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HIGHLY EFFICIENT SOLVERS FOR NONLINEAR EQUATIONS IN BANACH SPACE

Abstract. We introduce highly efficient solvers of nonlinear equations involving Banach space valued operators. The local convergence is based only on the first Fréchet derivative in contrast to earlier works using derivatives up to order seven to show the sixth order of convergence. Hence, we extend the applicability of these methods. Numerical examples are used to test the conditions of the theoretical results.

1. Introduction. One of the greatest challenges in computational mathematics is to find a solution x_* of the equation [1–13, 17, 18]

$$F(x) = 0, \tag{1.1}$$

where $F : \Omega \to \mathcal{E}_2$ is a Fréchet differentiable operator. Here and below, $\Omega \subset \mathcal{E}_1$ is a nonempty, open set, and $\mathcal{E}_1, \mathcal{E}_2$ are Banach spaces.

In this study, we are concerned with the local convergence of the sixth order method given as

$$\begin{aligned} x_0 \in \Omega, \qquad y_n &= x_n - \tfrac{2}{3}F'(x_n)^{-1}F(x_n) \\ z_n &= x_n - [a_1 I + a_2(F'(y_n)^{-1}F'(x_n))^2]F'(x_n)^{-1}F(y_n) \\ x_{n+1} &= z_n - [(b_2F'(x_n) + b_3F'(y_n))^{-1}(F'(x_n) + b_1F'(y_n))] \\ &\quad \cdot F'(x_n)^{-1}F(z_n) \end{aligned} \tag{1.2}$$

where a_1, a_2, b_1, b_2, b_3 are given real numbers satisfying

$$a_1 = 1 - a_2, \qquad a_2 = \tfrac{3}{8}, b_2 = b_1 - b_3 + 1, \qquad b_3 = \tfrac{1}{2}(5b_1 + 3),$$

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and $b_1 \neq -1$ is a free parameter. Method (1.2) was studied in [20], but for the case $\mathcal{E}_1 = \mathcal{E}_2 = \mathbb{R}^k$ (k a natural number). Using conditions on high order derivative, and Taylor series (although these derivatives do not appear in method (1.2)), convergence order was established in [20]. The hypotheses on higher order derivatives limit the usage of method (1.2).

As an academic example, let $\mathcal{E}_1 = \mathcal{E}_2 = \mathbb{R}$ and $\Omega = [-1/2, 3/2]$. Define F on Ω by

$$F(x) = x^3 \log x^2 + x^5 - x^4.$$

Then we have $x_* = 1$, and

$$\begin{aligned} F'(x) &= 3x^2 \log x^2 + 5x^4 - 4x^3 + 2x^2, \\ F''(x) &= 6x \log x^2 + 20x^3 - 12x^2 + 10x, \\ F'''(x) &= 6 \log x^2 + 60x^2 - 24x + 22. \end{aligned}$$

Obviously $F'''(x)$ is not bounded on Ω. So, the convergence of method (1.2) is not guaranteed by the analysis in [17, 18, 20].

Other problems with the usage of method (1.2) are: no information on how to choose x_0; bounds on $\|x_n - x_*\|$ and information on the location of x_*. All these are addressed in this paper by only using conditions on the first derivative, and in the more general setting of Banach space valued operators. To avoid the use of Taylor series and high convergence order derivatives, we rely on the computational order of convergence (COC) or the approximate computational order of convergence (ACOC) [1, 2, 4].

The rest of the paper includes: the local convergence analysis in Section 2, and an example in Section 3.

2. Local convergence analysis. We base the local convergence analysis of method (1.2) on some real functions and parameters. Consider a continuous increasing function $w_0 : [0, \infty) \to [0, \infty)$ satisfying $w_0(0) = 0$. Assume that the equation

$$w_0(t) = 1 \tag{2.1}$$

has a minimal positive solution denoted by ρ_0. Consider also continuous increasing functions $w : [0, \rho_0) \to [0, \infty)$ and $v : [0, \rho_0) \to [0, \infty)$ with $w(0) = 0$. Define functions g_1 and h_1 on the interval $[0, \rho_0)$ by

$$g_1(t) = \frac{\int_0^1 w((1-\theta)t)\, d\theta + \frac{1}{3}\int_0^1 v(\theta)\, d\theta}{1 - w_0(t)}, \qquad h_1(t) = g_1(t) - 1.$$

Assume that

$$\tfrac{1}{3} v(0) - 1 < 0. \tag{2.2}$$

By these definitions and (2.1), we have $h_1(0) = v(0)/3 - 1 < 0$ and $h_1(t) \to \infty$ as $t \to \rho_0^-$. The mean value theorem then ensures that the equation $h_1(t) = 0$

has at least one solution in the interval $(0, \rho_0)$. Denote the minimal such solution by r_1. Assume that the equation

$$w_0(g_1(t)t) = 1 \tag{2.3}$$

has a minimal positive solution ρ. Set $\rho_1 = \min\{\rho_0, \rho\}$. Moreover, define functions g_2 and h_2 on $[0, \rho_1)$ by

$$g_2(t) = \frac{\int_0^1 w((1-\theta)t)\,d\theta}{1-w_0(t)} + |a_1 - 1| \frac{(w_0(t) + w_0(g_1(t)t))(v(t) + v(g_1(t)t)) \int_0^1 v(\theta t)\,d\theta}{(1-w_0(t))(1-w_0(g_1(t)t))^2}$$

and $h_2(t) = g_2(t) - 1$. By these definitions $h_2(0) = -1$ and $h_2(t) \to \infty$ as $t \to \rho^-$. Denote by r_2 the minimal solution of $h_2(t) = 0$ in $(0, \rho_1)$. Assume that the equations

$$w_0(g_2(t)t) = 1 \tag{2.4}$$

and

$$p(t) = 1 \tag{2.5}$$

have minimal positive solutions ρ_2, and ρ_p, respectively, where

$$p(t) = \frac{l}{|b_2 + b_3|}\big(|b_2| w_0(t) + b_3 w_0(g_1(t)t)\big), \qquad b_2 + b_3 \neq 0.$$

Set $\rho_3 = \min\{\rho_1, \rho_2, \rho_P\}$. Furthermore, define functions g_3 and h_3 on $[0, \rho_3)$ by

$$\begin{aligned} g_3(t) = \Bigg[& \frac{\int_0^1 w((1-\theta)g_2(t)t)\,d\theta}{1 - w_0(g_2(t)t)} \\ & \times \frac{(w_0(t) + w_0(g_2(t)t)) \int_0^1 v(\theta g_2(t)t)\,d\theta}{(1-w_0(t))(1-w_0(g_2(t)t))} + \frac{|b_2 - 1|}{|b_2 + b_3|} \\ & \times \frac{(w_0(t) + w_0(g_1(t)t)) \int_0^1 v(\theta g_2(t)t)\,d\theta}{(1-w_0(t))(1-p(t))} \Bigg] g_2(t), \\ h_3(t) = {} & g_3(t) - 1. \end{aligned}$$

By these definitions, $h_3(0) = -1$ and $h_3(t) \to \infty$ as $t \to \rho_3^-$. Denote by r_3 the minimal solution of the equation $h_3(t) = 0$ in $(0, \rho_3)$. Finally, define a radius of convergence denoted by r as

$$r = \min\{r_j\}, \qquad j = 1, 2, 3. \tag{2.6}$$

It follows that

$$0 \leq w_0(t) < 1, \tag{2.7}$$

$$0 \leq w_0(g_1(t)t) < 1, \tag{2.8}$$

(2.9) $$0 \le w_0(g_2(t)t) < 1,$$

(2.10) $$0 \le p(t) < 1,$$

(2.11) $$0 \le g_j(t) < 1,$$

for all $t \in [0, r)$.

The local convergence analysis of method (1.2) also uses the following *conditions* (A):

(A_1) $F : \Omega \to \mathcal{E}_2$ is continuously differentiable and there exists $x_* \in \Omega$ such that $F(x_*) = 0$ and $F'(x_*)^{-1} \in \mathcal{L}(\mathcal{E}_2, \mathcal{E}_1)$.

(A_2) The function $w_0 : [0, \infty) \to [0, \infty)$ is continuous, increasing and satisfies $w_0(0) = 0$.

Set $\Omega_0 = \Omega \cap S(x_*, \rho_0)$, where ρ_0 is given in (2.1).

(A_3) The functions $w, v : [0, \rho_0) \to [0, \infty)$ are continuous, increasing, $w(0)=0$, and for each $x, y \in \Omega_0$,

$$\|F'(x_*)^{-1}(F'(x) - F'(y))\| \le w(\|x - y\|),$$
$$\|F'(x_*)F'(x)\| \le v(\|x - x_*\|).$$

(A_4) $\bar{S}(x_*, r) \subseteq \Omega, \rho_0, \rho, \rho_1, \rho_2, \rho_P, \rho_3, r$ exist, and are defined as previously.

(A_5) There exists $\bar{r} \ge r$ such that $\int_0^1 w_0(\theta \bar{r})\, d\theta < 1$.

Set $\Omega_1 = \Omega \cap \bar{S}(x_*, \bar{r})$.

Next, we present the local convergence analysis of method (1.2) using conditions (A) and the preceding notation.

THEOREM 2.1. *Under conditions* (A) *and for* $x_0 \in S(x_*, r) - \{x_*\}$, *the following hold:*

(2.12) $$\{x_n\} \subseteq S(x_*, r),$$

(2.13) $$\lim_{n\to\infty} x_n = x_*,$$

(2.14) $$\|y_n - x_*\| \le g_1(\|x_n - x_*\|)\|x_n - x_*\| \le \|x_n - x_*\| < r,$$

(2.15) $$\|z_n - x_*\| \le g_2(\|x_n - x_*\|)\|x_n - x_*\| \le \|x_n - x_*\|,$$

(2.16) $$\|x_{n+1} - x_*\| \le g_3(\|x_n - x_*\|)\|x_n - x_*\| \le \|x_n - x_*\|,$$

and x_* *is the only solution of the equation* $F(x)$ *in the set* Ω_1 *where* r, g_j *and* Ω_1 *are defined as previously.*

Proof. By conditions (A_2), (2.6), (2.7), (A_4) and $x \in S(x_*, r)$, we get

(2.17) $$\|F'(x_*)^{-1}(F'(x) - F'(x_*)\| \le w_0(\|x - x_*\|) < w_0(r) < 1,$$

which together with the Banach lemma on invertible operators [4,15] implies that $F'(x)^{-1} \in \mathcal{L}(\mathcal{E}_2, \mathcal{E}_1)$, and

(2.18) $$\|F'(x)^{-1}F'(x_*)\| \le \frac{1}{1 - w_0(\|x - x_*\|)}.$$

The point y_0 is well defined by the condition $x_0 \in S(x_*, r) - \{x_*\}$, and for $x = x_0$.

Using the first substep of method (1.2), (A_1), (2.6), (2.11) (for $j = 1$), (A_3) and (2.18) for $x = x_0$, we get

$$\begin{aligned}(2.19)\qquad \|y - x_*\| &= \|x_0 - x_* - F'(x_0)^{-1}F(x_0) + \tfrac{1}{3}F'(x_0)^{-1}F(x_0)\| \\ &\le \|F'(x_0)^{-1}F'(x_*)\| \\ &\quad \times \left\|\int_0^1 F'(x_*)^{-1}\big(F'(x_* + t(x_0 - x_*)) - F(x_0)\big)\, d\theta\,(x_0 - x_*)\right\| \\ &\quad \times \tfrac{1}{3}\|F'(x_0)^{-1}F'(x_*)\|\,\|F'(x_*)^{-1}F(x_0)\| \\ &\le \frac{\int_0^1 w((1-\theta)\|x_0 - x_*\|)\, d\theta + \frac{1}{2}\int_0^1 v(\theta\|x_0 - x_*\|)\, d\theta}{1 - w_0(\|x_0 - x_*\|)}\|x_0 - x_*\| \\ &\le g_1(\|x_0 - x_*\|)\|x_0 - x_*\| \le \|x_0 - x_*\| < r,\end{aligned}$$

which shows (2.14) for $n = 0$ and $y_0 \in S(x_*, r)$, where we have also used

$$\begin{aligned}(2.20)\qquad \|F'(x_*)^{-1}F(x)\| &= \|F'(x_*)^{-1}(F(x) - F(x_*))\| \\ &= \left\|\int_0^1 F'(x_*)^{-1}F'(x_* + \theta(x - x_*))\, d\theta\,(x - x_*)\right\| \\ &= \int_0^1 r(\theta\|x - x_*\|)\, d\theta\, \|x - x_*\|\end{aligned}$$

for $x = x_0$.

By the second substep of method (1.2), for $n = 0$,

$$\begin{aligned}(2.21)\qquad z_0 - x_* &= x_0 - x_* - F'(x_0)^{-1}F(x_0) + F'(x_0)^{-1}F(x_0) - a_1F'(x_0)^{-1}F(x_0) \\ &= (x_0 - x_* - F'(x_0)^{-1}F(x_0)) \\ &\quad + (a_1 - 1)F'(y_0)^{-1}[(F'(x_0) - F'(x_*)) + (F'(x_*) - F'(y_0))] \\ &\quad \times F'(y_0)^{-1}(F'(x_0) + F'(y_0))F'(x_0)^{-1}F(x_0),\end{aligned}$$

where we have used $a_1 = 1 - a_2$, and

$$\begin{aligned}&(2.22) \\ &F'(x_n)^{-1}F(x_0) - a_1F'(x_0)^{-1}F(x_0) - a_2(F'(y_0)^{-1}F'(x_0))^2F'(x_0)^{-1}F(x_0) \\ &\quad = (a_1 - 1)[(F'(y_0)^{-1}F'(x_0))^2 - I]F'(x_0)^{-1}F(x_0) \\ &\quad = (a_1 - 1)[F'(y_0)^{-1}F'(x_0) - I][F'(y_0)^{-1}F'(x_0) + I]F'(x_0)^{-1}F(x_0) \\ &\quad = (a_1 - 1)F'(y_0)^{-1}[(F'(x_0) - F'(x_*)) + (F'(x_*) + F'(y_0))] \\ &\qquad \times (F'(x_0) + F'(y_0))F'(x_0)^{-1}F(x_0).\end{aligned}$$

As in (2.18) for $x = y_0$, and using (2.8), (2.19) we get $F'(y_0)^{-1} \in \mathcal{L}(\mathcal{E}_2, \mathcal{E}_1)$

and

$$\|F'(y_0)^{-1}F'(x_*)\| \le \frac{1}{1-w_0(\|y_0-x_*\|)} \le \frac{1}{1-w_0(g_1(\|x_0-x_*\|)\|x_0-x_*\|}, \tag{2.23}$$

and z_0 is well defined by the second substep of method (1.2). Then, using (2.6), (2.11) (for $j=2$), (2.18) (for $x=x_0$), (2.19), (2.20) (for $x=x_0,y_0$) and (2.23), we obtain in turn

$$\begin{aligned}
&\|z_0-x_*\| \\
&\le \|x_0-x_*-F'(x_0)^{-1}F(x_0)\| \\
&\quad + |a_1-1|\|F'(y_0)^{-1}F'(x_*)\|\big[\|F'(x_*)^{-1}(F'(x_0)-F'(x_*)\| \\
&\quad + \|F'(x_*)^{-1}(F'(x_*)-F'(y_0))\|\big]\|F'(y_0)^{-1}F'(x_*)\| \\
&\quad \times \big[\|F'(x_*)^{-1}F'(x_0)\| + \|F'(x_*)^{-1}F'(y_0)\|\big]\|F'(x_0)^{-1}F'(x_*)\| \\
&\quad \times \|F'(x_*)^{-1}F(x_0)\| \\
&\le \Bigg[\frac{\int_0^1 w((1-\theta)\|x_0-x_*\|)\,d\theta}{1-w_0(\|x_0-x_*\|)} + |a_1-1| \\
&\quad \times \frac{(w_0(\|x_0-x_*\|)+w_0\|y_0-x_*\|)(v(\|x_0-x_*\|)+v(\|y_0-x_*\|))}{(1-w_0(g_1\|x_0-x_*\|)\|x_0-x_*\|)^2(1-w_0\|x_0-x_*\|)} \\
&\quad \times \int_0^1 v(\theta\|x_0-x_*\|)\,d\theta\Bigg]\|x_0-x_*\| \\
&\le g_2(\|x_0-x_*\|)\|x_0-x_*\| \le \|x_0-x_*\|,
\end{aligned} \tag{2.24}$$

so $z_0 \in S(x_*,r)$, and (2.15) holds for $n=0$.

Next, we show that $(b_2F'(x_0)+b_3F'(y_0))^{-1} \in \mathcal{L}(\mathcal{E}_2,\mathcal{E}_1)$.

Indeed by (2.6), (2.10), (A_2) and $b_2+b_3 \ne 0$, we have in turn

$$\begin{aligned}
&\|((b_2+b_3)F'(x_*))^{-1}(b_2F'(x_0)+b_3F'(y_0)-b_2F'(x_*)-b_3F'(x_*))\| \\
&\le \frac{1}{|b_2+b_3|}\big[|b_2|\|F'(x_*)^{-1}(F'(x_0)-F'(x_*))\| \\
&\qquad + |b_3|\|F'(x_*)^{-1}(F'(y_0)-F'(x_*))\|\big] \\
&\le \frac{1}{|b_2+b_3|}\big[|b_2|w_0(\|x_0-x_*\|)+|b_3|w_0(\|y_0-x_*\|)\big] \le p(\|x_0-x_*\|) < 1,
\end{aligned} \tag{2.25}$$

so

$$\|(b_2F'(x_0)+b_3F'(y_0))^{-1}F'(x_*)\| \le \frac{1}{|b_2+b_3|(1-p(\|x_0-x_*\|))}. \tag{2.26}$$

Similarly by (2.6), (2.9), (2.18) (for $x = z_0$) and (2.24), we have $F'(z_0)^{-1} \in \mathcal{L}(\mathcal{E}_2, \mathcal{E}_1)$, and

$$\begin{aligned}\|F'(z_0)^{-1}F'(x_*)\| &\leq \frac{1}{1 - w_0(\|z_0 - x_*\|)}\\ &\leq \frac{1}{1 - w_0(g_2(\|x_0 - x_*\|)\|x_0 - x_*\|)},\end{aligned}$$

so x_1 is well defined by the third substep of method (1.2). Next, using (2.6), (2.11) (for $j = 3$), (2.18) (for $x = x_0, y_0, z_0$), (2.19), (2.24), (2.25), (2.26), $b_2 = b_1 - b_3 + 1$ and the triangle inequality

$$\begin{aligned}(2.27)\qquad x_1 - x_* &= (z_0 - x_* - F'(z_0)^{-1}F(z_0)) + F'(z_0)^{-1}F(z_0)\\ &\quad - (b_2F'(x_0) + b_3F'(y_0))^{-1}(F'(x_0) + b_1F'(y_0))F'(x_0)^{-1}F(x_0)\\ &= (z_0 - x_* - F'(z_0)^{-1}F(z_0)) + (F'(z_0)^{-1}F(z_0) - F'(x_0)^{-1}F(z_0))\\ &\quad + [I - (b_2F'(x_0) + b_3F'(y_0))^{-1}(F'(x_0) + b_1F'(y_0))F'(x_0)^{-1}F(z_0)\\ &= (z_0 - x_* - F'(z_0)^{-1}F(z_0)) + F'(z_0)^{-1}F(z_0) - F'(x_0)^{-1}F(z_0)\\ &\quad + [I - (b_2F'(x_0) + b_3F'(y_0))^{-1}(F'(x_0) + b_1F'(y_0))]F'(x_0)^{-1}F(z_0)\\ &= (z_0 - x_* - F'(z_0)^{-1}F(z_0)) + F'(z_0)^{-1}(F'(x_0) - F'(z_0))F'(x_0)^{-1}F(z_0)\\ &\quad + (b_2F'(x_0) + b_3F'(y_0)^{-1})[(b_2 - 1)F'(x_0) + (b_2 - b_1)F'(y_0)]F'(x_0)^{-1}F(z_0)\\ &= (z_0 - x_* - F'(z_0)^{-1}F(z_0)) + F'(z_0)^{-1}(F'(x_0) - F'(z_0))F'(x_0)^{-1}F(z_0)\\ &\quad + (b_2 - 1)(b_2F'(x_0) + b_3F'(y_0))^{-1}[(F'(x_0) - F'(x_*))\\ &\quad + (F'(x_*) - F'(y_0))]F'(x_0)^{-1}F(z_0)\end{aligned}$$

leading to

$$\begin{aligned}(2.28)\qquad \|x_1 - x_*\| &\leq \Bigg[\frac{\int_0^1 w((1-\theta)\|z_0 - x_*\|)\,d\theta}{1 - w_0(\|z_0 - x_*\|)}\\ &\quad + \frac{(w_0\|x_0 - x_*\| + w_0\|z_0 - x_*\|)\int_0^1 v(\theta\|z_0 - x_*\|)\,d\theta}{(1 - w_0\|z_0 - x_*\|)(1 - w_0\|x_0 - x_*\|)}\\ &\quad + \frac{|b_2 - 1|}{|b_2 + b_3|}\,\frac{(w_0\|x_0 - x_*\| + w_0\|y_0 - x_*\|)\int_0^1 v(\theta\|z - x_*\|)\,d\theta}{(1 - p\|x_0 - x_*\|)(1 - w_0\|x_0 - x_*\|)}\Bigg]\|z_0 - x_*\|\\ &\leq g_3(\|x_0 - x_*\|)\|x_0 - x_*\| \leq \|x_0 - x_*\|,\end{aligned}$$

so $x_1 \in S(x_*, r)$ and (2.16) holds for $n = 0$. The induction for estimations (2.14)–(2.16) is terminated if x_i, y_i, z_i, x_{i+1} replace x_0, y_0, z_0, x_1 respectively in the preceding computations. Then, by the estimations

$$(2.29)\qquad \|x_{i+1} - x_*\| \leq \varsigma\|x_i - x_*\| < r,$$

where $\varsigma = g_3(\|x_0 - x_*\|) \in [0,1)$, we conclude that $\lim_{i\to\infty} x_i = x_*$ and $x_{i+1} \in S(x_*, r)$.

Consider $y_* \in \Omega_1$ with $F(y_*) = 0$. Set $T = \int_0^1 F'(y_* + \theta(x_* - y_*))\, d\theta$. By (A_2) and (A_3), we get

$$\|F'(x_*)^{-1}(T - F'(x_*))\| \le \int_0^1 w_0(\theta\|y_* - x_*\|)\, d\theta \le \int_0^1 w_0(\theta\bar{r})\, d\theta < 1,$$

so $T^{-1} \in \mathcal{L}(\mathcal{E}_2, \mathcal{E}_1)$. Finally, by the identity $0 = F(x_*) - F(y_*) = T(x_* - y_*)$, we deduce that $x_* = y_*$. ∎

REMARK 2.2. (a) In the case when $w_0(t) = L_0 t, w(t) = Lt$, the radius $\rho_{\mathrm{A}} = \frac{2}{2L_0+L}$ was obtained by Argyros [2] as the convergence radius for Newton's method under conditions (2.7)–(2.9). Notice that the convergence radius for Newton's method given independently by Rheinboldt [16] and Traub [21] is given by

$$\rho_{\mathrm{TR}} = \frac{2}{3L_1} < \rho_{\mathrm{A}},$$

where L_1 is the Lipschitz constant on Ω. As an example, let us consider the function $F(x) = e^x - 1$. Then $x_* = 0$. Set $\Omega = B(0,1)$. Then we have $L_0 = e - 1 < L = e^{1/L_0} < L_1 = e$, so $\rho_{\mathrm{TR}} = 0.24252961 < \rho_{\mathrm{A}} = 0.3826919122323857$.

(b) The local results can be used for projection methods such as Arnoldi's method, the generalized minimum residual method (GMREM), the generalized conjugate method (GCM) for combined Newton/finite projection methods and in connection to the mesh independence principle in order to develop the cheapest and most efficient mesh refinement strategy [2–4, 11].

(c) The results can also be used to solve equations where the operator F' satisfies the autonomous differential equation [11]:

$$F'(x) = P(F(x)),$$

where $P : \mathcal{E}_2 \to \mathcal{E}_2$ is a known continuous operator. Since $F'(x_*) = P(F(x_*)) = P(0)$, we can apply the results without actually knowing the solution x_*. As an example let $F(x) = e^x - 1$. Then we can choose $P(x) = x + 1$ and $x_* = 0$.

(d) It is worth noticing that method (1.2) does not change when we use the conditions of the preceding theorem instead of the stronger conditions used in [18, 20]. Moreover, we can compute the computational order of convergence (COC) defined as

$$\xi = \lim_{n\to\infty} \left[\ln\left(\frac{\|x_{n+1} - x_*\|}{\|x_n - x_*\|} \right) / \ln\left(\frac{\|x_n - x_*\|}{\|x_{n-1} - x_*\|} \right) \right]$$

or the approximate computational order of convergence (ACOC) [6]

$$\xi_1 = \lim_{n\to\infty}\left[\ln\left(\frac{\|x_{n+1}-x_n\|}{\|x_n-x_{n-1}\|}\right)/\ln\left(\frac{\|x_n-x_{n-1}\|}{\|x_{n-1}-x_{n-2}\|}\right)\right].$$

This way we obtain in practice the order of convergence, but no higher order derivatives are used.

3. Numerical example. We present the following example to test the convergence criteria.

EXAMPLE 3.1. Let $\mathcal{E}_1 = \mathcal{E}_2 = \mathbb{R}^3$, $\Omega = U(0,1)$, $x_* = (0,0,0)^T$ and define F on Ω by

$$F(x) = F(u_1,u_2,u_3) = \left(e^{u_1}-1, \frac{e-1}{2}u_2^2 + u_2, u_3\right)^T. \tag{3.1}$$

For $u = (u_1,u_2,u_3)^T$, the Fréchet derivative is given by

$$F'(u) = \begin{pmatrix} e^{u_1} & 0 & 0 \\ 0 & (e-1)u_2+1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Using the norm of the maximum of the rows $x_* = (0,0,0)^T$ and since $F'(x_*) = \operatorname{diag}(1,1,1)$, by conditions (H) we get $w_0(t) = (e-1)t$, $w(t) = e^{\frac{1}{e-1}}t$, $w_1(t) = v(t) = e^{\frac{1}{e-1}}$,

$$r_1 = 0.0154407, \quad r_2 = 0.461425, \quad r_3 = 0.00179951, \quad r = r_3.$$

EXAMPLE 3.2. Let $\mathcal{E}_1 = \mathcal{E}_2 = C[0,1]$, $\Omega = \bar{U}(0,1)$. Define a function F on Ω by

$$F(w)(x) = w(x) - 5\int_0^1 x\theta w(\theta)^3\,d\theta.$$

The Fréchet derivative is given by

$$F'(w(\xi))(x) = \xi(x) - 15\int_0^1 x\theta w(\theta)^2\xi(\theta)\,d\theta \quad \text{for each } \xi\in\Omega.$$

Then we have $x^* = 0$, $w_0(t) = L_0t$, $w(t) = Lt$, $w_1(t) = v(t) = 2$, $L_0 = 7.5 < L = 15$. Thus, the radii of convergence are given by

$$r_1 = 0.02222, \quad r_2 = 0.0678682, \quad r_3 = 0.0313983, \quad r = r_1.$$

EXAMPLE 3.3. Returning to the motivational example of the introduction, we can choose $w_0(t) = w(t) = 96.662907t$, $w_1(t) = v(t) = 1.0631$. Then the radii of convergence are given by

$$r_1 = 0.00445282, \quad r_2 = 0.00689712, \quad r_3 = 0.00355681, \quad r = r_3.$$

Example 3.4. Consider the Hammerstein integral equation (see [14, pp. 19–20]) defined by

$$x(s) = \frac{1}{5}\int_0^1 S(s,t)x(t)^3\,dt, \qquad x \in C[0,1], \qquad s,t \in [0,1], \tag{3.2}$$

where the kernel S is

$$S(s,t) = \begin{cases} s(1-t), & s \le t, \\ (1-s)t, & t \le s. \end{cases}$$

We use $\int_0^1 \phi(t)\,dt \simeq \sum_{k=1}^8 w_k\phi(t_k)$ in (3.2), where t_k and w_k are the abscissas and weights, respectively. Denoting the approximations of $x(t_i)$ by x_i ($i = 1,\dots,8$), we get the following 8×8 system of nonlinear equations:

$$5x_i - 5 - \sum_{k=1}^{8} a_{ik}x_k^3 = 0, \qquad i = 1,\dots,8,$$

$$a_{ik} = \begin{cases} w_k t_k(1-t_i), & k \le i, \\ w_k t_i(1-t_k), & i < k. \end{cases}$$

By the Gauss–Legendre quadrature formula, we obtain the values of t_k and w_k when $k = 8$, listed in Table 1. We follow the stopping critera for pro-

Table 1. Abscissas and weights for $k = 8$

j	t_j	w_j
1	0.01985507175123188415821957...	0.05061426814518812957626567...
2	0.10166676129318663020422303...	0.11119051722668723527217800...
3	0.23723379504183550709113047...	0.15685332293894364366898110...
4	0.40828267875217509753026193...	0.18134189168918099148257522...
5	0.59171732124782490246973807...	0.18134189168918099148257522...
6	0.76276620495816449290886952...	0.15685332293894364366898110...
7	0.89833323870681336979577696...	0.11119051722668723527217800...
8	0.98014492824876811584178043...	0.05061426814518812957626567...

graming $\|F(x_n)\| \le 10^{-100}$ and $\|x_{n+1} - x_n\| \le 10^{-100}$; the solution after seven iterations with precision 10^{-6} is

$$x_* = (1.002096\dots, 1.009900\dots, 1.019727\dots, 1.026436\dots, 1.026436\dots, 1.019727\dots, 1.009900\dots, 1.002096\dots)^T,$$

$\xi = 5.9432173$ and $\xi_1 = 5.6203419$. We get $w_0(t) = w(t) = \frac{3}{40}t$ and $v(t) = 1 + w_0(t)$. The radius of convergence is given by

$$r_1 = 8.88889, \qquad r_2 = 7.12148, \qquad r_3 = 6.73523, \qquad r = 1,$$

since r can be at most 1.

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