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Quasimöbius invariance of uniform domains

by

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Abstract. We study quasimöbius invariance of uniform domains in Banach spaces. We first investigate implications of certain geometric properties of domains in Banach spaces, such as (diameter) uniformity, δ-uniformity and the min-max property. Then we show that all of these conditions are equivalent if the domain is ψ-natural. As applications, we partially answer an open question proposed by Väisälä, and provide a new method to prove a recent result of Huang et al. (2013), which also gives an answer to another question raised by Väisälä.

1. Introduction and main results. Many results of the classical function theory have their counterparts in the setting of quasiconformal mappings in n-dimensional Euclidean spaces. To further extend the scope of this theory, since the late 1980's Väisälä has developed the theory of (dimension) free quasiconformal mappings in Banach spaces [18, 19, 20, 21, 22]. The main advantage of this approach is avoiding the use of volume integrals and conformal modulus, which allows one to study the quasiconformality of mappings in infinite-dimensional Banach spaces and other metric spaces without volume measures. Recently, this area is a subject of several investigations (see e.g. [5, 4, 6, 9, 11, 15, 26] and references therein).

In this paper, we consider the relationship between uniform domains and relative quasimöbius maps in real Banach spaces. The main objective is to study two open questions formulated by Väisälä [19], [22]. We will give a partial solution to one of these questions and provide a new method to answer the other one. Following [22], we assume that E is a real Banach space of dimension at least two, a proper domain $G \subsetneq E$ is a nonempty connected open set, and $d_G(x) = \operatorname{diam}(x, \partial G)$ for $x \in G$. Let us begin with the definition of uniform domains in Banach spaces.

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Definition 1.1. Let E be a real Banach space and $G \subsetneq E$ a domain, and let $c \geq 1$. We say that G is *c-uniform* if each pair of points x, y in G can be connected by an arc γ in G satisfying

(1) $\ell(\gamma) \leq c|x-y|$, and
(2) $\min\{\ell(\gamma[x,z]), \ell(\gamma[z,y])\} \leq c d_G(z)$ for all $z \in \gamma$,

where $\ell(\gamma)$ denotes the length of γ, and $\gamma[x,z]$ is the part of γ between x and z. Such an arc γ is said to be a *c-uniform arc*.

Remark 1.1. In 1978, Martio and Sarvas [14] introduced the twisted interior cone condition in connection with global injectivity properties of locally injective mappings. Since then, many other characterizations of uniform domains have been established [1, 2, 13]. Uniform domains can be understood as a class of domains developed in the context of generalizing the Riemann mapping theorem to quasiconformal maps in $\mathbb{R}^n$ with $n \geq 3$, a problem that still remains open. This class of domains has numerous geometric and function-theoretic properties that make it useful in many fields of modern mathematical analysis (see e.g. [3, 8, 16, 17, 22]).

In [13], Martio studied the quasiconformal invariance of uniform domains in $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$. He obtained the following result by showing several equivalent conditions for uniform domains.

Theorem A ([13, Theorems 5.4 and 6.2]). *Let $c, K \geq 1$. If $G \subsetneq \overline{\mathbb{R}^n}$ is a c-uniform domain and $f : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ is K-quasiconformal, then $f(G)$ is c_1-uniform with c_1 depending only on c, K and n.*

Subsequently, Väisälä [16] introduced the concept of *quasimöbius* maps (see Definition 2.9) and investigated the relation between this class of mappings and quasiconformal maps on uniform domains.

Theorem B ([16, Theorem 5.6]). *Let $c, K \geq 1$ and $n \geq 2$. If $G \subsetneq \overline{\mathbb{R}^n}$ is a c-uniform domain and $f : G \to G' \subsetneq \overline{\mathbb{R}^n}$ is a K-quasiconformal homeomorphism, then G' is c_1-uniform if and only if f is η-quasimöbius.*

Moreover, Väisälä has generalized Theorem B to real Banach spaces by using the free quasiconformality (FQC) theory in [19].

Theorem C ([19, Theorem 7.18]). *Let G and G' be domains in Banach spaces E and E', respectively. If G is c-uniform and $f : G \to G'$ is φ-FQC, then G' is c_1-uniform if and only if f is η-quasimöbius.*

The main motivation of this paper is the following two questions posed by Väisälä, the second of which is still open. For the definition of (M, C)-CQH and relative quasimöbius mapping see Definitions 2.9 and 2.10.

QUESTION 1.1 ([19, Question 7.19]). Suppose that G is a c-uniform domain and that $f : G \to G'$ is (M, C)-CQH. If f extends to a homeomorphism $\overline{f} : \overline{G} \to \overline{G'}$ and $\overline{f}$ is θ-quasimöbius relative to ∂G, is G' c'-uniform?

REMARK 1.2. It is worth mentioning that Huang, Li, Vuorinen, and Wang [4] answered Question 1.1 affirmatively. Their proofs are based on several concepts and results in the free quasiworld [22], such as coarse quasihyperbolic length and solid arcs, the equivalence of uniform and φ-uniform domains, the diameter cigar theorems for uniform domains.

QUESTION 1.2 ([22, 13.2.10]). Suppose that G is a c-uniform domain and that a homeomorphism $f : \overline{G} \to \overline{G'}$ is η-quasimöbius relative to ∂G and maps G onto G'. Is G' c'-uniform with $c' = c'(c, \eta)$?

We focus our attention on Question 1.2. Indeed, Question 1.2 is more difficult than Question 1.1 because one does not assume that f is coarsely quasihyperbolic. To deal with this question, we first prove the following implications between certain geometric properties of domains in Banach spaces, such as diameter uniformity, the min-max property and δ-uniformity for some $0 < \delta < 1$ (see Subsection 2.2 for the definitions).

THEOREM 1.1. *Suppose that $G \subsetneq E$ is a domain. Then we have the following implications: G is c-uniform $\Rightarrow$ G satisfies the min-max property $\Rightarrow$ G is diameter c_1-uniform $\Leftrightarrow$ G is δ-uniform for some* $0 < \delta < 1$.

REMARK 1.3. The min-max property for domains in $\mathbb{R}^n$ was introduced by Gehring and Hag [1]. They extended the properties of hyperbolic geodesics in $\mathbb{B}^n$ to more general domains, and studied the relationship between this property, uniformity and the quasiconformal extension property. In [13], Martio introduced the concept of δ-uniformity in terms of the cross ratio of four points for domains in $\mathbb{R}^n$. By using this condition, he obtained certain general properties of uniform domains.

As an application of Theorem 1.1, we prove the following quasimöbius invariance result for diameter uniform domains and δ-uniform $(0 < \delta < 1)$ domains.

COROLLARY 1.1. *Let G and G' be proper domains in Banach spaces E and E', respectively. Let $f : G \to G'$ be a θ-quasimöbius homeomorphism.*

(1) *If G is diameter c-uniform, then G' is diameter c'-uniform with some $c' = c'(c, \theta)$;*
(2) *If G is δ-uniform with $0 < \delta < 1$, then G' is δ'-uniform with some $\delta' = \delta'(\delta, \theta) \in (0, 1)$.*

Next, we consider the converse of Theorem 1.1. One easily sees that every c-uniform domain is diameter c-uniform. For the converse, it follows from [13, Theorem 4.5] that a diameter c-uniform domain in $\mathbb{R}^n$ is c_1-uniform with some c_1 depending on c and n. It is natural to ask whether this holds in infinite-dimensional Banach spaces. Our second main result is an attempt in this direction.

THEOREM 1.2. *Let $G \subsetneq E$ be a domain. Then G is c-uniform if and only if G is diameter c_1-uniform and ψ-natural.*

REMARK 1.4. ψ-natural domains are defined in Definition 2.3. We note that every proper domain in $\mathbb{R}^n$ is ψ-natural with $\psi = \psi(n)$ [24, Corollary 2.18]. In an infinite dimensional Hilbert space, the broken tube construction in [23] provides an example of a domain which is not natural. Moreover, it is diameter c-uniform but not c_1-uniform for any $c_1 \geq 1$.

There are several applications of Theorems 1.1 and 1.2 in studying the relationship between uniform domains and (relative) quasimöbius maps. The following is immediate because a ψ-uniform domain (see (2.2) for the definition) is ψ-natural.

COROLLARY 1.2. *Let $G \subsetneq E$ be a domain. Then G is c-uniform if and only if G is diameter c_1-uniform and ψ-uniform. In particular, a diameter uniform convex domain in a Banach space is uniform.*

As the second application, we partially answer Question 1.2 as follows.

THEOREM 1.3. *Let G and G' be proper domains in Banach spaces E and E', respectively. Suppose that G is c-uniform and that a homeomorphism $f : \overline{G} \to \overline{G'}$ is η-quasimöbius relative to ∂G and maps G onto G'. Then G' is diameter c'-uniform with some $c' = c'(c, \eta)$. If in addition G' is ψ-natural, then G' is c_1'-uniform with some $c_1' = c_1'(c, \eta, \psi)$.*

Moreover, we show that Theorem 1.3 gives a way to affirmatively answer Question 1.1. Our method is quite different from the one used in [4].

THEOREM 1.4. *Let G and G' be proper domains in Banach spaces E and E', respectively. Suppose that $f : G \to G'$ is (M, C)-CQH, and that f extends to a homeomorphism $f : \overline{G} \to \overline{G'}$ which is η-quasimöbius relative to ∂G.*

(1) *If G is ψ-natural, then G' is ψ'-natural with some $\psi' = \psi'(\psi, M, C, \eta)$.*
(2) *If G is c-uniform, then G' is c'-uniform with some $c' = c'(c, M, C, \eta)$.*

By Theorems 1.2 and 1.4, we further obtain the following consequence.

COROLLARY 1.3. *Let G and G' be proper domains in Banach spaces E and E', respectively. Suppose that $f : G \to G'$ is (M, C)-CQH, and that f extends to a homeomorphism $f : \overline{G} \to \overline{G'}$ which is η-quasimöbius relative to ∂G. If G is diameter c-uniform and G' is ψ-natural, then both G and G' are c_1-uniform with some $c_1 = c_1(c, M, C, \theta, \psi) \geq 1$.*

The rest of this paper is organized as follows. In Section 2, we recall the necessary definitions and preliminary results. The proofs of Theorems 1.1 and 1.2 are given in Section 3. Section 4 is devoted to the proofs of Theorems 1.3 and 1.4, and the proofs of Corollaries 1.1, 1.2 and 1.3 are presented in Section 5.

2. Preliminaries and notations

2.1. Notation. Let letters $A, B, C, \ldots$ denote positive numerical constants. Similarly, $C(a, b, c, \ldots)$ denotes universal positive functions of the parameters $a, b, c, \ldots$. Sometimes we write $C = C(a, b, c, \ldots)$ to emphasize the parameters on which C depends and abbreviate $C(a, b, c, \ldots)$ to C.

Following the notation and terminology of [4, 22], we use E and E' to denote real Banach spaces of dimension at least 2. The norm of a vector x in E is written as $|x|$. For every pair of points z_1, z_2 in E, the distance between them is denoted by $|z_1 - z_2|$, and the closed line segment with endpoints z_1 and z_2 by $[z_1, z_2]$. The one-point extension of E is the Hausdorff space $\dot{E} = E \cup \{\infty\}$, where the neighborhoods of ∞ are the complements of closed bounded subsets of E.

For a set A in E, we use $\overline{A}$ to denote the closure of A and $\partial A = \overline{A} \setminus A$ to be its norm boundary. For a bounded set A in E, $\operatorname{diam} A$ is the diameter of A. Let

$$B(x, r) = \{z \in E : \ |z - x| < r\}, \quad \overline{B}(x, r) = \{z \in E : |z - x| \le r\},$$
$$S(x, r) = \{z \in E : |z - x| = r\}.$$

Let X be a metric space. A *curve* is a continuous function $\gamma : [a, b] \to X$. The length of γ is defined by

$$\ell(\gamma) = \sup\Big\{ \sum_{i=1}^{n} |\gamma(t_i) - \gamma(t_{i-1})| \Big\},$$

where the supremum is over all partitions $a = t_0 < t_1 < \cdots < t_n = b$. The curve is *rectifiable* if $\ell(\gamma) < \infty$. The metric space X is *rectifiably connected* if each pair of points can be connected by a rectifiable curve.

The length function associated with a rectifiable curve γ: $[a, b] \to X$ is $s_\gamma : [a, b] \to [0, \ell(\gamma)]$, defined by $s_\gamma(t) = \ell(\gamma|_{[a,t]})$ for $t \in [a, b]$. For any rectifiable curve $\gamma : [a, b] \to X$, there is a unique parametrization $\gamma_s : [0, \ell(\gamma)] \to X$ such that $\gamma = \gamma_s \circ s_\gamma$. Obviously, $\ell(\gamma_s|_{[0,t]}) = t$ for $t \in [0, \ell(\gamma)]$. The parametrization γ_s is called the *arclength parametrization* of γ. For a rectifiable curve γ in X, the line integral over γ of a Borel function $\varrho : X \to [0, \infty)$ is

$$\int_\gamma \varrho\, ds = \int_0^{\ell(\gamma)} \varrho \circ \gamma_s(t)\, dt.$$

2.2. Domains in Banach spaces. In this part, we assume that E is a real Banach space of dimension at least two, and $G \subsetneq E$ is a domain. We begin with the definition of the quasihyperbolic metric. This metric, first introduced by Gehring and Palka [3] for proper domains in $\mathbb{R}^n$, has recently been used by many authors in the study of quasiconformal mappings and related questions [2, 10, 12, 15, 27].

Recall that the *quasihyperbolic length* of a rectifiable curve α in the norm metric in G is the number (cf. [2, 3])

$$\ell_k(\alpha) = \int_\alpha \frac{|dz|}{d_G(z)}.$$

For each pair of points z_1, z_2 in G, the *quasihyperbolic distance* $k_G(z_1, z_2)$ between z_1 and z_2 is defined in the usual way:

$$k_G(z_1, z_2) = \inf \ell_k(\alpha),$$

where the infimum is over all rectifiable curves α joining z_1 and z_2 in G. An arc α from z_1 to z_2 is a *quasihyperbolic geodesic* if

$$\ell_k(\alpha) = k_G(z_1, z_2).$$

It is known that a quasihyperbolic geodesic between every pair of points in G exists if the dimension of E is finite [2, Lemma 1]. However, this is not true in infinite-dimensional Banach spaces [18, Example 2.9].

Let us remark that the second author with Talponen [15] further investigated the properties of quasihyperbolic geodesics in Banach spaces. They demonstrated that in a strictly convex Banach space with the Radon–Nikodym property, the quasihyperbolic geodesics are unique. Moreover, they provided an example to show that for a convex domain in a nonreflexive Banach space, it is possible that there is no quasihyperbolic geodesic between any given pair of points in the domain.

In order to overcome this shortcoming, Väisälä [19] introduced the following concept.

Definition 2.1. Let $G \subsetneq E$ be a domain and let $c \geq 1$. An arc $\alpha \subseteq G$ is a *c-neargeodesic* if $\ell_k(\alpha[x, y]) \leq c k_G(x, y)$ for all $x, y \in \alpha$.

Moreover, Väisälä [19] proved the following property concerning the existence of neargeodesics in domains of Banach spaces.

Theorem D ([19, Theorem 3.3]). *Let $G \subsetneq E$ be a domain. Then for all $z_1, z_2 \in G$ and for any $c > 1$, there is a c-neargeodesic α joining z_1 and z_2 in G.*

We also record the following elementary inequality (cf. [22]):

$$k_G(z_1, z_2) \geq \left| \log \frac{d_G(z_2)}{d_G(z_1)} \right| \quad \text{for all } z_1, z_2 \text{ in } G. \tag{2.1}$$

We recall the definition of φ-uniform domains presented in [25].

Definition 2.2. Let $\varphi : [0,\infty) \to [0,\infty)$ be a strictly increasing homeomorphism. A domain $G \subsetneq E$ is called *φ-uniform* if for all x, y in G,

$$k_G(x,y) \le \varphi(r_G(x,y)), \quad \text{where} \quad r_G(x,y) = \frac{|x-y|}{\min\{d_G(x), d_G(y)\}}. \tag{2.2}$$

In order to give a simple criterion for φ-uniform domains, consider domains G satisfying the following property [25, Examples 2.50(1)]: there exists a constant $C \ge 1$ such that each pair of points $x, y \in G$ can be joined by a rectifiable curve $\gamma \in G$ with $\ell(\gamma) \le C|x-y|$ and $\min\{d_G(x), d_G(y)\} \le C \operatorname{diam}(\gamma, \partial G)$. Then G is φ-uniform with $\varphi(t) = C^2 t$. In particular, every convex domain is φ-uniform with $\varphi(t) = t$. However, in general, (unbounded) convex domains need not be uniform.

Suppose that $\emptyset \ne A \subseteq G \subsetneq E$. We write

$$r_G(A) = \sup\{r_G(x,y) : x \in A,\ y \in A\},$$
$$k_G(A) = \sup\{k_G(x,y) : x \in A,\ y \in A\}.$$

Definition 2.3. Let $\psi : [0,\infty) \to [0,\infty)$ be an increasing function. A domain $G \subsetneq E$ is called *ψ-natural* if

$$k_G(A) \le \psi(r_G(A))$$

for every nonempty connected set $A \subseteq G$ with $r_G(A) < \infty$.

This notion, due to Väisälä [20], has also been studied by Vuorinen in [24]. We note that a φ-uniform domain is φ-natural, and every convex domain is ψ-natural with $\psi(t) = t$ [20, Theorems 2.8 and 2.9]. In fact, the next two results show that the class of natural domains is fairly large.

Lemma E ([24, Corollary 2.18]). *Every proper domain in $\mathbb{R}^n$ is ψ_n-natural with ψ_n depending only on n.*

Lemma F ([20, Theorem 2.8]). *Let $c \ge 1$. A c-uniform domain G in a Banach space E is ψ-natural with $\psi = \psi(c)$.*

But in an infinite-dimensional Hilbert space, the broken tube construction in [23, 2.3] provides an example of a domain which is not natural.

Definition 2.4. Let $G \subsetneq E$ be a domain and let $c \ge 1$. We say that G is *c-John* if each pair of points x, y in G can be connected by an arc γ in G such that

$$\min\{\ell(\gamma[x,z]), \ell(\gamma[z,y])\} \le c\, d_G(z)$$

for all $z \in \gamma$. Moreover, the arc γ is then called a *c-cone arc.*

Remark 2.1. The concept of John domains in Euclidean spaces was first introduced in 1961 by John [7] in connection with his work in elasticity. Recently, Li, Vuorinen and the first author [11] studied several equivalent conditions for John metric spaces. Indeed, their method is used in our proof of Theorem 1.2.

DEFINITION 2.5. Let $G \subsetneq E$ be a domain and let $c \geq 1$. Then G is called *diameter c-uniform* if each pair of points x_1, x_2 in G can be joined by an arc α in G satisfying

(1) $\min_{j=1,2} \operatorname{diam}(\alpha[x_j, x]) \leq c d_G(x)$ for all $x \in \alpha$, and
(2) $\operatorname{diam} \alpha \leq c|x_1 - x_2|$.

Moreover, α satisfying the above conditions is said to be a *diameter uniform arc.*

REMARK 2.2. It is easy to see that a c-uniform domain is diameter c-uniform. For the converse, it follows from [13, Theorem 4.5] that a diameter c-uniform domain in $\mathbb{R}^n$ is c_1-uniform with c_1 depending on c and n. Note that in Banach spaces, Väisälä [23] constructed a broken tube domain which is diameter c-uniform but not c_1-uniform for any constant $c_1 \geq 1$.

DEFINITION 2.6. Let $G \subsetneq E$ be a domain. We say that G has the *min-max property* if there exists a family $\varGamma$ of curves in G and a constant $c \geq 1$ such that any pair of points in G can be joined by a curve $\gamma \in \varGamma$ such that

$$\frac{1}{c} \min_{j=1,2} |x_j - y| \leq |x - y| \leq c \max_{j=1,2} |x_j - y| \tag{2.3}$$

for each ordered triple of points $x_1, x, x_2 \in \gamma$ and each $y \in \partial G$.

DEFINITION 2.7. Let $G \subsetneq E$ be a domain and let $0 < \delta < 1$. Then G is called *δ-uniform* if each pair of points x_1, x_2 in G can be joined by an arc α in G such that their cross ratio satisfies

$$\tau(x, x_i, y, x_j) = \frac{|x-y|}{|x-x_i|} \cdot \frac{|x_i - x_j|}{|x_j - y|} \geq \delta, \qquad i \neq j \in \{1, 2\}, \tag{2.4}$$

for all $x \in \alpha \setminus \{x_1, x_2\}$ and $y \in E \setminus G$.

2.3. Mappings on metric spaces. Let X be a metric space and $\dot{X} = X \cup \{\infty\}$. By a *triple in X* we mean an ordered sequence $T = (x, y, z)$ of three distinct points in X. The ratio of T is the number

$$\rho(T) = \frac{|y-x|}{|z-x|}.$$

If $f : X \to Y$ is an injective map, the image of a triple $T = (x, y, z)$ is the triple $f(T) = (f(x), f(y), f(z))$.

Suppose that $A \subseteq X$. A triple $T = (x, y, z)$ in X is said to be a *triple in the pair* (X, A) if $x \in A$ or if $\{y, z\} \subseteq A$. Equivalently, both $|y - x|$ and $|z - x|$ are distances from a point in A.

DEFINITION 2.8. Let X and Y be metric spaces, and let $\eta : [0, \infty) \to [0, \infty)$ be a homeomorphism. Suppose $A \subset X$. An embedding $f : X \to Y$ is said to be *η-quasisymmetric* relative to A, or η-QS rel A, if $\rho(f(T)) \le \eta(\rho(T))$ for each triple T in (X, A).

Notice that an embedding $f : X \to Y$ is η-QS relative to A if and only if $\rho(T) \le t$ implies that $\rho(f(T)) \le \eta(t)$ for each triple T in (X, A) and $t \ge 0$ (cf. [22]). Obviously, quasisymmetric relative to X is equivalent to usual quasisymmetric.

A *quadruple* in X is an ordered sequence $Q = (x, y, z, w)$ of four distinct points in X. The *cross ratio* of Q is defined to be the number

$$\tau(Q) = \tau(x, y, z, w) = \frac{|x-z|}{|x-y|} \cdot \frac{|y-w|}{|z-w|}.$$

Observe that the definition can be extended in the usual manner to the case where one of the points is ∞. For example,

$$\tau(x, y, z, \infty) = \frac{|x-z|}{|x-y|}.$$

If $X_0 \subseteq \dot{X}$ and if $f : X_0 \to \dot{Y}$ is an injective map, then the image of a quadruple Q in X_0 is the quadruple $f(Q) = (f(x), f(y), f(z), f(w))$. Suppose that $A \subseteq X_0$. We say that a quadruple $Q = (x, y, z, w)$ in X_0 is a *quadruple in the pair* (X_0, A) if $\{x, w\} \subseteq A$ or $\{y, z\} \subseteq A$. Equivalently, all four distances in the definition of $\tau(Q)$ are (at least formally) distances from a point in A.

Definition 2.9. Let X and Y be metric spaces and let $\eta : [0, \infty) \to [0, \infty)$ be a homeomorphism. Suppose $A \subseteq \dot{X}$. An embedding $f : \dot{X} \to \dot{Y}$ is said to be *η-quasimöbius* relative to A, or η-QM rel A, if $\tau(f(Q)) \le \eta(\tau(Q))$ for each quadruple in (X, A).

Apparently, η-QM relative to X is equivalent to η-quasimöbius.

We conclude this section by recalling the definition of CQH maps and by presenting a useful result.

Definition 2.10. Let G and G' be domains in Banach spaces E and E', respectively. Let $M \ge 1$ and $C \ge 0$. A homeomorphism $f : G \to G'$ is said to be *C-coarsely M-quasihyperbolic*, or (M, C)-CQH, if for all $x, y \in G$,

$$\frac{k_G(x, y) - C}{M} \le k_{G'}(f(x), f(y)) \le M k_G(x, y) + C.$$

Lemma G ([20, Lemma 2.14]). *Suppose that $f : \overline{G} \to \overline{G'}$ is η-quasimöbius relative to ∂G and that $f(G) = G'$. Suppose also that $x, y \in G$ with $r_G(x, y) \le t$. Then $r_{G'}(f(x), f(y)) \le \mu(t, \eta) < \infty$.*

3. Proofs of Theorems 1.1 and 1.2

3.1. In this subsection, we prove Theorem 1.1 by dividing the arguments into several lemmas. We use the methods from the proofs of [1, Theorem 2.7] and [13, Theorem 5.4].

Lemma 3.1. *Suppose that $G \subsetneq E$ is a c-uniform domain. Then G satisfies the min-max property.*

Proof. Set

$$\varGamma := \{\alpha_{u,v} : u, v \in G,\ \alpha_{u,v} \text{ is a 2-neargeodesic with end points } u, v \text{ in } G\}.$$

It follows from Theorem D that $\varGamma$ is not empty. Fix $\alpha \in \varGamma$ and for an ordered triple of points $x_1, x, x_2 \in \gamma$ and $y \in \partial G$, it suffices to verify (2.3) for these points.

First, by [22, Theorem 10.17], G is quasihyperbolic c_1-uniform with $c_1 = c_1(c)$; for the definition see [22, 10.2]. Moreover, it follows from the Cigar Theorem [22, 10.9] that the subarc $\gamma[x_1, x_2]$ is c_2-uniform with $c_2 = c_2(c)$. Thus we have

$$\min_{j=1,2} |x_j - x| \leq \min_{j=1,2} \ell(\gamma[x_j, x]) \leq c_2 d_G(x) \leq c_2|x - y|.$$

This yields

$$\min_{j=1,2} |x_j - y| \leq |x - y| + \min_{j=1,2} |x_j - x| \leq (1 + c_2)|x - y|. \tag{3.1}$$

On the other hand, by the uniformity of $\gamma[x_1, x_2]$,

$$|x-y| \leq |x-x_1|+|x_1-y| \leq c_2|x_1-x_2|+|x_1-y| \leq (2c_2+1) \max_{j=1,2} |x_j-y|. \tag{3.2}$$

From (3.1) and (3.2) we see that G satisfies the min-max property for the family $\varGamma$.

Lemma 3.2. *Suppose that a domain $G \subsetneq E$ satisfies the min-max property. Then G is diameter c_1-uniform.*

Proof. Fix $x_1, x_2 \in G$. If $|x_1 - x_2| \leq \frac{1}{2} d_G(x_1)$, then the line segment $[x_1, x_2]$ is the desired diameter uniform arc.

If $|x_1 - x_2| > \frac{1}{2} d_G(x_1)$, take $y \in \partial G$ with $|x_1 - y| \leq 2d_G(x_1)$. Then

$$|x_1 - y| \leq 4|x_1 - x_2|. \tag{3.3}$$

Because G satisfies the min-max property, there exists a curve γ joining x_1 and x_2 in G and a constant $c \geq 1$ such that (2.3) holds. Then for all $x \in \gamma$, we deduce from (3.3) that

$$|x - y| \leq c \max_{j=1,2} |x_j - y| \leq c(|x_1 - x_2| + |x_1 - y|) \leq 5c|x_1 - x_2|,$$

which implies

$$\operatorname{diam} \gamma \leq 10c|x_1 - x_2|. \tag{3.4}$$

Fix $z \in \gamma$ and take another point $z \in \partial G$ with $|x - z| \leq 2d_G(x)$. We claim that for $j = 1$ or $j = 2$,

$$\gamma[x_j, x] \subseteq B(z, 2cd_G(x)). \tag{3.5}$$

Otherwise, there are $u_j \in \gamma[x_j, x]$ for $j = 1, 2$ such that

$$\min_{j=1,2} |u_j - z| > 2cd_G(x) \geq c|x - z|,$$

which contradicts (2.3). Therefore, we see from (3.5) that

$$\min_{j=1,2} \operatorname{diam} \gamma[x_j, x] \leq 4cd_G(x). \tag{3.6}$$

Hence it follows from (3.4) and (3.6) that γ is the required arc. ∎

Lemma 3.3. *Let $G \subsetneq E$ be a diameter c-uniform domain with $c \geq 1$. Then G is δ-uniform for some constant $\delta = \delta(c) \in (0, 1)$.*

Proof. Fix two distinct points $x_1, x_2 \in G$. Because G is diameter c-uniform, we can choose a diameter c-uniform arc γ connecting x_1 and x_2 in G. Thus for all $x \in \gamma$ and $y \in E \setminus G$, we may assume without loss of generality that

$$\operatorname{diam} \gamma[x_1, x] \leq \operatorname{diam} \gamma[x_2, x].$$

In order to show that G is δ-uniform, we need to find lower bounds for the cross ratios $\tau(x, x_1, y, x_2)$ and $\tau(x, x_2, y, x_1)$.

We first check that

$$\tau(x, x_1, y, x_2) \geq \frac{1}{2c^2} \tag{3.7}$$

by considering two cases. If $|x_2 - y| \leq 2c|x_1 - x_2|$, we have

$$\tau(x, x_1, y, x_2) = \frac{|x - y|\,|x_1 - x_2|}{|x - x_1|\,|x_2 - y|} \geq \frac{\operatorname{diam} \gamma[x_1, x]\,|x_1 - x_2|}{c|x - x_1|\,|x_2 - y|} \geq \frac{1}{2c^2}.$$

If $|x_2 - y| > 2c|x_1 - x_2|$, the diameter uniformity of γ yields

$$\max_{j=1,2} |x_j - x| \leq \operatorname{diam} \gamma[x_1, x_2] \leq c|x_1 - x_2|,$$

and thus

$$\frac{|x - y|}{|x_2 - y|} \geq \frac{|x_2 - y| - |x - x_2|}{|x_2 - y|} \geq \frac{1}{2}.$$

This implies that

$$\tau(x, x_1, y, x_2) = \frac{|x - y|\,|x_1 - x_2|}{|x - x_1|\,|x_2 - y|} \geq \frac{1}{2c},$$

and again we obtain (3.7).

It remains to find a lower bound for $\tau(x, x_2, y, x_1)$. Towards this end, we check that

$$|x - y| \geq \frac{1}{c + 1}|y - x_1|. \tag{3.8}$$

Suppose on the contrary that $|x - y| < \frac{1}{c+1}|y - x_1|$. Then

$$|x_1 - x| \geq |y - x_1| - |y - x| > \frac{c}{c + 1}|y - x_1|$$

and thus

$$|x-y| > \frac{1}{c}\operatorname{diam}\gamma[x_1,x] \geq \frac{1}{c}|x-x_1| > \frac{1}{c+1}|y-x_1|,$$

which is a contradiction, showing (3.8). Moreover, we infer from (3.8) that

$$\tau(x,x_2,y,x_1) = \frac{|x-y|\,|x_1-x_2|}{|x-x_2|\,|x_1-y|} \geq \frac{1}{c(c+1)}. \tag{3.9}$$

By (3.7) and (3.9) we find that G is δ-uniform with $\delta = 1/(2c^2+2c)$. ∎

Lemma 3.4. *Let $G \subsetneq E$ be a δ-uniform domain with $\delta \in (0,1)$. Then G is diameter c-uniform for some constant $c = c(\delta) \geq 1$.*

Proof. Fix $u, v \in G$ distinct and take $t = \frac{1}{16}|u-v|$. If $|u-v| < \max\{d_G(u), d_G(v)\}$, then $[u,v] \subseteq G$ is a diameter uniform arc. Thus we may assume that

$$|u-v| \geq \max\{d_G(u), d_G(v)\}. \tag{3.10}$$

Because G is δ-uniform, we can join u and v by an arc α satisfying the δ-uniformity condition (2.4). Denote by α_0 the subarc of α contained in $G \setminus \big(B(u,t) \cup B(v,t)\big)$ and such that α_0 connects the spheres $S(u,t)$ and $S(v,t)$. Let $u_0 = \alpha_0 \cap S(u,t)$ and $v_0 = \alpha_0 \cap S(v,t)$.

Next, we are going to construct a curve β joining u to u_0 and satisfying

$$\beta[u,x] \subseteq B(x, C d_G(x)) \tag{3.11}$$

for all $x \in \beta$, where $C = C(\delta) \geq 1$ will be determined below.

To this end, choose a sequence of points u_i such that $u_i \in \alpha \cap S(u, t/2^i)$ for all $i \geq 1$. Moreover, for each $i \geq 0$, pick $y_i \in \partial G$ with

$$|y_i - u_i| \leq 2 d_G(u_i).$$

We consider two possibilities.

If there is some $i \geq 0$ such that $|v - y_i| < \frac{1}{2}|u-v|$, then

$$\begin{aligned} d_G(u) &\geq d_G(u_i) - |u-u_i| \geq \tfrac{1}{2}|u_i - y_i| - |u-u_i| \\ &\geq \tfrac{1}{2}|u-v| - \tfrac{3}{2}|u-u_i| - \tfrac{1}{2}|v-y_i| \\ &> \tfrac{1}{4}|u-v| - \tfrac{3}{2}|u-u_0| > 2|u-u_0|, \end{aligned}$$

where the last inequality follows from the choice of u_0 in α. It follows that we can take the line segment $[u,u_0] = \beta$ satisfying (3.11) for $C = 1$, as desired.

Now assume that $|v - y_i| \geq \frac{1}{2}|u-v|$ for all $i \geq 0$. In this case for each $i \geq 0$, we take an arc β_i connecting u_i to u_{i+1} satisfying the δ-uniformity condition. Thus $\beta := \bigcup_{i=0}^{\infty} \beta_i$ is a curve joining u to u_0. To show that β is as required, we need some estimates.

By our choices of u_i and y_i, the δ-uniformity of α implies that

$$d_G(u_i) \geq \frac{1}{2}|y_i - u_i| \geq \frac{\delta|v - y_i|}{2|u - v|}|u_i - u| = \frac{\delta}{2^{i+5}}|v - y_i| \geq \frac{\delta}{2^{i+6}}|u - v| =: r_i \tag{3.12}$$

for all $i \geq 0$.

Moreover, we claim that for all $x \in \beta_i$ and for every $y \in \partial G$,

$$|x - y| \geq \frac{\delta^3}{2^{i+12}}|u - v|. \tag{3.13}$$

This can be seen as follows. If $x \in B\big(u_i, \frac{1}{2}r_i\big) \cup B\big(u_{i+1}, \frac{1}{2}r_{i+1}\big)$, by (3.12) we have

$$|x - y| \geq |y - u_{i+1}| - |x - u_{i+1}| \geq \tfrac{1}{2}r_{i+1} = \frac{\delta}{2^{i+8}}|u - v|,$$

as needed.

If $x \notin B\big(u_i, \frac{1}{2}r_i\big) \cup B\big(u_{i+1}, \frac{1}{2}r_{i+1}\big)$, we have $|x - u_i| \geq \frac{1}{2}r_i$, $|y - u_{i+1}| \geq r_{i+1}$ and

$$|u_i - u_{i+1}| \leq |u_i - u| + |u_{i+1} - u| \leq \frac{t}{2^{i-1}}.$$

Because β_i satisfies the δ-uniformity property, the estimates above produce

$$|x - y| \geq \frac{\delta|x - u_i|\,|y - u_{i+1}|}{|u_i - u_{i+1}|} \geq \frac{\delta^3}{2^{i+12}}|u - v|,$$

as desired. This shows (3.13).

We verify (3.11) by using (3.13). Fix $x \in \beta$. Then there is an index $k = 0, 1, \ldots$ such that $x \in \beta_k$. For all $z \in \beta_i$ with $i \geq k$, the δ-uniformity of β_i yields

$$|z - u_i| \leq \frac{1}{\delta}|u_i - u_{i+1}| \leq \frac{t}{\delta 2^{i-1}},$$

and therefore

$$|z - u| \leq |z - u_i| + |u_i - u| \leq \frac{t}{\delta 2^{i-2}}.$$

Consequently, by (3.13) we find that

$$|z - x| \leq |z - u| + |u - x| \leq \frac{t}{\delta 2^{k-3}} \leq \frac{2^{11}}{\delta^4}d_G(x),$$

which implies (3.11) by letting $C = 2^{11}\delta^{-4}$.

A similar argument shows that there is a curve γ connecting v_0 and v such that

$$\gamma[v, x] \subset B(x, Cd_G(x)) \tag{3.14}$$

for all $x \in \gamma$.

Let $\alpha' = \beta \cup \alpha_0 \cup \gamma$. We shall prove that α' satisfies the desired diameter uniform condition. For all $x \in \beta$, we deduce from (3.10) and (3.11) that

$$\frac{1}{C}|x-u_0| \le d_G(u_0) \le d_G(u) + |u-u_0| \le 2|u-v|,$$

and thus

$$\operatorname{diam}\beta \le 4C|u-v|. \tag{3.15}$$

Similarly, it follows from (3.10) and (3.14) that

$$\operatorname{diam}\gamma \le 4C|u-v|. \tag{3.16}$$

Moreover, for all $x \in \alpha_0$, by the δ-uniformity of α, we have $|u-x| \le |u-v|/\delta$, which implies

$$\operatorname{diam}\alpha_0 \le \frac{2}{\delta}|u-v|. \tag{3.17}$$

Therefore, we see from (3.15)–(3.17) that

$$\operatorname{diam}\alpha' \le \operatorname{diam}\beta + \operatorname{diam}\alpha_0 + \operatorname{diam}\gamma \le (8C + 2/\delta)|u-v|. \tag{3.18}$$

It follows from (3.11), (3.14) and (3.18) that we only need to show that there exists a constant $A \ge 1$ such that

$$\min\{\operatorname{diam}\alpha'[u,x], \operatorname{diam}\alpha'[v,x]\} \le A d_G(x)$$

for all $x \in \alpha_0$. Because α satisfies the δ-uniformity condition, for all $y \in \partial G$ we have

$$\begin{aligned} |x-y| &\ge \max\left\{\frac{\delta|x-u|}{|u-v|}|y-v|, \frac{\delta|x-v|}{|u-v|}|y-u|\right\} \\ &\ge \frac{\delta}{16}\max\{|y-v|, |y-u|\} \ge \frac{\delta}{32}|u-v|. \end{aligned}$$

This together with (3.18) yields

$$\operatorname{diam}\alpha' \le \left(8C + \frac{2}{\delta}\right)\frac{32}{\delta} d_G(x),$$

as desired. Hence we obtain the conclusion of Lemma 3.4. ∎

Proof of Theorem 1.1. The conclusion follows immediately from Lemmas 3.1–3.4. ∎

3.2. In this part, we give the proof of Theorem 1.2. Note that the necessity part follows from Lemma F. To prove the sufficiency, we first develop certain intermediate results. The idea of our arguments for Lemmas 3.5 and 3.6 below comes from [11].

Lemma 3.5. *Assume that $G \subsetneq E$ is a ψ-natural and diameter c-uniform domain. If $x, x_0 \in G$ with $d_G(x_0) \le 2d_G(x)$ and $|x-x_0| \le c_0 d_G(x_0)$ for* *some constant $c_0 \geq 1$, then there is an arc α in G connecting x and x_0 such that*

$$\ell_k(\alpha[x, x_0]) \leq 2\psi(2cc_0(1+c)),$$

where k is the quasihyperbolic metric of G.

Proof. Because G is diameter c-uniform, there is a diameter c-uniform arc γ in G joining x and x_0. Because $d_G(x_0) \leq 2d_G(x)$, we thus have

$$\begin{aligned} \tfrac{1}{2}d_G(x_0) &\leq \min\{d_G(x), d_G(x_0)\} \\ &\leq d_G(z) + \min\{|z-x|, |z-x_0|\} \\ &\leq (1+c)d_G(z) \end{aligned} \tag{3.19}$$

for all $z \in \gamma$. Moreover, because G is ψ-natural and $|x - x_0| \leq c_0 d_G(x_0)$, we see by (3.19) that

$$\begin{aligned} k(x, x_0) &\leq \operatorname{diam}_k(\gamma) \leq \psi\left(\frac{\operatorname{diam}\gamma}{\operatorname{diam}(\gamma, \partial G)}\right) \\ &\leq \psi\left(\frac{2c(1+c)|x-x_0|}{d_G(x_0)}\right) \leq \psi(2cc_0(1+c)), \end{aligned} \tag{3.20}$$

where $\operatorname{diam}_k(\gamma)$ denotes the quasihyperbolic diameter of γ.

Next, choose an arc α connecting x and x_0 in G such that $\ell_k(\alpha) \leq 2k(x, x_0)$. Then (3.20) yields

$$\ell_k(\alpha) \leq 2k(x, x_0) \leq 2\psi(2cc_0(1+c)),$$

as desired. ∎

LEMMA 3.6. *Assume that $G \subsetneq E$ is a ψ-natural and diameter c-uniform domain. If $x, x_0 \in G$ with $d_G(x_0) > 2d_G(x)$, then there is an arc α in G connecting x and x_0 such that*

$$\ell_k(\alpha[x, y]) \leq 2\psi(8c^2(1+c)),$$

where y is the first point of α such that $d_G(y) = 2d_G(x)$ when traversing α from x to x_0 and k is the quasihyperbolic metric of G.

Proof. Because G is diameter c-uniform, there is a diameter c-uniform arc γ in G joining x and x_0. Denote by y_0 the first point of γ such that

$$d_G(y_0) = 2d_G(x) < d_G(x_0).$$

Let $\beta = \gamma[x, y_0]$ and $\beta' = \gamma[y_0, x_0]$. Then we show that

$$k(x, y_0) \leq \psi(8c^2(1+c)). \tag{3.21}$$

We consider two possibilities. If $\operatorname{diam}\beta \leq \operatorname{diam}\beta'$, thus by the uniformity of γ we have

$$\operatorname{diam}\beta \leq c d_G(y_0) = 2c d_G(x)$$

and

$$d_G(x) \leq |x - z| + d_G(z) \leq (c+1)d_G(z)$$

for all $z \in \beta$. Because G is ψ-natural, this implies that

$$k(x, y_0) \le \operatorname{diam}_k(\beta) \le \psi\left(\frac{\operatorname{diam}\beta}{\operatorname{diam}(\beta, \partial G)}\right) \le \psi(2c(1+c)),$$

as required.

If $\operatorname{diam}\beta > \operatorname{diam}\beta'$, choose $u \in \beta$ with $\operatorname{diam}\beta[x, u] = \operatorname{diam}\beta/2$. Again we see from the uniformity of β that

$$|x - x_0| \le 2\operatorname{diam}\beta \le 4\operatorname{diam}\beta[x, u] \le 4cd_G(u) \le 8cd_G(x),$$

and

$$\begin{aligned} d_G(x) &= \min\{d_G(x), d_G(x_0)\} \le d_G(z) + \min\{|z - x|, |z - x_0|\} \\ &\le (c+1)d_G(z) \end{aligned}$$

for any $z \in \beta$. Because G is ψ-natural, this yields

$$\begin{aligned} k(x, y_0) \le \operatorname{diam}_k(\beta) &\le \psi\left(\frac{\operatorname{diam}\beta}{\operatorname{diam}(\beta, \partial G)}\right) \le \psi\left(\frac{c(c+1)|x - x_0|}{d_G(x)}\right) \\ &\le \psi(8c^2(1+c)), \end{aligned}$$

which is (3.21).

To conclude the proof, we choose an arc α_0 connecting x and y_0 in G such that $\ell_k(\alpha_0) \le 2k(x, y_0)$. Moreover, let $y \in \alpha_0$ be the first point such that $d_G(y) = 2d_G(x)$. Then $\alpha = \alpha_0 \cup \gamma[y_0, x_0]$ is as desired. ∎

LEMMA 3.7. *Assume that $G \subsetneq E$ is a ψ-natural and diameter c-uniform domain. Then G is c_1-John for some $c_1 = c_1(\psi, c)$.*

Proof. Fix $x, y \in G$. Because G is diameter c-uniform, there is a diameter c-uniform arc γ in G connecting x and y. Choose $x_0 \in \gamma$ such that $\operatorname{diam}\gamma[x, x_0] = \operatorname{diam}\gamma[x_0, y]$. Then

$$\operatorname{diam}\gamma[x, x_0] = \operatorname{diam}\gamma[x_0, y] \le cd_G(x_0). \tag{3.22}$$

In order to show that G is c_1-John, by symmetry, we only need to show that there is a constant $c_1 > 0$ and a curve σ joining x and x_0 such that

$$\ell(\sigma[x, z]) \le c_1 d_G(z) \tag{3.23}$$

for all $z \in \sigma$.

We divide the proof of (3.23) into two cases. Let $a_1 = 2\psi(8c^2(1+c))$, $c_0 = 2e^{a_1} + c$ and $a_2 = 2\psi(2cc_0(1+c))$.

CASE 1: $d_G(x) \ge d_G(x_0)/2$. By (3.22) we have

$$|x - x_0| \le cd_G(x_0).$$

Moreover, from Lemma 3.5 we see that there is an arc σ joining x and x_0 such that

$$\ell_k(\sigma) \le 2\psi(2c^2(1+c)) \le a_1. \tag{3.24}$$

Then for all $z \in \sigma$, we find by (2.1) and (3.24) that

$$\left|\log \frac{d_G(z)}{d_G(x_0)}\right| \le k(z, x_0) \le \ell_k(\sigma) \le a_1,$$

and therefore

$$e^{-a_1} d_G(x_0) \le d_G(z) \le e^{a_1} d_G(x_0).$$

This yields

$$\ell(\sigma) \le e^{\ell_k(\sigma)} d_G(x_0) \le e^{a_1} d_G(z), \tag{3.25}$$

as desired.

CASE 2: $d_G(x) < d_G(x_0)/2$. Let $n \ge 1$ be the unique integer such that

$$2^{n-1} d_G(x) < d_G(x_0) \le 2^n d_G(x).$$

We construct a sequence of points $x = x_1, x_2, \dots, x_n, x_{n+1} = x_0$ and curves β_i as follows.

First, we see from Lemma 3.6 that there is an arc α_1 joining x_1 and x_0 with $\ell_k(\alpha_1[x_1, x_2]) \le a_1$, where x_2 is the first point of α_1 (when traversing α_1 from x_1 towards x_0) with $d_G(x_2) = 2d_G(x_1)$. If $d_G(x_2) \ge d_G(x_0)/2$, again we connect x_2 to x_0 by an arc α_2 such that the conclusion of Lemma 3.6 holds for this arc. Then we stop with $n = 2$, $\beta_2 = \alpha_2$ and $x_3 = x_0$. Otherwise we continue the process by letting $\beta_i = \alpha_i[x_i, x_{i+1}]$ where x_{i+1} is the first point of α_i with $d_G(x_{i+1}) = 2d_G(x_i)$, and where α_i is the arc joining x_i to x_0 such that the conclusion of Lemma 3.6 holds. Since $d_G(x_i) = 2^{i-1} d_G(x_1)$, we find that $d_G(x_i) \ge d_G(x_0)/2$ as soon as $i \ge \log_2 \frac{d_G(x_0)}{d_G(x_1)}$. Thus the above process stops with $i = n$, $\beta_n = \alpha_n$ and $x_{n+1} = x_0$.

We shall verify that the curve $\beta = \bigcup_{i=1}^n \beta_i$ has the desired property. To this end, note that by Lemma 3.6 we have

$$\ell_k(\beta_i) \le a_1 \tag{3.26}$$

for $1 \le i \le n-1$. By using a similar argument to (3.25), this yields

$$\ell(\beta_i) \le e^{a_1} d_G(x_i). \tag{3.27}$$

For $i = n$, we claim that

$$\ell_k(\beta_n) \le a_2. \tag{3.28}$$

Indeed, because $|x - x_0| \le c d_G(x_0)$, by (3.27) we have

$$\begin{aligned} |x_n - x_0| &\le |x_n - x_{n-1}| + \cdots + |x_2 - x_1| + |x_1 - x_0| \\ &\le \sum_{i=1}^{n-1} \ell(\beta_i) + |x - x_0| \le \sum_{i=1}^{n-1} e^{a_1} d_G(x_i) + |x - x_0| \\ &\le (2e^{a_1} + c) d_G(x_0) = c_0 d_G(x_0). \end{aligned}$$

This together with Lemma 3.5 implies (3.28).

Then from (3.28) and a similar argument with (3.25) it follows that

$$\ell(\beta_n) \le e^{a_2} d_G(x_n). \tag{3.29}$$

Without loss of generality we may assume $a_2 \ge a_1$. Fix $z \in \beta$. Then there is $1 \le j \le n$ such that $z \in \beta_j$. Moreover, by (2.1) and (3.26),

$$\left|\log \frac{d_G(a_j)}{d_G(z)}\right| \le k(z, a_j) \le \ell_k(\beta_j) \le a_2$$

and thus

$$d_G(x_j) \le e^{a_2} d_G(z). \tag{3.30}$$

Therefore, we see from (3.27), (3.29) and (3.30) that

$$\ell(\beta[x,z]) \le \sum_{i=1}^{j} \ell(\beta_j) \le e^{a_2} \sum_{i=1}^{j} d_G(x_j) \le 2e^{a_2} d_G(x_j) \le 2e^{2a_2} d_G(z),$$

which shows (3.23). Hence the proof of Lemma 3.7 is complete. ■

Finally, we are in a position to show the sufficiency part of Theorem 1.2. Note that by Lemma 3.7, we only need to prove the following result.

LEMMA 3.8. *Assume that $G \subsetneq E$ is a ψ-natural, c_1-John and diameter c-uniform domain. Then G is b-uniform for some $b = b(\psi, c_1, c)$.*

Proof. Fix $x, y \in G$. Let k be the quasihyperbolic metric of G. If $|x-y| < d_G(x)$, the line segment $[x,y]$ is the desired uniform arc.

In the following, we may assume that $|x-y| \ge d_G(x)$. Since G is c_1-John, there is a c_1-cone arc α connecting x and y in G. Choose two points x' and y' in α such that

$$\tfrac{1}{2}|x-y| = \ell(\alpha[x,x']) = \ell(\alpha[y,y']) \le c_1 \min\{d_G(x'), d_G(y')\}. \tag{3.31}$$

Then by (3.31) we have

$$\begin{aligned} |x'-y'| &\le |x'-x| + |x-y| + |y-y'| \le 2|x-y| \\ &\le 4c_1 \min\{d_G(x'), d_G(y')\}. \end{aligned} \tag{3.32}$$

Moreover, we may join x' and y' by a diameter c-uniform arc β in G because G is a diameter c-uniform domain. It follows from the uniformity of β and (3.32) that

$$\operatorname{diam}\beta \le c|x'-y'| \le 4c_1 c \min\{d_G(x'), d_G(y')\} \tag{3.33}$$

and

$$\min\{d_G(x'), d_G(y')\} \le d_G(z) + \min\{|x'-z|, |z-y'|\} \le (1+c) d_G(z) \tag{3.34}$$

for all $z \in \beta$. Because G is ψ-natural, we see by (3.33) and (3.34) that

$$k(x',y') \le \operatorname{diam}_k(\beta) \le \psi\left(\frac{\operatorname{diam}\beta}{\operatorname{diam}(\beta, \partial G)}\right) \le \psi(4c_1 c(1+c)) =: \lambda. \tag{3.35}$$

It thus follows from (3.35) that there is an arc γ connecting x' and y' in G such that

$$\ell_k(\gamma) \le 2k(x', y') \le 2\lambda. \tag{3.36}$$

Denote $\sigma = \alpha[x, x'] \cup \gamma \cup \alpha[y', y]$. We show that σ satisfies the uniformity condition. Because $d_G(x) \le |x - y|$, by (3.31) we have

$$d_G(x') \le d_G(x) + |x' - x| \le 2|x - y|. \tag{3.37}$$

Moreover, we see from (3.36) that

$$\log\left(1 + \frac{\ell(\gamma)}{d_G(x')}\right) \le \ell_k(\gamma) \le 2\lambda,$$

which together with (3.37) implies that

$$\ell(\gamma) \le 2e^{2\lambda}|x - y|.$$

Therefore, by (3.31),

$$\ell(\sigma) = \ell(\alpha[x, x']) + \ell(\gamma) + \ell(\alpha[y', y]) \le (1 + 2e^{2\lambda})|x - y|. \tag{3.38}$$

It remains to show that σ satisfies the cone condition. Fix $z \in \sigma$. If $z \in \alpha[x, x'] \cup \alpha[y', y]$, then we use the fact that α is c_1-cone. If $z \in \gamma$, again by (2.1) and (3.36) we have

$$\left|\log \frac{d_G(x')}{d_G(z)}\right| \le k(z, x') \le \ell_k(\gamma) \le 2\lambda,$$

and therefore $d_G(x') \le e^{2\lambda} d_G(z)$. By (3.38) and (3.31),

$$\ell(\sigma) \le (1 + 2e^{2\lambda})|x - y| \le 2c_1(1 + 2e^{2\lambda}) d_G(x') \le 2c_1(1 + 2e^{2\lambda}) e^{2\lambda} d_G(z),$$

as desired. This proves Lemma 3.8. ■

4. Proofs of Theorems 1.3 and 1.4. In this section, we start by proving certain auxiliary results. We first show that the diameter uniformity of domains is preserved under quasisymmetric mappings.

LEMMA 4.1. *Let G and G' be proper domains in Banach spaces E and E', respectively. Suppose that G is diameter c-uniform and that a homeomorphism $f : G \to G'$ is η-quasisymmetric. Then G' is diameter c'-uniform with $c' = c'(c, \eta)$.*

Proof. For any x', y' in G', denote $x = f^{-1}(x')$ and $y = f^{-1}(y')$. Because G is diameter c-uniform, there is a diameter c-uniform arc γ connecting x and y in G. We show that $f(\gamma) = \gamma'$ is the desired uniform arc in G'. Without loss of generality, we may assume that $f : \overline{G} \to \overline{G'}$ is also η-quasisymmetric by [22, Theorem 6.12].

On the one hand, for all $u \in \gamma$, the diameter uniformity of γ implies that

$$|x - u| \le c|x - y|.$$

Because f is η-quasisymmetric, we see that

$$|f(x)-f(u)| \le \eta(c)|f(x)-f(y)|,$$

and thus

$$\operatorname{diam}\gamma' \le 2\eta(c)|f(x)-f(y)|. \tag{4.1}$$

On the other hand, for any $z' \in \gamma'$ with $f(z)=z'$, without loss of generality we may assume that

$$\operatorname{diam}\gamma[x,z] \le \operatorname{diam}\gamma[y,z].$$

Because γ is diameter c-uniform, for all $v\in\gamma[x,z]$ and $w\in\partial G$ we have

$$|z-v| \le c|z-w|.$$

By the quasisymmetry proeprty of f, we obtain

$$|f(z)-f(v)| \le \eta(c)|f(z)-f(w)|.$$

This yields

$$\operatorname{diam}\gamma'[x',z'] \le 2\eta(c)d_G(z'). \tag{4.2}$$

Thus, from (4.1) and (4.2) we see that γ' is as required. ∎

Next, we prove that the δ-uniformity condition is preserved under quasimöbius maps.

LEMMA 4.2. *Let G and G' be proper domains in Banach spaces E and E', respectively. Let $0<\delta<1$. Suppose that G is δ-uniform and that a homeomorphism $f: G\to G'$ is η-quasimöbius. Then G' is δ'-uniform with $\delta'=\delta'(\delta,\eta)\in(0,1)$.*

Proof. We want to reduce the situation to the quasisymmetric case by using auxiliary inversions. To this end, consider the one-point extensions $\dot{E}=E\cup\{\infty\}$ and $\dot{E}'=E'\cup\{\infty\}$. For all $x\in\dot{E}$, we define the inversion $u:\dot{E}\to\dot{E}$ as

$$u(x)=\frac{x}{|x|^2}$$

with $u(0)=\infty$ and $u(\infty)=0$. Similarly, for every $x'\in\dot{E}'$, the inversion $u':\dot{E}'\to\dot{E}'$ is defined by

$$u'(x')=\frac{x'}{|x'|^2}$$

with $u'(0)=\infty$ and $u'(\infty)=0$. It follows from [22, Theorem 6.22] that u and u' are both θ-quasimöbius with $\theta(t)=81t$. Since G is δ-uniform, we see that $u(G)$ is δ_1-uniform with $\delta_1=\delta/81$. Similarly, G' is δ'-uniform if and only if $u'(G')$ is δ_1'-uniform with the constants δ' and δ_1' depending only on each other.

By [22, Theorem 6.24] we may assume that $f:\overline{G}\to\overline{G'}$ is η-quasimöbius as well. By auxiliary translations, we may also assume that $0\in\partial G$ and

$f(0) = 0 \in \partial G'$, or $f(0) = \infty \in \partial G'$. Now we consider two possibilities. If $f(0) = 0$, we define

$$g := u' \circ f \circ u^{-1} : u(G) \to u'(G').$$

If $f(0) = \infty$, we define

$$g := f \circ u^{-1} : u(G) \to G'.$$

Thus, g is η_1-quasimöbius with $\eta_1(t) = 81\eta(81t)$. Because in both cases $g(\infty) = \infty$, we see from [16, Theorem 3.20] that g is η_1-quasisymmetric.

Furthermore, because $u(G)$ is δ_1-uniform, by Lemma 3.4 we find that $u(G)$ is diameter c-uniform with $c = c(\delta_1)$. Then Lemma 4.1 shows that $g \circ u(G)$ is diameter c'-uniform with $c' = c'(c, \eta)$. Therefore by using Lemma 3.3, we conclude that G' is δ'-uniform with $\delta' = \delta'(\delta, \eta)$, as desired. ∎

Lemma 4.3. *Let G and G' be proper domains in Banach spaces E and E', respectively. Suppose that G is diameter c-uniform and that a homeomorphism $f : G \to G'$ is η-quasimöbius. Then G' is diameter c'-uniform with $c' = c'(c, \eta)$.*

Proof. This follows immediately from Lemmas 3.3, 3.4 and 4.2. ∎

Lemma 4.4. *Let G and G' be proper domains in Banach spaces E and E', respectively. Suppose that G has the min-max property and that a homeomorphism $f : \overline{G} \to \overline{G'}$ is η-quasisymmetric relative to ∂G and maps G onto G'. Then G' also has the min-max property.*

Proof. Suppose that there exists a family Γ of curves in G and a constant $c \geq 1$ such that any pair of points in G can be joined by a curve $\gamma \in \Gamma$ and (2.3) holds for each ordered triplet $x_1, x, x_2 \in \gamma$ and for all $y \in \partial G$.

Because f is η-quasisymmetric relative to ∂G, it is not difficult to see that any pair of points in G' can be joined by a curve $\gamma' \in \Gamma' = f(\Gamma)$ such that (2.3) holds with the constant $\eta(c)$, for each ordered triplet $x'_1, x', x'_2 \in \gamma'$ and for all $y' \in \partial G'$. This completes the proof. ∎

Now, we are ready to complete the proofs of Theorems 1.3 and 1.4.

Proof of Theorem 1.3. We know from Theorem 1.2 that the second assertion of Theorem 1.3 follows from the first. Thus we only need to show that G' is diameter uniform. By using a similar argument to that for Lemma 4.2, we reduce the situation to the case of relative quasisymmetry by using auxiliary inversions. For completeness, we give the details.

Consider the one-point extensions $\dot{E} = E \cup \{\infty\}$ and $\dot{E}' = E' \cup \{\infty\}$. By auxiliary translations, we may assume that $0 \in \partial G$ and $f(0) = 0 \in \partial G'$, or $f(0) = \infty \in \partial G'$. Now we consider two possibilities. If $f(0) = 0$, we define

$$g := u' \circ f \circ u^{-1} : u(G) \to u'(G').$$

If $f(0)=\infty$, we define

$$g := f\circ u^{-1} : u(G)\to G'.$$

By [22, Theorem 6.22], g is η_1-quasimöbius relative to $\partial(u(G))$ with $\eta_1(t)=81\eta(81t)$. Because in both cases $g(\infty)=\infty$, we see that g is η_1-quasisymmetric relative to $\partial(u(G))$. Moreover, since G is c-uniform, Theorem C shows that $u(G)$ is c_1-uniform with $c_1=c_1(c)$. Thus by Lemma 3.1, $u(G)$ has the min-max property. Furthermore, it follows from Lemma 4.4 that $g\circ u(G)$ also satisfies the min-max property. Therefore, we may use Lemma 3.2 to conclude that $g\circ u(G)$ is diameter c_1'-uniform with c_1' depending only on c and η.

Hence by Lemma 4.3, G' is diameter c'-uniform with c' depending only on c and η. This completes the proof of Theorem 1.3. ∎

Proof of Theorem 1.4. We first prove (1). Let A' be a nonempty connected set in G' with $r_{G'}(A')=t$ and $A=f^{-1}(A')\subseteq G$. Because f is η-quasimöbius relative to ∂G, we find that $f^{-1}:\overline{G}\to\overline{G'}$ is η'-quasimöbius relative to $\partial G'$ with $\eta'(t)=\eta^{-1}(t^{-1})^{-1}$ (cf. [22]). Then, by Lemma G, there is an increasing function $\mu:[0,\infty)\to[0,\infty)$ depending only on η such that

$$r_G(A)\le \mu(t). \tag{4.3}$$

Moreover, because f is (M,C)-CQH and G is ψ-natural, (4.3) yields

$$k_{G'}(A')\le M\,\mathrm{diam}_k(A)+C\le M\psi(r_G(A))+C\le M\psi\circ\mu(t)+C.$$

This implies that G' is ψ'-natural by taking $\psi'(t)=M\psi\circ\mu(t)+C$, as desired.

Next, we verify (2). Assume that G is c-uniform with $c\ge 1$. By Lemma F, we see that G is ψ-natural with ψ depending only on c. Moreover, (1) shows that G' is ψ'-natural. On the other hand, by Theorem 1.3, G' is c'-uniform. Hence the proof of Theorem 1.4 is complete. ∎

5. Proofs of Corollaries 1.1, 1.2 and 1.3

Proof of Corollary 1.1. Corollary 1.1 follows immediately from Lemmas 4.2 and 4.3. ∎

Proof of Corollary 1.2. Because the ψ-uniformity of domains implies their ψ-naturality, the sufficiency in Corollary 1.2 follows from Theorem 1.2. The necessity is proved by using [19, Theorem 6.16] because a c-uniform domain is clearly diameter c-uniform. The second assertion follows from the fact that a convex domain in Banach space is φ-uniform with $\varphi(t)=t$. ∎

Proof of Corollary 1.3. By Theorem 1.4, G is ψ_1-natural. Because G is diameter c-uniform, we know from Theorem 1.2 that G is c_0-uniform. Again, it follows from Theorem 1.4 that G' is c_1-uniform. ∎

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References

[1] F. W. Gehring and K. Hag, *Remarks on uniform and quasiconformal extension domains*, Complex Variables Theory Appl. 9 (1987), 175–188.

[2] F. W. Gehring and B. G. Osgood, *Uniform domains and the quasi-hyperbolic metric*, J. Anal. Math. 36 (1979), 50–74.

[3] F. W. Gehring and B. P. Palka, *Quasiconformally homogeneous domains*, J. Anal. Math. 30 (1976), 172–199.

[4] M. Huang, Y. Li, M. Vuorinen and X. Wang, *On quasimöbius maps in real Banach spaces*, Israel J. Math. 198 (2013), 467–486.

[5] X. Huang and J. Liu, *Quasihyperbolic metric and quasisymmetric mappings in metric spaces*, Trans. Amer. Math. Soc. 367 (2015), 6225–6246.

[6] M. Huang, A. Rasila, X. Wang and Q. Zhou, *Semisolidity and locally weak quasisymmetry of homeomorphisms in metric spaces*, Studia Math. 242 (2018), 267–301.

[7] F. John, *Rotation and strain*, Comm. Pure Appl. Math. 14 (1961), 391–413.

[8] P. W. Jones, *Extension theorems for BMO*, Indiana Univ. Math. J. 29 (1980), 41–66.

[9] Y. Li, S. Ponnusamy and Q. Zhou, *Sphericalization and flattening preserve uniform domains in nonlocally compact metric spaces*, J. Austral. Math. Soc. (online, 2020).

[10] Y. Li, M. Vuorinen and Q. Zhou, *Weakly quasisymmetric maps and uniform spaces*, Comput. Methods Funct. Theory 18 (2018), 689–715.

[11] Y. Li, M. Vuorinen and Q. Zhou, *Characterizations of John spaces*, Monatsh. Math. 188 (2019), 547–559.

[12] Y. Li, M. Vuorinen and Q. Zhou, *Apollonian metric, uniformity and Gromov hyperbolicity*, Complex Var. Elliptic Equations 65 (2020), 215–228.

[13] O. Martio, *Definitions of uniform domains*, Ann. Acad. Sci. Fenn. Ser. A I Math. 5 (1980), 197–205.

[14] O. Martio and J. Sarvas, *Injectivity theorems in plane and space*, Ann. Acad. Sci. Fenn. Ser. A I Math. 4 (1978), 383–401.

[15] A. Rasila and J. Talponen, *On quasihyperbolic geodesics in Banach spaces*, Ann. Acad. Sci. Fenn. Ser. A I Math. 39 (2014), 163–173.

[16] J. Väisälä, *Quasimöbius maps*, J. Anal. Math. 44 (1985), 218–234.

[17] J. Väisälä, *Uniform domains*, Tohoku Math. J. 40 (1988), 101–118.

[18] J. Väisälä, *Free quasiconformality in Banach spaces. I*, Ann. Acad. Sci. Fenn. Ser. A I Math. 15 (1990), 355–379.

[19] J. Väisälä, *Free quasiconformality in Banach spaces. II*, Ann. Acad. Sci. Fenn. Ser. A I Math. 16 (1991), 255–310.

[20] J. Väisälä, *Free quasiconformality in Banach spaces. III*, Ann. Acad. Sci. Fenn. Ser. A I Math. 17 (1992), 393–408.

[21] J. Väisälä, *Free quasiconformality in Banach spaces. IV*, in: Analysis and Topology, 697–717, World Sci., River Edge, NJ, 1998.

[22] J. Väisälä, *The free quasiworld, freely quasiconformal and related maps in Banach spaces*, in: Quasiconformal Geometry and Dynamics (Lublin 1996), Banach Center Publ. 48, Inst. Math., Polish Acad. Sci., Warszawa, 1999, 55–118.
[23] J. Väisälä, *Broken tubes in Hilbert spaces*, Analysis (Munich) 24 (2004), 227–238.
[24] M. Vuorinen, *Capacity densities and angular limits of quasiregular mappings*, Trans. Amer. Math. Soc. 263 (1981), 343–354.
[25] M. Vuorinen, *Conformal invariants and quasiregular mappings*, J. Anal. Math. 45 (1985), 69–115.
[26] X. Wang, Q. Zhou, T. Guan, M. Huang and A. Rasila, *A note on the relationship between quasi-symmetric mappings and φ-uniform domains*, J. Math. Anal. Appl. 445 (2017), 1114–1119.
[27] Q. Zhou, A. Rasila and Y. Li, *Free distance ratio mappings in Banach spaces*, Monatsh. Math. 191 (2020), 843–856.

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