

Monolithic spaces of measures

by

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Abstract. For a compact space K we consider the space $P(K)$ of probability regular Borel measures on K , equipped with the weak* topology inherited from $C(K)^*$. We discuss possible characterizations of those compact spaces K for which $P(K)$ is \aleph_0 -monolithic. The main result states that under \diamond there exists a nonseparable Corson compact space K such that $P(K)$ is \aleph_0 -monolithic but K supports a measure of uncountable type.

1. Introduction. A compact space K is \aleph_0 -monolithic if every separable subspace of K is metrizable. Typical example of such spaces are those closely related to functional analysis: Eberlein compacta and, more generally, Corson compacta, i.e. spaces that can be embedded into

$$\Sigma(\mathbb{R}^\kappa) = \{x \in \mathbb{R}^\kappa : |\{\alpha : x_\alpha \neq 0\}| \leq \omega\},$$

for some κ . In fact, there is a monotone version of monolithicity that implies Corson compactness (see Gruenhage [11]).

Given a compact space K , we denote by $P(K)$ the space of probability regular Borel measures on K and we always equip $P(K)$ with the weak* topology inherited from $C(K)^*$, the dual space of the space $C(K)$ of continuous functions. This means that the topology on $P(K)$ is determined by continuity of all the mappings

$$P(K) \ni \mu \mapsto \int_K g \, d\mu, \quad g \in C(K).$$

We investigate here for which compact spaces K , the space $P(K)$ is \aleph_0 -monolithic. It is easy to check that $P(K)$ is \aleph_0 -monolithic if and only if $B_{C(K)^*}$, the dual unit ball, is \aleph_0 -monolithic in its weak* topology. Monolithicity of dual unit balls of Banach spaces emerged quite naturally in a

2020 *Mathematics Subject Classification*: Primary 28A33, 46E27, 03E50.

Key words and phrases: \aleph_0 -monolithic, space of measures, Martin's axiom, Jensen's principle.

Received 18 March 2020; revised 12 August 2020 and 11 September 2020.

Published online 14 January 2021.

number of papers devoted to investigating some isomorphic properties of Banach spaces related to the space c_0 ; see Kalenda and Kubiś [13], Ferrer, Koszmider and Kubiś [9], Correa and Tausk [6], Ferrer [8], Correa [5].

Recall that a measure $\mu \in P(K)$ is *of type κ* if κ is the density of the space $L_1(\mu)$ of integrable functions. Equivalently, the (Maharam) type of $\mu \in P(K)$ can be defined as the minimal cardinality of a family \mathcal{C} of Borel subsets of K having the property that

$$\inf\{\mu(B \triangle C) : C \in \mathcal{C}\} = 0 \quad \text{for every } B \in \text{Bor}(K).$$

In the next section we collect a number of essentially known results; put together, they will explain that monolithicity of spaces of measures is easy to handle under Martin's axiom $\text{MA}(\omega_1)$. Then, at the end of Section 2 we ask whether \aleph_0 -monolithicity of $P(K)$ can be characterized by the property that every $\mu \in P(K)$ is supported by a metrizable subspace of K . Our main objective is to demonstrate that this is not the case; assuming the Diamond Principle, we construct a counterexample in Section 4 and analyze the resulting space in Section 5. Section 3 contains some auxiliary results on measures on Boolean algebras.

2. Monolithicity under Martin's axiom. Given a compact space K and $\mu \in P(K)$, by the *support* of μ we mean the uniquely determined smallest closed subset of K of measure 1. We first recall the following well-known fact.

LEMMA 2.1. *If a compact space K is the support of a measure $\mu \in P(K)$ of type ω then the space $P(K)$ is separable.*

The question for which compacta K the space $P(K)$ is separable was investigated by Talagrand [20] and Mägerl and Namioka [16]. Lemma 2.1 follows from the fact that if a measure $\mu \in P(K)$ of countable type is positive on every nonempty open subset of K then one can easily define a sequence of $\mu_n \in P(K)$ such that for every nonempty open $U \subseteq K$ there is n with $\mu_n(U) > 1/2$. In turn, this gives rise to a positive isomorphic embedding $C(K) \rightarrow \ell_\infty$ and then the dual operator maps the weak* separable space $P(\beta\omega)$ onto $P(K)$.

It is perhaps worth recalling (though not needed later) that Talagrand [20] constructed under CH two examples showing that the implication of Lemma 2.1 cannot be reversed and that separability of $P(K)$ does not follow from separability of $C(K)^*$. Such examples were later constructed in the usual set theory (see [7] and [3]).

The observations given in this section build on the following two results.

THEOREM 2.2 (Arkhangel'skiĭ and Shapirovskiĭ [2]). *Under $\text{MA}(\omega_1)$, every compact \aleph_0 -monolithic ccc space is metrizable.*

THEOREM 2.3 (Fremlin [10]). *Under $\text{MA}(\omega_1)$, if a compact space K carries a measure of uncountable type then K can be continuously mapped onto $[0, 1]^{\omega_1}$.*

We start by noting some basic facts.

LEMMA 2.4. *Let K be a compact space.*

- (a) *If $P(K)$ is \aleph_0 -monolithic then so is K .*
- (b) *If the support of every measure $\mu \in P(K)$ is metrizable then the space $P(K)$ is \aleph_0 -monolithic.*

Proof. Clause (a) follows from the fact that K embeds into $P(K)$ via the mapping

$$K \ni x \mapsto \delta_x \in P(K),$$

where δ_x is the Dirac measure at x .

To check (b) take any sequence of $\mu_n \in P(K)$ and consider the measure

$$\nu = \sum_{n=1}^{\infty} 2^{-n} \mu_n \in P(K).$$

Then the support S of ν is metrizable and hence $P(S)$ is metrizable too; moreover, $\{\mu_n : n = 1, 2, \dots\} \subseteq P(S)$, and we are done. ■

PROPOSITION 2.5. *Suppose that K is a compact space such that $P(K)$ is \aleph_0 -monolithic.*

- (a) *Then every $\mu \in P(K)$ is of type $\leq \omega_1$.*
- (b) *If $\mu \in P(K)$ is of type ω then the support of μ is metrizable.*
- (c) *Under $\text{MA}(\omega_1)$, the support of every $\mu \in P(K)$ is metrizable.*

Proof. Talagrand [21] showed in ZFC (see also [18]) that if K admits a measure of type $\geq \omega_2$ then $P(K)$ can be continuously mapped onto $[0, 1]^{\omega_2}$. But then $P(K)$ cannot be \aleph_0 -monolithic, as this property is preserved by taking continuous images of compacta. Hence, (a) follows from Talagrand's result.

To check (b), let $S \subseteq K$ be the support of a measure μ of countable type. Then $P(S)$ can be seen as a subspace of $P(K)$; $P(S)$ is separable by Lemma 2.1. Consequently, $P(S)$ is metrizable and S is also metrizable since S embeds into $P(S)$.

Now to check (c), it is enough to prove that under Martin's axiom, K cannot carry a measure of type ω_1 . This follows from Theorem 2.3: otherwise, there is a continuous surjection $K \rightarrow [0, 1]^{\omega_1}$; since $[0, 1]^{\omega_1}$ is not \aleph_0 -monolithic, K cannot be \aleph_0 -monolithic, so neither can $P(K)$. ■

COROLLARY 2.6. *Under Martin's axiom $\text{MA}(\omega_1)$, the following are equivalent for a compact space K :*

- (i) $P(K)$ is \aleph_0 -monolithic;
- (ii) K is \aleph_0 -monolithic;
- (iii) the support of every $\mu \in P(K)$ is metrizable.

Proof. (i) \Rightarrow (ii) and (iii) \Rightarrow (i) hold by Lemma 2.4.

To verify (ii) \Rightarrow (iii) recall that the support of $\mu \in P(K)$ is ccc and apply Theorem 2.2. Alternatively, we can use Theorem 2.3 again. ■

REMARK 2.7. It seemed natural to recall Fremlin's result in our context. Let us remark, however, that Proposition 2.5(c) could also be derived from Theorem 2.2 alone: if $S \subseteq K$ is the support of $\mu \in P(K)$ then one can check that $P(S)$ is also a ccc subspace of $P(K)$ and it follows that $P(S)$ is \aleph_0 -monolithic.

The implication (ii) \Rightarrow (i) of Corollary 2.6 is not provable in the usual set theory. Kunen [14] constructed under CH a nonseparable compact space K which is Corson compact (hence \aleph_0 -monolithic) and such that K supports a measure $\mu \in P(K)$ of countable type (see [14, remark on p. 287]). Then $P(K)$ is separable but nonmetrizable, so $P(K)$ is not \aleph_0 -monolithic. In fact, it can be derived from a result due to Talagrand [19] that under CH there is a Corson compact space K such that $P(K)$ contains a copy of $\beta\omega$, so the monolithicity of $P(K)$ is dramatically violated. The status of (i) \Rightarrow (iii) of Corollary 2.6 seemed to be unclear; to state this explicitly we arrive at the following question.

PROBLEM 2.8. *Can one prove in ZFC that if $P(K)$ is \aleph_0 -monolithic then the support of every $\mu \in P(K)$ is metrizable?*

Problem 2.8 was communicated to us a couple of years ago by Wiesław Kubiś in connection with [13] and [9]. More recently, the same question was asked by Claudia Correa who noted that a positive answer would provide a handy characterization of those compact K for which $P(K)$ is \aleph_0 -monolithic (see [5]). We shall show, however, that the answer is negative.

THEOREM 2.9. *Under \diamond , there is a nonmetrizable Corson compact space K such that $P(K)$ is \aleph_0 -monolithic but K supports a measure of type ω_1 .*

The construction that is behind our main result is a variant of Kunen's construction from [14] done in the spirit of [17]. We should recall that Kunen's primary construction from [14] gave a K supporting a measure of uncountable type. However, it seems that one needs to add a number of new ingredients to the inductive process to guarantee that $P(K)$ is indeed \aleph_0 -monolithic. Moreover, our construction requires \diamond and we do not know if 2.9 follows from CH. It is worth recalling that Kunen's construction was also used by Brandsma and van Mill [4] to give an example of a compact HL space with a nonmonolithic hyperspace.

Recall finally that for a Corson compact space K , the space $P(K)$ is Corson compact if and only if the support of every $\mu \in P(K)$ is metrizable (see [1]). Hence, the space $P(K)$ announced in Theorem 2.9 is \aleph_0 -monolithic but not Corson compact.

3. Measures on some Boolean algebras. In this section we discuss properties of finitely additive measures on Boolean algebras. If \mathcal{G} is a subset of a Boolean algebra \mathfrak{A} then $[\mathcal{G}]$ denotes the smallest subalgebra of \mathfrak{A} containing \mathcal{G} .

Let us fix a Boolean algebra \mathfrak{A} for a while; we denote by $P(\mathfrak{A})$ the space of all finitely additive probability measures on \mathfrak{A} . If we consider $K = \text{ult}(\mathfrak{A})$, the Stone space of \mathfrak{A} , then we can speak of $P(\mathfrak{A})$ rather than of $P(K)$. Indeed, every measure on K is uniquely determined by its restriction to the algebra $\text{Clop}(K)$ of clopen subsets of K , which is isomorphic to \mathfrak{A} . In the other direction, every $\mu \in P(\mathfrak{A})$ uniquely defines the measure $\widehat{\mu} \in P(K)$, where $\widehat{\mu}(\widehat{a}) = \mu(a)$ for $a \in \mathfrak{A}$. Here $a \mapsto \widehat{a}$ denotes the Stone isomorphism between \mathfrak{A} and $\text{Clop}(K)$. Then the weak* topology on $P(K)$ becomes the topology on $P(\mathfrak{A})$ of convergence on elements of \mathfrak{A} .

LEMMA 3.1. *Given an algebra \mathfrak{A} , the space $P(\mathfrak{A})$ is monolithic if and only if for every countable set $E \subseteq P(\mathfrak{A})$ there is a countable subalgebra \mathfrak{A}_0 of \mathfrak{A} such that every sequence of $\mu_n \in E$ converging on \mathfrak{A}_0 converges also on \mathfrak{A} .*

Proof. It is easy to see that the condition is necessary. For the sufficiency, note that if \mathfrak{A}_0 is such a test subalgebra for a set E then elements of \mathfrak{A}_0 separate the set of measures \overline{E} .

Indeed, take $\mu, \nu \in \overline{E}$ that agree on \mathfrak{A}_0 and consider any $b \in \mathfrak{A}$. Since \mathfrak{A}_0 is countable, there are $\mu_n, \nu_n \in E$ such that $\mu_n \rightarrow \mu$ on $[\mathfrak{A}_0 \cup \{b\}]$ and $\nu_n \rightarrow \nu$ on $[\mathfrak{A}_0 \cup \{b\}]$. Then the sequence $\mu_1, \nu_1, \mu_2, \nu_2, \dots$ converges to $\mu|_{\mathfrak{A}_0} = \nu|_{\mathfrak{A}_0}$ on \mathfrak{A}_0 , so it converges on \mathfrak{A} ; in particular, $\mu(b) = \nu(b)$. ■

Denote by λ the usual product measure on the Cantor cube 2^{ω_1} defined on the product σ -algebra $\text{Ba}(\omega_1)$ (of Baire subsets of 2^{ω_1}). The algebra $\text{Ba}(\omega_1)$ is σ -generated by the algebra $\text{Clop}(\omega_1)$ of clopen sets. It will be convenient to use the following notation.

NOTATION 3.2. Given a subalgebra \mathfrak{A} of $\text{Ba}(\omega_1)$ and $I \subseteq \omega_1$, we write $\mathfrak{A}(I)$ for the family of those $A \in \mathfrak{A}$ which are determined by coordinates in the set I .

In particular, if $\alpha < \omega_1$ then $\text{Ba}(\alpha)$ is the family of all Baire sets determined by coordinates in $\{\beta : \beta < \alpha\}$. Accordingly, $\text{Clop}(\alpha)$ is the family of closed-and-open subsets of 2^{ω_1} that are determined by coordinates below α .

We shall frequently use the fact that λ is a product measure: if $A \in \text{Ba}(I)$ and $B \in \text{Ba}(J)$, where $I \cap J = \emptyset$, then $\lambda(A \cap B) = \lambda(A) \cdot \lambda(B)$.

We collect below some preliminary facts concerning measures on $\text{Ba}(\omega_1)$ and on its subalgebras. Recall that, given two finitely additive measures μ and λ defined on a Boolean algebra \mathfrak{A} , we say that μ is *absolutely continuous* with respect to λ ($\mu \ll \lambda$) if for every $\varepsilon > 0$ there is $\delta > 0$ such that for all $a \in \mathfrak{A}$, if $\lambda(a) < \delta$ then $\mu(a) < \varepsilon$.

The first lemma just rephrases the classical Radon–Nikodym theorem.

LEMMA 3.3. *Let \mathfrak{A} be any subalgebra of $\text{Ba}(\omega_1)$. If μ is a finitely additive finite measure on \mathfrak{A} which is absolutely continuous with respect to λ on \mathfrak{A} then there is a function $g : 2^{\omega_1} \rightarrow \mathbb{R}$, measurable with respect to the σ -algebra generated by \mathfrak{A} , such that $\mu(A) = \int_A g \, d\lambda$ for every $A \in \mathfrak{A}$.*

LEMMA 3.4. *Suppose that for $\beta < \alpha$ and algebras $\mathfrak{B}, \mathfrak{F} \subseteq \text{Ba}(\omega_1)$ we have $\mathfrak{B}(\alpha) \subseteq [\mathfrak{B}(\beta) \cup \mathfrak{F}(\omega_1 \setminus \alpha)]$. Then every $B \in \mathfrak{B}(\alpha)$ is a finite disjoint union of sets of the form $B_1 \cap B_2$, where $B_1 \in \mathfrak{B}(\beta)$ and $B_2 \in \mathfrak{F}(\omega_1 \setminus \alpha)$.*

Proof. This is standard: it is enough to check that every set from the algebra $[\mathfrak{B}(\beta) \cup \mathfrak{F}(\omega_1 \setminus \alpha)]$ has the required property. ■

LEMMA 3.5. *Let $\mathfrak{B} \subseteq \mathfrak{F}$ be subalgebras of $\text{Ba}(\omega_1)$ such that for some cofinal set $S \subseteq \omega_1$ the following are satisfied:*

- (i) $\mathfrak{B}(\alpha)$ is countable for every $\alpha \in S$;
- (ii) if $\beta, \alpha \in S$ then $\mathfrak{B}(\alpha) \subseteq [\mathfrak{B}(\beta) \cup \mathfrak{F}(\alpha \setminus \beta)]$.

Suppose that, for every n , $\mu_n \in P(\mathfrak{F})$ is such a measure that $\mu_n|_{\mathfrak{F}(\omega_1 \setminus \xi_n)}$ is absolutely continuous with respect to λ for some $\xi_n \in S$. Then the closure of $\{\mu_n|_{\mathfrak{B}} : n < \omega\}$ in $P(\mathfrak{B})$ is metrizable.

Proof. We consider first a single measure $\mu \in P(\mathfrak{F})$ that is absolutely continuous with respect to λ on $\mathfrak{F}(\omega_1 \setminus \xi)$ for some $\xi \in S$.

CLAIM. *There is $\xi < \eta_0 < \omega_1$ such that if $\eta \in S \setminus \eta_0$, $A \in \mathfrak{B}(\xi)$, $B \in \mathfrak{F}(\eta \setminus \xi)$, $C \in \mathfrak{F}(\omega_1 \setminus \eta)$ then $\mu(A \cap B \cap C) = \mu(A \cap B) \cdot \lambda(C)$.*

Consequently, $\mu(G \cap C) = \mu(G) \cdot \lambda(C)$ for $G \in \mathfrak{B}(\eta)$, $C \in \mathfrak{F}(\omega_1 \setminus \eta)$.

To check the claim we take any $A \in \mathfrak{B}(\xi)$ and apply Lemma 3.3 to the measure

$$\mathfrak{F}(\omega_1 \setminus \xi) \ni B \mapsto \mu(A \cap B).$$

By Lemma 3.3 there is a $\text{Ba}(\omega_1 \setminus \xi)$ -measurable function $g_A : 2^{\omega_1} \rightarrow \mathbb{R}$ such that $\mu(A \cap B) = \int_B g_A \, d\lambda$ for every $B \in \mathfrak{F}(\omega_1 \setminus \xi)$. Since every $\text{Ba}(\omega_1)$ -measurable function is determined by countably many coordinates (and since $\mathfrak{B}(\xi)$ is countable), there is $\eta_0 \in S$ such that $\xi < \eta_0 < \omega_1$ and g_A is $\text{Ba}(\eta_0 \setminus \xi)$ -measurable for every $A \in \mathfrak{B}(\xi)$. If we now take any $\eta \geq \eta_0$, $A \in \mathfrak{B}(\xi)$,

$B \in \mathfrak{F}(\eta \setminus \xi)$, $C \in \mathfrak{F}(\omega_1 \setminus \eta)$ then

$$\mu(A \cap B \cap C) = \int_{B \cap C} g_A \, d\lambda = \lambda(C) \cdot \mu(A \cap B),$$

by stochastic independence. This proves the first statement. The second one follows from Lemma 3.4 and (ii)—such a set G is a finite union of sets of the form $A \cap B$ where $A \in \mathfrak{B}(\xi)$ and $B \in \mathfrak{F}(\eta \setminus \xi)$.

Coming back to a sequence of measures μ_n as in the assumption, it follows from the Claim that there is a single $\eta < \omega_1$ such that for every n we have

$$(*) \quad \mu_n(G \cap C) = \mu_n(G) \cdot \lambda(C) \quad \text{whenever } G \in \mathfrak{B}(\eta), C \in \mathfrak{F}(\omega_1 \setminus \eta).$$

To conclude the argument, in view of Lemma 3.1 it suffices to check that any subsequence of μ_n 's converges on \mathfrak{B} whenever it converges on the countable algebra $\mathfrak{B}(\eta)$. This follows from $(*)$ and the fact that every $H \in \mathfrak{B}$ belongs to some $\mathfrak{B}(\alpha)$ for α large enough, so H is a finite union of intersections $G \cap C$ as in $(*)$. ■

4. Construction. Write $\text{Lim}(\omega_1)$ for the set of all limit ordinals in ω_1 . Recall that Jensen's diamond principle \diamond declares the existence of a sequence $\langle S_\alpha : \alpha < \omega_1 \rangle$ with $S_\alpha \subseteq \alpha$ such that the set $\{\alpha < \omega_1 : X \cap \alpha = S_\alpha\}$ is stationary for every $X \subseteq \omega_1$ ([15, §7] or [12, p. 191]).

We shall use \diamond in the following form.

LEMMA 4.1. *Under \diamond , there is a sequence $\langle \nu_\alpha : \alpha \in \text{Lim}(\omega_1) \rangle$ of finitely additive measures on $\text{Ba}(\omega_1)$ such that for every continuous increasing sequence $\langle \mathfrak{F}_\alpha : \alpha < \omega_1 \rangle$ of countable subalgebras of $\text{Ba}(\omega_1)$ and for every $\mu \in P(\text{Ba}(\omega_1))$ the set*

$$\{\alpha \in \text{Lim}(\omega_1) : \mu|_{\mathfrak{F}_\alpha} = \nu_\alpha|_{\mathfrak{F}_\alpha}\}$$

is stationary.

Proof. Since \diamond implies CH and $|\text{Ba}(\omega_1)| = \mathfrak{c}$, we can write $\text{Ba}(\omega_1)$ as a union $\bigcup_{\alpha < \omega_1} \mathfrak{B}_\alpha$ of a continuous increasing chain of countable algebras. By the standard coding using \diamond , we can find $\langle \nu_\alpha : \alpha \in \text{Lim}(\omega_1) \rangle$ such that for every $\mu \in P(\text{Ba}(\omega_1))$ the set

$$S = \{\alpha \in \text{Lim}(\omega_1) : \mu|_{\mathfrak{B}_\alpha} = \nu_\alpha|_{\mathfrak{B}_\alpha}\}$$

is stationary (compare [15, Exercise 51]).

Consider any continuous increasing chain $\langle \mathfrak{F}_\alpha : \alpha < \omega_1 \rangle$ of countable algebras with union \mathfrak{F} . It is easy to check that the set

$$T = \{\alpha \in \text{Lim}(\omega_1) : \mathfrak{F}_\alpha = \mathfrak{B}_\alpha \cap \mathfrak{F}\}$$

is closed and unbounded in ω_1 , so $S \cap T$ is stationary, and we are done. ■

Below we define an increasing chain $\langle \mathfrak{A}_\alpha : \alpha \in \text{Lim}(\omega_1) \rangle$ of countable algebras $\mathfrak{A}_\alpha \subseteq \text{Ba}(\alpha)$ so that

$$\mathfrak{A} = \bigcup_{\alpha \in \text{Lim}(\omega_1)} \mathfrak{A}_\alpha$$

will be the Boolean algebra we are looking for. At each step α we choose a countable family \mathcal{G}_α (of new generators) and define \mathfrak{A}_α to be the algebra generated by $\bigcup_{\beta < \alpha} \mathfrak{A}_\beta \cup \mathcal{G}_\alpha$. We also use the auxiliary algebras

$$\mathfrak{B}_\alpha = [\mathfrak{A}_\alpha \cup \text{Clop}(\alpha)]$$

(to which we shall apply Lemma 3.5).

To state the inductive assumptions we need another piece of notation. Given any $G \subseteq 2^{\omega_1}$ and $\alpha < \omega_1$ we write

$$\text{cyl}_\alpha(G) = \pi_\alpha^{-1} [\pi_\alpha[G]],$$

where $\pi_\alpha : 2^{\omega_1} \rightarrow 2^\alpha$ is the usual projection. Note that for $\beta < \alpha$, if $G = A \cap B$ where $A \in \text{Ba}(\alpha)$ and $\emptyset \neq B \in \text{Ba}(\alpha \setminus \beta)$ then $\text{cyl}_\beta(G) = A$.

Here is the list of requirements (in what follows, $\alpha, \beta, \dots \in \text{Lim}(\omega_1)$ and i, j, k, n are natural numbers):

- R(1) $\lambda(A) > 0$ for every nonempty $A \in \mathfrak{A}_\alpha$;
- R(2) every \mathcal{G}_α is enumerated as $\mathcal{G}_\alpha = \{G(\alpha, n) : n < \omega\}$, where $G(\alpha, n) \subseteq G(\alpha, n+1)$ for every n and $\lim_n \lambda(G(\alpha, n)) = 1$;
- R(3) if $\beta < \alpha$ then for every i there is j such that $G(\alpha, i) \subseteq G(\beta, j)$;
- R(4) if $\beta < \alpha < \omega_1$ then for almost all i the set $G(\alpha, i)$ is of the form $G(\alpha, i) = A \cap B$ with $A \in \mathfrak{A}_\beta$, $B \in \text{Ba}(\alpha \setminus \beta)$;
- R(5) if $\beta < \beta' < \alpha$ then for every i ,

$$\text{cyl}_\beta(G(\beta', i)) \subseteq \text{cyl}_\beta(G(\alpha, j)) \quad \text{for almost all } j;$$

- R(6) if $\beta < \alpha$ then $\mathfrak{B}_\alpha \subseteq [\mathfrak{B}_\beta \cup \text{Ba}(\alpha \setminus \beta)]$.

We now fix the guessing sequence $\langle \nu_\alpha : \alpha \in \text{Lim}(\omega_1) \rangle$ of Lemma 4.1. Our construction is modelled by those measures ν_α considered on the continuous increasing chain of algebras $\langle \mathfrak{F}_\alpha : \alpha \in \text{Lim}(\omega_1) \rangle$ that we define along the construction. The role of \mathfrak{F}_ξ (for $\xi \in \text{Lim}(\omega_1)$) is to remember what happened below ξ : we assume that \mathfrak{F}_ξ contains $\bigcup_{\alpha < \xi} \mathfrak{B}_\alpha$ and all the witnesses for R(6) below ξ so that

$$\mathfrak{B}_\alpha \subseteq [\mathfrak{B}_\beta \cup \mathfrak{F}_\xi(\alpha \setminus \beta)] \quad \text{for } \beta < \alpha < \xi.$$

We also put $\mathfrak{F} = \bigcup_{\xi \in \text{Lim}(\omega_1)} \mathfrak{F}_\xi$.

We start by examining our freedom at the limit step of the construction.

LEMMA 4.2. *Suppose that we are given $\mathfrak{A}_\beta, \mathcal{G}_\beta$ for $\beta < \alpha$ satisfying (an appropriate part of) R(1)–R(6). Suppose also that $\alpha = \sup_n \beta_n$ for an increasing sequence of $\beta_n \in \text{Lim}(\omega_1)$. Then there is a function $\varphi : \omega \rightarrow \omega$ such*

that the sets

$$G^\varphi(i) = \bigcap_{n \geq i} G(\beta_n, \varphi(n))$$

and the corresponding algebras satisfy R(1)–R(6) (with $G(\alpha, i) = G^\varphi(i)$).

Proof. Using R(2) we can define $\varphi_0 : \omega \rightarrow \omega$ so that $\lambda(G(\beta_n, \varphi_0(n))) > 1 - 1/2^{n+2}$ for all n . Then, by elementary calculations, $\lambda(G^\varphi(i)) > 1 - 1/2^{i+1}$ for every function $\varphi \geq \varphi_0$, so the sets $G^\varphi(i)$ satisfy R(2). Note that in particular $G^\varphi(0) \neq \emptyset$.

Fix a bijection $g : \alpha \rightarrow \omega$; we inductively define a function $\varphi : \omega \rightarrow \omega$ so that $\varphi \geq \varphi_0$ and the following are satisfied:

(a) if $g(\beta) < n$, $\beta < \beta_n$ then

$$G(\beta_n, \varphi(n)) = A_n \cap B_n, \quad \text{where } A_n \in \mathfrak{A}_\beta, B_n \in \text{Ba}(\beta_n \setminus \beta);$$

(b) if $g(\beta) < n$, $\beta < \beta_k < \beta_n$ then

$$\text{cyl}_\beta(G(\beta_n, \varphi(n))) \supseteq \text{cyl}_\beta(G(\beta_k, \varphi(k)));$$

(c) if $\beta < \beta'$, $g(\beta) < n$, $g(\beta') < n$, $k < n$ then

$$\text{cyl}_\beta(G(\beta_n, \varphi(n))) \supseteq \text{cyl}_\beta(G(\beta', k)).$$

Note that such a φ can be defined by inductive assumptions since (a)–(c) require fulfilling only a finite number of conditions at each step. We first check that the sets $G^\varphi(i)$ satisfy R(2)–R(6). Note that R(2) follows from $\varphi \geq \varphi_0$.

Given $\beta < \alpha$ and $i < \omega$, we have $\beta < \beta_n < \alpha$ for some $n > i$; then by R(3) there is j such that $G(\beta_n, \varphi(n)) \subseteq G(\beta, j)$; hence $G^\varphi(i) \subseteq G(\beta_n, \varphi(n)) \subseteq G(\beta, j)$; this shows that R(3) is preserved.

To check that the sets $G^\varphi(i)$ satisfy R(4) fix $\beta < \alpha$. If we take any i with $\beta < \beta_i$ and $g(\beta) < i$ then (using the notation as in (a))

$$G^\varphi(i) = \bigcap_{n \geq i} G(\beta_n, \varphi(n)) = \bigcap_{n \geq i} A_n \cap B_n = A_i \cap \bigcap_{n \geq i} B_n,$$

since $A_n \supseteq A_i$ for $n \geq i$ by (b). The formula above shows that $G^\varphi(i)$ has the required form for almost all i .

We check R(5) and R(6) in a similar manner: for instance, given $\beta < \beta' < \alpha$ and any i , $\text{cyl}_\beta(G^\varphi(j)) \supseteq \text{cyl}_\beta(G(\beta', i))$ for almost all j by (c).

To treat R(1) note first that we can additionally demand that the function φ satisfy $\lambda(G^\varphi(i+1) \setminus G^\varphi(i)) > 0$ for every i . Then R(1) follows easily from the following observation.

Suppose that λ is strictly positive on some \mathfrak{A}_β . Let $G = A \cap B$, where $A \in \mathfrak{A}_\beta$, $B \in \text{Ba}(\alpha \setminus \beta)$ and $\lambda(A), \lambda(B) > 0$. Then λ is strictly positive on the algebra generated by \mathfrak{A}_β and G . ■

REMARK 4.3. Lemma 4.2 is stated in the form presenting the main idea of a diagonal argument. However, what we really need to know at the limit step of the construction in 4.4 is slightly more complicated. Suppose that in the setting of 4.2 we are additionally given a sequence of Baire sets $C_n \in \text{Ba}(\beta_{n+1} \setminus \beta_n)$ (with $\lambda(C_n)$ growing fast to 1). Then the above proof shows, after minor changes, that the sets

$$G^\varphi(i) = \bigcap_{n \geq i} C_n \cap G(\beta_n, \varphi(n))$$

also satisfy the assertion of 4.2 for some φ .

CONSTRUCTION 4.4. Suppose that the construction has been done for $\beta \leq \alpha'$ and consider the next limit ordinal $\alpha = \alpha' + \omega$. We use this step simply to add new sets to $\mathfrak{A}_{\alpha'}$: Choose any strictly increasing sequence of $C_n \in \text{Clop}(\alpha \setminus \alpha')$ such that $\lim_n \lambda(C_n) = 1$. Define $G(\alpha, n) = G(\alpha', n) \cap C_n$ for $n < \omega$ and set

$$\mathcal{G}_\alpha = \{G(\alpha, n) : n < \omega\}.$$

Checking that R(1)–R(6) are preserved is fairly standard; for future reference note the following.

REMARK 4.5. There is $B \in \mathfrak{A}_\alpha$ such that $\inf\{\lambda(A \triangle B) : A \in \mathfrak{A}_{\alpha'}\} > 0$. Indeed, this holds whenever we take $B = G(\alpha, n) = G(\alpha', n) \cap C_n$ with $\lambda(C_n) > 0$ since C_n is independent from all elements in \mathfrak{A}' .

Suppose now that the construction has been done below α which is a limit ordinal in $\text{Lim}(\omega_1)$.

Let us say that *Limit Case* (1) happens if the following holds: there are $\varepsilon > 0$ and $\beta_n, \alpha_n \in \text{Lim}(\omega_1)$, where

$$\beta_0 < \alpha_0 \leq \beta_1 < \alpha_2 \leq \cdots < \alpha,$$

and there are $C_n \in \mathfrak{F}_{\alpha_n}(\alpha_n \setminus \beta_n)$ such that $\lim_n \lambda(C_n) = 1$ while $\nu_\alpha(C_n) \leq 1 - \varepsilon$ for every n .

When *Limit Case* (1) happens, we proceed as follows.

Note first that, passing to a subsequence if necessary, we can assume that $\lambda(C_n) > 1 - 1/2^{n+2}$ for every n . Then we put

$$G(\alpha, i) = \bigcap_{n \geq i} C_n \cap G(\beta_n, \varphi(n)),$$

where φ is chosen as in Lemma 4.2, which guarantees that the new generators satisfy R(1)–R(6). Note that the appearance of C_n 's in the formula above (those sets were not mentioned in 4.2) does not change much since every C_n is determined by coordinates in $\alpha_n \setminus \beta_n$ (see Remark 4.3).

We say that *Limit Case* (2) happens in the remaining case. We then perform the previous construction in a simpler form, taking above $C_n = 2^{\omega_1}$ for every n .

5. Analyzing the resulting Stone space. We shall now prove Theorem 2.9 by analyzing the Stone space $K = \text{ult}(\mathfrak{A})$ of the Boolean algebra \mathfrak{A} constructed in the previous section.

LEMMA 5.1. *For every $\alpha \in \text{Lim}(\omega_1)$ the set*

$$M_\alpha = K \setminus \bigcup_n \widehat{G(\alpha, n)}$$

is a metrizable subspace of K .

Proof. Indeed, if $\xi > \alpha$ then for every k we have

$$G(\xi, k) \cap M_\alpha = \emptyset$$

by R(3); it follows that $\{\widehat{A} \cap M_\alpha : A \in \mathfrak{A}_\alpha\}$ is a countable base of M_α . ■

LEMMA 5.2. *The space K is a nonseparable Corson compact space supporting a strictly positive measure of type ω_1 .*

Proof. Since $\lambda(A) > 0$ for every nonempty $A \in \mathfrak{A}$, the measure $\widehat{\lambda}$, uniquely determined by the formula $\widehat{\lambda}(\widehat{A}) = \lambda(A)$ for $A \in \mathfrak{A}$, is strictly positive on K .

To see that $\widehat{\lambda}$ is a measure of uncountable type, note that by Remark 4.5, for every α there is $B \in \mathfrak{A}$ such that

$$\inf\{\lambda(A \triangle B) : A \in \mathfrak{A}_\alpha\} > 0,$$

so no countable subfamily of \mathfrak{A} can be Δ -dense in \mathfrak{A} with respect to λ . In particular, K is not metrizable, as it carries a measure of uncountable type.

In order to check that $K = \text{ult}(\mathfrak{A})$ is Corson compact it suffices to find a family $\mathcal{G} \subseteq \mathfrak{A}$ such that $\mathfrak{A} = [\mathcal{G}]$, and having the property that every centered $\mathcal{G}_0 \subseteq \mathcal{G}$ is countable. Indeed, in that case we have an embedding

$$\Phi : \text{ult}(\mathfrak{A}) \ni x \mapsto \langle \chi_{\widehat{G}}(x) : G \in \mathcal{G} \rangle$$

into $\Sigma(2^{\mathcal{G}})$. Here $\chi_{\widehat{G}}$ denotes the characteristic function of the clopen set $\widehat{G} \subseteq K$, so Φ is clearly continuous; the injectivity of Φ follows from the fact that \mathcal{G} generates \mathfrak{A} .

In our case we take

$$\mathcal{G} = \{G(\alpha, n) : \alpha \in \text{Lim}(\omega_1), n < \omega\}.$$

If $\mathcal{G}_0 \subseteq \mathcal{G}$ is centered then there is a 0-1 measure μ on $\text{Ba}(\omega_1)$ such that $\mu(G) = 1$ for $G \in \mathcal{G}_0$. Then $\mu|_{\mathfrak{F}_\alpha}$ was guessed at some limit step α by ν_α . Then, necessarily, Limit Case (1) happened (recall that, in particular, \mathfrak{F}_α contains $\text{Clop}(\alpha)$), so $\nu_\alpha(G(\alpha, n)) < 1$ and thus $\nu_\alpha(G(\alpha, n)) = 0$ for every n . By R(3), $G(\beta, n) \notin \mathcal{G}_0$ whenever $\beta \geq \alpha$ and $n \in \omega$, so \mathcal{G}_0 is indeed countable.

Finally, K is nonseparable since every separable Corson compactum is metrizable. ■

LEMMA 5.3. *Let $\mu \in P(\text{Ba}(\omega_1))$ be a measure such that $\widehat{\mu|_{\mathfrak{A}}}$ defines a regular Borel measure in $P(K)$ vanishing on all closed metrizable subsets of K . Then $\mu|_{\mathfrak{F}(\omega_1 \setminus \alpha_0)}$ is absolutely continuous with respect to λ for some $\alpha_0 < \omega_1$.*

Proof. Recall that the measures ν_α often guess all the other measures on the algebras \mathfrak{F}_α . Hence, by Lemma 4.1 the set S of those $\alpha \in \text{Lim}(\omega_1)$ for which μ agrees with ν_α on \mathfrak{F}_α is stationary.

CLAIM. *Limit Case (1) happened for no $\alpha \in S$.*

Suppose otherwise, that is, (1) occurred for some $\alpha \in S$ and $\varepsilon > 0$. Then, in the notation of the construction, $\mu(G(\alpha, n)) \leq 1 - \varepsilon$ for every n . In other words, if we take

$$M_\alpha = K \setminus \bigcup_n \widehat{G(\alpha, n)},$$

then $\widehat{\mu}(M_\alpha) \geq \varepsilon$. To arrive at a contradiction, it is sufficient to note that M_α is metrizable by Lemma 5.1.

Fix some $\varepsilon > 0$. We know from the Claim that

$$\begin{aligned} (\forall \alpha \in S)(\exists \xi^\varepsilon(\alpha) < \alpha)(\forall \xi^\varepsilon(\alpha) < \beta < \alpha)(\exists n)(\forall A \in \mathfrak{F}_\beta(\beta \setminus \xi^\varepsilon(\alpha))) \\ \lambda(A) < 1/n \Rightarrow \mu(A) < \varepsilon. \end{aligned}$$

By the pressing down lemma, there is $\alpha_0^\varepsilon < \omega_1$ such that $\xi^\varepsilon(\alpha) \leq \alpha_0^\varepsilon$ for stationary many $\alpha \in S$. Repeating this argument for every $\varepsilon = 1/k$, we conclude that there is $\alpha_0 < \omega_1$ such that, for all $\varepsilon > 0$, $\xi^\varepsilon(\alpha) \leq \alpha_0$ for α from some stationary set $T_\varepsilon \subseteq S$.

It follows that μ is absolutely continuous with respect to λ on $\mathfrak{F}_\beta(\beta \setminus \alpha_0)$ for every $\beta > \alpha_0$. Hence, $\mu \ll \lambda$ on $\mathfrak{F}(\omega_1 \setminus \alpha_0)$, as required. ■

We are finally ready to verify the main point.

THEOREM 5.4. *The space $P(K)$ is monolithic.*

Proof. Given any measure μ in $P(K)$, consider

$$c = \sup\{\mu(L) : L \subseteq K, L \text{ is closed and metrizable}\}.$$

Take a sequence of closed metrizable subspaces L_n with $\mu(L_n) > c - 1/n$. Then $L = \overline{\bigcup_n L_n}$ is again metrizable because, by Lemma 5.2, K is Corson compact, hence monolithic. We have $\mu(L) = c$, so $\mu(M) = 0$ for every closed metrizable $M \subseteq K \setminus L$.

It follows that every measure from $P(K)$ is a convex combination of a measure concentrated on a metrizable subspace of K and a measure vanishing on all closed metrizable subspaces of K . Therefore, it is sufficient to check that the closure of $\{\mu_n : n < \omega\} \subseteq P(K)$ in $P(K)$ is metrizable whenever every μ_n vanishes on metrizable subsets of K . This is a direct consequence of Lemmas 5.3 and 3.5. ■

Acknowledgements. I wish to thank Witold Marciszewski for our discussion on \aleph_0 -monolithic spaces. I am indebted to the referee for his/her careful reading and several comments that enabled me to clarify the main idea and improve the presentation.

The research has been supported by the grant 2018/29/B/ST1/00223 from National Science Centre, Poland.

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