

## Fourier multipliers on the Heisenberg group revisited

by

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**Abstract.** We give explicit expressions of differential-difference operators that satisfy the hypothesis of the general Fourier multiplier theorem associated to the Heisenberg group proved by Mauceri and De Michele, for one dimension, and Lin, for higher dimensions. We also give a much shorter proof of the above mentioned theorem. Moreover, we obtain a sharp weighted estimate for Fourier multipliers on the Heisenberg group.

**1. Introduction and the main results.** Given a bounded function  $m$  on  $\mathbb{R}^n$ , consider the operator  $T_m$  defined as follows:

$$\mathcal{F}(T_m f)(\xi) = m(\xi)\mathcal{F}(f)(\xi).$$

Here  $\mathcal{F}$  stands for the Fourier transform on  $\mathbb{R}^n$ . By the Plancherel theorem, one can immediately see that  $T_m$  is bounded on  $L^2(\mathbb{R}^n)$ . We say that  $m$  (or equivalently  $T_m$ ) is an  $L^p$  multiplier if  $T_m$  can be extended to  $L^p(\mathbb{R}^n)$  as a bounded linear operator. A sufficient condition for being an  $L^p$  multiplier was given by Hörmander (see for instance [11, Corollary 8.11]):

**THEOREM 1.1** (Hörmander). *Let  $k = [n/2] + 1$ . If  $m \in C^k$  away from the origin and*

$$\sup_R R^{|\beta|-n/2} \left( \int_{R < |\xi| < 2R} |D^\beta m(\xi)|^2 d\xi \right)^{1/2} \leq C$$

for all  $|\beta| \leq k$ , then  $T_m$  is an  $L^p$  multiplier.

One can also define a Fourier multiplier corresponding to the group Fourier transform on the Heisenberg group  $H^n$ . Let  $M = \{M(\lambda) \in B(L^2(\mathbb{R}^n)) : \lambda \in \mathbb{R}^*\}$  be a family of uniformly bounded operators. Then the operator  $T_M$  is defined

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as follows:

$$\widehat{(T_M f)}(\lambda) = M(\lambda)\hat{f}(\lambda).$$

Here  $\hat{f}$  stands for the group Fourier transform on the Heisenberg group. Again by the Plancherel formula for the group Fourier transform it is immediate that  $T_M$  is bounded on  $L^2(H^n)$ . We are interested in when  $T_M$  can be extended to a bounded linear operator on  $L^p(H^n)$ . In [10], Mauceri and De Michele gave a first sufficient condition for  $n = 1$ . Later Chin Cheng Lin [21] generalized their result to other dimensions. See also [2] for some other interesting properties of Fourier multipliers on the Heisenberg group. The spectral multipliers associated to the sublaplacian are the most important examples of Fourier multipliers on the Heisenberg group. The best possible result for them is obtained in [12] and [23].

In [21], Lin decomposed  $M(\lambda)$  in terms of certain partial isometries which form a basis of the space of all Hilbert–Schmidt operators acting on the Fock spaces. Then he expressed the “difference-differential” operators in terms of those decompositions. But such difference-differential operators looked very complicated. Also, the proof of Lemma 2 in [21], which is the key point in his work, involved very long and technical calculations, even though he gave a proof only for some particular type of polynomials.

The goals of this paper are the following: firstly we will find explicit expressions for the difference-differential operators. Secondly we will give a much simpler proof of the multiplier theorem on the Heisenberg group and will also reduce the number of difference-differential operators. Thirdly, we will prove a quantitative weighted estimate. In order to state our results we first set up some notation.

Let us consider the annihilation and creation operators

$$A_j(\lambda) = \frac{\partial}{\partial \xi_j} + |\lambda|\xi_j, \quad A_j^*(\lambda) = -\frac{\partial}{\partial \xi_j} + |\lambda|\xi_j$$

where  $j = 1, \dots, n$ . They are very common in quantum mechanics. The non-commutative derivations of any operator  $m$  are given by

$$\delta_j(\lambda)m = |\lambda|^{-1/2}[m, A_j(\lambda)], \quad \bar{\delta}_j(\lambda)m = |\lambda|^{-1/2}[A_j^*(\lambda), m]$$

for  $j = 1, \dots, n$ . For  $\alpha, \beta \in \mathbb{N}^n$ , define

$$\delta^\alpha(\lambda) = \delta_1^{\alpha_1}(\lambda) \dots \delta_n^{\alpha_n}(\lambda), \quad \bar{\delta}^\beta(\lambda) = \bar{\delta}_1^{\beta_1}(\lambda) \dots \bar{\delta}_n^{\beta_n}(\lambda).$$

Given a family of operators  $\{m(\lambda) : \lambda \in \mathbb{R}\}$ , we now consider a new operator

$\Theta(\lambda)$  defined as follows:

$$\begin{aligned} \Theta(\lambda)m(\lambda) &= \frac{d}{d\lambda}m(\lambda) + \frac{1}{2\lambda}[m(\lambda), \xi \cdot \nabla] \\ &+ \frac{1}{4\lambda\sqrt{|\lambda|}} \sum_{j=1}^n (\delta_j(\lambda)m(\lambda)A_j^*(\lambda) + \bar{\delta}_j(\lambda)m(\lambda)A_j(\lambda)). \end{aligned}$$

Though the expression of  $\Theta(\lambda)$  may look complicated, it corresponds to the difference-differential operator related to the  $t$ -variable defined in [10] and [21]. In fact, if  $g$  is a Schwartz class function on  $H^n$ , one can check that  $\widehat{(itg)}(\lambda) = \Theta(\lambda)\hat{g}(\lambda)$ . The operator  $\Theta(\lambda)$  also appears implicitly in some other works. For example, the operator  $A$  appearing in [18, Proposition 2.4] coincides with  $\Theta(\lambda)$ .

An operator-valued function  $M : \mathbb{R} \setminus \{0\} \rightarrow B(L^2(\mathbb{R}^n))$  is said to be in  $E^k(\mathbb{R} \setminus \{0\})$  if  $\delta^\alpha(\lambda)\bar{\delta}^\beta(\lambda)\Theta^s(\lambda)$  is in  $B(L^2(\mathbb{R}^n))$  for all  $|\alpha| + |\beta| + 2s \leq k$  and  $\lambda \in \mathbb{R} \setminus \{0\}$ . We also need the dyadic projections

$$\chi_N(\lambda) = \sum_{2^N \leq (2k+n)|\lambda| < 2^{N+1}} P_k(\lambda)$$

where  $P_k(\lambda)$  is the projection on the eigenspace corresponding to the eigenvalue  $(2k+n)|\lambda|$  of the scaled Hermite operator  $H(\lambda) = -\Delta + \lambda^2|x|^2$ .

We will prove the following result.

**THEOREM 1.2.** *Let  $M$  be an operator-valued function which belongs to  $E^k(\mathbb{R} \setminus \{0\})$ ,  $k \geq 2[(n+3)/2]$ . Also, assume*

$$\sup_{\lambda \in \mathbb{R} \setminus \{0\}} \|M(\lambda)\| \leq C.$$

If

$$\sup_{N>0} 2^{N(l-n-1)} \int_{-\infty}^{\infty} \left\| |\lambda|^{-\frac{|\alpha|+|\beta|}{2}} \delta^\alpha(\lambda)\bar{\delta}^\beta(\lambda)\Theta^s(\lambda)M(\lambda)\chi_N(\lambda) \right\|_{\text{HS}}^2 |\lambda|^n d\lambda \leq C$$

for all  $\alpha, \beta \in \mathbb{N}^n$  and  $s \in \mathbb{N}$  satisfying  $|\alpha| + |\beta| + 2s = l \leq 2[(n+3)/2]$ , then  $T_M$  is of weak type  $(1, 1)$  and bounded for  $1 < p < 2$ .

The  $A_2$  conjecture (now a theorem) was one of the well-known conjectures in harmonic analysis until T. Hytönen [13] solved it for all standard Calderón–Zygmund operators, showing the sharp quantitative  $L^2(w)$  bound with a linear dependence in the  $A_2$  constant  $[w]_{A_2}$ . For historical developments, see [6, 25, 24, 5, 17, 9, 20, 1]. Recently, M. T. Lacey [16] extended Hytönen’s result to Dini-continuous operators and A. Lerner [20] found a simple proof of Lacey’s result. In this article we prove similar results for multipliers on the Heisenberg group.

A weight  $w$  is a nonnegative locally integrable function on  $H^n$ . Given  $1 < p < \infty$ , the Muckenhoupt class  $A_p$  consists of all weights  $w$  satisfying

$$[w]_{A_p} := \sup_Q \langle w \rangle_Q \langle \sigma \rangle_Q^{p-1} < \infty, \quad \sigma := w^{1-p'},$$

where the supremum is taken over all cubes  $Q$  in  $H^n$ . Here,  $\langle w \rangle_Q = |Q|^{-1} \int_Q w$ . We will prove the following result.

**THEOREM 1.3.** *Let  $M$  be an operator-valued function with each entry in  $E^k(\mathbb{R} \setminus \{0\})$ ,  $k \geq 2[(n+3)/2]$ . Also, assume*

$$\sup_{\lambda \in \mathbb{R} \setminus \{0\}} \|M(\lambda)\| \leq C.$$

If

$$\sup_{N>0} 2^{N(l-n-1)} \int_{-\infty}^{\infty} \left\| |\lambda|^{-\frac{|\alpha|+|\beta|}{2}} \delta^\alpha(\lambda) \bar{\delta}^\beta(\lambda) \Theta^s(\lambda) M(\lambda) \chi_N(\lambda) \right\|_{\text{HS}}^2 |\lambda|^n d\lambda \leq C$$

for all  $\alpha, \beta \in \mathbb{N}^n$  and  $s \in \mathbb{N}$  satisfying  $|\alpha| + |\beta| + 2s = l \leq 2[(n+3)/2]$ , then

$$\|T_M f\|_{L^p(w)} \leq C [w]^{\max\{1, \frac{1}{p-2}\}} \|f\|_{L^p(w)}$$

for all  $w \in A_{p/2}(H^n)$  with  $2 < p < \infty$ .

We notice that if  $n$  is even then the number of derivatives required in the theorems stated above equals  $n+2$ . Also, when  $n$  is odd, we have to consider one more derivative in the hypothesis for some technical reasons. Therefore one could expect that the results are not sharp. In fact, for spectral multipliers associated to the sub-Laplacian, one can prove Theorem 1.3 using  $n+2$  derivatives (see [4]) for any given  $n$ . Proving the above theorems for  $n+1$  derivatives is an interesting open problem.

The paper is structured as follows. In Section 2 we will discuss some preliminaries about Heisenberg groups. In Section 3 we will show that the differential-difference operators defined here are actually similar to those of [21]. Section 4 is devoted to proving some crucial estimates and also proving Theorem 1.2. In Section 5, Theorem 1.3 will be proved.

**2. Preliminaries.** Let us consider the Heisenberg group  $H^n = \mathbb{C}^n \times \mathbb{R}$  equipped with the group operation

$$(z, t)(w, s) = \left( z + w, t + s + \frac{1}{2} \Im z \cdot \bar{w} \right).$$

It is a two-step nilpotent Lie group whose center is  $\{(0, t) : t \in \mathbb{R}\}$ . The Haar measure on  $H^n$  is simply the Lebesgue measure  $dz dt$  on  $\mathbb{C}^n \times \mathbb{R}$ . The homogeneous norm  $|(z, t)|$  is given by  $(\frac{1}{16}(\sum_{i=1}^n |z_i|^2)^2 + t^2)^{1/4}$ . We will use the notation  $\rho(z, t) = |(z, t)|^4$ .

The representation theory of  $H^n$  is well-studied due to the Stone-von Neumann theorem. The representations which are trivial at the center are

merely one-dimensional. On the other hand, the representations which are non-trivial at the center are called *Schrödinger representations* and for each  $\lambda \in \mathbb{R} \setminus \{0\}$  they are explicitly given by

$$\pi_\lambda(z, t)\phi(\xi) = e^{i\lambda t} e^{i\lambda(x \cdot \xi + \frac{1}{2}x \cdot y)} \phi(\xi + y)$$

where  $\phi \in L^2(\mathbb{R}^n)$ , the corresponding Hilbert space. The group Fourier transform of a function  $f \in L^1(H^n)$  is given by

$$\hat{f}(\lambda) = \int_{H^n} f(z, t) \pi_\lambda(z, t) dz dt.$$

If  $f^\lambda$  stands for the inverse Fourier transform of  $f$  in the last variable, then the group Fourier transform can be written as

$$\hat{f}(\lambda) = \int_{\mathbb{C}^n} f^\lambda(z) \pi_\lambda(z) dz,$$

where  $\pi_\lambda(z) = \pi_\lambda(z, 0)$ . This leads us to define the *Weyl transform* of a function  $f$  on  $L^1(\mathbb{C}^n)$  in the following way:

$$W_\lambda(f) = \int_{\mathbb{C}^n} f(z) \pi_\lambda(z) dz.$$

Thus we have the following relation between the group Fourier transform on the Heisenberg group and the Weyl transform:

$$\hat{f}(\lambda) = W_\lambda(f^\lambda).$$

For a given  $g \in L^1 \cap L^2(\mathbb{C}^n)$ , it can be shown that  $W_\lambda(g)$  is a Hilbert–Schmidt operator satisfying

$$\|g\|_{L^2}^2 = (2\pi)^{-n} |\lambda|^n \|W_\lambda(g)\|_{\text{HS}}^2.$$

In fact, the map  $g \mapsto W_\lambda(g)$  can be extended to an isometric isomorphism from  $L^2(\mathbb{C}^n)$  to  $S_2$ , the space of all Hilbert–Schmidt operators on  $L^2(\mathbb{R}^n)$ .

From the relation between the Fourier transform on the Heisenberg group and the Weyl transform it is clear that for any given  $f \in L^1 \cap L^2(H^n)$  and for any  $\lambda \in \mathbb{R}^*$ ,  $\hat{f}(\lambda)$  is also a Hilbert–Schmidt operator. The map  $f \mapsto \hat{f}(\lambda)$  extends to an isometric isomorphism from  $L^2(H^n)$  to  $L^2(\mathbb{R}^*, S_2, (2\pi)^{-n-1} |\lambda|^n d\lambda)$  and the Plancherel theorem can be read as

$$\|f\|_{L^2(H^n)}^2 = (2\pi)^{-n-1} \int_{-\infty}^{\infty} \|\hat{f}(\lambda)\|_{\text{HS}}^2 |\lambda|^n d\lambda.$$

Now let  $m \in B(L^2(\mathbb{R}^n))$ . Consider the operator  $T_m^\lambda$  defined as

$$W_\lambda(T_m^\lambda f) = m W_\lambda(f).$$

From the Plancherel formula, it is clear that  $T_m^\lambda$  is bounded on  $L^2(\mathbb{C}^n)$ . If  $T_m^\lambda$  can be extended to a bounded linear operator on  $L^p(\mathbb{C}^n)$  then  $m$  is called a

(left) *Weyl multiplier* for  $L^p(\mathbb{C}^n)$ . Weyl multipliers are studied in [22] and [2]; see also [27] for more details on the Heisenberg group.

In the Introduction we already defined the Fourier multipliers on the Heisenberg group associated to a uniformly bounded family of operators. If  $M = \{M(\lambda) \in B(L^2(\mathbb{R}^n)) : \lambda \in \mathbb{R}^*\}$  is a family of operators which are uniformly bounded, it is shown in [2] that

$$T_M f(z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda t} T_{M(\lambda)}^\lambda f^\lambda(z) d\lambda$$

is bounded on  $L^2(H^n)$ , where  $T_{M(\lambda)}^\lambda$  is the Weyl multiplier associated to  $M(\lambda)$ .

The left invariant vector fields on the Heisenberg group given by

$$T = \frac{\partial}{\partial t}, \quad X_j = \frac{\partial}{\partial x_j} + \frac{1}{2} y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - \frac{1}{2} x_j \frac{\partial}{\partial t}$$

give rise to families of unbounded operators  $Z_j(\lambda)$  and  $\bar{Z}_j(\lambda)$  defined via

$$e^{-i\lambda t} Z_j(\lambda) f(z) = \frac{1}{2} (X_j - iY_j)(e^{-i\lambda t} f(z))$$

and

$$e^{-i\lambda t} \bar{Z}_j(\lambda) f(z) = \frac{1}{2} (X_j + iY_j)(e^{-i\lambda t} f(z)).$$

They have the following explicit forms:

$$Z_j(\lambda) = \frac{\partial}{\partial z_j} + \frac{\lambda}{4} \bar{z}_j, \quad \bar{Z}_j(\lambda) = \frac{\partial}{\partial \bar{z}_j} - \frac{\lambda}{4} z_j.$$

The following lemma is well-known.

LEMMA 2.1. *For any  $\lambda > 0$  and  $f \in L^2(\mathbb{C}^n)$  we have*

- (1)  $W_\lambda(Z_j(\lambda)f) = -\frac{i}{2} W_\lambda(f) A_j^*(\lambda)$ ,  $W_\lambda(\bar{Z}_j(\lambda)f) = -\frac{i}{2} W_\lambda(f) A_j(\lambda)$ ,
- (2)  $\lambda W_\lambda(z_j f) = i[W_\lambda(f), A_j(\lambda)]$ ,  $\lambda W_\lambda(\bar{z}_j f) = i[A_j^*(\lambda), W_\lambda(f)]$ .

Note that for  $\lambda < 0$ , we have to replace  $A_j(\lambda)$  by  $-A_j^*(\lambda)$  and  $A_j^*(\lambda)$  by  $-A_j(\lambda)$  in the above lemma.

The next lemma gives an expression for the derivatives of  $T_{m(\lambda)}^\lambda$  which will be used later.

LEMMA 2.2.

$$\begin{aligned} & \frac{d^k}{d\lambda^k} T_{m(\lambda)}^\lambda f(z) \\ &= \sum_{|\alpha|+|\beta|+|\gamma|+|\rho|+2s=2k} C_{\alpha,\beta} |\lambda|^{-\frac{|\alpha|+|\beta|}{2}} T_{\delta^\alpha(\lambda)\bar{\delta}^\beta(\lambda)\Theta^s(\lambda)m(\lambda)}^\lambda (z^\gamma \bar{z}^\rho f)(z). \end{aligned}$$

*Proof.* We will assume  $\lambda > 0$  and prove this lemma only for  $k = 1, 2$ . For general  $k$ , it can be proved by induction. We will show that

$$(1) \quad \frac{d}{d\lambda} T_{m(\lambda)}^\lambda = T_{\Theta(\lambda)m(\lambda)}^\lambda + \sum_{j=1}^n \left( \frac{i}{4\sqrt{\lambda}} T_{\delta_j(\lambda)m(\lambda)}^\lambda (\bar{z}_j f^\lambda) - \frac{i}{4\sqrt{\lambda}} T_{\bar{\delta}_j(\lambda)m(\lambda)}^\lambda (z_j f) \right).$$

Now in [2, Lemma 2.4] it was found that

$$(2) \quad \frac{d}{d\lambda} T_{m(\lambda)}^\lambda = T_{\frac{d}{d\lambda} m(\lambda)}^\lambda + T_{\frac{1}{2\lambda}[m(\lambda), \xi \cdot \nabla]}^\lambda + \frac{1}{2\lambda} [B, T_{m(\lambda)}^\lambda]$$

where  $B = \sum_{j=1}^n (z_j \frac{\partial}{\partial z_j} + \bar{z}_j \frac{\partial}{\partial \bar{z}_j})$ . As

$$z_j \frac{\partial}{\partial z_j} + \bar{z}_j \frac{\partial}{\partial \bar{z}_j} = z_j Z_j(\lambda) + \bar{z}_j \bar{Z}_j(\lambda),$$

we have

$$[B, T_{m(\lambda)}^\lambda] = \sum_{j=1}^n [z_j Z_j(\lambda) + \bar{z}_j \bar{Z}_j(\lambda), T_{m(\lambda)}^\lambda].$$

Using Lemma 2.1 one can easily see that

$$\begin{aligned} W_\lambda(z_j Z_j(\lambda) T_{m(\lambda)}^\lambda f) &= \frac{1}{2\sqrt{\lambda}} \delta_j(\lambda) (m(\lambda) W_\lambda(f) A_j^*(\lambda)), \\ W_\lambda(\bar{z}_j \bar{Z}_j(\lambda) T_{m(\lambda)}^\lambda f) &= \frac{1}{2\sqrt{\lambda}} \bar{\delta}_j(\lambda) (m(\lambda) W_\lambda(f) A_j(\lambda)), \\ W_\lambda(T_{m(\lambda)}^\lambda (z_j Z_j(\lambda) f)) &= \frac{1}{2\sqrt{\lambda}} m(\lambda) \delta_j(\lambda) (W_\lambda(f) A_j^*(\lambda)), \\ W_\lambda(T_{m(\lambda)}^\lambda (\bar{z}_j \bar{Z}_j(\lambda) f)) &= \frac{1}{2\sqrt{\lambda}} m(\lambda) \bar{\delta}_j(\lambda) (W_\lambda(f) A_j(\lambda)). \end{aligned}$$

Putting together the above relations we get

$$\begin{aligned} & W_\lambda \left( \left[ z_j \frac{\partial}{\partial z_j} + \bar{z}_j \frac{\partial}{\partial \bar{z}_j}, T_{m(\lambda)}^\lambda \right] f \right) \\ &= \frac{1}{2\sqrt{\lambda}} (\bar{\delta}_j(\lambda) m(\lambda)) W_\lambda(f) A_j(\lambda) + \frac{1}{2\sqrt{\lambda}} (\delta_j(\lambda) m(\lambda)) W_\lambda(f) A_j^*(\lambda) \\ &= \frac{1}{2} (\bar{\delta}_j(\lambda) m(\lambda)) \delta_j(\lambda) W_\lambda(f) + \frac{1}{2\sqrt{\lambda}} (\bar{\delta}_j(\lambda) m(\lambda)) A_j(\lambda) W_\lambda(f) \\ &\quad - \frac{1}{2} (\delta_j(\lambda) m(\lambda)) \bar{\delta}_j(\lambda) W_\lambda(f) + \frac{1}{2\sqrt{\lambda}} (\delta_j(\lambda) m(\lambda)) A_j^*(\lambda) W_\lambda(f). \end{aligned}$$

Using (2) and the above equation we get the required result.

Now we will prove the lemma for  $k = 2$ . From (1) we have

$$\begin{aligned} \frac{d^2}{d\lambda^2} T_{m(\lambda)}^\lambda &= \frac{d}{d\lambda} T_{\Theta(\lambda)m(\lambda)}^\lambda \\ &+ \sum_{j=1}^n \frac{d}{d\lambda} \left( \frac{i}{4\sqrt{\lambda}} T_{\delta_j(\lambda)m(\lambda)}^\lambda(\bar{z}_j f^\lambda) - \frac{i}{4\sqrt{\lambda}} T_{\delta_j(\lambda)m(\lambda)}^\lambda(z_j f) \right). \end{aligned}$$

The first term of the right hand side can be dealt with similarly to the case  $k = 1$ . So, it is enough to consider  $\frac{d}{d\lambda} \left( \frac{1}{\sqrt{\lambda}} T_{\delta_j(\lambda)m(\lambda)}^\lambda(z_j f) \right)$ . Observe that

$$\frac{1}{\sqrt{\lambda}} T_{\delta_j(\lambda)m(\lambda)}^\lambda(z_j f)(z) = -i\bar{z}_j T_{m(\lambda)}^\lambda(z_j f)(z) + iT_{m(\lambda)}^\lambda(z_j \bar{z}_j f)(z).$$

Hence,

$$\frac{d}{d\lambda} \left( \frac{1}{\sqrt{\lambda}} T_{\delta_j(\lambda)m(\lambda)}^\lambda(z_j f) \right)(z) = -i\bar{z}_j \frac{d}{d\lambda} T_{m(\lambda)}^\lambda(z_j f)(z) + i \frac{d}{d\lambda} T_{m(\lambda)}^\lambda(z_j \bar{z}_j f)(z).$$

Both terms can be handled similarly to the case  $k = 1$ , and this leads to the desired result. ■

Let  $\Phi_\mu$ ,  $\mu \in \mathbb{N}^n$ , stand for the normalized Hermite functions and

$$\Phi_\mu^\lambda(\xi) = |\lambda|^{n/4} \Phi_\mu(|\lambda|^{1/2} \xi).$$

Then it is well-known that

$$A_j(\lambda) \Phi_\mu^\lambda = \sqrt{2\mu_j |\lambda|} \Phi_{\mu - e_j}^\lambda \quad \text{and} \quad A_j^*(\lambda) \Phi_\mu^\lambda = \sqrt{(2\mu_j + 2)|\lambda|} \Phi_{\mu + e_j}^\lambda.$$

Now, we will consider the Hermite multipliers. For any bounded function  $a$  on  $\mathbb{N} \times \mathbb{R}^*$ , they can be defined as

$$a(H(\lambda)) = \sum_{k=0}^{\infty} a(k, \lambda) P_k(\lambda)$$

where  $P_k(\lambda)$  is the projection on the eigenspace of  $H(\lambda)$  corresponding to the eigenvalue  $(2k+n)|\lambda|$ . Let us also consider the finite difference operators acting on  $a$ ,

$$\begin{aligned} \Delta_+ a(k, \lambda) &= a(k+1, \lambda) - a(k, \lambda), \\ \Delta_- a(k, \lambda) &= a(k, \lambda) - a(k-1, \lambda). \end{aligned}$$

The next theorem is the  $\lambda$ -version of [22, Lemma 2.1] and will be used to prove the crucial estimate in this article.

LEMMA 2.3. *Let  $p, q \in \mathbb{N}^n$ . Then there exist constants  $C_{p,q,r}$  such that*

$$\begin{aligned} \delta^p(\lambda) \bar{\delta}^q(\lambda) a(H(\lambda)) \\ = \sum C_{p,q,r} |\lambda|^{-\frac{|q|+2|r|-|p|}{2}} (A^*(\lambda))^{q+r-p} A^r(\lambda) (\Delta_-^{|r|} \Delta_+^{|q|} a)(H(\lambda)) \end{aligned}$$

where the sum is over the set of  $r \in \mathbb{N}^n$  such that  $0 \leq r \leq p \leq q+r$ .



**3. The relation between  $\delta_j(\lambda)$ ,  $\bar{\delta}(\lambda)$ ,  $\Theta(\lambda)$  and the differential-difference operators defined in [21] and [10].** In this section we will show that the differential-difference operators that we have defined earlier are similar to those in [21]. The only difference is that in our case we have realized the operators on  $L^2(\mathbb{R}^n)$  whereas C. C. Lin considered operators on Fock spaces, which are actually isomorphic to  $L^2(\mathbb{R}^n)$ .

In order to define the partial isometries let us first set up some notation. Let  $(\lambda, m, \alpha) \in \mathbb{R}^* \times \mathbb{Z}^n \times \mathbb{N}^n$ . Define

$$\begin{aligned} m_i^+ &= \max\{m_i, 0\}, & m_i^- &= -\min\{m_i, 0\}, \\ m^+ &= (m_1^+, \dots, m_n^+), & m^- &= (m_1^-, \dots, m_n^-). \end{aligned}$$

As defined in [21] and [10], the partial isometries on  $L^2(\mathbb{R}^n)$  can be defined as follows:

$$V_\alpha^m(\lambda)\Phi_\mu^\lambda = (-1)^{|m^+|}\delta_{\alpha+m^+, \mu}\Phi_{\alpha+m^-}^\lambda \quad \text{when } \lambda > 0$$

and

$$V_\alpha^m(\lambda) = [V_\alpha^m(-\lambda)]^* \quad \text{when } \lambda < 0.$$

Here,  $\delta_{\alpha, \beta}$  stands for the Kronecker delta. Now consider an operator  $M(\lambda)$  which is a finite linear combination of the partial isometries, that is,

$$M(\lambda) = \sum_{m, \alpha} B(\lambda, m, \alpha) V_\alpha^m(\lambda)$$

where the sum runs over a finite subset of  $\mathbb{Z}^n \times \mathbb{N}^n$ . We will calculate  $\delta_j(\lambda)M(\lambda)$ ,  $\bar{\delta}_j(\lambda)M(\lambda)$  and for  $\lambda > 0$ ,  $\Theta(\lambda)M(\lambda)$ . We have

$$\begin{aligned} \delta_j(\lambda)M(\lambda)\Phi_\mu^\lambda &= |\lambda|^{-1/2}[M(\lambda), A_j(\lambda)]\Phi_\mu^\lambda \\ &= \sqrt{2} \sum_{m, \alpha} B(\lambda, m, \alpha)(\alpha_j + m_j^+ + 1)^{1/2}(-1)^{|m^+|}\delta_{\alpha+m^+, \mu}\Phi_{\alpha+m^-}^\lambda \\ &\quad - \sqrt{2} \sum_{m, \alpha} B(\lambda, m, \alpha)(\alpha_j + m_j^-)^{1/2}(-1)^{|m^+|}\delta_{\alpha+m^+, \mu}\Phi_{\alpha+m^- - e_j}^\lambda. \end{aligned}$$

Now, if  $m_j \geq 1$  for all  $m$  appearing in the sum, then the above equals

$$(3) \quad \begin{aligned} &\sqrt{2} \sum_{m, \alpha} B(\lambda, m - e_j, \alpha + e_j)(\alpha_j + 1)^{1/2}V_\alpha^m(\lambda)\Phi_\mu^\lambda \\ &\quad - \sqrt{2} \sum_{m, \alpha} B(\lambda, m - e_j, \alpha)(\alpha_j + m_j)^{1/2}V_\alpha^m(\lambda)\Phi_\mu^\lambda, \end{aligned}$$

whereas if  $m_j \leq 0$ , then  $\delta_j(\lambda)M(\lambda)\Phi_\mu^\lambda$  equals

$$\begin{aligned} &\sqrt{2} \sum_{m, \alpha} B(\lambda, m - e_j, \alpha - e_j)\alpha_j^{1/2}V_\alpha^m(\lambda)\Phi_\mu^\lambda \\ &\quad - \sqrt{2} \sum_{m, \alpha} B(\lambda, m - e_j, \alpha)(\alpha_j - m_j + 1)^{1/2}V_\alpha^m(\lambda)\Phi_\mu^\lambda. \end{aligned}$$

From the above calculation we can easily see that  $|\lambda|^{-1/2}\delta_j(\lambda)M(\lambda)$  is actually similar to  $\Delta_{z_j}$  defined in [21]. Similarly we can show that  $|\lambda|^{-1/2}\bar{\delta}_j(\lambda)M(\lambda)$  is similar to  $\Delta_{\bar{z}_j}$  defined in [21].

We now look at  $\Theta(\lambda)M(\lambda)\Phi_\mu$ . First assume  $m_j \geq 1$  for all  $j$ . Let us first calculate  $\frac{1}{4\lambda\sqrt{\lambda}}\delta_j(\lambda)M(\lambda)A^*(\lambda)\Phi_\mu^\lambda$ , which equals

$$\frac{1}{4\lambda}(2\mu_j + 2)^{1/2}\delta_j(\lambda)M(\lambda)\Phi_{\mu+e_j}^\lambda.$$

On account of (3) the above equals

$$\begin{aligned} & \frac{1}{2\lambda} \sum_{m,\alpha} (\alpha_j + m_j + 1) B(\lambda, m, \alpha) V_\alpha^m(\lambda) \Phi_\mu^\lambda \\ & - \frac{1}{2\lambda} \sum_{m,\alpha} \sqrt{(\alpha_j + 1)(\alpha_j + m_j + 1)} B(\lambda, m, \alpha + e_j) V_\alpha^m(\lambda) \Phi_\mu^\lambda. \end{aligned}$$

Similarly, we can show that

$$\begin{aligned} \frac{1}{4\lambda\sqrt{\lambda}}\bar{\delta}_j(\lambda)M(\lambda)A(\lambda)\Phi_\mu^\lambda &= \frac{1}{2\lambda} \sum_{m,\alpha} \sqrt{\alpha_j(\alpha_j + m_j)} B(\lambda, m, \alpha - e_j) V_\alpha^m(\lambda) \Phi_\mu^\lambda \\ & - \frac{1}{2\lambda} \sum_{m,\alpha} (\alpha_j + m_j) B(\lambda, m, \alpha) V_\alpha^m(\lambda) \Phi_\mu^\lambda. \end{aligned}$$

Thus, if  $m_j \geq 1$  for all  $m \in \mathbb{Z}^n$  appearing in the sum of  $M(\lambda)$ , we have

$$\begin{aligned} (4) \quad & \frac{1}{4\lambda\sqrt{\lambda}}\delta_j(\lambda)M(\lambda)A^*(\lambda)\Phi_\mu^\lambda + \frac{1}{4\lambda\sqrt{\lambda}}\bar{\delta}_j(\lambda)M(\lambda)A(\lambda)\Phi_\mu^\lambda \\ &= \frac{1}{2\lambda} \sum_{m,\alpha} \sqrt{\alpha_j(\alpha_j + |m_j|)} B(\lambda, m, \alpha - e_j) V_\alpha^m(\lambda) \Phi_\mu^\lambda \\ & - \frac{1}{2\lambda} \sum_{m,\alpha} \sqrt{(\alpha_j + 1)(\alpha_j + |m_j| + 1)} B(\lambda, m, \alpha + e_j) V_\alpha^m(\lambda) \Phi_\mu^\lambda \\ & + \frac{1}{2\lambda} \sum_{m,\alpha} B(\lambda, m, \alpha) V_\alpha^m(\lambda). \end{aligned}$$

Similarly we can check that if  $m_j \leq 0$  for all  $m \in \mathbb{Z}^n$  appearing in the sum of  $M(\lambda)$ , we will get the same result.

In order to calculate  $\frac{d}{d\lambda}M(\lambda)$ , let us observe that

$$M(\lambda) = \delta_{\sqrt{\lambda}} \sum_{m,\alpha} B(\lambda, m, \alpha) V_\alpha^m(1) \delta_{\sqrt{\lambda}}^{-1},$$

where for any function  $f$  on  $\mathbb{R}^n$ ,  $\delta_\lambda(f)(\xi) = f(\lambda\xi)$ . Hence, it is easy to see that (see [15])

$$(5) \quad \frac{d}{d\lambda}M(\lambda) = \sum_{m,\alpha} \frac{\partial}{\partial \lambda} B(\lambda, m, \alpha) V_\alpha^m(\lambda) - \frac{1}{2\lambda} [M(\lambda), x \cdot \nabla].$$

From (4), (5) and the definition of  $\Theta(\lambda)$  we have

$$(6) \quad \begin{aligned} \Theta(\lambda)M(\lambda) &= \sum_{m,\alpha} \frac{\partial}{\partial \lambda} B(\lambda, m, \alpha) V_\alpha^m(\lambda) + \frac{n}{2\lambda} \sum_{m,\alpha} B(\lambda, m, \alpha) V_\alpha^m(\lambda) \\ &\quad + \frac{1}{2\lambda} \sum_{m,\alpha} \sum_{j=1}^n \sqrt{\alpha_j(\alpha_j + |m_j|)} B(\lambda, m, \alpha - e_j) V_\alpha^m(\lambda) \\ &\quad - \frac{1}{2\lambda} \sum_{m,\alpha} \sum_{j=1}^n \sqrt{(\alpha_j + 1)(\alpha_j + |m_j| + 1)} B(\lambda, m, \alpha + e_j) V_\alpha^m(\lambda), \end{aligned}$$

which is similar to  $\Delta_t$  defined in [21].

**4. Some kernel estimates and proof of Theorem 1.2.** As in [10] and [21], to prove Theorem 1.2 we will also use [8, Theorem 3.1]. For that we need to find a well-behaved approximate identity which satisfies a certain estimate. Let us consider the function

$$\varphi_k^{n-1}(z) = L_k^{n-1}\left(\frac{1}{2}|z|^2\right) e^{-\frac{1}{4}|z|^2}$$

where  $L_k^{n-1}$  are the usual Laguerre polynomials of type  $n-1$  and degree  $k$ . Also, define  $\varphi_{k,\lambda}^{n-1}(z) = \varphi_k^{n-1}(\sqrt{|\lambda|}z)$ . Let  $\phi_r(z, t)$  be the Fourier transform in the  $\lambda$ -variable of the function

$$\phi_r^\lambda(z) = C_n \sum_k e^{-2r(2k+n)|\lambda|} |\lambda|^n \varphi_{k,\lambda}^{n-1}(z),$$

where  $C_n^{-1} = \int_{H^n} \phi_1(z, t) dz dt$ . The functions  $\phi_r$  will play the role of approximate identity. In fact, they satisfy the following properties.

LEMMA 4.1. *For each  $r > 0$ , let  $\phi_r$  be defined above. Then*

- (i)  $\phi_r(z, t) = r^{-n+1} \phi_1(r^{-\frac{1}{2}}z, r^{-1}t)$ ,  $r > 0$ ,
- (ii)  $\int_{H^n} |\phi_r(z, t)| \left(1 + \frac{\rho(z, t)}{r^2}\right)^\eta dz dt \leq C$  for some  $\eta > 0$ ,
- (iii)  $\int_{H^n} \phi_r(z, t) dz dt = 1$ ,
- (iv)  $\phi_r * \phi_s = \phi_s * \phi_r$ ,
- (v)  $\int_{H^n} |\phi_r((z, t)(z_0, t_0)^{-1}) - \phi_r(z, t)| dz dt \leq C(\rho(z_0, t_0)/r^2)^\eta$ , where  $(z_0, t_0)$  is in  $H^n$ ,
- (vi)  $\phi_r(z, t) = \phi_r(-z, -t)$ .

The proof is the same as that of [21, Lemma 1]. Once we have the above approximate identity, we define  $\psi_r = \phi_{r/2} - \phi_r$ . Then [8, Theorem 3.1] tells us that in order to prove Theorem 1.2 we only have to prove that there exists  $\epsilon > 0$  such that

$$(7) \quad \int_{H^n} |T_M \psi_r(z, t)| \left(1 + (\rho(z, t)/r^2)^\epsilon\right) dz dt < C.$$

As shown in [21], we can also show that to prove (7) it is enough to show

$$(8) \quad \int_{H^n} |T_M \psi_r(x)|^2 \rho(x)^{[(n+3)/2]} dx \leq Cr^{2[(n+3)/2]-(n+1)}, \quad 0 < r < \infty.$$

We first need some results which will be used to prove (8).

**THEOREM 4.2.** *If  $f$  is a Schwartz class function in  $H^n$ , then*

$$(\widehat{if})(\lambda) = \Theta(\lambda) \hat{f}(\lambda).$$

*Proof.* For  $\alpha, \beta \in \mathbb{N}$ , consider the special Hermite functions on  $\mathbb{C}^n$  defined by

$$\Phi_{\alpha, \beta}^\lambda(z) = (2\pi)^{-n/2} (\pi_\lambda(z) \phi_\alpha^\lambda, \phi_\beta^\lambda)$$

where the  $\phi_\alpha^\lambda$  are Hermite functions and  $\pi_\lambda(z) = \pi_\lambda(z, 0)$ . Then using [26, Proposition 1.3.2] we can easily see that

$$(9) \quad |\lambda|^n W_\lambda(\Phi_{\alpha+m^+, \alpha+m^-}^\lambda) = (2\pi)^{n/2} (-1)^{|m^+|} V_\alpha^m(\lambda).$$

Let  $f$  be a Schwartz class function on  $H^n$ . Then for any  $\lambda \in \mathbb{R}^*$ ,

$$f^\lambda = \sum_{m, \alpha} B(\lambda, m, \alpha) |\lambda|^n \Phi_{\alpha+m^+, \alpha+m^-}^\lambda.$$

We now calculate  $\frac{\partial}{\partial \lambda} f^\lambda$ . Using the relation

$$\Phi_{\alpha+m^+, \alpha+m^-}^\lambda(z) = \Phi_{\alpha+m^+, \alpha+m^-}(\sqrt{\lambda} z)$$

one can easily see that

$$\begin{aligned} \frac{\partial}{\partial \lambda} f^\lambda &= \sum_{m, \alpha} \frac{\partial}{\partial \lambda} B(\lambda, m, \alpha) |\lambda|^n \Phi_{\alpha+m^+, \alpha+m^-}^\lambda \\ &\quad + \frac{n}{\lambda} \sum_{m, \alpha} B(\lambda, m, \alpha) |\lambda|^n \Phi_{\alpha+m^+, \alpha+m^-}^\lambda \\ &\quad + \frac{1}{2\lambda} \sum_{m, \alpha} B(\lambda, m, \alpha) |\lambda|^n \sum_{j=1}^n \left( \left( z_j \frac{\partial}{\partial z_j} + \bar{z}_j \frac{\partial}{\partial \bar{z}_j} \right) \Phi_{\alpha+m^+, \alpha+m^-} \right) (\sqrt{\lambda} \cdot). \end{aligned}$$

From [26, (1.3.17), (1.3.18)] and [26, (1.3.23)] we have

$$\begin{aligned} &2z_j \frac{\partial}{\partial z_j} \Phi_{\alpha+m^+, \alpha+m^-} \\ &= \frac{i}{2} z_j [(2(\alpha_j + m_j^+) + 2)^{1/2} \Phi_{\alpha+m^+ + e_j, \alpha+m^-} + (2(\alpha_j + m_j^-))^{1/2} \Phi_{\alpha, \alpha+m^- - e_j}] \end{aligned}$$

$$\begin{aligned}
 &= -\frac{1}{2}[(2(\alpha_j + m_j^+) + 2)\Phi_{\alpha+m^+, \alpha+m^-} \\
 &\quad - (2(\alpha_j + m_j^+) + 2)^{1/2}(2(\alpha_j + m_j^-) + 2)^{1/2}\Phi_{\alpha+m^++e_j, \alpha+m^-+e_j}] \\
 &\quad - \frac{1}{2}[(2(\alpha_j + m_j^+))^{1/2}(2(\alpha_j + m_j^-))^{1/2}\Phi_{\alpha+m^+-e_j, \alpha+m^- - e_j} \\
 &\quad \quad \quad - 2(\alpha_j + m_j^-)\Phi_{\alpha+m^+, \alpha+m^-}] \\
 &= (\alpha_j + m_j^+ + 1)^{1/2}(\alpha_j + m_j^- + 1)^{1/2}\Phi_{\alpha+m^++e_j, \alpha+m^-+e_j} \\
 &\quad - (\alpha_j + m_j^+)^{1/2}(\alpha_j + m_j^-)^{1/2}\Phi_{\alpha+m^+-e_j, \alpha+m^- - e_j} + (m_j - 1)\Phi_{\alpha+m^+, \alpha+m^-}.
 \end{aligned}$$

Similarly, we can also show that

$$\begin{aligned}
 2\bar{z}_j \frac{\partial}{\partial \bar{z}_j} \Phi_{\alpha+m^+, \alpha+m^-} &= (\alpha_j + m_j^+ + 1)^{1/2}(\alpha_j + m_j^- + 1)^{1/2}\Phi_{\alpha+m^++e_j, \alpha+m^-+e_j} \\
 &\quad - (\alpha_j + m_j^+)^{1/2}(\alpha_j + m_j^-)^{1/2}\Phi_{\alpha+m^+-e_j, \alpha+m^- - e_j} \\
 &\quad - (m_j + 1)\Phi_{\alpha, \alpha+m^-}.
 \end{aligned}$$

Combining the above two relations we get

$$\begin{aligned}
 \left( z_j \frac{\partial}{\partial z_j} + \bar{z}_j \frac{\partial}{\partial \bar{z}_j} \right) \Phi_{\alpha+m^+, \alpha+m^-} &= ((\alpha_j + m_j^+) + 1)^{1/2}((\alpha_j + m_j^-) + 1)^{1/2}\Phi_{\alpha+m^++e_j, \alpha+m^-+e_j} \\
 &\quad - (\alpha_j + m_j^+)^{1/2}(\alpha_j + m_j^-)^{1/2}\Phi_{\alpha+m^+-e_j, \alpha+m^- - e_j} - \Phi_{\alpha, \alpha+m^-}.
 \end{aligned}$$

Consequently,

$$\begin{aligned}
 \frac{\partial}{\partial \lambda} f^\lambda &= \sum_{m, \alpha} \frac{\partial}{\partial \lambda} B(\lambda, m, \alpha) |\lambda|^n \Phi_{\alpha+m^+, \alpha+m^-}^\lambda \\
 &\quad + \frac{n}{2\lambda} \sum_{m, \alpha} B(\lambda, m, \alpha) |\lambda|^n \Phi_{\alpha+m^+, \alpha+m^-}^\lambda \\
 &\quad + \frac{1}{2\lambda} \sum_{m, \alpha} B(\lambda, m, \alpha - e_j) \sqrt{\alpha_j(\alpha_j + |m_j|)} |\lambda|^n \Phi_{\alpha+m^+, \alpha+m^-}^\lambda \\
 &\quad - \frac{1}{2\lambda} \sum_{m, \alpha} B(\lambda, m, \alpha + e_j) \sqrt{(\alpha_j + 1)(\alpha_j + |m_j| + 1)} |\lambda|^n \Phi_{\alpha+m^+, \alpha+m^-}^\lambda.
 \end{aligned}$$

Using (6) and (9), we get the required result. ■

For any function  $b$  defined on  $\mathbb{R}$ , consider the function

$$b(L_\lambda) = \sum_{k=0}^{\infty} b((2k+n)|\lambda|) |\lambda|^n \varphi_{k, \lambda}^{n-1}.$$

Here  $L_\lambda$  stands for special Hermite operators with parameter  $\lambda$ . Then we have the following corollary.

COROLLARY 4.3. *For any Schwarz class function  $b$  defined on  $\mathbb{R}$ ,*

$$\begin{aligned} \frac{d}{d\lambda}b(L_\lambda) &= \sum_{k=0}^{\infty} (2k+n)b'((2k+n)|\lambda|)|\lambda|^n \varphi_{k,\lambda}^{n-1} \\ &\quad - \frac{1}{2\lambda} \sum_{k=0}^{\infty} k \Delta_- b((2k+n)|\lambda|)|\lambda|^n \varphi_{k,\lambda}^{n-1} \\ &\quad - \frac{1}{2\lambda} \sum_{k=0}^{\infty} (k+n) \Delta_+ b((2k+n)|\lambda|)|\lambda|^n \varphi_{k,\lambda}^{n-1}. \end{aligned}$$

*Proof.* As  $\varphi_{k,\lambda}^{n-1} = \sum_{|\alpha|=k} \Phi_{\alpha,\alpha}^\lambda$  from the proof of the above theorem we have

$$\begin{aligned} \frac{d}{d\lambda}b(L_\lambda) &= \sum_{k=0}^{\infty} (2k+n)b'((2k+n)|\lambda|)|\lambda|^n \varphi_{k,\lambda}^{n-1} \\ &\quad + \frac{n}{2\lambda} \sum_{k=0}^{\infty} b((2k+n)|\lambda|)|\lambda|^n \varphi_{k,\lambda}^{n-1} \\ &\quad + \frac{1}{2\lambda} \sum_{k=0}^{\infty} b((2k-2+n)|\lambda|)|\lambda|^n \sum_{|\alpha|=k} \sum_{j=1}^n \alpha_j \Phi_{\alpha,\alpha}(\sqrt{|\lambda|}z) \\ &\quad - \frac{1}{2\lambda} \sum_{k=0}^{\infty} b((2k+2+n)|\lambda|)|\lambda|^n \sum_{|\alpha|=k} \sum_{j=1}^n (\alpha_j+1) \Phi_{\alpha,\alpha}(\sqrt{|\lambda|}z) \\ &= \sum_{k=0}^{\infty} (2k+n)b'((2k+n)|\lambda|)|\lambda|^n \varphi_{k,\lambda}^{n-1} \\ &\quad - \frac{1}{2\lambda} \sum_{k=0}^{\infty} k \Delta_- b((2k+n)|\lambda|)|\lambda|^n \varphi_{k,\lambda}^{n-1} \\ &\quad - \frac{1}{2\lambda} \sum_{k=0}^{\infty} (k+n) \Delta_+ b((2k+n)|\lambda|)|\lambda|^n \varphi_{k,\lambda}^{n-1}. \end{aligned}$$

Hence the corollary is proved. ■

The next lemma is similar to [21, Lemma 2].

LEMMA 4.4. *The following inequality holds:*

$$(10) \quad \left\| |\lambda|^{-\frac{|\alpha|+|\beta|}{2}} \left\| \chi_N(\lambda) \delta^\alpha(\lambda) \bar{\delta}^\beta(\lambda) W_\lambda \left( \frac{\partial^l}{\partial \lambda^l} \psi_r^\lambda \right) \right\|_{\text{op}} \right\| \leq C 2^{-N(l+\frac{|\alpha|+|\beta|}{2})} f_{\alpha,\beta,l}(r2^N)$$

for all  $|\alpha| + |\beta| + 2l \leq 2[(n+3)/2]$ , where  $f_{\alpha,\beta,l}$  is a rapidly decreasing function.

REMARK 4.5. As mentioned earlier, in [21] Lin gave a detailed proof of Lemma 2 only for some very particular type of polynomials of the form  $P(z, t) = z_1^a \bar{z}_1^b$  with  $a \geq b$  and  $P(z, t) = z_1^a \bar{z}_2^b$  and that also involved very long and technical calculations. We have already discussed in Section 2 that  $|\lambda|^{-1/2} \delta_j(\lambda)$  and  $|\lambda|^{-1/2} \bar{\delta}_j(\lambda)$  are similar to the operators  $\Delta_{z_j}$  and  $\Delta_{\bar{z}_j}$  defined in [21], respectively. So, we can now handle the case associated with the operators  $|\lambda|^{-(|\alpha|+|\beta|)/2} \delta^\alpha(\lambda) \bar{\delta}^\beta(\lambda)$  very easily by using Lemma 2.3. It turns out to be a little more difficult when we have to consider the operator also involving  $\Theta(\lambda)$ .

*Proof of Lemma 4.4.* We will prove this lemma only for  $\lambda > 0$ ; the other case can be handled similarly. Let

$$b_r(x) = e^{-\frac{r}{2}x} - e^{-rx}.$$

From Lemma 4.3 we can see that the Hermite coefficient of  $W_\lambda\left(\frac{\partial}{\partial \lambda} b_r(L_\lambda)\right)$  is

$$\left( \frac{\partial}{\partial \lambda} - \frac{1}{2\lambda} k \Delta_- - \frac{1}{2\lambda} (k+n) \Delta_+ \right) b_r((2k+n)\lambda).$$

For convenience we set

$$\Gamma_\lambda^k := \frac{\partial}{\partial \lambda} - \frac{1}{2\lambda} k \Delta_- - \frac{1}{2\lambda} (k+n) \Delta_+.$$

Then  $\Gamma_\lambda^k b_r((2k+n)\lambda)$  equals

$$\begin{aligned} & (2k+n)b_r'((2k+n)\lambda) - k \int_0^1 b_r'((2(k-u)+n)\lambda) du \\ & \qquad \qquad \qquad - (k+n) \int_0^1 b_r'((2(k+u)+n)\lambda) du \\ & = -\frac{1}{2}(2k+n)\lambda \int_0^1 \int_0^u b_r''((2(k-v)+n)\lambda) du \\ & \qquad \qquad \qquad + \frac{1}{2}(2k+n)\lambda \int_0^1 \int_0^u b_r''((2(k+v)+n)\lambda) dv \\ & \qquad \qquad \qquad + \frac{n}{2} \int_0^1 b_r'((2(k-u)+n)\lambda) du - \frac{n}{2} \int_0^1 b_r'((2(k+u)+n)\lambda) du. \end{aligned}$$

Let  $Q \subset \mathbb{R}^2$  be the set enclosed by the three lines  $x = 0$ ,  $y = x$ ,  $y = 1$ , and let  $\sigma_1, \sigma_2$  be the functions on  $Q$  defined by  $\sigma_1(u, v) = -v$  and  $\sigma_2(u, v) = v$ . Also let  $\varsigma_1(w) = -w$  and  $\varsigma_2(w) = w$ , for  $w \in [0, 1]$ . Then the Hermite coefficient

of  $\frac{\partial}{\partial \lambda} b(H(\lambda))$  can be written as

$$-\frac{1}{2}(2k+n)\lambda \int_Q b_r''((2(k+\sigma_1(u,v))+n)\lambda) du dv + \frac{n}{2} \int_0^1 b_r'((2(k+\varsigma_1(w))+n)\lambda) dw$$

$$+\frac{1}{2}(2k+n)\lambda \int_Q b_r''((2(k+\sigma_2(u,v))+n)\lambda) du dv - \frac{n}{2} \int_0^1 b_r'((2(k+\varsigma_2(w))+n)\lambda) dw.$$

Now let  $Q^m$  be the cartesian product of  $m$  copies of  $Q$ . Let us use the notation  $(u, v)$  for any element  $(u_1, v_1, \dots, u_m, v_m)$  of  $Q^m$ . We claim that the Hermite coefficient of  $\frac{\partial^l}{\partial \lambda^l} b_r(H(\lambda))$  can be written as the sum of several terms (the number of terms in the sum depends only on  $n$  and  $l$ ) of the form

$$(2k+n)^m \lambda^m \int_{Q^m} \int_{[0,1]^{l-m}} g(u, v, w) b_r^{l+m}((2(k+\sigma(u,v)+\varsigma(w))+n)\lambda) du dv dw,$$

where  $0 \leq m \leq l$ . Here  $g$ ,  $\sigma$  and  $\varsigma$  are some bounded functions on  $Q^m \times [0, 1]^{l-m}$ ,  $Q^m$  and  $[0, 1]^{l-m}$  respectively where the bounds depend only on  $n$  and  $l$ . Also,  $b_r^{l+m}$  stands for the  $(l+m)$ th derivative of  $b_r$ .

We use induction on  $l$ . We have already shown that the result is true for  $l = 1$ . Suppose it is true for some  $l \in \mathbb{N}$ . Since  $\Gamma_\lambda^k$  satisfies the Leibniz rule and  $\Gamma_\lambda^k((2k+n)\lambda)$  vanishes, the Hermite coefficient of  $\frac{\partial^{l+1}}{\partial \lambda^{l+1}} b_r(H(\lambda))$  can be written as the sum of several terms of the form

$$\int_{Q^m} \int_{[0,1]^{l-m}} (2k+n)^m \lambda^m g(u, v, w) \Gamma_\lambda^k b_r^{l+m}((2(k+\sigma(u,v)+\varsigma(w))+n)\lambda) du dv dw$$

where  $0 \leq m \leq l$  and  $C$  is a constant depending only on  $n$  and  $l$ . Also  $g$ ,  $\sigma$  and  $\varsigma$  are bounded functions on  $Q^m \times [0, 1]^{l-m}$ ,  $Q^m$  and  $[0, 1]^{l-m}$  respectively. The above expression can be written as  $I_1 + I_2$  where

$$I_1 = \int_{Q^m \times [0,1]^{l-m}} (2k+n)(2k+n)^m \lambda^m g(u, v, w) b_r^{l+m+1} \times ((2(k+\sigma(u,v)+\varsigma(w))+n)\lambda) du dv dw$$

$$- \int_{Q^m \times [0,1]^{l-m}} k(2k+n)^m \lambda^m \times \int_0^1 g(u, v, w) b_r^{l+m+1}((2(k+u')+2\sigma(u,v)+2\varsigma(w)+n)\lambda) du' du dv dw$$

$$- \int_{Q^m \times [0,1]^{l-m}} (k+n)(2k+n)^m \lambda^m \times \int_0^1 g(u, v, w) b_r^{l+m+1}((2(k-u')+2\sigma(u,v)+2\varsigma(w)+n)\lambda) du' du dv dw$$



and

$$I_2 = \int_{Q^m \times [0,1]^{l-m}} (2k+n)^m \lambda^m 2(\sigma(u,v) + \varsigma(w)) g(u,v,w) b_r^{l+m+1} \\ \times ((2(k + \sigma(u,v) + \varsigma(w)) + n)\lambda) du dv dw.$$

$I_1$  can be dealt with similarly to the case  $l = 1$ . On the other hand,  $I_2$  can be written as

$$(2k+n)^m \lambda^m \int_{Q^m} \int_{[0,1]^{l-m}} \int_{[0,1]} \tilde{g}(u,v,w,w') b_r^{l+m+1} \\ \times ((2(k + \sigma(u,v) + \tilde{\varsigma}(w,w')) + n)\lambda) du dv dw dw',$$

where  $\tilde{g}(u,v,w,w') = 2(\alpha(u,v) + \beta(w))g(u,v,w)$  and  $\tilde{\varsigma}(w,w') = \varsigma(w)$ . Hence our claim is proved.

Now we are in a position to estimate  $\delta^\alpha(\lambda)\bar{\delta}^\beta(\lambda)W_\lambda(\frac{\partial^l}{\partial \lambda^l}\psi_r^\lambda)$ . From the above discussion and using Lemma 2.3 one can notice that  $\delta^\alpha(\lambda)\bar{\delta}^\beta(\lambda)W_\lambda(\frac{\partial^l}{\partial \lambda^l}\psi_r^\lambda)$  can be written as a sum of several operators of the form

$$\int_{Q^m} \int_{[0,1]^{l-m}} g(u,v,w) \lambda^{-\frac{|\beta|+2|\gamma|-\alpha}{2}} (A^*(\lambda))^{\alpha+\gamma-\beta} \\ \times A^\gamma(\lambda) D_-^{|\gamma|} D_+^{|\beta|} \tilde{b}_{r,u,v}^{l,m}(H(\lambda)) du dv dw,$$

with  $\gamma \in \mathbb{N}^n$  satisfying  $0 \leq \gamma \leq \alpha \leq \beta + \gamma$  and

$$\tilde{b}_{r,u,v}^{l,m}(k,\lambda) = (2k+n)^m \lambda^m b_r^{l+m} ((2(k + \sigma(u,v) + \varsigma(w)) + n)\lambda).$$

Of course, the total number of terms depends only on  $n, p, q$  and  $l$ . Therefore, to prove our lemma we only need to estimate the operator norm of operators of the form

$$(11) \quad \int_{Q^m} \int_{[0,1]^{l-m}} g(u,v,w) \lambda^{-\frac{|\beta|+2|\gamma|-\alpha}{2}} \chi_N(\lambda) (A^*(\lambda))^{\alpha+\gamma-\beta} A^\gamma(\lambda) \\ \times D_-^{|\gamma|} D_+^{|\beta|} \tilde{b}_{r,u,v}^{l,m}(H(\lambda)) du dv dw.$$

Since the finite difference operators can be estimated by derivatives, we have

$$|D_-^{|\gamma|} D_+^{|\beta|} \tilde{b}_{r,u,v}^{l,m}(k,\lambda)| \leq C |\partial_1^s \tilde{b}_{r,u,v}^{l,m}(k,\lambda)|,$$

where  $|\gamma| + |\beta| \leq [(n+3)/2]$  and  $\partial_1^s$  is the partial derivative of order  $s$  with respect to the first variable. The above can be further dominated by

$$(12) \quad \sum_{i=0}^s C_{r,m,l} (2k+n)^{m-i} \lambda^{m+s-i} |b_r^{l+m+s-i} ((2k+n + \sigma(u,v) + \varsigma(w))\lambda)|.$$

Now, if  $(2k+n)\lambda \sim 2^N$  and  $l+m = \vartheta$ , it can be shown [2, Lemma 2.2] that

$$|b_r^{l+m+s-i} ((2k+n + \sigma(u,v) + \varsigma(w))\lambda)| \lesssim 2^N r 2^{-N(\vartheta+s-i)} \tilde{f}_{\vartheta,s,i}(2^N r)$$

where  $\tilde{f}_{\vartheta,s,i}(x) = x^{\vartheta+s-i}e^{-cx} + x^{\vartheta+s-i-1}e^{-cx}$ , a rapidly decreasing function for each  $i$ . Hence, (12) is bounded by a constant multiple of

$$2^N r \lambda^s 2^{-N(l+s)} \tilde{f}_{l,s}(2^N r),$$

where  $\tilde{f}_{l,r}$  is a rapidly decreasing function.

Recall that

$$A_j^*(\lambda)\Phi_\mu^\lambda = (2\mu_j + 2)^{1/2} \lambda^{1/2} \Phi_{\mu+e_j}^\lambda \quad \text{and} \quad A_j(\lambda)\Phi_\mu^\lambda = (2\mu_j)^{1/2} \lambda^{1/2} \Phi_{\mu-e_j}^\lambda.$$

Hence, using the boundedness of  $g$ , the operator norm of (11) can be dominated by

$$\lambda^{-\frac{|\beta|+2|\gamma|-|\alpha|}{2}} 2^N \frac{|\alpha|+2|\gamma|-|\beta|}{2} 2^N r \lambda^{|\gamma|+|\beta|} 2^{-N(l+|\gamma|+|\beta|)} \tilde{f}_{l,r}(2^R t_{j+1}),$$

which is equal to

$$\lambda^{\frac{|\alpha|+|\beta|}{2}} 2^{-N(l+\frac{|\alpha|+|\beta|}{2})} f_{\alpha,\beta,l}(r 2^N),$$

where  $f_{\alpha,\beta,l}$  is a rapidly decreasing function. Hence the lemma is proved. ■

Now we are in a position to prove Theorem 1.2. As discussed earlier, we only have to prove (8).

LEMMA 4.6. *For  $l \in \mathbb{R}$  satisfying  $0 \leq l \leq [(n+3)/2]$ , we have*

$$\int_{H^n} |T_M \psi_r(z, t)|^2 \rho(z, t)^l dz dt \lesssim r^{2l-(n+1)}.$$

*Proof.* We will prove this only for  $l \in \mathbb{N}$ . For other  $l$  the estimate can be obtained easily by using the estimate of  $[l]$  and  $[l+1]$ .

As  $\rho(z, t)^l \lesssim (\sum_{i=1}^n |z_i|^2)^{2l} + t^{2l}$ , we will prove that

$$(13) \quad \int_{H^n} \left( \sum_{i=1}^n |z_i|^2 \right)^{2l} |T_M \psi_r(z, t)|^2 dz dt \lesssim r^{2l-(n+1)}$$

and

$$(14) \quad \int_{H^n} t^{2l} |T_M \psi_r(z, t)|^2 dz dt \lesssim r^{2l-(n+1)}.$$

We first prove (14). Using the Plancherel theorem in the  $t$ -variable we can observe that the left hand side of (14) is a constant multiple of

$$\int_{\mathbb{R}} \left\| \frac{\partial^l}{\partial \lambda^l} T_{M(\lambda)}^\lambda \psi_r^\lambda \right\|_2^2 d\lambda.$$

Thus by the Leibniz rule, it is enough to prove that

$$\int_{\mathbb{R}} \left\| \left( \frac{\partial^{l-l_1}}{\partial \lambda^{l-l_1}} T_{M(\lambda)}^\lambda \right) \left( \frac{\partial^{l_1}}{\partial \lambda^{l_1}} \psi_r^\lambda \right) \right\|_2^2 d\lambda \lesssim r^{2l-(n+1)}$$

for  $0 \leq l_1 \leq l$ . By Lemma 2.2, the left hand side can be dominated by

(15)

$$\sum_{|a|+|b|+|c|+|d|+2s=l-l_1} C_{a,b,c,d,s} \int_{\mathbb{R}} \left\| \frac{1}{|\lambda|^{\frac{|a|+|b|}{2}}} T_{\delta^a(\lambda)\bar{\delta}^b(\lambda)\Theta^s(\lambda)M(\lambda)} \left( z^c \bar{z}^d \frac{\partial^{l_1}}{\partial \lambda^{l_1}} \psi_r^\lambda \right) \right\|_{\mathbb{H}^2}^2 d\lambda.$$

Therefore, in view of the Plancherel theorem, it is enough to estimate

$$(16) \quad \int_{\mathbb{R}} \frac{1}{|\lambda|^{|a|+|\beta|+|c|+|d|}} \times \left\| \delta^a(\lambda)\bar{\delta}^b(\lambda)\Theta^s(\lambda)M(\lambda)\delta^c(\lambda)\bar{\delta}^d(\lambda)W_\lambda \left( \frac{\partial^{l_1}}{\partial \lambda^{l_1}} \psi_r^\lambda \right) \right\|_{\mathbb{H}^2}^2 |\lambda|^n d\lambda,$$

where  $|a| + |b| + |c| + |d| + 2s = l - l_1$ . Now we can write

$$\begin{aligned} & \delta^a(\lambda)\bar{\delta}^b(\lambda)\Theta^s(\lambda)M(\lambda)\delta^c(\lambda)\bar{\delta}^d(\lambda)W_\lambda \left( \frac{\partial^{l_1}}{\partial \lambda^{l_1}} \psi_r^\lambda \right) \\ &= \sum_{N=0}^{\infty} (\delta^a(\lambda)\bar{\delta}^b(\lambda)\Theta^s(\lambda)M(\lambda))\chi_N(\lambda) \cdot \chi_N(\lambda) \left( \delta^c(\lambda)\bar{\delta}^d(\lambda)W_\lambda \left( \frac{\partial^{l_1}}{\partial \lambda^{l_1}} \psi_r^\lambda \right) \right). \end{aligned}$$

So, (16) can be dominated by

$$(17) \quad \sum_{N=0}^{\infty} \int_{\mathbb{R}} \left\| \lambda^{-\frac{|a|+|b|}{2}} \delta^a(\lambda)\bar{\delta}^b(\lambda)\Theta^s(\lambda)M(\lambda)\chi_N(\lambda) \right\|_{\mathbb{H}^2}^2 \times \left\| \lambda^{-\frac{|c|+|d|}{2}} \chi_N(\lambda)\delta^c(\lambda)\bar{\delta}^d(\lambda)W_\lambda \left( \frac{\partial^{l_1}}{\partial \lambda^{l_1}} \psi_r^\lambda \right) \right\|_{\text{op}}^2 |\lambda|^n d\lambda.$$

By Lemma 4.4, we have

$$\left\| \chi_N(\lambda)\delta^c(\lambda)\bar{\delta}^d(\lambda)W_\lambda \left( \frac{\partial^{l_1}}{\partial \lambda^{l_1}} \psi_r^\lambda \right) \right\|_{\text{op}}^2 \lesssim |\lambda|^{(|c|+|d|)} 2^{-N(|c|+|d|+2l_1)} f_{c,d,l_1}(2^N r),$$

where  $f_{c,d,l_1}$  is a rapidly decreasing function. Using the above estimate and the hypothesis of Theorem 1.2 we get

$$\begin{aligned} & \int_{\mathbb{R}} \left\| \lambda^{-\frac{|a|+|b|+|c|+|d|}{2}} \delta^a(\lambda)\bar{\delta}^b(\lambda)\Theta^s(\lambda)M(\lambda)\delta^c(\lambda)\bar{\delta}^d(\lambda)W_\lambda \left( \frac{\partial^{l_1}}{\partial \lambda^{l_1}} \psi_r^\lambda \right) \right\|_{\mathbb{H}^2}^2 |\lambda|^n d\lambda \\ & \lesssim \sum_{N=0}^{\infty} 2^{N(n+1-|a|-|b|-2s)} 2^{-N(|c|+|d|+2l_1)} f_{c,d,l_1}(2^N t_{j+1}) \\ & \lesssim \sum_{N=0}^{\infty} 2^{N(n+1-2l)} f_{c,d,l_1}(2^N r) \lesssim r^{2l-(n+1)}. \end{aligned}$$

We can estimate (14) similarly by observing that it amounts to the estimate of (16) with  $s = l_1 = 0$ . ■

We conclude this section with the following lemma which will be used in Section 5 in order to prove Theorem 1.3.

LEMMA 4.7. *Let  $0 < r < 1$ . For  $l \leq [(n+1)/2]$  and  $i = 1, \dots, n$ , the following estimates are true:*

- (i)  $\int_{H^n} |\partial_t T_M \psi_r(z, t)|^2 \rho(z, t)^l dz dt \lesssim r^{2l-n-3}$ ,
- (ii)  $\int_{H^n} |X_i T_M \psi_r(z, t)|^2 \rho(z, t)^l dz dt \lesssim r^{2l-n-3}$ ,
- (iii)  $\int_{H^n} |Y_i T_M \psi_r(z, t)|^2 \rho(z, t)^l dz dt \lesssim r^{2l-n-3}$ .

*Proof.* We will first prove (i). Similarly to Lemma 4.6, here also we have to estimate

$$(18) \quad \int_{H^n} \left( \sum_{i=1}^n |z_i|^2 \right)^{2l} |\partial_t T_M \psi_r(z, t)|^2 dz dt \lesssim r^{2l-n-3},$$

$$(19) \quad \int_{H^n} t^{2l} |\partial_t T_M \psi_r(z, t)|^2 dz dt \lesssim r^{2l-n-3}.$$

We only prove (19); the estimate (18) can be proved similarly. By the Plancherel theorem in the  $t$ -variable, the left hand side of (17) is a constant multiple of

$$\int_{\mathbb{C}^n} \left\| \frac{\partial^l}{\partial \lambda^l} (\lambda T_{M(\lambda)}^\lambda) \psi_r^\lambda \right\|_2^2 d\lambda.$$

To estimate the above integral it is enough to estimate

$$\int_{\mathbb{C}^n} \left\| \lambda \frac{\partial^l}{\partial \lambda^l} T_{M(\lambda)}^\lambda \psi_r^\lambda \right\|_2^2 d\lambda \quad \text{and} \quad \int_{\mathbb{C}^n} \left\| \frac{\partial^{l-1}}{\partial \lambda^{l-1}} T_{M(\lambda)}^\lambda \psi_r^\lambda \right\|_2^2 d\lambda.$$

Using Lemma 4.6, we can see that the second integral satisfies the required estimate, so we only have to prove the same for the first integral. Observe that if  $2^N \leq (2k+n)|\lambda| < 2^{N+1}$ , then  $|\lambda| < 2^{N+1}$ . So,

$$\begin{aligned} & \int_{-\infty}^{\infty} |\lambda|^2 \left\| \lambda^{-\frac{\alpha+\beta}{2}} \delta^\alpha(\lambda) \bar{\delta}^\beta(\lambda) \Theta^s(\lambda) M(\lambda) \chi_N(\lambda) \right\|_{\text{HS}}^2 |\lambda|^n d\lambda \\ & \lesssim 2^N \int_{-2^{N+1}}^{2^{N+1}} \left\| \lambda^{-\frac{\alpha+\beta}{2}} \delta^\alpha(\lambda) \bar{\delta}^\beta(\lambda) \Theta^s(\lambda) M(\lambda) \chi_N(\lambda) \right\|_{\text{HS}}^2 |\lambda|^n d\lambda \\ & \lesssim 2^N \cdot 2^{N(n+1-l)} < 2^{N(n+2-l)}. \end{aligned}$$

Hence, we can get the required estimate by proceeding as in the proof of Lemma 4.6 and using the fact that  $r < 1$ .

Now we will prove (ii). Again we will only estimate

$$(20) \quad \int_{H^n} t^{2l} |X_i T_M \psi_r(z, t)|^2 dz dt \lesssim r^{2l-n-2}.$$

By the Plancherel theorem and the relation between  $X_i$ ,  $Z_i(\lambda)$  and  $\bar{Z}_i(\lambda)$ , it is enough to estimate

$$(21) \quad \int_{\mathbb{C}^n} \left\| \frac{\partial^l}{\partial \lambda^l} (Z_i(\lambda) T_{M(\lambda)}^\lambda \psi_r^\lambda) \right\|_2^2 d\lambda$$

and

$$(22) \quad \int_{\mathbb{C}^n} \left\| \frac{\partial^l}{\partial \lambda^l} (\bar{Z}_i(\lambda) T_{M(\lambda)}^\lambda \psi_r^\lambda) \right\|_2^2 d\lambda.$$

Since both can be estimated similarly, we will only estimate (21). It can be dominated by

$$(23) \quad \int_{\mathbb{C}^n} \left\| Z_i(\lambda) \frac{\partial^l}{\partial \lambda^l} (T_{M(\lambda)}^\lambda \psi_r^\lambda) \right\|_2^2 d\lambda + \frac{1}{4} \int_{\mathbb{C}^n} \left\| z_i \frac{\partial^{l-1}}{\partial \lambda^{l-1}} (T_{M(\lambda)}^\lambda \psi_r^\lambda) \right\|_2^2 d\lambda.$$

We can estimate the second term similarly to (13).

In order to estimate the first term of (23) notice that it is enough to estimate (see Lemma 4.6) the term

$$\int_{\mathbb{R}} |\lambda|^{-(|a|+|b|)} \left\| Z_i(\lambda) T_{\delta^a(\lambda)\bar{\delta}^b(\lambda)\Theta^s(\lambda)M(\lambda)} \left( z^c z^d \frac{\partial^{l_1}}{\partial \lambda^{l_1}} \psi_r^\lambda \right) \right\|_2^2 d\lambda,$$

where  $|\alpha| + |\beta| + |\gamma| + |\delta| + 2s = l - l_1$ .

Since  $W_\lambda(Z_i(\lambda)f) = iW_\lambda(f)A_j(\lambda)$ , one can dominate this by

$$(24) \quad \sum_{N=0}^{\infty} \int_{\mathbb{R}} |\lambda|^{-(|a|+|b|+|c|+|d|)} \|\delta^a(\lambda)\bar{\delta}^b(\lambda)\Theta^s(\lambda)M(\lambda)\chi_N(\lambda)\|_{\text{HS}}^2 \\ \times \left\| \chi_N(\lambda)\delta^c(\lambda)\bar{\delta}^d(\lambda)W_\lambda \left( \frac{\partial^{l_1}}{\partial \lambda^{l_1}} \psi_r^\lambda \right) A_i(\lambda) \right\|_{\text{op}}^2 |\lambda|^{n-(l-l_1)/2} d\lambda.$$

Since

$$A_i(\lambda)\Phi_\alpha^\lambda = (2\alpha_j)^{1/2}|\lambda|^{1/2}\Phi_{\alpha-e_j}^\lambda,$$

we can estimate  $\|\chi_N(\lambda)\delta^c(\lambda)\bar{\delta}^d(\lambda)W_\lambda \left( \frac{\partial^{l_1}}{\partial \lambda^{l_1}} \psi_r^\lambda \right) A_i(\lambda)\|_{\text{op}}^2$  as in Lemma 10 to obtain

$$\left\| \chi_N(\lambda)\delta^c(\lambda)\bar{\delta}^d(\lambda)W_\lambda \left( \frac{\partial^{l_1}}{\partial \lambda^{l_1}} \psi_r^\lambda \right) A_i(\lambda) \right\|_{\text{op}}^2 \leq C2^{-N(2l+|\alpha|+|\beta|-1)} g_{\alpha,\beta,l}(r2^N).$$

As we get an extra  $2^N$  on the right hand side, (24) can be bounded by  $r^{l-(n+2)}$ , as in Lemma 4.6. Thus (20) is proved.

The third assertion of the lemma is proved similarly to (ii). ■

**5. Proof of Theorem 1.3.** We start this section with some definitions. We first describe the dyadic Heisenberg cubes from [7] and [14].

**THEOREM 5.1.** *There exists a collection  $\mathcal{D} = \{Q_\alpha^j \subset H^n : j \in \mathbb{Z}^n, \alpha \in I_j\}$  of open sets and absolute constants  $0 < \eta < 1$ ,  $a > 0$ ,  $b > 0$  and  $\epsilon > 0$  such that*

- (i)  $|H^n \setminus \bigcup_\alpha Q_\alpha^j| = 0$  for all  $j$ .
- (ii) For  $l \geq j$  and any  $\alpha, \beta$ , either  $Q_\alpha^l$  is contained in  $Q_\beta^j$  or they do not intersect.
- (iii) For each  $j, \alpha$  there exists a  $\beta$  such that

$$Q_\beta^{j+1} \subset Q_\alpha^j.$$

$Q_\beta^{j+1}$  is called a child of  $Q_\alpha^j$ .

- (iv) For each  $j, \alpha$  there exist unique  $\beta$  such that

$$Q_\alpha^j \subset Q_\beta^{j-1}.$$

$Q_\beta^{j-1}$  is called the parent of  $Q_\alpha^j$ .

- (v) If  $Q_\alpha^j$  is a child of  $Q_\beta^{j-1}$ , then

$$|Q_\alpha^j| \geq \epsilon |Q_\beta^{j-1}|.$$

- (vi) There is a point  $(z_\alpha^j, t_\alpha^j)$  such that  $B((z_\alpha^j, t_\alpha^j), \eta^j) \subset Q_\alpha^j \subset B((z_\alpha^j, t_\alpha^j), a\eta^j)$ .

We will call  $Q_\alpha^j$  cubes of side length  $\eta^j$  and center  $(z_\alpha^j, t_\alpha^j)$ . For any  $\gamma > 0$  the dilation of such a cube is defined as follows:

$$\gamma Q_\alpha^j = B((z_\alpha^j, t_\alpha^j), a\gamma\eta^j).$$

A collection  $\mathcal{S}$  of cubes in  $H^n$  is said to be  $\eta$ -sparse if there are sets  $\{E_S \subset S : S \in \mathcal{S}\}$  which are pairwise disjoint and satisfy  $|E_S| > \eta|S|$  for all  $S \in \mathcal{S}$ . Corresponding to a dyadic grid  $\mathcal{D}$  and a sparse family  $\mathcal{S}$ , for  $1 \leq r < \infty$ , we can consider a sparse operator, which is defined as follows.

$$\mathcal{A}_{r,\mathcal{S}}f(z, t) = \sum_{Q \in \mathcal{S}} \left( \frac{1}{|Q|} \int_Q |f|^r \right)^{1/r} \chi_Q(z, t).$$

Let  $t_j = 2^{-j}$ ,  $j \in \mathbb{N}$ . Let us define the following operators:

$$T_j f(z, t) = T\psi_{t_j} * f(z, t).$$

For  $N \in \mathbb{N}$ , consider

$$T^N f = \sum_{j=1}^N T_j f.$$

It is easy to see that  $T^N f$  tends to  $Tf$  in  $L^2$  as  $N$  tends to  $\infty$ . Let  $k_j$  and  $K^N$  be the kernels of  $T_j$  and  $T^N$  respectively.

For any cube  $Q \subset H^n$ , consider the operators

$$\begin{aligned}\mathcal{T}_{j,Q}f(z,t) &= \left( \int_{H^n \setminus 3Q} k_j((z,t)(w,s)^{-1})f(w,s) dt \right) \chi_Q(z,t), \\ \mathcal{T}_Q^N f(z,t) &= \sum_{j=1}^N \mathcal{T}_{j,Q}f(z,t).\end{aligned}$$

Also, consider the operator

$$\mathcal{T}^{N*}f(z,t) = \sup_{Q \ni (z,t)} |\mathcal{T}_Q^N f(z,t)|.$$

We will prove the following theorem.

**THEOREM 5.2.** *Suppose  $M$  satisfies the hypothesis of Theorem 1.3. Then, for  $f \in C_c^\infty(H^n)$  and  $(z_0, t_0) \in H^n$ ,*

$$(25) \quad \mathcal{T}^{N*}f(z_0, t_0) \leq C(\Lambda(\mathcal{T}^N f)(z_0, t_0) + \Lambda_2 f(z_0, t_0))$$

where the constant does not depend on  $N$ . Here  $\Lambda$  stands for the maximal function associated to the Heisenberg group. Also,

$$(26) \quad \|\mathcal{T}^{N*}\|_{L^{2,\infty}(H^n)} \leq C\|f\|_{L^2(H^n)}.$$

*Proof.* We closely follow [3, proof of Theorem 3.1]. As  $\mathcal{T}^N$  are uniformly bounded in  $L^2(\mathbb{R}^n)$  and  $\Lambda$  and  $\Lambda_2$  both satisfy a weak type (2, 2) estimate, (26) is an easy consequence of (25). So, we will prove (25). Fix a cube  $Q$  which contains  $(z_0, t_0)$ . Define  $f_1 = f\chi_{3Q}$  and  $f_2 = f - f_1$ . Let us also consider  $\phi \in C_0^\infty(H^n)$  supported in the homogeneous ball  $\{(z, t) : \rho(z, t) < 1\}$  and satisfying  $\phi((z, t)) = 1$  whenever  $\rho(z, t) < 1/2$ . Define

$$\begin{aligned}K_1^N((z, t), (w, s)) &:= K^N((z, t), (w, s))\phi((z, t)(w, s)^{-1}), \\ K_2^N((z, t), (w, s)) &:= K^N((z, t), (w, s))(1 - \phi((z, t)(w, s)^{-1})), \\ k_{j,1}((z, t), (w, s)) &:= k_j((z, t), (w, s))\phi((z, t)(w, s)^{-1}), \\ k_{j,2}((z, t), (w, s)) &:= k_j((z, t), (w, s))(1 - \phi((z, t)(w, s)^{-1})).\end{aligned}$$

Also, let  $T_1^N$  and  $T_2^N$  be the integral operators corresponding to the kernels  $K_1^N$  and  $K_2^N$ , respectively.

Since  $f_2$  is supported outside  $3Q$ , we have

$$\begin{aligned}T^N(f_2)((z_0, t_0)) &= \int_{H^n} K^N((z_0, t_0)(w, s)^{-1})f_2(w, s) dw ds \\ &= \int_{H^n \setminus 3Q} K^N((z_0, t_0)(w, s)^{-1})f(w, s) dw ds = \mathcal{T}_Q^N f(z_0, t_0).\end{aligned}$$

Now, let  $(z, t) \in Q$ . We will first prove that

$$(27) \quad |T_1^N f_2(z, t) - T_1^N f_2(z_0, t_0)| \leq C\Lambda_2 f(z_0, t_0),$$

where  $C$  is independent of  $N$ . We will make use of the estimates obtained in Section 4. We have

$$\begin{aligned} & |T_1^N f_2(z, t) - T_1^N f_2(z_0, t_0)| \\ & \lesssim \int_{H^n \setminus 3Q} |K_1^N((z, t)(w, s)^{-1}) - K_1^N((z_0, t_0)(w, s)^{-1})| |f(w, s)| dw ds. \end{aligned}$$

By the Cauchy–Schwarz inequality the above term can be dominated by

$$\begin{aligned} & \left( \int_{H^n \setminus 3Q} \rho((z, t)(w, s)^{-1})^{\frac{n+3/2}{2}} \right. \\ & \quad \times |K_1^N((z, t)(w, s)^{-1}) - K_1^N((z_0, t_0)(w, s)^{-1})|^2 dw ds \Big)^{1/2} \\ & \quad \times \left( \int_{H^n \setminus 3Q} \frac{|f(w, s)|^2}{\rho((z, t)(w, s)^{-1})^{\frac{n+3/2}{2}}} dw ds \right)^{1/2}. \end{aligned}$$

We claim that

$$(28) \quad \int_{H^n \setminus 3Q} \rho((z, t)(w, s)^{-1})^{\frac{n+3/2}{2}} \times |K_1^N((z, t)(w, s)^{-1}) - K_1^N((z_0, t_0)(w, s)^{-1})|^2 dw ds \lesssim l(Q).$$

If the claim is true, then

$$\begin{aligned} & |\mathcal{T}_N^1 f_2(z, t) - \mathcal{T}_N^1 f_2(z_0, t_0)| \\ & \lesssim l(Q)^{1/2} \left( \sum_{k=1}^{\infty} \int_{3^{k+1}Q \setminus 3^kQ} \frac{|f(w, s)|^2}{\rho((z, t)(w, s)^{-1})^{\frac{n+3/2}{2}}} dw ds \right)^{1/2} \\ & \lesssim l(Q)^{1/2} \left( \sum_{k=1}^{\infty} \frac{1}{(a3^k l(Q))^{2(n+3/2)}} \int_{3^{k+1}Q} |f(w, s)|^2 dw ds \right)^{1/2} \\ & \lesssim \left( \sum_{k=1}^{\infty} 3^{-k} \right)^{1/2} A_2 f(z_0, t_0) \lesssim A_2 f(z_0, t_0). \end{aligned}$$

Hence (27) is proved.

In order to prove (28), it is enough to prove

$$(29) \quad \int_{H^n \setminus 3Q} \rho((z, t)(w, s)^{-1})^{\frac{n+3/2}{2}} \times |k_{j,1}((z, t)(w, s)^{-1}) - k_{j,1}((z_0, t_0)(w, s)^{-1})|^2 dw ds \lesssim l(Q) \min \left\{ \frac{t_{j+1}^{1/2}}{l(Q)}, \frac{l(Q)}{t_{j+1}^{1/2}} \right\}.$$

As  $(z, t) \in Q$  and  $(w, s) \in H^n \setminus 3Q$ ,  $\rho((z_0, t_0)(w, s)^{-1})$  and  $\rho((z, t)(w, s)^{-1})$



are comparable. So using Lemma 4.6 we have

$$\begin{aligned} & \int_{H^n \setminus 3Q} \rho((z, t)(w, s)^{-1})^{\frac{n+3/2}{2}} |k_{j,1}((z, t)(w, s)^{-1}) - k_{j,1}((z_0, t_0)(w, s)^{-1})|^2 dw ds \\ & \lesssim \int_{H^n \setminus 3Q} \rho((z, t)(w, s)^{-1})^{\frac{n+3/2}{2}} |k_{j,1}((z, t)(w, s)^{-1})|^2 dw ds \lesssim t_{j+1}^{\frac{1}{2}}. \end{aligned}$$

Therefore, we only have to show that

$$\begin{aligned} & \int_{H^n \setminus 3Q} \rho((z_0, t_0)(w, s)^{-1})^{\frac{n+3/2}{2}} \\ & \quad \times |k_{j,1}((z, t)(w, s)^{-1}) - k_{j,1}((z_0, t_0)(w, s)^{-1})|^2 dw ds \\ & \lesssim \frac{l(Q)^2}{t_{j+1}^{1/2}}. \end{aligned}$$

By a change of variable, it is enough to prove

$$\int_{H^n \setminus 2Q_0} \rho(w, s)^{\frac{n+3/2}{2}} |k_{j,1}((z, t)(z_0, t_0)^{-1}(w, s)) - k_{j,1}(w, s)|^2 dw ds \lesssim \frac{l(Q)^2}{t_{j+1}^{1/2}},$$

where  $2Q_0$  is the cube with center at the origin and radius the same as that of  $2Q$ . Let  $(z, t)(z_0, t_0)^{-1} = (u, \tilde{t})$ . We only consider the  $\tilde{t} = 0$  case. For general  $\tilde{t}$  one can follow the proof of [21, Lemma 1(iv)]. Let  $L$  be the left invariant vector field corresponding to the curve  $\gamma(\alpha) = \alpha \frac{(u, 0)}{|u|}$ ,  $\alpha \in \mathbb{R}$ . Then from the fundamental theorem of calculus, we have

$$\begin{aligned} & \left( \int_{H^n \setminus 2Q_0} \rho(w, s)^{\frac{n+3/2}{2}} |k_{j,1}((u, 0)(w, s)) - k_{j,1}(w, s)|^2 dw ds \right)^{1/2} \\ & \lesssim \left( \int_{H^n \setminus 2Q_0} \rho(w, s)^{\frac{n+3/2}{2}} \left| \int_0^{|u|} Lk_{j,1}(\gamma(\alpha)(w, s)) d\alpha \right|^2 dw ds \right)^{1/2} \\ & \lesssim \int_0^{|u|} \left( \int_{H^n \setminus 2Q_0} \rho(w, s)^{\frac{n+3/2}{2}} |Lk_{j,1}(\gamma(\alpha)(w, s))|^2 dw ds \right)^{1/2} d\alpha. \end{aligned}$$

As  $\rho(w, s)$  and  $\rho((u, 0)(w, s))$  are comparable,  $\rho(w, s)$  and  $\rho(\gamma(\alpha)(w, s))$  are also comparable. Hence the above can be dominated by

$$\int_0^{|u|} \left( \int_{H^n \setminus 2Q_0} \rho(\gamma(\alpha)(w, s))^{\frac{n+3/2}{2}} |Lk_{j,1}(\gamma(\alpha)(w, s))|^2 dw ds \right)^{1/2} d\alpha.$$

From the definition of  $k_{j,1}$ , we have  $Lk_{j,1}(w, s) = Lk_j(w, s)\phi(w, s) + k_j(w, s)L\phi(w, s)$ . Now as  $|L\phi(w, s)|$  is bounded, the corresponding integral

associated to that term can be dominated by

$$\int_0^{|u|} \left( \int_{H^n \setminus 2Q_0} \rho(\gamma(\alpha)(w, s))^{\frac{n+3/2}{2}} |k_j(\gamma(\alpha)(w, s))|^2 dw ds \right)^{1/2} d\alpha.$$

Using Lemma 4.6 this can be further bounded by

$$t_{j+1}^{-1/4} |u| \lesssim a l(Q) t_{j+1}^{-1/4} \lesssim l(Q)^{1/2} \frac{l(Q)^{1/2}}{t_{j+1}^{1/4}}.$$

Using the boundedness of  $\phi(w, s)$ , we can dominate the integral associated to the term  $Lk_j(w, s)\phi(w, s)$  by

$$C \int_0^{|u|} \left( \int_{H^n \setminus 2Q_0} \rho(\gamma(\alpha)(w, s))^{\frac{n+3/2}{2}} |Lk_{j,1}(\gamma(\alpha)(w, s))|^2 dw ds \right)^{1/2} d\alpha.$$

As  $k_{j,1}$  is supported in the homogeneous ball of radius 1 and  $\rho(w, s)$  and  $\rho(\gamma(\alpha)(w, s))$  are comparable, the above is less than or equal to

$$\int_0^{|u|} \left( \int_{H^n \setminus 2Q_0} \rho(\gamma(\alpha)(w, s))^{\frac{n+5/2}{2}} |Lk_j(\gamma(\alpha)(w, s))|^2 dw ds \right)^{1/2} d\alpha.$$

By Lemma 4.7 the above integral is again bounded by

$$C \frac{|u|}{t_{j+1}^{1/4}} \leq Cl(Q)^{1/2} \frac{l(Q)^{1/2}}{t_{j+1}^{1/4}}.$$

Hence (29) is proved and this completes the proof of (27).

We will now prove that

$$(30) \quad |T_2^N f_2(z, t)| \leq CA_2 f(z', t')$$

for any  $(z, t), (z', t') \in Q$ .

Let  $s_j = t_j^{1/2}$ . Since for all  $(w, s) \in H^n \setminus 3Q$ ,  $\rho((z, t)(w, s)^{-1})$  and  $\rho((z', t')(w, s)^{-1})$  are comparable, by the Cauchy–Schwarz inequality we can dominate  $|T_2^N f(z, t)|$  by

$$\sum_{j=1}^N \left( \int_{H^n \setminus 3Q} (1 + s_{j+1}^{-2} \rho((z, t)(w, s)^{-1}))^{\frac{n+3/2}{2}} |k_{j,2}((z, t)(w, s)^{-1})|^2 dw ds \right)^{1/2} \\ \times \left( \int_{H^n \setminus 3Q} \frac{|f(w, s)|^2}{(1 + s_{j+1}^{-2} \rho((z', t')(w, s)^{-1}))^{\frac{n+3/2}{2}}} dw ds \right)^{1/2}.$$

Since  $k_{j,2}$  is supported outside the homogeneous ball of radius 1 and center

at the origin and  $s_{j+1} > 1$ , using Lemma 4.6 we have

$$\begin{aligned} \int_{H^n \setminus 3Q} (1 + s_{j+1}^{-2} \rho((z, t)(w, s)^{-1}))^{\frac{n+3/2}{2}} |k_{j,2}((z, t)(w, s)^{-1})|^2 dw ds \\ \leq s_{j+1}^{-n-3/2} \int_{H^n} \rho((z, t)(w, s)^{-1})^{\frac{n+3/2}{2}} |k_j((z, t)(w, s)^{-1})|^2 dw ds \\ \leq C s_{j+1}^{-(n+3/2)} t_{j+1}^{1/2} \leq C s_{j+1}^{-(n+1/2)}. \end{aligned}$$

It is well-known that  $(\int_{H^n \setminus 3Q} \frac{|f(w, s)|^2}{(1 + s_{j+1}^{-2} \rho((z', t')(w, s)^{-1}))^{\frac{n+3/2}{2}}} dw ds)^{1/2}$  is dominated by  $s_{j+1}^{(n+2)/2} \Lambda_2 f(z', t')$ . Hence  $|T_2^N f_2(z, t)|$  is bounded by

$$C \sum_{j=1}^N s_{j+1}^{3/4} \Lambda_2 f(z, t) \leq C \Lambda_2 f(z', t')$$

where the constant is independent of  $N$ . Thus, we have proved (30).

Now, let  $(z, t) \in Q$ . Then

$$\begin{aligned} |\mathcal{T}_Q^N f(z_0, t_0)| &\leq |T^N f_2(z_0, t_0)| \\ &\leq |T^N f_2(z_0, t_0) - T^N f_2(z, t)| + |T^N f_2(z, t)| \\ &\leq |T_1^N f_2(z_0, t_0) - T_1^N f_2(z, t)| + |T_2^N f_2(z_0, t_0)| \\ &\quad + |T_2^N f_2(z, t)| + |T^N f(z, t)| + |T^N f_1(z, t)| \\ &\leq C(\Lambda_2 f(z_0, t_0) + |T^N f(z, t)| + |T^N f_1(z, t)|). \end{aligned}$$

Taking the integral over the region  $(z, t) \in Q$  on both sides and dividing by  $|Q|$  we have

$$|\mathcal{T}_Q^N f(z_0, t_0)| \leq C \left( \Lambda_2 f(z_0, t_0) + \Lambda(T^N f)(z_0, t_0) + \frac{1}{|Q|} \int_Q |T^N f_1(z, t)| \right).$$

By using the  $L^2$  boundedness of  $T^N$  we can estimate

$$\begin{aligned} \frac{1}{|Q|} \int_Q |T^N f_1(z, t)| dz dt &\leq \left( \frac{1}{|Q|} \int_Q |T^N f_1(z, t)|^2 dz dt \right)^{1/2} \\ &\leq \left( \frac{1}{|Q|} \int_{H^n} |T^N f_1(z, t)|^2 dz dt \right)^{1/2} \\ &\leq \left( \frac{1}{|Q|} \int_{3Q} |f(z, t)|^2 dz dt \right)^{1/2} \leq \Lambda_2 f(z_0, t_0). \end{aligned}$$

So, finally we have

$$|\mathcal{T}_Q^N f(z_0, t_0)| \leq C(\Lambda_2 f(z_0, t_0) + \Lambda(T^N f)(z_0, t_0)).$$

Taking the supremum over all cubes containing  $(z_0, t_0)$  we get

$$\mathcal{T}^{N*} f(z_0, t_0) \leq C(\Lambda_2 f(z_0, t_0) + \Lambda(T^N f)(z_0, t_0)).$$

Hence (25) is proved. Inequality (26) follows from (25) by using the weak (2, 2) boundedness of  $\Lambda_2$  and  $T^N$ . ■

Let us consider the following maximal function

$$\mathcal{M}_{T^N} f(z, t) = \sup_{(z,t) \in Q} \operatorname{ess\,sup}_{(\xi,s) \in Q} |T^N(f\chi_{H^n \setminus 3Q})(\xi, s)|.$$

Here the supremum is taken over all the cubes which contain  $(z, t)$ . Also for any cube  $Q_0$  define

$$\mathcal{M}_{T^N_{Q_0}} f(z, t) = \sup_{(z,t) \in Q, Q \subset Q_0} \operatorname{ess\,sup}_{(\xi,s) \in Q} |T^N(f\chi_{H^n \setminus 3Q})(\xi, s)|.$$

The next lemma will be used to prove 1.3.

LEMMA 5.3. *Suppose  $M$  satisfies the hypothesis of Theorem 1.3. Then*

(i) *For a.e.  $(z, t) \in Q$ ,*

$$|T^N(f\chi_{3Q_0})(z, t)| \leq C(f(z, t) + \mathcal{M}_{T^N_{Q_0}} f(z, t)).$$

(ii)  $\mathcal{M}_{T^N} f(z, t) \leq C(\Lambda_2 f(z, t) + \mathcal{T}^{N*} f(z, t))$  *for any  $(z, t) \in H^n$ .*

*Here  $C$  depends only on  $n$  and  $T$ , and not on  $N$ .*

*Proof.* The proof of (i) is the same as the proof of [20, Lemma 3.2(i)]. So, we prove (ii). Let us consider a dyadic cube  $Q$  containing  $(z, t) \in H^n$ . Let  $(\xi, s) \in Q$ . Then

$$\begin{aligned} & |T^N f\chi_{H^n \setminus 3Q}(\xi, s)| \\ & \leq |T^N f\chi_{H^n \setminus 3Q}(\xi, s) - T^N f\chi_{H^n \setminus 3Q}(z, t)| + |T^N f\chi_{H^n \setminus 3Q}(z, t)|. \end{aligned}$$

Now,  $|T^N f\chi_{H^n \setminus 3Q}(\xi, s) - T^N f\chi_{H^n \setminus 3Q}(z, t)|$  can be dominated by

$$\begin{aligned} & |T_1^N f\chi_{H^n \setminus 3Q}(\xi, s) - T_1^N f\chi_{H^n \setminus 3Q}(z, t)| \\ & \quad + |T_2^N f\chi_{H^n \setminus 3Q}(\xi, s)| + |T_2^N f\chi_{H^n \setminus 3Q}(z, t)|. \end{aligned}$$

We have seen in the proof of Theorem 5.2 that all the above terms can be dominated by  $\Lambda_2 f(z, t)$ . On the other hand,  $|T^N f\chi_{H^n \setminus 3Q}(z, t)|$  is bounded by  $\mathcal{T}^{N*} f(z, t)$ . Hence,

$$\mathcal{M}_{T^N} f(z, t) \leq C(\Lambda_2 f(z, t) + \mathcal{T}^{N*} f(z, t)). \quad \blacksquare$$

*Proof of Theorem 1.3.* Since  $\mathcal{T}^{N*}$  is of weak type (2, 2), by our previous lemma  $\mathcal{M}_{T^N}$  is of weak type (2, 2). Let  $Q$  be a cube in  $H^n$ . We shall find a sparse family  $\mathcal{F} \subset \mathcal{D}(Q_0)$  such that

$$(31) \quad |T(f\chi_{3Q_0})(z, t)| \leq C \sum_{Q \subset \mathcal{F}} \left( \frac{1}{|3Q|} \int_{3Q} |f|^2 \right)^{1/2} \chi_Q(z, t)$$

for a.e.  $(z, t) \in Q_0$ . This can be proved using the idea of [20, Theorem 3.1]. To prove (31), we will use a recursive method. We will first find pairwise disjoint cubes  $Q_j^1 \in \mathcal{D}(Q_0)$  such that  $\sum |Q_j^1| \leq \frac{1}{2}|Q_0|$  and

$$(32) \quad |T(f\chi_{3Q_0})(z, t)|\chi_{Q_0}(z, t) \leq C \left( \frac{1}{|3Q_0|} \int_{3Q_0} |f|^2 \right)^{1/2} \chi_{Q_0} + \sum_j |T(f\chi_{3Q_j^1})|\chi_{Q_j^1}$$

for a.e.  $(z, t) \in H^n$ . Once we prove this, we will apply the same process on each  $Q_j^1$  and continue the recursive process, which finally leads to (31).

As  $\mathcal{M}_{TN}$  is of weak type  $(2, 2)$ , we can choose  $\alpha \in \mathbb{R}$  (depending only on  $n$ ) such that the measure of the set

$$E = \left\{ (z, t) \in Q_0 : |f(z, t)| > \alpha \left( \frac{1}{|3Q_0|} \int_{3Q_0} |f|^2 \right)^{1/2} \right\} \cup \left\{ (z, t) \in Q_0 : \mathcal{M}_{T_{Q_0}^N} f(z, t) > \alpha \left( \frac{1}{|3Q_0|} \int_{3Q_0} |f|^2 \right)^{1/2} \right\}$$

is less than  $\frac{1}{4\epsilon}|Q_0|$ , where the constant  $\epsilon$  comes from the definition of the dyadic cubes and  $\eta$ .

Applying Calderón–Zygmund decomposition to the function  $\chi_E$  on  $Q_0$  at height  $\epsilon/2$  we get pairwise disjoint cubes  $Q_j^1 \in \mathcal{D}(Q_0)$  such that

- (a)  $\frac{\epsilon}{2}|Q_j^1| \leq |Q_j^1 \cap E| \leq \frac{1}{2}|Q_j^1|$ ,
- (b)  $|E \setminus \bigcup_j Q_j^1| = 0$ .

From (a), we have  $Q_j^1 \cap E^c \neq \emptyset$ . Using  $|E| \leq \frac{\epsilon}{4}|Q_0|$ , one can observe that

$$\sum_j |Q_j^1| \leq \frac{1}{2}|Q_0|.$$

Therefore

$$\operatorname{ess\,sup}_{(\xi, s) \in Q_j^1} |T(f\chi_{3Q_0 \setminus 3Q_j^1})(\xi, s)| \leq C \left( \frac{1}{|3Q_0|} \int_{3Q_0} |f|^2 \right)^{1/2}.$$

Also, using Lemma 5.3 we can easily see that if  $x \in Q_0 \setminus \bigcup_j Q_j^1$ , then

$$|T(f\chi_{3Q_0})(x)| \leq C \left( \frac{1}{|3Q_0|} \int_{3Q_0} |f|^2 \right)^{1/2}.$$

Consequently,

$$\begin{aligned}
& |T(f\chi_{3Q_0})|\chi_{Q_0} \\
& \leq |T(f\chi_{3Q_0})|\chi_{Q_0 \setminus Q_j^1} + \sum_j |T(f\chi_{3Q_0})|\chi_{Q_j^1} \\
& \leq |T(f\chi_{3Q_0})|\chi_{Q_0 \setminus Q_j^1} + \sum_j |T(f\chi_{3Q_0 \setminus 3Q_j^1})|\chi_{Q_j^1} + \sum_j |T(f\chi_{3Q_j^1})|\chi_{Q_j^1} \\
& \leq \left( \frac{1}{|3Q_0|} \int_{3Q_0} |f|^2 \right)^{1/2} + \sum_j |T(f\chi_{3Q_j^1})|\chi_{Q_j^1}.
\end{aligned}$$

Hence, (32) is proved.

Now, assume that  $f$  is compactly supported, and let  $B$  be any ball which contains  $\text{supp } f$ . By [14, Theorem 4.1], there exist a finite number of dyadic decompositions  $\{S^l : l = 1, \dots, L\}$  and a cube  $Q \in S^l$  for some  $l = 1, \dots, L$  such that  $B \subset Q$ . As  $Q$  contains  $\text{supp } f$ , from (31) we have

$$\begin{aligned}
\|T^N f(z, t)\|_{L^p(B, w)} & \leq \|T^N f(z, t)\|_{L^p(Q, w)} \\
& \leq C \left\| \sum_{Q \subset \mathcal{F}} \left( \frac{1}{|3Q|} \int_{3Q} |f|^2 \right)^{1/2} \right\|_{L^p(Q, w)} \leq C \|f\|_{L^p(w)}.
\end{aligned}$$

Since the above is true for any cube  $B$  which contains  $\text{supp } f$ , we have

$$\|T^N f(z, t)\|_{L^p(w)} \leq C \|f\|_{L^p(w)}$$

for all  $f \in C_c^\infty(H^n)$  and hence for all  $f \in L^p(w)$ .

Of course, the constant appearing here is independent of  $N$ . Now, as  $T^N$  converges to  $T_M$  in  $L^2$ , there exists a subsequence which converges to  $T_M$  almost everywhere. Thus, we can conclude that

$$\|T_M f(z, t)\|_{L^p(w)} \leq C \|f\|_{L^p(w)}$$

for  $w \in A_{p/2}$ ,  $2 < p < \infty$ . Hence Theorem 1.3 is proved. ■

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