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Arc-meromorphous functions

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Abstract. We introduce arc-meromorphous functions, which are continuous functions representable as quotients of semialgebraic arc-analytic functions, and develop the theory of arc-meromorphous sheaves on Nash manifolds. Our main results are Cartan's theorems A and B for quasi-coherent arc-meromorphous sheaves.

0. Introduction. In this note, building on the theory of *arc-analytic functions* initiated by the second named author [16], we introduce *arc-meromorphous functions* and *arc-meromorphous sheaves* on Nash manifolds. Arc-meromorphous functions are analogs for *regulous* and *Nash regulous functions* studied in [8] and [13], respectively. The term "regulous" is derived from "regular" and "continuous", whereas "meromorphous" comes from "meromorphic" and "continuous". Our theory of arc-meromorphous sheaves is developed in parallel to the theories of regulous sheaves [8] (see also the recent survey [14]) and Nash regulous sheaves [13]. It is established in [8] and [13] that Cartan's theorems A and B hold for quasi-coherent regulous sheaves and quasi-coherent Nash regulous sheaves. Our main results are Theorem 2.4 (Cartan's theorem A) and Theorem 2.5 (Cartan's theorem B) for quasi-coherent arc-meromorphous sheaves. Recall that Cartan's theorems A and B fail for coherent real algebraic sheaves [6, Example 12.1.5], [7, Theorem 1] and coherent Nash sheaves [11].

We refer to [6] for the general theory of semialgebraic sets, semialgebraic functions, and related concepts. Recall that a *Nash manifold* is an analytic submanifold $X \subset \mathbb{R}^n$, for some n, which is also a semialgebraic set. A real-valued function on X is called a *Nash function* if it is both analytic and semialgebraic. By [22, Theorem VI.2.1, Remark VI.2.11], each Nash manifold is Nash isomorphic to a nonsingular algebraic set in $\mathbb{R}^m$, for some m.

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A function $f\colon X \to \mathbb{R}$ is said to be *arc-analytic* if for every analytic arc $\gamma\colon (-1,1) \to X$ the composite $f \circ \gamma$ is an analytic function. Without additional assumptions, arc-analyticity is a fairly weak condition. For example, an arc-analytic function on $\mathbb{R}^n$ need not be continuous [5] or subanalytic [17], and its nonanalyticity locus can be nondiscrete even if $n = 2$ [18]. However, arc-analytic functions that are also semialgebraic proved to be very useful in real algebraic and analytic geometry (see [1, 2, 4, 8, 12, 13, 15, 19, 20, 21] and the references therein). By [16, Proposition 5.1], every semialgebraic arc-analytic function is continuous. A standard example of a semialgebraic arc-analytic function is $f\colon \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \sqrt{x^4+y^4}$; this function is neither regulous nor Nash regulous.

Unless explicitly stated otherwise, we consider Nash manifolds and their subsets endowed with the Euclidean topology induced by the usual metric on $\mathbb{R}$. However, in the subsequent sections the so called $\mathcal{AR}$ topology, defined in [16], plays the key role.

1. From arc-analytic to arc-meromorphous functions. Let X be a Nash manifold in $\mathbb{R}^n$. The collection $\mathcal{A}(X)$ of all semialgebraic arc-analytic functions from X into $\mathbb{R}$ forms a commutative ring with identity. As demonstrated in [1, 2, 16], the ring $\mathcal{A}(X)$ has remarkable algebraic and geometric properties (in the cited papers the ring in question is denoted by $\mathcal{A}_a(X)$). We need some preparation to continue the study of $\mathcal{A}(X)$.

Given a family F of real-valued functions on X, we set

$$Z(F) := \{x \in X\colon f(x) = 0 \text{ for all } f \in F\}$$

and write $Z(f)$ if $F = \{f\}$.

A subset Z of X is said to be *arc-symmetric* if for every analytic arc $\gamma\colon (-1,1) \to X$ with $\gamma((-1,0)) \subseteq Z$ we have $\gamma((-1,1)) \subseteq Z$. By [16, Théorème 1.4], the semialgebraic arc-symmetric sets in X are precisely the closed sets of a certain Noetherian topology on X, called the $\mathcal{AR}$ *topology* (a topology is said to be *Noetherian* if every descending chain of closed sets is stationary). Thus a subset $Z \subseteq X$ is $\mathcal{AR}$-closed if and only if it is semialgebraic and arc-symmetric. The zero locus $Z(F)$ of any family $F \subseteq \mathcal{A}(X)$ is an $\mathcal{AR}$-closed set by [16, Proposition 5.1(ii)]. In view of [1, Theorem 1], each $\mathcal{AR}$-closed set $Z \subseteq X$ is of the form $Z = Z(f)$ for some $f \in \mathcal{A}(X)$. To be precise, [16] and [1] deal with $X = \mathbb{R}^n$. However, the general case follows immediately. Indeed, choose a Nash embedding $e\colon X \to \mathbb{R}^p$, for some p, such that its image $e(X)$ is a closed subset of $\mathbb{R}^p$ in the Euclidean topology. Clearly, $e(X)$ is a semialgebraic set that is arc-symmetric in $\mathbb{R}^p$. It suffices to note that a subset Z of X is semialgebraic and arc-symmetric in X if and only if the subset $e(Z)$ of $\mathbb{R}^p$ is semialgebraic and arc-symmetric in $\mathbb{R}^p$. Moreover, if X is closed in $\mathbb{R}^n$ in the Euclidean topology, then the $\mathcal{AR}$ topology on X is induced by the $\mathcal{AR}$ topology on $\mathbb{R}^n$ (take $X = \mathbb{R}^n \setminus \{0\}$ to see that "closed" cannot be dropped).

We summarize this discussion as follows.

THEOREM 1.1. *Let X be a Nash manifold. For a subset $Z \subseteq X$, the following conditions are equivalent:*

(a) *Z is $\mathcal{AR}$-closed.*
(b) *$Z = Z(f)$ for some function $f \in \mathcal{A}(X)$.*
(c) *$Z = Z(F)$ for some family $F \subseteq \mathcal{A}(X)$.* ∎

For any subset $E \subseteq X$, we denote by $J(E)$ the ideal of the ring $\mathcal{A}(X)$ comprised of all functions vanishing on E,

$$J(E) := \{f \in \mathcal{A}(X) \colon f(x) = 0 \text{ for all } x \in E\}.$$

The radical of an ideal I of $\mathcal{A}(X)$ will be denoted by $\mathrm{Rad}(I)$. In [1, Proposition 1] the following variant of the Nullstellensatz is established (see also the comments preceding Theorem 1.1).

THEOREM 1.2. *Let X be a Nash manifold and let I be an ideal of the ring $\mathcal{A}(X)$. Then*

$$J(Z(I)) = \mathrm{Rad}(I). \ \blacksquare$$

A straightforward consequence of Theorems 1.1 and 1.2 is the following.

COROLLARY 1.3. *Let X be a Nash manifold. Then the assignment*

$$X \supseteq E \mapsto J(E) \subseteq \mathcal{A}(X)$$

gives rise to one-to-one correspondences:

(i) *between the $\mathcal{AR}$-closed subsets of X and the radical ideals of $\mathcal{A}(X)$;*
(ii) *between the irreducible $\mathcal{AR}$-closed subsets of X and the prime ideals of $\mathcal{A}(X)$;*
(iii) *between the points of X and the maximal ideals of $\mathcal{A}(X)$.* ∎

For a function f in $\mathcal{A}(X)$, we let $\mathrm{Rad}(f)$ denote the radical of the principal ideal generated by f.

The following is a direct consequence of Theorem 1.2.

COROLLARY 1.4. *Let X be a Nash manifold and let f and g be functions in the ring $\mathcal{A}(X)$. Then $\mathrm{Rad}(f) = \mathrm{Rad}(g)$ if and only if $Z(f) = Z(g)$.* ∎

The radical of an arbitrary ideal can be described as follows.

COROLLARY 1.5. *Let X be a Nash manifold and let I be an ideal of the ring $\mathcal{A}(X)$. Then*

$$\mathrm{Rad}(I) = \mathrm{Rad}(f)$$

for some $f \in I$. Furthermore, the equality of radicals holds if and only if $Z(I) = Z(f)$.

Proof. By Theorem 1.1, $Z(I) = Z(f)$ for some $f \in I$. Hence the proof is complete in view of Theorem 1.2. ■

We next introduce a new concept.

DEFINITION 1.6. Let X be a Nash manifold. A function $f\colon U \to \mathbb{R}$, defined on an open subset $U \subseteq X$, is said to be *arc-meromorphous* if it is continuous and there exist two functions φ, ψ in $\mathcal{A}(X)$ such that ψ is not identically 0 on any connected component of X (hence $Z(\psi)$ is nowhere dense in X) and $f = \varphi/\psi$ on $U \setminus Z(\psi)$.

The set $\mathcal{A}^0_X(U)$ of all arc-meromorphous functions on U is a commutative ring with identity. By convention, $\mathcal{A}^0_X(\emptyset) = 0$. The notion of arc-meromorphous function corresponds to those of regulous function [8] and Nash regulous function [13]. However, the following result has no counterpart for regulous or Nash regulous functions.

THEOREM 1.7. *For any Nash manifold X, the rings $\mathcal{A}^0_X(X)$ and $\mathcal{A}(X)$ are identical.*

Proof. Clearly, $\mathcal{A}(X) \subseteq \mathcal{A}^0_X(X)$, so it remains to prove that $\mathcal{A}^0_X(X) \subseteq \mathcal{A}(X)$. We may assume that $X \subseteq \mathbb{R}^n$ is a nonsingular algebraic set. Given a function f in $\mathcal{A}^0_X(X)$, we choose two functions φ, ψ in $\mathcal{A}(X)$ such that ψ is not identically 0 on any connected component of X and $f = \varphi/\psi$ on $X \setminus Z(\psi)$. By [4, Theorem 1.1], there exists a regular map $\pi\colon X' \to X$, which is the composite of a finite sequence of blowups with nonsingular Zariski closed nowhere dense centers, such that both $\varphi \circ \pi$ and $\psi \circ \pi$ are Nash functions on X'. In view of [13, Proposition 3.1], the function $f \circ \pi$ is arc-analytic and semialgebraic. Consequently, f is an arc-analytic semialgebraic function in $\mathcal{A}(X)$ since every analytic arc $\gamma\colon (-1,1) \to X$ can be lifted to an analytic arc $\gamma'\colon (-1,1) \to X'$ with $\pi \circ \gamma' = \gamma$. ■

We are indebted to the anonymous referee for pointing out that Theorem 1.7 with $X = \mathbb{R}^n$ is already contained in [3, Lemma 2.4].

COROLLARY 1.8. *Let X be a Nash manifold and let $U \subseteq X$ be a semialgebraic open set. Then $\mathcal{A}^0_X(U) \subseteq \mathcal{A}(U)$. In other words, every function in $\mathcal{A}^0_X(U)$ is arc-analytic and semialgebraic.*

Proof. Clearly, $\mathcal{A}^0_X(U) \subseteq \mathcal{A}^0_U(U)$. By Theorem 1.7, $\mathcal{A}^0_U(U) = \mathcal{A}(U)$, which completes the proof. ■

The inclusion of rings in Corollary 1.8 cannot be replaced by the equality.

EXAMPLE 1.9. The function $f\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$, defined by $f(x) = \|x\|$ for $x \in \mathbb{R}^n \setminus \{0\}$, is a Nash function and hence it belongs to $\mathcal{A}(\mathbb{R}^n \setminus \{0\})$. Clearly, f is not in $\mathcal{A}^0_{\mathbb{R}^n}(\mathbb{R}^n \setminus \{0\})$.

PROPOSITION 1.10. *Let X be a Nash manifold, f a function in $\mathcal{A}(X)$, and g a function in $\mathcal{A}^0_X(X \setminus Z(f))$. Then, for each integer N large enough, the function $h\colon X \to \mathbb{R}$ defined by*

$$h = f^N g \quad \text{on } X \setminus Z(f) \quad \text{and} \quad h = 0 \quad \text{on } Z(f)$$

belongs to $\mathcal{A}(X)$.

Proof. The functions f and g are semialgebraic and continuous. Hence, according to [6, Proposition 2.6.4], h is continuous if N is large enough. It follows that h belongs to $\mathcal{A}^0_X(X)$. The proof is complete in view of Theorem 1.7. ∎

As usual, for f in $\mathcal{A}(X)$, we let $\mathcal{A}(X)_f$ denote the localization of the ring $\mathcal{A}(X)$ with respect to the multiplicatively closed subset

$$\{1\} \cup \{f^k \colon k = 1, 2, \ldots\}.$$

In particular, $\mathcal{A}(X)_f = 0$ if f is identically 0 on X.

PROPOSITION 1.11. *Let X be a Nash manifold and let f be a function in $\mathcal{A}(X)$. Then the restriction homomorphism $\mathcal{A}(X) \to \mathcal{A}^0_X(X \setminus Z(f))$ induces an isomorphism between the localization $\mathcal{A}(X)_f$ and $\mathcal{A}^0_X(X \setminus Z(f))$.*

Proof. By Proposition 1.10, the induced homomorphism

$$\mathcal{A}(X)_f \to \mathcal{A}^0_X(X \setminus Z(f))$$

is surjective.

If an element $g/f^k \in \mathcal{A}(X)_f$ is sent to $0 \in \mathcal{A}^0_X(X \setminus Z(f))$, then $fg = 0$ in $\mathcal{A}(X)$. Hence $g/f^k = 0$, which means that the induced homomorphism is injective. ∎

2. Arc-meromorphous sheaves. By analogy to regulous sheaves [8] and Nash regulous sheaves [13], we introduce and investigate meromorphous sheaves.

NOTATION AND CONVENION 2.1. Throughout this section, X stands for a Nash manifold. We consider X endowed with the $\mathcal{AR}$ *topology* (Section 1). Thus, a subset $Z \subseteq X$ is closed if and only if $Z = Z(f)$ for some function $f \in \mathcal{A}(X)$ (Theorem 1.1). We set $X(f) \coloneqq X \setminus Z(f)$.

For every open subset U of X, the ring $\mathcal{A}^0_X(U)$ of arc-meromorphous functions is defined (Definition 1.6). Since the $\mathcal{AR}$ topology is Noetherian, it readily follows that the assignment

$$\mathcal{A}^0_X \colon U \mapsto \mathcal{A}^0_X(U)$$

is a sheaf of rings on X. For each point $p \in X$ the stalk $\mathcal{A}^0_{X,p}$ of the sheaf $\mathcal{A}^0_X$ is a local ring; the unique maximal ideal of $\mathcal{A}^0_{X,p}$ consists of all germs vanishing at p. Thus $(X, \mathcal{A}_X^0)$ is a locally ringed space. In Section 3 we also exploit the fact that $\mathcal{A}_X^0$ is a sheaf of $\mathbb{R}$-algebras.

There is a close connection between the ringed space $(X, \mathcal{A}_X^0)$ and the affine scheme $\operatorname{Spec}(\mathcal{A}(X))$. We first describe relationships between the underlying topological spaces.

By definition, a subset $V \subseteq \operatorname{Spec}(\mathcal{A}(X))$ is closed if and only if $V = V(F)$ for some collection $F \subseteq \mathcal{A}(X)$, where

$$V(F) := \{\mathfrak{p} \in \operatorname{Spec}(\mathcal{A}(X)) \colon F \subseteq \mathfrak{p}\}.$$

If I is the ideal generated by F, then $V(F) = V(\operatorname{Rad}(I))$. Hence, according to Corollary 1.5, $V(F) = V(f)$ for some $f \in I$; here $V(f) := V(\{f\})$. Consequently, each open subset of $\operatorname{Spec}(\mathcal{A}(X))$ is of the form

$$D(f) := \operatorname{Spec}(\mathcal{A}(X)) \setminus V(f)$$

for some $f \in \mathcal{A}(X)$. In particular, each open subset of $\operatorname{Spec}(\mathcal{A}(X))$ is affine.

Define a map

$$\iota \colon X \to \operatorname{Spec}(\mathcal{A}(X))$$

by $\iota(x) = \mathfrak{m}_x$ for all $x \in X$, where

$$\mathfrak{m}_x := \{f \in \mathcal{A}(X) \colon f(x) = 0\}$$

($\mathfrak{m}_x = J(\{x\})$ in the notation of Section 1).

Proposition 2.2. *The map ι is a topological embedding of X onto the subspace* $\operatorname{Max}(\mathcal{A}(X))$ *of* $\operatorname{Spec}(\mathcal{A}(X))$ *consisting of the maximal ideals of* $\mathcal{A}(X)$. *Furthermore:*

(1) $Z(f) = \iota^{-1}(V(f))$ *for every* $f \in \mathcal{A}(X)$.
(2) *For each closed subset* $Z \subseteq X$ *there exists a unique closed subset* $\tilde{Z} \subseteq \operatorname{Spec}(\mathcal{A}(X))$ *such that* $Z = \iota^{-1}(\tilde{Z})$.
(3) $X(f) = \iota^{-1}(D(f))$ *for every* $f \in \mathcal{A}(X)$.
(4) *For each open subset* $U \subseteq X$ *there exists a unique open subset* $\tilde{U} \subseteq \operatorname{Spec}(\mathcal{A}(X))$ *such that* $U = \iota^{-1}(\tilde{U})$.

Proof. By Corollary 1.3, ι induces a bijection of X onto $\operatorname{Max}(\mathcal{A}(X))$. It follows immediately that (1) holds.

Suppose that $Z \subseteq X$ and $V \subseteq \operatorname{Spec}(\mathcal{A}(X))$ are closed subsets with $Z = \iota^{-1}(V)$. Then $Z = Z(f)$ and $V = V(g)$ for some $f, g \in \mathcal{A}(X)$. Hence $Z(f) = Z(g)$ and, by Corollary 1.4, $\operatorname{Rad}(f) = \operatorname{Rad}(g)$. Consequently, $V(f) = V(g) = V$, which proves (2).

Conditions (3) and (4) follow from (1) and (2), respectively. It is now clear that ι induces a homeomorphism between X and $\operatorname{Max}(\mathcal{A}(X))$. ■

By [16, p. 462], the ring $\mathcal{A}(X)$ is not Noetherian if $\dim X \geq 2$. However, the following holds.

Corollary 2.3. *The topological space* $\operatorname{Spec}(\mathcal{A}(X))$ *is Noetherian.*

Proof. This follows from Proposition 2.2 since the $\mathcal{AR}$ topology on X is Noetherian. ∎

It should be mentioned that Proposition 2.2 and Corollary 2.3 are analogs for [8, Théorème 5.29, Corollaires 5.30, 5.31, 5.32] and [13, Proposition 4.2, Corollary 4.3].

Next we study sheaves on the spaces under consideration. For any function $f \in \mathcal{A}(X)$, we identify the rings $\mathcal{A}(X)_f$ and $\mathcal{A}_X^0(X(f))$ via the canonical isomorphism described in Proposition 1.11. Denoting by $\tilde{\mathcal{A}}_X$ the structure sheaf on $\operatorname{Spec}(\mathcal{A}(X))$, we get

$$\iota_* \mathcal{A}_X^0 = \tilde{\mathcal{A}}_X \quad \text{and} \quad \mathcal{A}_X^0 = \iota^{-1} \tilde{\mathcal{A}}_X.$$

In view of Proposition 2.2, the category of sheaves of Abelian groups (resp. sheaves of $\mathcal{A}_X^0$-modules) on X is equivalent to the category of sheaves of Abelian groups (resp. sheaves of $\tilde{\mathcal{A}}_X$-modules) on $\operatorname{Spec}(\mathcal{A}(X))$. The equivalence is effected by the direct image functor $\mathcal{F} \mapsto \iota_* \mathcal{F}$, whose inverse is the inverse image functor $\mathcal{G} \mapsto i^{-1}\mathcal{G}$. Via these equivalences, quasi-coherent sheaves of $\mathcal{A}_X^0$-modules on X correspond to quasi-coherent sheaves of $\tilde{\mathcal{A}}_X$-modules on $\operatorname{Spec}(\mathcal{A}(X))$.

Henceforth, by an *arc-meromorphous* sheaf on X we mean a sheaf of $\mathcal{A}_X^0$-modules. Our goal is to establish basic properties of quasi-coherent arc-meromorphous sheaves.

For clarity of exposition, it is convenient to recall Cartan's theorem A for affine schemes [9, Theorem 7.16, Corollary 7.17]. For a commutative ring R, consider the affine scheme $\mathbb{X} = \operatorname{Spec}(R)$ with structure sheaf $\mathcal{O}_{\mathbb{X}}$. Any R-module M determines a quasi-coherent sheaf $\tilde{M}$ of $\mathcal{O}_{\mathbb{X}}$-modules on $\mathbb{X}$. The functor $M \mapsto \tilde{M}$ gives an equivalence of categories between the category of R-modules and the category of quasi-coherent sheaves of $\mathcal{O}_{\mathbb{X}}$-modules on $\mathbb{X}$. Its inverse is the global section functor $\mathcal{G} \mapsto \mathcal{G}(\mathbb{X})$. In particular, every quasi-coherent sheaf of $\mathcal{O}_{\mathbb{X}}$-modules on $\mathbb{X}$ is generated by its global sections.

Returning to our main topic, for an $\mathcal{A}(X)$-module M, we define a presheaf $\tilde{M}_X$ of $\mathcal{A}_X^0$-modules on X by

$$\tilde{M}_X(U) := M \otimes_{\mathcal{A}(X)} \mathcal{A}_X^0(U)$$

for every open subset $U \subseteq X$. Since $U = X(f)$ for some $f \in \mathcal{A}(X)$, we get

$$\tilde{M}_X(U) = \tilde{M}_X(X(f)) = M \otimes_{\mathcal{A}(X)} \mathcal{A}(X)_f = \tilde{M}(D(f)),$$

where the last equality is the canonical identification. Hence $\tilde{M}_X$ is actually a sheaf and

$$\tilde{M}_X = \iota^{-1} \tilde{M}.$$

We immediately get the following variant of Cartan's theorem A.

Theorem 2.4. *The functor* $M \mapsto \tilde{M}_X$ *gives an equivalence of categories between the category of* $\mathcal{A}(X)$*-modules and the category of quasi-coherent arc-meromorphous sheaves on* X*. Its inverse is the global section functor* $\mathcal{F} \mapsto \mathcal{F}(X)$*. In particular, every quasi-coherent arc-meromorphous sheaf on* X *is generated by its global sections.* ∎

According to Cartan's theorem B for affine schemes, if $\mathcal{G}$ is a quasi-coherent sheaf of $\mathcal{O}_{\mathbb{X}}$-modules on $\mathbb{X} = \mathrm{Spec}(R)$, then $H^i(\mathbb{X}, \mathcal{G}) = 0$ for $i \geq 1$ [10, Théorème 1.3.1].

The equivalence of the categories of sheaves on X and on $\mathrm{Spec}(\mathcal{A}(X))$ via the functors i_* and ι^{-1} yields the following variant of Cartan's theorem B.

Theorem 2.5. *If* $\mathcal{F}$ *is a quasi-coherent arc-meromorphous sheaf on* X*, then*

$$H^i(X, \mathcal{F}) = 0 \quad \text{for all } i \geq 1. \ \blacksquare$$

It should be mentioned that Theorems 2.4 and 2.5 are counterparts of the results on quasi-coherent k-regulous sheaves [8] and on quasi-coherent Nash k-regulous sheaves [13].

3. Arc-meromorphous maps and morphisms. Let X and Y be Nash manifolds. A map $f\colon X \to Y$ is said to be *arc-analytic* if for every analytic arc $\gamma\colon (-1,1) \to X$ the composite $f \circ \gamma\colon (-1,1) \to Y$ is an analytic arc. If Y is a Nash manifold in $\mathbb{R}^n$, then $f\colon X \to Y$ is an arc-analytic map precisely when $y_i \circ f$ is an arc-analytic function for $i = 1, \ldots, n$, where $y_i\colon \mathbb{R}^n \to \mathbb{R}$ is the ith coordinate function. Every semialgebraic arc-analytic map from X into Y is continuous for the Euclidean topology and the $\mathcal{AR}$ topology. Moreover, we have the following.

Proposition 3.1. *Any semialgebraic arc-analytic map* $f\colon X \to Y$ *of Nash manifolds induces a natural morphism of locally ringed spaces*

$$(f, f^{\#})\colon (X, \mathcal{A}^0_X) \to (Y, \mathcal{A}^0_Y),$$

where $f^{\#}\colon \mathcal{A}^0_Y \to f_*\mathcal{A}_X$ *is a morphism of sheaves of* $\mathbb{R}$*-algebras,*

$$f^{\#}(V)\colon \mathcal{A}^0_Y(V) \to (f_*\mathcal{A}^0_X)(V) = \mathcal{A}^0_X(f^{-1}(V)), \quad \psi \mapsto \psi \circ f|_{f^{-1}(V)},$$

for every $\mathcal{AR}$*-open subset* V *of* Y*.*

Proof. Let $V \subseteq Y$ be an $\mathcal{AR}$-open set and let ψ be a function in $\mathcal{A}^0_Y(V)$. It suffices to show that the composite $\psi \circ f|_{f^{-1}(V)}$ belongs to $\mathcal{A}^0_X(f^{-1}(V))$. We have $V = Y(\beta)$ for some arc-analytic semialgebraic function β in $\mathcal{A}(Y)$, and hence $f^{-1}(V) = X(\beta \circ f)$. Moreover,

$$\mathcal{A}^0_Y(V) = \mathcal{A}(Y)_\beta \quad \text{and} \quad \mathcal{A}^0_X(f^{-1}(V)) = \mathcal{A}(X)_{\beta\circ f},$$

which completes the proof. ∎

We call a morphism of locally ringed spaces

$$(f, f^{\#})\colon (X, \mathcal{A}^0_X) \to (Y, \mathcal{A}^0_Y)$$

an $\mathbb{R}$-*morphism* if $f^{\#}\colon \mathcal{A}^0_Y \to f_*\mathcal{A}^0_X$ is a morphism of sheaves of $\mathbb{R}$-algebras. By Proposition 3.1, every semialgebraic arc-analytic map from X into Y induces an $\mathbb{R}$-morphism. Conversely, we also have the following.

PROPOSITION 3.2. *Let X, Y be Nash manifolds, and let*

$$(f, f^{\#})\colon (X, \mathcal{A}^0_X) \to (Y, \mathcal{A}^0_Y)$$

be an $\mathbb{R}$-morphism. Then f is a semialgebraic arc-analytic map, and the morphism $(f, f^{\#})$ is induced by f as in Proposition 3.1.

Proof. Let V be an $\mathcal{AR}$-open subset of Y and let ψ be a function in $\mathcal{A}^0_Y(V)$. We claim that $f^{\#}(V)(\psi) = \psi \circ f|_V$. Indeed, pick a point p in $f^{-1}(V)$ and consider the induced local homomorphism of $\mathbb{R}$-algebras

$$f^{\#}_p\colon \mathcal{A}^0_{Y,f(p)} \to \mathcal{A}^0_{X,p}.$$

Set $c := f^{\#}_p(\psi_{f(p)})(p)$, where $\psi_{f(p)}$ is the germ of ψ at $\psi(p)$. Clearly, the germ $f^{\#}_p(\psi_{f(p)}) - c$ belongs to the maximal ideal of the local ring $\mathcal{A}^0_{X,p}$. Since

$$f^{\#}_p(\psi_{f(p)} - c) = f^{\#}_p(\psi_{f(p)}) - c,$$

the germ $\psi_{f(p)} - c$ belongs to the maximal ideal of the local ring $\mathcal{A}^0_{Y,f(p)}$. Thus

$$\psi_{f(p)}(f(p)) = c = f^{\#}_p(\psi_{f(p)})(p).$$

The claim follows, the point $p \in f^{-1}(V)$ being arbitrary.

It remains to show that the map f is semialgebraic and arc-analytic. We may assume that Y is a Nash manifold in $\mathbb{R}^n$. If $y_i\colon \mathbb{R}^n \to \mathbb{R}$ is the ith coordinate function, then the composite $y_i \circ f$ belongs to $\mathcal{A}^0_X(X)$ in view of the claim. By Theorem 1.7, $\mathcal{A}^0_X(X) = \mathcal{A}(X)$, and hence $y_i \circ f$ is a semialgebraic arc-analytic function. Consequently, f is a semialgebraic arc-analytic map. ∎

Next we describe homomorphisms between $\mathbb{R}$-algebras of semialgebraic arc-analytic functions.

PROPOSITION 3.3. *Let X, Y be Nash manifolds, and let $\alpha\colon \mathcal{A}(Y) \to \mathcal{A}(X)$ be a homomorphism of $\mathbb{R}$-algebras. Then there exists a unique semialgebraic arc-analytic map $f\colon X \to Y$ such that $\alpha(\psi) = \psi \circ f$ for all functions ψ in $\mathcal{A}(Y)$.*

Proof. We may assume that Y is a nonsingular algebraic subset of $\mathbb{R}^n$. Let $y_i\colon \mathbb{R}^n \to \mathbb{R}$ be the ith coordinate function, $i = 1, \ldots, n$. Set $f_i := \alpha(\bar{y}_i)$, where $\bar{y}_i\colon Y \to \mathbb{R}$ is the restriction of y_i. If $F\colon \mathbb{R}^n \to \mathbb{R}$ is a polynomial function and p is a point in X, then

$$F(f_1(p),\dots,f_n(p)) = \alpha(F(\bar{y}_1,\dots,\bar{y}_n))(p).$$

Assuming that F vanishes on Y, we see that the function $F(\bar{y}_1,\dots,\bar{y}_n)$ is identically 0 on Y. Consequently,

$$f \coloneqq (f_1,\dots,f_n)\colon X \to Y \subseteq \mathbb{R}^n$$

is a well-defined semialgebraic arc-analytic map.

Consider the maximal ideals

$$\mathfrak{m}_{X,p} \coloneqq \{\varphi \in \mathcal{A}(X)\colon \varphi(p) = 0\},$$
$$\mathfrak{m}_{Y,f(p)} \coloneqq \{\psi \in \mathcal{A}(Y)\colon \psi(f(p)) = 0\}.$$

The ideal $\alpha^{-1}(\mathfrak{m}_{Y,f(p)})$ of the ring $\mathcal{A}(X)$ contains the functions $\bar{y}_i - f_i(p)$ since

$$\alpha(\bar{y}_i - f_i(p))(p) = \alpha(\bar{y}_i)(p) - f_i(p) = 0.$$

Hence the zero set $Z(\alpha^{-1}(\mathfrak{m}_{Y,f(p)}))$ is the singleton $\{f(p)\}$. Since $\alpha^{-1}(\mathfrak{m}_{Y,f(p)})$ is a prime ideal, it follows from Theorem 1.2 that

$$\alpha^{-1}(\mathfrak{m}_{Y,f(p)}) = \mathfrak{m}_{X,p}.$$

Consequently, for every function ψ in $\mathcal{A}(Y)$, we get

$$\alpha(\psi)(p) - \psi(f(p)) = \alpha(\psi - \psi(f(p)))(p) = 0.$$

Thus $\alpha(\psi) = \psi \circ f$, the point $p \in X$ being arbitrary. Clearly, f is uniquely determined. ∎

The results established in Propositions 3.1–3.3 can be summarized as follows.

Theorem 3.4. *Let X, Y be Nash manifolds, and let $\mathcal{A}(X,Y)$ be the set of all semialgebraic arc-analytic maps from X into Y. Then there are natural one-to-one correspondences:*

(i) $\mathcal{A}(X,Y) \to \mathrm{Mor}_{\mathbb{R}}((X,\mathcal{A}^0_X),(Y,\mathcal{A}^0_Y))$;
(ii) $\mathcal{A}(X,Y) \to \mathrm{Hom}_{\mathbb{R}}(\mathcal{A}(Y),\mathcal{A}(X))$;
(iii) $\mathcal{A}(X,Y) \to \mathrm{Mor}_{\mathbb{R}}(\mathrm{Spec}(\mathcal{A}(X)),\mathrm{Spec}(\mathcal{A}(Y)))$.

Here

$$\mathrm{Mor}_{\mathbb{R}}((X,\mathcal{A}^0_X),(Y,\mathcal{A}^0_Y))$$

is the set of $\mathbb{R}$-morphisms from $(X,\mathcal{A}^0_X)$ into $(Y,\mathcal{A}^0_Y)$,

$$\mathrm{Hom}_{\mathbb{R}}(\mathcal{A}(Y),\mathcal{A}(X))$$

is the set of homomorphisms of $\mathbb{R}$-algebras from $\mathcal{A}(Y)$ into $\mathcal{A}(X)$, and

$$\mathrm{Mor}_{\mathbb{R}}(\mathrm{Spec}(\mathcal{A}(X)),\mathrm{Spec}(\mathcal{A}(Y)))$$

is the set of morphisms of $\mathbb{R}$-schemes (= schemes over $\mathrm{Spec}(\mathbb{R})$*) from* $\mathrm{Spec}(\mathcal{A}(X))$ *into* $\mathrm{Spec}(\mathcal{A}(Y))$.

Proof. Clearly, (i) follows from Propositions 3.1 and 3.2, whereas Proposition 3.3 implies (ii). Existence of a natural one-to-one correspondence

$$\mathrm{Hom}_{\mathbb{R}}(\mathcal{A}(Y),\mathcal{A}(X)) \to \mathrm{Mor}_{\mathbb{R}}(\mathrm{Spec}(\mathcal{A}(X)),\mathrm{Spec}(\mathcal{A}(Y)))$$

is a standard fact from the theory of schemes, and hence (iii) holds. ∎

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