\mathcal{P}_\kappa(\lambda)$, where $\kappa = \mu^+$, is consistent relative to an almost-huge cardinal, for any choice of regular cardinals $\mu < \kappa \leq \lambda$.

2. The regularity dichotomy. This section is devoted to a proof of Theorem 1.2. We will prove some more general facts about the regularity of normal ideals on $\mathcal{P}(\lambda)$ and show how they imply the desired results about κ-complete ideals on successor cardinals κ.

Our notations are mostly standard. By $\mathcal{P}_\kappa(\lambda)$ we mean $\{z \subseteq \lambda : |z| < \kappa\}$. If x is a set of ordinals, then $\mathrm{ot}(x)$ denotes its ordertype. The notations $[\lambda]^\kappa$ and $[\lambda]^{<\kappa}$ stand for $\{z \subseteq \lambda : \mathrm{ot}(z) = \kappa\}$ and $\{z \subseteq \lambda : \mathrm{ot}(z) < \kappa\}$ respectively.

The following facts can be found in [6]. Recall that an ideal I on $Z \subseteq \mathcal{P}(X)$ is *normal* when for all $x \in X$, $\hat{x} := \{z \in Z : x \in z\} \in I^*$, and for all sequences $\langle A_x : x \in X\rangle \subseteq I$, the diagonal union $\nabla_{x \in X} A_x := \bigcup_{x \in X} (A_x \cap \hat{x})$ belongs to I. This is equivalent to the statement that for every $A \in I^+$ and every $f : A \to X$ such that $f(z) \in z$ for all $z \in A$, there is $B \in I^+$ such that f is constant on B.

The smallest normal ideal on a set Z is the *nonstationary ideal on* Z, which is the dual ideal to the *club filter* (closed-unbounded filter) generated by sets of the form $\{z \in Z : f[z^{<\omega}] \subseteq z\}$, where f is a function from $X^{<\omega}$ to X. As the name suggests, positive sets for the nonstationary ideal are called *stationary*. Consequently, if there is a (proper) normal ideal on $Z \subseteq \mathcal{P}(X)$, then Z is stationary.

A normal ideal I on $Z \subseteq \mathcal{P}(X)$ is *δ-saturated* for a cardinal δ if there is no sequence $\langle A_\alpha : \alpha < \delta\rangle$ such that $A_\alpha \cap A_\beta \in I$ for $\alpha < \beta$, and simply *saturated* if it is $|X|^+$-saturated. If I is saturated, then $\mathcal{P}(Z)/I$ is a complete

Boolean algebra, with suprema given by diagonal unions. If $\langle A_x : x \in X\rangle$ is an antichain, then we can use normality to refine it to a pairwise disjoint sequence of I-positive sets by replacing A_x with $A_x \cap \hat{x} \setminus \bigcup_{y \neq x}(A_y \cap \hat{y})$.

The idea behind the following lemma is taken from [1].

Lemma 2.1. *Suppose I is an ideal on $Z \subseteq \mathcal{P}(\lambda)$, $\delta \leq \lambda$, and I is either normal or δ-complete. If there is no $A \in I^+$ such that $I \restriction A$ is δ^+-saturated, then I is $(2, \delta)$-regular.*

Proof. Let $\langle A_\alpha : \alpha < \delta\rangle \subseteq I^+$, and for each A_α, choose a sequence of I-positive sets $\langle B_\alpha^\beta : \beta < \delta^+\rangle$ such that each $B_\alpha^\beta \subseteq A_\alpha$ and $B_\alpha^\beta \cap B_\alpha^{\beta'} \in I$ when $\beta < \beta' < \delta^+$. For each $\alpha < \delta$, let $f(\alpha) \leq \alpha$ be the minimal ordinal such that $|\{\beta : A_\alpha \cap B_{f(\alpha)}^\beta \in I^+\}| = \delta^+$. We can find $\xi < \delta^+$ such that for all $\alpha < \delta$, all $\alpha' < f(\alpha)$, and all $\beta \geq \xi$, $A_\alpha \cap B_{\alpha'}^\beta \in I$.

Recursively choose a refinement $\langle C_\alpha : \alpha < \delta\rangle$ of $\langle A_\alpha : \alpha < \delta\rangle$ and an increasing sequence of ordinals $\langle \beta_\alpha : \alpha < \delta\rangle \subseteq \delta^+$ as follows. Let $C_0 = B_0^\xi$ and $\beta_0 = \xi$. Given $\langle C_{\alpha'} : \alpha' < \alpha\rangle$, let C_α be an I-positive set of the form $A_\alpha \cap B_{f(\alpha)}^{\beta_\alpha}$, where $\beta_\alpha \geq \sup_{\alpha' < \alpha}(\beta_{\alpha'} + 1)$. Note that whenever $\alpha \neq \alpha'$ are less than δ, it is ensured that $C_\alpha \cap C_{\alpha'} \in I$. This is because if $f(\alpha) = f(\alpha') = \eta$, then $B_\eta^{\beta_{\alpha'}} \cap B_\eta^{\beta_\alpha} \in I$ by construction, and if $f(\alpha) < f(\alpha')$, then $B_{f(\alpha)}^{\beta_\alpha} \cap A_{\alpha'} \in I$ since $\beta_\alpha \geq \xi$.

Finally, we refine $\langle C_\alpha : \alpha < \delta\rangle$ to a pairwise disjoint sequence $\langle D_\alpha : \alpha < \delta\rangle$. If I is normal, we put $D_\alpha = C_\alpha \cap \hat{\alpha} \setminus \bigcup_{\alpha' \neq \alpha}(C_{\alpha'} \cap \hat{\alpha}')$. If I is δ-complete, we put $D_\alpha = C_\alpha \setminus \bigcup_{\alpha' < \alpha} C_{\alpha'}$. ∎

Proposition 2.2. *A normal ideal on $\mathcal{P}(\lambda)$ is (ω, λ)-regular if and only if it is $(2, \lambda)$-regular.*

Proof. Suppose I is an (ω, λ)-regular normal ideal on $\mathcal{P}(\lambda)$. Let $\langle A_\alpha : \alpha < \lambda\rangle \subseteq I^+$. We may assume that $A_\alpha \subseteq \hat{\alpha}$ for each $\alpha < \lambda$. Let $\langle B_\alpha : \alpha < \lambda\rangle \subseteq I^+$ be a refinement such that $\bigcap_{\alpha \in x} B_\alpha = \emptyset$ whenever $x \subseteq \lambda$ is infinite. For each $z \in \bigcup_{\alpha < \lambda} B_\alpha$, let $s(z)$ be the finite set $\{\alpha : z \in B_\alpha\}$. Note that $s(z) \subseteq z$.

Using the normality of I, for each $\alpha < \lambda$, there is an I-positive $C_\alpha \subseteq B_\alpha$ such that s is constant on C_α with value $g(\alpha) \in [\lambda]^{<\omega}$. Note that if $g(\alpha) \neq g(\beta)$, then $C_\alpha \cap C_\beta = \emptyset$.

Now since I is (ω, λ)-regular and countably complete, there exists no $A \in I^+$ such that the dual of $I \restriction A$ is an ultrafilter. Therefore every I-positive set can be partitioned into two disjoint I-positive sets, and thus infinitely many. Hence for every $x \in [\lambda]^{<\omega}$, the sequence $\langle C_\alpha : \alpha \in x\rangle$ has a disjoint refinement $\langle D_\alpha^x : \alpha \in x\rangle \subseteq I^+$ by Lemma 2.1. Finally, let $E_\alpha = D_\alpha^{g(\alpha)}$. Then $\langle E_\alpha : \alpha < \lambda\rangle$ is a pairwise disjoint refinement of $\langle A_\alpha : \alpha < \lambda\rangle$. ∎

The above argument can be generalized to show that for a κ-complete normal ideal on $\mathcal{P}(\lambda)$ and $\mu < \kappa$, (μ, λ)-regularity implies $(2, \lambda)$-regularity if the ideal concentrates on $z \subseteq \lambda$ such that $|z|^{<\mu} = |z|$. However, we ultimately want to prove something more general about successor κ. The following lemma contains the key combinatorial idea.

LEMMA 2.3. *Suppose I is a normal ideal on $\mathcal{P}(\lambda)$, μ is a cardinal such that $\{z \subseteq \lambda : \mathrm{cf}(\sup z) \geq \mu\} \in I^*$, and for all $A \in I^+$ and $\delta < \lambda$, $I \restriction A$ is not δ^+-saturated. If I is (μ, λ)-regular, then I is regular.*

Proof. Let $\langle A_\alpha : \alpha < \lambda \rangle \subseteq I^+$. We may assume that for all $\alpha < \lambda$ and all $z \in A_\alpha$, $\mathrm{cf}(\sup z) \geq \mu$ and $\alpha \in z$. Since I is (μ, λ)-regular, there is a sequence of I-positive sets $\langle B_\alpha : \alpha < \lambda \rangle$ such that $B_\alpha \subseteq A_\alpha$ for all α, and for all z, $s(z) := \{\alpha : z \in B_\alpha\}$ has size $< \mu$. Note that $s(z) \subseteq z$. For all $z \in \bigcup_{\alpha<\lambda} B_\alpha$, since $\mathrm{cf}(\sup z) \geq \mu$, $s(z)$ is not cofinal in z. Thus let $f : \bigcup_{\alpha<\lambda} B_\alpha \to \lambda$ be such that $s(z) \subseteq f(z) \in z$. By normality, for all $\alpha < \lambda$, there is an I-positive $C_\alpha \subseteq B_\alpha$ on which f is constant. Let $g(\alpha)$ be this constant value, and note that $g(\alpha) > \alpha$.

For each $\alpha < \lambda$, choose a pairwise disjoint refinement $\langle D^\alpha_\beta : \beta < \alpha \rangle \subseteq I^+$ of $\langle C_\beta : \beta < \alpha \rangle$, using Lemma 2.1. Then let $E_\alpha = D^{g(\alpha)}_\alpha$. If $g(\alpha_0) = g(\alpha_1)$, then $E_{\alpha_0} \cap E_{\alpha_1} = \emptyset$ by construction. If $g(\alpha_0) \neq g(\alpha_1)$, then $E_{\alpha_0} \cap E_{\alpha_1} = \emptyset$, since for $i < 2$ and $z \in E_{\alpha_i}$, $f(z) = g(\alpha_i)$. ∎

LEMMA 2.4. *Suppose μ, λ are regular cardinals, and I is a normal ideal on $\mathcal{P}(\lambda)$ such that $\{z \subseteq \lambda : \mathrm{cf}(\sup z) = \mu\} \in I^*$. Then I is $(\mu + 1, \lambda)$-regular. If the function $z \mapsto \sup z$ is $\leq\delta$-to-one on a set in I^*, then I is (δ, μ, λ)-regular.*

Proof. Let $\langle A_i : i < \lambda \rangle \subseteq I^+$. Let $Z = \{z \subseteq \lambda : \mathrm{cf}(\sup z) = \mu\}$. For each $z \in Z$, let $c_z \subseteq z$ be a cofinal subset of ordertype μ. By induction, we build an increasing sequence $\langle \alpha_i : i < \lambda \rangle \subseteq \lambda$ and a refinement $\langle B_i : i < \lambda \rangle \subseteq I^+$ of $\langle A_i : i < \lambda \rangle$ as follows. Given $\langle \alpha_i : i < j \rangle$, $\sup z > \sup_{i<j} \alpha_i$ for I-almost all $z \in A_j$. For such z, let $\sup_{i<j} \alpha_i < \alpha_j(z) \in c_z$. Let $B_j \subseteq A_j$ be an I-positive set on which the function $z \mapsto \alpha_j(z)$ is constant, and let α_j be this constant value. For each z, let $s(z) = \{i < \lambda : z \in B_i\}$. Note that $z \in B_i$ implies $\alpha_i \in c_z$, so $\mathrm{ot}(s(z)) \leq \mu$. This establishes the claim that I is $(\mu+1, \lambda)$-regular. For the second claim, note that if $\mathrm{ot}(s(z)) = \mu$, then $s(z)$ is cofinal in c_z and hence in z. Thus, if $z \mapsto \sup z$ is $\leq\delta$-to-one on a set in I^*, then we may take the sequence $\langle B_i : i < \lambda \rangle$ such that $|\bigcap_{i\in x} B_i| \leq \delta$ whenever $\mathrm{ot}(x) = \mu$. ∎

The following result was independently observed by Burke–Matsubara [2] and Foreman–Magidor [8]. Its proof uses deep results of Shelah [11] and Cummings [4].

LEMMA 2.5. *Suppose I is a normal saturated ideal on $\mathcal{P}(\lambda)$. Then $\{z : \mathrm{cf}(\sup z) = \mathrm{cf}(|z|)\} \in I^*$.*

The following basic fact can be proved in multiple ways, for example via Ulam matrices or via generic ultrapowers (see [6]).

LEMMA 2.6. *If κ is a successor cardinal, then no κ-complete ideal on κ is κ-saturated, and no κ-complete normal ideal on $\mathcal{P}_\kappa(\lambda)$ is λ-saturated.*

THEOREM 2.7. *Suppose $\kappa = \mu^+$ and I is a κ-complete normal ideal on $\mathcal{P}_\kappa(\lambda)$. If I is $(\mathrm{cf}(\mu), \lambda)$-regular, then I is regular. If λ is a regular cardinal, then I is $(\mathrm{cf}(\mu) + 1, \lambda)$-regular.*

Proof. Let $\langle A_\alpha : \alpha < \lambda\rangle \subseteq I^+$. We first separate the saturated and non-saturated parts. We choose an initial refinement by putting $B_\alpha = A_\alpha$ if there is no $B \subseteq A_\alpha$ such that $I\restriction B$ is saturated, and otherwise choose $B_\alpha \subseteq A_\alpha$ such that $I\restriction B_\alpha$ is saturated. Let Y_0 be the ordinals below λ falling into the first case, and Y_1 those falling into the second. Note that whenever $\alpha \in Y_0$ and $\beta \in Y_1$, we have $B_\alpha \cap B_\beta \in I$. As in the proof of Lemma 2.1, we may refine to a sequence $\langle C_\alpha : \alpha < \lambda\rangle$ such that $C_\alpha \cap C_\beta = \emptyset$ whenever at least one of α, β is in Y_0. If we put $C = \nabla_{\alpha\in Y_1} C_\alpha$, then $I\restriction C$ is saturated, since if $\langle D_\alpha : \alpha < \lambda^+\rangle$ were an antichain in $\mathcal{P}(C)/I$, then for some $\beta < \lambda$, there would be λ^+-many α such that $C_\beta \cap D_\alpha \in I^+$, in contradiction to the fact that $I\restriction C_\beta$ is saturated.

We may assume $\mathrm{cf}(\sup z) = \mathrm{cf}(\mu)$ for all $z \in C$. Since $I\restriction A$ is not λ-saturated for any $A \in I^+$, Lemma 2.3 implies that if I is $(\mathrm{cf}(\mu), \lambda)$-regular, then there is a disjoint refinement of $\langle C_\alpha : \alpha \in Y_1\rangle$ into I-positive sets $\langle D_\alpha : \alpha \in Y_1\rangle$. Putting this together with $\langle C_\alpha : \alpha \in Y_0\rangle$, we have a disjoint refinement of the original sequence into I-positive sets.

If λ is a regular cardinal, then by Lemma 2.4, there is a refinement $\langle E_\alpha : \alpha \in Y_1\rangle \subseteq I^+$ of $\langle C_\alpha : \alpha \in Y_1\rangle$ such that $\bigcap_{\alpha\in x} E_\alpha = \emptyset$ whenever $\mathrm{ot}(x) > \mathrm{cf}(\mu)$, showing that I is $(\mathrm{cf}(\mu) + 1, \lambda)$-regular. ■

In order to prove Theorem 1.2, we use some results from [12] which allow a reduction to normal ideals:

LEMMA 2.8 (Taylor). *Let I be a κ-complete ideal on κ.*

(1) *Suppose every sequence $\langle A_i : i < \kappa\rangle \subseteq I^+$ has a refinement $\langle B_i : i < \kappa\rangle \subseteq I^+$ such that $I\restriction B_i$ is (α, β, κ)-regular for each i. Then I is (α, β, κ)-regular.*
(2) *If $\kappa = \mu^+$ and I is κ^+-saturated, then there is $A \in I^+$ and a bijection $f : \kappa \to \kappa$ such that $\{f[X] : X \in I\restriction A\}$ is a normal ideal on κ.*

Let I be a κ-complete ideal on $\kappa = \mu^+$. First let us show part (1) of Theorem 1.2. By Lemmas 2.1 and 2.6, I is $(2, \delta)$-regular for $\delta < \kappa$. For the other regularity properties, let $\langle A_\alpha : \alpha < \kappa\rangle \subseteq I^+$. Let $B_\alpha \subseteq A_\alpha$ be an I-positive set such that $I\restriction B_\alpha$ is κ^+-saturated if there is such a B_α. In such a case, Lemma 2.8(2) implies that we can find an I-positive $C_\alpha \subseteq B_\alpha$ such that $I\restriction C_\alpha$ is isomorphic to a normal ideal. By Lemmas 2.4 and 2.5, $I\restriction C_\alpha$ is $(\mathrm{cf}(\mu)+1,\kappa)$-regular and $(1,\mathrm{cf}(\mu),\kappa)$-regular whenever C_α is defined. If C_α is undefined, then $I\restriction A_\alpha$ is regular by Lemma 2.1. Lemma 2.8(1) then shows that I is $(\mathrm{cf}(\mu)+1,\kappa)$-regular and $(1,\mathrm{cf}(\mu),\kappa)$-regular.

Now let us show part (2) of Theorem 1.2. Let $\langle A_\alpha : \alpha<\kappa\rangle \subseteq I^+$, and choose sets C_α exactly as above. If I is $(\mathrm{cf}(\mu),\kappa)$-regular, then so is each $I\restriction C_\alpha$ when C_α is defined, and thus $I\restriction C_\alpha$ is regular by Theorem 2.7. Again by Lemmas 2.1 and 2.8(1), I is regular in this case.

To show part (3) of Theorem 1.2, we introduce an extension of Taylor's three-variable notion of regularity. Let us say an ideal I is *(I,α,β)-regular* if every sequence $\langle A_i : i<\beta\rangle \subseteq I^+$ has a refinement $\langle B_i : i<\beta\rangle \subseteq I^+$ such that $\bigcap_{i\in x} B_i \in I$ whenever $\mathrm{ot}(x)=\alpha$. If I is a κ-complete ideal on κ, then (I,β,κ)-regularity is a weakening of (α,β,κ)-regularity for every $\alpha<\kappa$.

LEMMA 2.9. *Suppose $\kappa=\mu^+$ and I is a κ-complete ideal on κ. If I is (I,ξ,κ)-regular, where $\mu^\xi=\mu$, then I is regular.*

Proof. Let $\langle A_\alpha : \alpha<\kappa\rangle \subseteq I^+$, and let $\langle B_\alpha : \alpha<\kappa\rangle \subseteq I^+$ be a refinement such that $B_\alpha \subseteq \hat{\alpha}$ for all α, and $\bigcap_{\alpha\in x} B_\alpha \in I$ whenever $\mathrm{ot}(x)\geq\xi$. For every $\alpha<\kappa$ we can define an I-positive $C_\alpha \subseteq B_\alpha$ by

$$C_\alpha = B_\alpha \setminus \bigcup_{x\in[\alpha]^\xi} \bigcap_{\beta\in x} B_\beta.$$

If x is a subset of κ of ordertype $\xi+1$, then let $\alpha=\max(x)$. If $\beta\in C_\alpha$, then $\beta\notin\bigcap_{\gamma\in x\cap\alpha} C_\gamma$. This shows that I is $(\xi+1,\kappa)$-regular. Since $\mu^\xi=\mu$ implies $\xi<\mathrm{cf}(\mu)$, I is regular by Theorem 1.2(2). ∎

3. Consistency results. This section is devoted to a proof of Theorem 1.3. If $V\subseteq W$ are models of set theory and $I\in V$ is an ideal, then in W we can generate an ideal $\bar{I}$ from I by taking all sets which are covered by a set from I. Let us first show the preservation of nonregular ideals by forcings with a strong enough chain condition, as a consequence of Theorem 2.7.

LEMMA 3.1. *Suppose $\kappa=\mu^+$, $\lambda\geq\kappa$, and I is a nonregular, κ-complete, normal ideal on $Z\subseteq\mathcal{P}_\kappa(\lambda)$. If $\mathbb{P}$ is $\mathrm{cf}(\mu)$-c.c., then in $V^{\mathbb{P}}$, the ideal $\bar{I}$ generated by I is nonregular.*

Proof. Let us show the contrapositive, that if $\bar{I}$ is regular in a $\mathbb{P}$-generic extension, then I is regular in V. Let $\langle A_\alpha : \alpha<\lambda\rangle\subseteq I^+$ be in V. If $p\Vdash \bar{I}$ is regular, then there is a $\mathbb{P}$-name for a refinement $\langle \dot{B}_\alpha : \alpha<\lambda\rangle$ such that each $z\in Z$ is forced by p to be in at most one B_α. In V, for each α let $C_\alpha=\{z\in A_\alpha : (\exists q\leq p)\ q\Vdash z\in\dot{B}_\alpha\}$. Since $p\Vdash \dot{B}_\alpha\subseteq\check{C}_\alpha$, each C_α is I-positive. By the chain condition, for each z, the set

$$s(z) := \{\alpha : (\exists q\leq p)\ q\Vdash z\in\dot{B}_\alpha\} = \{\alpha : z\in C_\alpha\}$$

has size $< \mathrm{cf}(\mu)$. This shows that I is $(\mathrm{cf}(\mu), \lambda)$-regular in V, and thus regular by Theorem 2.7. ■

If I is a κ-complete normal ideal and $\mathbb{P}$ is a κ-c.c. forcing, then it is easy to show that the ideal generated by I is also κ-complete and normal in $V^{\mathbb{P}}$. If I is saturated, then Foreman's Duality Theorem [7] allows us to say much more. This is connected to the forcing properties of the quotient algebra and generic elementary embeddings.

The following facts can be found in [6]. If I is an ideal on Z and $G \subseteq \mathcal{P}(Z)/I$ is generic, then in $V[G]$, we can form the ultrapower embedding $j : V \to V^Z/G$. If $Z \subseteq \mathcal{P}(\lambda)$ and I is normal, then the pointwise image of λ under j is represented in the ultrapower by the identity function on Z, i.e. $[\mathrm{id}]_G = j[\lambda]$. If I is κ-complete, $\kappa = \mu^+$, and $Z \subseteq \mathcal{P}_\kappa(\lambda)$, then κ is the critical point of j, and $V^Z/G \models |j[\lambda]| < j(\kappa)$. Consequently, $V[G] \models |\lambda| = |\mu|$. This implies that there is no condition $A \in I^+$ such that $I \restriction A$ is λ-saturated. Thus in this context, I being saturated is the same as $\mathcal{P}(Z)/I$ having the best possible chain condition. If this occurs, then I is *precipitous*, meaning that whenever $G \subseteq \mathcal{P}(Z)/I$ is generic, V^Z/G is well-founded and thus isomorphic to a transitive class $M \subseteq V[G]$.

THEOREM 3.2 (Foreman [7]). *Suppose I is a κ-complete precipitous ideal on Z, and $\mathbb{P}$ is a κ-c.c. forcing. In $V^{\mathbb{P}}$, let $\bar{I}$ denote the ideal generated by I, and let j denote a generic ultrapower embedding obtained from forcing with $\mathcal{P}(Z)/I$. Then there is an isomorphism*

$$\iota : \mathcal{B}(\mathbb{P} * \dot{\mathcal{P}(Z)}/\bar{I}) \cong \mathcal{B}(\mathcal{P}(Z)/I * j(\dot{\mathbb{P}}))$$

given by $\iota(p, \dot{A}) = \|[\mathrm{id}] \in j(\dot{A})\| \wedge (1, j(\dot{p}))$.

The next proposition shows the relevance of the cardinal arithmetic assumption in Lemma 2.9. For example, we can produce a model in which CH fails and there is a nonregular ideal I on ω_2 which is (I, ω, ω_2)-regular.

PROPOSITION 3.3. *Suppose $\kappa = \mu^+$, $\nu \le \mu$ is such that $\nu^{<\nu} = \nu$, and I is a saturated, nonregular, κ-complete ideal on κ. If $G \subseteq \mathrm{Add}(\nu, \kappa)$ is generic, then in $V[G]$, $\bar{I}$ is $(\bar{I}, \nu, \kappa)$-regular.*

Proof. Since $\nu^{<\nu} = \nu$, $\mathrm{Add}(\nu, \kappa)$ is ν^+-c.c. By Theorem 3.2, in $V[G]$, there is an isomorphism

$$\sigma : \mathcal{P}(\kappa)/\bar{I} \cong \mathcal{B}(\mathcal{P}(\kappa)^V/I \times \mathrm{Add}(\nu, \kappa^+)).$$

If $\langle A_\alpha : \alpha < \kappa \rangle \subseteq \bar{I}^+$, choose for each α some $(B_\alpha, p_\alpha) \le \sigma(A_\alpha)$. Let $\beta < \kappa^+$ be such that $\operatorname{dom} p_\alpha \subseteq \beta \times \nu$ for all α. Let $q_\alpha = \{((\beta + \alpha, 0), 0)\}$ for $\alpha < \kappa$, and choose $C_\alpha \le \sigma^{-1}(B_\alpha, p_\alpha \wedge q_\alpha)$. The intersection of any ν-many C_α is in $\bar{I}$, since there is no lower bound to ν-many q_α. ■

Lemma 3.4. *Suppose I is a normal ideal on $Z \subseteq \mathcal{P}(X)$. Then I is $|X|^+$-saturated if and only if every normal $J \supseteq I$ is equal to $I \restriction A$ for some $A \subseteq Z$.*

Proof. Suppose I is $|X|^+$-saturated. Let $Y \subseteq X$ and $\{A_x : x \in Y\}$ be such that $\{[A_x]_I : x \in Y\}$ is an antichain in $\mathcal{P}(Z)/I$ of I-positive sets that are J-measure-zero, and is maximal among all such collections. Then $[\nabla A_x]_I$ is the $\subseteq_I$-largest element of $J \cap I^+$, so $J = I \restriction (Z \setminus \nabla A_x)$. Now suppose I is not $|X|^+$-saturated, and let $\{A_\alpha : \alpha < \delta\}$ be a maximal antichain where $\delta \geq |X|^+$. Let J be the ideal generated by $\bigcup \{\Sigma_{\alpha \in Y}[A_\alpha] : Y \in \mathcal{P}_{|X|^+}(\delta)\}$. Then J is a proper normal ideal extending I. If J were equal to $I \restriction A$ for some $A \in I^+$, there would be some α such that $A \cap A_\alpha \in I^+$. Although $A \cap A_\alpha \in J$ by construction, every I-positive subset of A is $(I \restriction A)$-positive. ∎

A partial order is said to be *κ-dense* if it has a dense subset of size $\leq \kappa$. It is said to be *nowhere κ-dense* if it is not κ-dense below any condition. An ideal is said to be λ-dense or nowhere λ-dense when its associated Boolean algebra has these properties.

Lemma 3.5. *Suppose $\mu^+ = \kappa \leq \lambda$, $\nu \leq \mu$ is such that $\nu^{<\nu} = \nu$, and $Z \subseteq \mathcal{P}_\kappa(\lambda)$ is stationary. Let $\mathbb{P} = \mathrm{Add}(\nu, \theta)$ for some $\theta \geq \kappa$. Then in $V^{\mathbb{P}}$, there are no normal, κ-complete, λ-dense ideals on Z.*

Proof. Since a λ-dense ideal is λ^+-saturated, it suffices to consider $\mathbb{P}$-names for λ^+-saturated normal ideals on Z. Suppose $p \Vdash \dot{J}$ is a κ-complete, λ^+-saturated, normal ideal on Z. Let $I = \{X \subseteq Z : p \Vdash X \in \dot{J}\}$. It is easy to check that I is normal and κ-complete. The map $\sigma : \mathcal{P}(Z)/I \to \mathcal{B}(\mathbb{P} \restriction p) * \mathcal{P}(Z)/\dot{J}$ that sends X to $(\|\check{X} \in \dot{J}^+\|, [\dot{X}]_J)$ is order-preserving and antichain-preserving. Since $\nu^{<\nu} = \nu$, $\mathbb{P}$ is κ-c.c., so the two-step iteration $\mathbb{P} \restriction p * \mathcal{P}(Z)/\dot{J}$ is λ^+-c.c. Thus I is λ^+-saturated.

Let H be $\mathbb{P}$-generic over V with $p \in H$. Since $\mathbb{P}$ is κ-c.c., $\bar{I}$ remains normal. By Theorem 3.2, the map $e : q \mapsto (1, \dot{j(q)})$ is a regular embedding of $\mathbb{P}$ into $\mathcal{P}(Z)/I * \dot{j(\mathbb{P})}$. Thus in $V[H]$, $\mathcal{P}(Z)/\bar{I} \cong \mathcal{P}^V(Z)/I * \mathrm{Add}(\nu, \dot{\eta})$, where $\Vdash \dot{\eta} = \mathrm{ot}(j(\theta) \setminus j[\theta])$. By the saturation of I, $\Vdash j(\kappa) = \lambda^+$, so $\Vdash \dot{\eta} \geq \lambda^+$.

Observe that $\bar{I}$ is normal and λ^+-saturated, and $\bar{I} \subseteq J$. By Lemma 3.4, there is $A \in \bar{I}^+$ such that $J = \bar{I} \restriction A$. Since $\mathrm{Add}(\nu, \eta)$ is nowhere λ-dense, $\mathcal{P}(Z)/\bar{I}$ is nowhere λ-dense. Thus J is not λ-dense. ∎

Thus we may rid the universe of dense ideals that concentrate on $\mathcal{P}_\kappa(\lambda)^V$. This finishes the job if $\kappa = \lambda$, but not necessarily in other cases. For example, Gitik [9] showed that if $V \subseteq W$ are models of set theory, $\kappa < \lambda$ are regular in W, and there is a real number in $W \setminus V$, then $\mathcal{P}_\kappa(\lambda)^W \setminus \mathcal{P}_\kappa(\lambda)^V$ is stationary. In order to take care of such problems, we use some arguments of Laver and Hajnal–Juhász that are reproduced in [6].

The notation

$$\binom{\alpha}{\beta} \to \binom{\gamma}{\delta}_{\eta}$$

stands for the assertion that for every $f : \alpha \times \beta \to \eta$ there are $A \in [\alpha]^{\gamma}$ and $B \in [\beta]^{\delta}$ such that f is constant on $A \times B$. As usual with arrow notations, if ordinals on the left side are increased and ordinals on the right side are decreased, then we get a weaker statement.

LEMMA 3.6. *Suppose there is a λ-dense, κ-complete, normal ideal I on $\mathcal{P}_\kappa(\lambda)$ such that every I-positive set has cardinality $\geq \eta$. Then for $\mu, \nu < \kappa$,*

$$\binom{\lambda^+}{\lambda^{<\kappa}} \to \binom{\mu}{\eta}_{\nu}.$$

Proof. Let $\theta = \lambda^{<\kappa}$, enumerate $\mathcal{P}_\kappa(\lambda)$ as $\langle z_\alpha : \alpha < \theta \rangle$, and let $f : \lambda^+ \times \theta \to \nu$. By κ-completeness, for each $\alpha < \lambda^+$, there is $\gamma < \nu$ such that $X_\alpha := \{z_\beta : f(\alpha, \beta) = \gamma\} \in I^+$. By λ-density, there is a set $S \in [\lambda^+]^{\lambda^+}$, a set $D \in I^+$, and a $\gamma^* < \nu$ such that for all $\alpha \in S$, $D \subseteq_I X_\alpha$ and $f(\alpha, \beta) = \gamma^*$ for $z_\beta \in X_\alpha$. Let $A \subseteq S$ have size μ. Since $\bigcap_{\alpha \in A} X_\alpha$ is I-positive, there is a set $B \subseteq \theta$ of size $\geq \eta$ such that for all $\alpha \in A$ and all $\beta \in B$, $f(\alpha, \beta) = \gamma^*$. ∎

LEMMA 3.7. *Suppose θ is regular and $\mu < \theta$ is such that $\mu^{<\mu} = \mu$. If $G \subseteq \mathrm{Add}(\mu, \theta)$ is generic, then in $V[G]$,*

$$\binom{\theta^+}{\theta} \nrightarrow \binom{\mu}{\theta}_{2}.$$

Proof. In V, choose an almost-disjoint family $\{X_\alpha : \alpha < \theta^+\} \subseteq \mathcal{P}(\theta)$, and for each α, let $\langle \gamma^\alpha_\beta : \beta < \theta \rangle$ enumerate X_α in increasing order. In $V[G]$, let $f : \theta^+ \times \theta \to 2$ be defined by $f(\alpha, \beta) = G(\gamma^\alpha_\beta, 0)$. Let $A \subseteq \theta^+$ be a set of size μ in $V[G]$. By the chain condition, there is a $\zeta < \theta$ such that $G = G_0 \times G_1$, where G_0 is $\mathrm{Add}(\mu, \zeta)$-generic, and $A \in V[G_0]$. In $V[G_0]$, let δ be such that $\zeta < \delta < \theta$ and $\{X_\alpha \setminus \delta : \alpha \in A\}$ is pairwise disjoint. For any $p \in \mathrm{Add}(\mu, \theta \setminus \zeta)$ and any $\eta \geq \delta$, there are $q \leq p$ and $\alpha, \beta \in A$ such that $q(\gamma^\alpha_\eta, 0) \neq q(\gamma^\beta_\eta, 0)$. Since G_1 is generic, we see that for all $\eta \geq \delta$, there are $\alpha, \beta \in A$ such that $f(\alpha, \eta) \neq f(\beta, \eta)$. Thus there is no $B \subseteq \theta$ of size θ such that f is constant on $A \times B$. ∎

We can now prove Theorem 1.3. Suppose that in V, I is a nonregular, κ-complete, normal ideal on $\mathcal{P}_\kappa(\lambda)$, where $\kappa = \mu^+$ and $\mathrm{cf}(\mu)$ is uncountable. Let $\theta \geq \lambda^\mu$ be regular and such that $\theta^\mu = \theta$. Let $G \subseteq \mathrm{Add}(\omega, \theta)$ be generic. By Lemma 3.1, $\bar{I}$ is nonregular in $V[G]$. Suppose $Z \subseteq \mathcal{P}_\kappa(\lambda)$ has cardinality $< \theta$. Then there is $\zeta < \theta$ such that $G = G_0 \times G_1$, where G_0 is $\mathrm{Add}(\omega, \zeta)$-generic, and $Z \in V[G_0]$. By Lemma 3.5, there is no λ-dense, κ-complete, normal ideal concentrating on Z in $V[G]$. For κ-complete normal ideals on

$\mathcal{P}_\kappa(\lambda)$ in $V[G]$ that do not have any positive set of size $< \theta$, Lemma 3.6 implies that if such an ideal were λ-dense, then we would have

$$\binom{\lambda^+}{\theta} \to \binom{\mu}{\theta}_\mu$$

since $\lambda^\mu = \theta$ in $V[G]$. But Lemma 3.7 implies that the weaker relation

$$\binom{\theta^+}{\theta} \to \binom{\omega}{\theta}_2$$

fails in $V[G]$. Thus $V[G]$ has no λ-dense, κ-complete, normal ideal on $\mathcal{P}_\kappa(\lambda)$.

References

[1] J. E. Baumgartner, A. Hajnal and A. Máté, *Weak saturation properties of ideals*, in: Infinite and Finite Sets, Colloq. Math. Soc. János Bolyai 10, North-Holland, Amsterdam, 1975, Vol. I, 137–158.

[2] D. R. Burke and Y. Matsubara, *The extent of strength in the club filters*, Israel J. Math. 114 (1999), 253–263.

[3] C. C. Chang and H. J. Keisler, *Model Theory*, 3rd ed., Stud. Logic Found. Math. 73, North-Holland, Amsterdam, 1990.

[4] J. Cummings, *Collapsing successors of singulars*, Proc. Amer. Math. Soc. 125 (1997), 2703–2709.

[5] M. Eskew, *Measurability properties on small cardinals*, Ph.D. thesis, UC Irvine, 2014.

[6] M. Foreman, *Ideals and generic elementary embeddings*, in: Handbook of Set Theory, M. Foreman and A. Kanamori (eds.), Vol. 2, Springer, Dordrecht, 2010, 885–1147.

[7] M. Foreman, *Calculating quotient algebras of generic embeddings*, Israel J. Math. 193 (2013), 309–341.

[8] M. Foreman and M. Magidor, *Mutually stationary sequences of sets and the non-saturation of the non-stationary ideal on $P_\varkappa(\lambda)$*, Acta Math. 186 (2001), 271–300.

[9] M. Gitik, *Nonsplitting subset of $\mathcal{P}_\kappa(\kappa^+)$*, J. Symbolic Logic 50 (1985), 881–894.

[10] H. J. Keisler, *On cardinalities of ultraproducts*, Bull. Amer. Math. Soc. 70 (1964), 644–647.

[11] S. Shelah, *Proper Forcing*, Lecture Notes in Math. 940, Springer, Berlin, 1982.

[12] A. D. Taylor, *Regularity properties of ideals and ultrafilters*, Ann. Math. Logic 16 (1979), 33–55.

Monroe Eskew
Kurt Gödel Research Center
Institut für Mathematik
Universität Wien
Augasse 2-6, UZA 1 – Building 2
1090 Wien, Austria
E-mail: monroe.eskew@univie.ac.at