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Linear operators with range in a vector-valued Hölder function space

by

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Abstract. We give necessary and sufficient conditions for boundedness and (weak) compactness of linear operators from Banach spaces into vector-valued Hölder function spaces.

1. Introduction. Let (X,d) and (Y,ρ) be metric spaces. A map $f: X \to Y$ is said to be *Lipschitz* if

$$p(f) = \sup\left\{\frac{\rho(f(x), f(y))}{d(x,y)} : x, y \in X,\, x \neq y\right\} < \infty,$$

and *supercontractive* if

$$\lim_{d(x,y)\to 0} \frac{\rho(f(x), f(y))}{d(x,y)} = 0.$$

Constant maps are Lipschitz and supercontractive. But, for example, if $x_0 \in X$ and $f(x) = d(x,x_0)$ for all $x \in X$, then f is Lipschitz, but not supercontractive. A supercontractive Lipschitz function is often called a *little Lipschitz* function. Let X be a metric space and F be a Banach space. The vector-valued Lipschitz space, denoted by $\mathrm{Lip}(X,F)$, is the Banach space of all bounded Lipschitz functions $f: X \to F$ endowed with the Lipschitz norm

$$\|f\| = \|f\|_\infty + p(f),$$

where $\|f\|_\infty = \sup_{x\in X} \|f(x)\|$ is the uniform norm. Also the vector-valued little Lipschitz space $\mathrm{lip}(X,F)$ is the closed subspace of $\mathrm{Lip}(X,F)$ formed by all little Lipschitz functions. If $F = \mathbb{C}$ is the scalar field of complex numbers, to simplify the notation we put $\mathrm{Lip}(X) = \mathrm{Lip}(X,\mathbb{C})$ and $\mathrm{lip}(X) = \mathrm{lip}(X,\mathbb{C})$. These spaces have been extensively investigated, starting from the works

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by de Leeuw [13] and Sherbert [14, 15]. The interested reader is referred to [1, 10, 12, 16] for background. There are little Lipschitz spaces containing only the constant functions, for instance, $\mathrm{lip}([0,1])$ with the usual metric [15], but there also exist some large classes of little Lipschitz spaces which separate points. Also, $\mathrm{lip}(X_\alpha, F)$ has the separation property, where $X_\alpha = (X, d^\alpha)$ for some $0 < \alpha < 1$ [16, Chapter 3]. Lipschitz functions on X_α are called Hölder functions of exponent α. The spaces $\mathrm{Lip}(X)$ and $\mathrm{lip}(X_\alpha)$ for $0 < \alpha < 1$ are Banach function algebras which are called Lipschitz algebras and their character spaces (maximal ideal spaces) coincide with X [14]. In this paper we are concerned with the spaces $\mathrm{lip}(X_\alpha, F)$ for $0 < \alpha < 1$.

In the recent years, quite a lot of attention has been given to operators on vector-valued Lipschitz functions, in particular, isometries on these spaces have been studied by several authors; see, e.g., [2, 3, 4, 6, 7, 8, 9]. Boundedness and compactness of weighted composition operators between spaces of vector-valued Lipschitz functions were studied by Esmaeili and Mahyar [6]. They have given necessary and sufficient conditions for the boundedness and compactness of weighted composition operators between these spaces; also, they showed that every bounded separating linear operator between these spaces is a weighted composition operator. In [11], Johnson proved that if (X, d) is a compact metric space and E is a Banach space, then the space of all bounded linear operators from E into $\mathrm{Lip}(X)$ is isomorphic to $\mathrm{Lip}(X, E^*)$. He also showed that the space of all compact linear operators from E into $\mathrm{lip}(X)$ is isomorphic to $\mathrm{lip}(X, E^*)$.

In this paper, we extend these results to spaces of vector-valued Lipschitz functions on a not necessarily compact metric space X. In addition, we investigate the well-definedness and weak compactness of such operators.

2. Results. Let (X, d) be a metric space and F be a Banach space. The algebraic and the topological duals of F are denoted by F' and F^* respectively. For each $u \in F$, $\hat{u}$ is the image of u in F^{**} under the canonical embedding, and $\tilde{u} \in F''$ is defined by $\tilde{u}(\lambda) = \lambda(u)$ for all $\lambda \in F'$. For each $x \in X$ the linear operator e_x is the evaluation mapping at x from $\mathrm{lip}(X_\alpha, F)$ into F. For every $x \in X$ and $\lambda \in F^*$ the notation $\lambda \otimes e_x$ stands for the linear functional in $\mathrm{lip}(X_\alpha, F)^*$ defined by $(\lambda \otimes e_x)(f) = \lambda(f(x))$. Indeed $\lambda \otimes e_x$ is the composition of λ and e_x. In [11, Theorem 5.3], it is proved that the linear span of $\{\lambda \otimes e_x : \lambda \in F^*, x \in X\}$ is dense in $\mathrm{lip}(X_\alpha, F)^*$ provided that X is finitely compact.

Let E be a Banach space and $T : E \to \mathrm{lip}(X_\alpha, F)$ be a linear operator. Set $\Sigma = \{\lambda \in F^* : \|\lambda\| = 1\}$ endowed with the metric induced by the norm on F^*. Define $\varphi : X \times \Sigma \to E'$ by

$$\varphi(x, \lambda) = \lambda \otimes e_x \circ T.$$

Indeed, φ is the restriction of the algebraic adjoint $T' : \mathrm{lip}(X_\alpha, F)^* \to E'$ of T to the set $\{\lambda \otimes e_x : \lambda \in \Sigma,\ x \in X\}$. For every $\lambda \in \Sigma$, let φ_λ be the restriction of φ to $X \times \{\lambda\}$ and for every $x \in X$, let φ_x be the restriction of φ to $\{x\} \times \Sigma$. Also, the metric on the product of any two metric spaces will always be the sum of the metrics on each of them.

THEOREM 2.1. *Suppose E' and Σ are equipped with the topology induced by E and by the weak*-topology of F^*, respectively. If $T : E \to \mathrm{lip}(X, F)$ is a linear operator, then:*

(1) *The range of φ is a bounded subset of E'.*
(2) *For every $x \in X$, φ_x is continuous.*
(3) *When $d(x, y) \to 0$, the family $\{\frac{\varphi_\lambda(x)-\varphi_\lambda(y)}{d(x,y)}\}_{\lambda\in\Sigma}$ uniformly converges to zero.*
(4) *For every $x \in X$, $s, t > 0$ and $\lambda, \mu \in \Sigma$,*

$$\varphi_x(t\lambda + s\mu) = t\varphi_x(\lambda) + s\varphi_x(\mu)$$

provided that $t\lambda + s\mu \in \Sigma$ and $\varphi_x(z\lambda) = z\varphi_x(\lambda)$ for all scalar z with $|z| = 1$.

Conversely, if a function $\varphi : X \times \Sigma \to E'$ satisfies the above conditions, then there exists a linear operator $T : E \to \mathrm{lip}(X, F)$ such that

$$\widetilde{Tv}(\lambda \otimes e_x) = \tilde{v} \circ \varphi(x, \lambda) \qquad (v \in E,\ x \in X,\ \lambda \in \Sigma).$$

Proof. Suppose that T is a linear operator from E into $\mathrm{lip}(X, F)$.

We first prove (1). For $v \in E$, let $U_v = \{\mu \in E' : |\mu(v)| < 1\}$. Then for every $\lambda \in \Sigma$ and $x \in X$,

$$|\varphi(x, \lambda)(v)| = |\lambda(Tv(x))| \le \|Tv(x)\| \le \|Tv\|_\infty.$$

Hence for $t_0 = \|Tv\|_\infty$ and for every $t > t_0$ we have $\varphi_x(\Sigma) \subseteq tU_v$. Therefore (1) holds.

To prove (2), let $\{\lambda_i\}_{i\in I}$ be a net in Σ which converges to $\lambda_0 \in \Sigma$. For every $x \in X$ and $v \in E$, $Tv \in \mathrm{lip}(X, F)$ and so $Tv(x) \in F$. Thus

$$\lim_{i\in I} (\varphi_x(\lambda_i))(v) = \lim_{i\in I} \lambda_i(Tv(x)) = \lambda_0(Tv(x)) = (\varphi_x(\lambda_0))(v).$$

Hence for every $x \in X$, φ_x is continuous, which proves (2).

Let us now prove (3). Given any $v \in E$, $\lambda \in \Sigma$ and $x, y \in X$ with $x \neq y$, we get

$$\left|\frac{\varphi_\lambda(x) - \varphi_\lambda(y)}{d(x, y)}(v)\right| = \left|\frac{\lambda(Tv(x)) - \lambda(Tv(y)))}{d(x, y)}\right| = \left|\lambda\left(\frac{Tv(x) - Tv(y)}{d(x, y)}\right)\right| \le \frac{\|Tv(x) - Tv(y)\|}{d(x, y)}.$$

Now, as $\lim_{d(x,y)\to 0} \frac{\|Tv(x)-Tv(y)\|}{d(x,y)} = 0$ since $Tv \in \mathrm{lip}(X, F)$, (3) follows.

Condition (4) is obvious.

Conversely, suppose that a function $\varphi : X \times \Sigma \to E'$ satisfies (1)–(4). For every $x \in X$ and $v \in E$, let

$$f_{x,v} : F^* \to \mathbb{C}, \qquad f_{x,v}(\lambda) = \begin{cases} \|\lambda\| \tilde{v} \circ \varphi_x(\lambda/\|\lambda\|), & \lambda \neq 0, \\ 0, & \lambda = 0. \end{cases}$$

Condition (4) implies that $f_{x,v}$ is linear. Now, we show that $f_{x,v}$ is weak*-continuous. Suppose for contradiction that there exists a net $\{\lambda_i\}_{i\in I}$ in F^* such that $\lambda_i \to 0$ in the weak*-topology and $f_{x,v}(\lambda_i) \nrightarrow 0$. Then there exists $\varepsilon > 0$ and subnet $\{\lambda_{i_j}\}_{j\in J}$ such that for every $j \in J$, $|f_{x,v}(\lambda_{i_j})| \geq \varepsilon$ and so

$$\|\lambda_{i_j}\| \, |\tilde{v} \circ \varphi_x(\lambda_{i_j}/\|\lambda_{i_j}\|)| \geq \varepsilon. \tag{2.1}$$

Let $U = U(v, \varepsilon) = \{\mu \in E' : |\mu(v)| < \varepsilon\}$. Since $\varphi(X \times \Sigma)$ is bounded, there exists $t_0 > 0$ such that $\varphi(X \times \Sigma) \subseteq tU$ for every $t \geq t_0$. Thus for every $\lambda \in F^* \setminus \{0\}$, $\varphi_x(\lambda/\|\lambda\|) \in t_0 U$ and so

$$|\tilde{v} \circ \varphi_x(\lambda/\|\lambda\|)| = |(\varphi_x(\lambda/\|\lambda\|))(v)| < t_0\varepsilon.$$

Consequently, by (2.1), for every $j \in J$,

$$\|\lambda_{i_j}\| \geq \frac{\varepsilon}{|\tilde{v} \circ \varphi_x(\lambda_{i_j}/\|\lambda_{i_j}\|)|} > \frac{1}{t_0}.$$

Therefore, for every $u \in F$,

$$\left|\frac{\lambda_{i_j}}{\|\lambda_{i_j}\|}(u)\right| = \frac{|\lambda_{i_j}(u)|}{\|\lambda_{i_j}\|} \leq t_0 |\lambda_{i_j}(u)| \to 0.$$

Thus $\lambda_{i_j}/\|\lambda_{i_j}\| \to 0$ in the weak*-topology. Hence by (2), $\tilde{v} \circ \varphi_x(\lambda_{i_j}/\|\lambda_{i_j}\|) \to 0$, and by (2.1) one concludes that $\lim_j \|\lambda_{i_j}\| = \infty$, which contradicts the uniform boundedness principle. Therefore, for every $x \in X$ and $v \in E$, $f_{x,v}$ is a weak*-continuous linear functional on F^*. Consequently, for every $x \in X$ and $v \in E$ there exists $u_{x,v} \in F$ such that $f_{x,v} = \widehat{u}_{x,v}$. Therefore by (3), for every $v \in E$, the function

$$f_v : X \to F, \qquad f_v(x) = u_{x,v},$$

belongs to the $\operatorname{lip}(X, F)$. Indeed, pick any $v \in E$. Then

$$\begin{aligned} \lim_{d(x,y)\to 0} \frac{\|f_v(x) - f_v(y)\|}{d(x,y)} &= \lim_{d(x,y)\to 0} \sup_{\lambda\in\Sigma} \left|\frac{\lambda(f_v(x) - f_v(y))}{d(x,y)}\right| \\ &= \lim_{d(x,y)\to 0} \sup_{\lambda\in\Sigma} \left|\frac{f_{x,v}(\lambda) - f_{y,v}(\lambda)}{d(x,y)}\right| \\ &= \lim_{d(x,y)\to 0} \sup_{\lambda\in\Sigma} \left|\frac{\varphi(x,\lambda) - \varphi(y,\lambda)}{d(x,y)}(v)\right| \\ &= \lim_{d(x,y)\to 0} \sup_{\lambda\in\Sigma} \left|\frac{\varphi_\lambda(x) - \varphi_\lambda(y)}{d(x,y)}(v)\right| = 0. \end{aligned}$$

In addition, since $\varphi(X\times\Sigma)$ is bounded there exists, for every $v\in E$, a positive constant t_v such that $\varphi(X\times\Sigma)\subseteq tU(v,1)$ for all $t\geq t_v$. Thus

$$\begin{aligned}\|f_v\|_\infty &= \sup_{x\in X}\|f_v(x)\| = \sup_{x\in X}\|f_{x,v}\| = \sup_{x\in X}\sup_{\lambda\in\Sigma}|f_{x,v}(\lambda)| \\ &= \sup_{\substack{x\in X\\ \lambda\in\Sigma}}|\tilde v\circ\varphi_x(\lambda)| = \sup_{\substack{x\in X\\ \lambda\in\Sigma}}|\varphi(x,\lambda)(v)| \leq t_v.\end{aligned}$$

Since f_v is supercontractive and bounded it follows that f_v is Lipschitz, so $f_v\in\operatorname{lip}(X,F)$. Therefore, the linear operator

$$T: E\to\operatorname{lip}(X,F),\qquad Tv=f_v,$$

is well-defined and $\widetilde{Tv}(\lambda\otimes e_x)=\tilde v\circ\varphi(x,\lambda)$ for all $x\in X$, $\lambda\in\Sigma$ and $v\in F$. This completes the proof. ■

THEOREM 2.2. *If $T: E\to\operatorname{Lip}(X,F)$ is a linear operator, then the following are equivalent:*

(1) *T is bounded.*
(2) *$\varphi\in\operatorname{Lip}(X\times\Sigma,E^*)$.*
(3) *The family $\{\varphi_\lambda\}_{\lambda\in\Sigma}$ is a bounded subset of $\operatorname{Lip}(X,E^*)$.*

Moreover,

$$\tfrac12\|T\|\leq\sup_{\lambda\in\Sigma}\|\varphi_\lambda\|\leq\|\varphi\|\leq 2\|T\|.$$

Proof. (1)⇒(2). Suppose that T is a bounded linear operator. For any distinct $(x,\lambda),(y,\mu)\in X\times\Sigma$,

$$\begin{aligned}\frac{\|\varphi(x,\lambda)-\varphi(y,\mu)\|}{\rho((x,\lambda),(y,\mu))} &= \sup_{\substack{v\in E\\ \|v\|=1}}\frac{|(\varphi(x,\lambda)-\varphi(y,\mu))(v)|}{\rho((x,\lambda),(y,\mu))} \\ &= \sup_{\substack{v\in E\\ \|v\|=1}}\frac{|\lambda(Tv(x))-\mu(Tv(y))|}{\rho((x,\lambda),(y,\mu))} \\ &\leq \sup_{\substack{v\in E\\ \|v\|=1}}\left(\frac{|\lambda(Tv(x))-\lambda(Tv(y))|}{d(x,y)+\|\lambda-\mu\|}+\frac{|\lambda(Tv(y))-\mu(Tv(y))|}{d(x,y)+\|\lambda-\mu\|}\right) \\ &= \sup_{\substack{v\in E\\ \|v\|=1}}\left(\left|\lambda\left(\frac{Tv(x)-Tv(y)}{d(x,y)+\|\lambda-\mu\|}\right)\right|+\left|\frac{\lambda-\mu}{d(x,y)+\|\lambda-\mu\|}(Tv(y))\right|\right) \\ &\leq \sup_{\substack{v\in E\\ \|v\|=1}}\left(\frac{\|Tv(x)-Tv(y)\|}{d(x,y)+\|\lambda-\mu\|}+\frac{\|\lambda-\mu\|}{d(x,y)+\|\lambda-\mu\|}\|Tv(y)\|\right) \\ &\leq \sup_{\substack{v\in E\\ \|v\|=1}}(p(T(v))+\|T(v)\|_\infty) = \sup_{\substack{v\in E\\ \|v\|=1}}\|T(v)\| = \|T\|.\end{aligned}$$

Moreover,

$$\|\varphi\|_\infty = \sup_{(x,\lambda)\in X\times\Sigma} \|\varphi(x,\lambda)\| = \sup_{\substack{x\in X\\ \lambda\in\Sigma}} \|\lambda\otimes e_x\circ T\| \le \|T\|.$$

Thus $\varphi \in \mathrm{Lip}(X\times\Sigma, E^*)$ and $\|\varphi\| \le 2\|T\|$.

(2)⇒(3). This is obvious.

(3)⇒(1). For every $v\in E$,

$$\begin{aligned}\|Tv\|_\infty &= \sup_{x\in X}\|Tv(x)\| = \sup_{x\in X}\sup_{\lambda\in\Sigma}|\lambda(Tv(x))|\\ &= \sup_{x\in X}\sup_{\lambda\in\Sigma}|\varphi_\lambda(x)(v)| \le \sup_{x\in X}\sup_{\lambda\in\Sigma}\|\varphi_\lambda(x)\|\,\|v\| \le \sup_{\lambda\in\Sigma}\|\varphi_\lambda\|_\infty\|v\|.\end{aligned}$$

Also,

$$\begin{aligned}p(Tv) &= \sup_{\substack{x,y\in X\\ x\ne y}} \frac{\|Tv(x)-Tv(y)\|}{d(x,y)}\\ &= \sup_{\substack{x,y\in X\\ x\ne y}}\sup_{\lambda\in\Sigma}\frac{|\lambda(Tv(x)-Tv(y))|}{d(x,y)} = \sup_{\substack{x,y\in X\\ x\ne y}}\sup_{\lambda\in\Sigma}\frac{|(\varphi_\lambda(x)-\varphi_\lambda(y))(v)|}{d(x,y)}\\ &\le \sup_{\substack{x,y\in X\\ x\ne y}}\sup_{\lambda\in\Sigma}\frac{\|\varphi_\lambda(x)-\varphi_\lambda(y)\|\,\|v\|}{d(x,y)} \le \sup_{\lambda\in\Sigma} p(\varphi_\lambda)\|v\|.\end{aligned}$$

Therefore, $\|Tv\| \le 2\sup_{\lambda\in\Sigma}\|\varphi_\lambda\|\,\|v\|$, which implies that T is bounded and $\|T\| \le 2\sup_{\lambda\in\Sigma}\|\varphi_\lambda\|$. ■

COROLLARY 2.3. *If $T : E \to \mathrm{lip}(X,F)$ is a linear operator, then the following are equivalent:*

(1) *T is bounded.*
(2) *$\varphi \in \mathrm{Lip}(X\times\Sigma, E^*)$.*
(3) *The family $\{\varphi_\lambda\}_{\lambda\in\Sigma}$ is a bounded subset of $\mathrm{Lip}(X,E^*)$.*

Moreover,

$$\tfrac{1}{2}\|T\| \le \sup_{\lambda\in\Sigma}\|\varphi_\lambda\| \le \|\varphi\| \le 2\|T\|.$$

Now we want to investigate the compactness of linear operators with range in a space of vector-valued little Lipschitz functions.

THEOREM 2.4. *Let $0<\alpha<1$ and $T : E\to \mathrm{lip}(X_\alpha, F)$ be a linear operator. The following conditions are equivalent:*

(1) *The operator T is compact.*
(2) *$\varphi \in \mathrm{lip}(X_\alpha\times\Sigma_\alpha, E^*)$ and the range of φ is a relatively compact subset of E^*.*

(3) *The family* $\{\frac{\varphi_\lambda(x)-\varphi_\lambda(y)}{d^\alpha(x,y)}\}_{\lambda\in\Sigma}$ *tends to zero uniformly with respect to the norm when* $d(x,y)\to 0$*, and the range of* φ *is a relatively compact subset of* E^*.

Proof. (3)⇒(1). For compactness of T, we assume that $\{v_k\}_{k\in\mathbb{N}}$ is a sequence in $\mathbb{B}$, where $\mathbb{B}$ is the open unit ball in E. Since the range of φ is a relatively compact subset of E^*, $\{\widehat{v_k}\}_{k\in\mathbb{N}}$ is a uniformly bounded equicontinuous sequence of complex-valued functions on the closure of the range of φ which is a compact metric space. Therefore by the Arzelà–Ascoli theorem, there exists a subsequence $\{v_{k_n}\}_{n\in\mathbb{N}}$ such that $\{\widehat{v_{k_n}}\}_{n\in\mathbb{N}}$ is uniformly Cauchy on the range of φ.

We proceed to show that $\{T\widehat{v_{k_n}}\}_{n\in\mathbb{N}}$ is Cauchy in $\mathrm{lip}(X_\alpha,F)$. Take $\varepsilon>0$ arbitrary. By assumption, $\{\frac{\varphi_\lambda(x)-\varphi_\lambda(y)}{d^\alpha(x,y)}\}_{\lambda\in\Sigma}$ tends to zero uniformly in norm when $d(x,y)\to 0$, so there exists $0<\delta<1$ such that for every $\lambda\in\Sigma$ and every $x,y\in X$ with $0<d(x,y)<\delta$, we have $\|\varphi_\lambda(x)-\varphi_\lambda(y)\|/d^\alpha(x,y)$ $<\varepsilon/4$. Since $\{\widehat{v_{k_n}}\}_{n\in\mathbb{N}}$ is uniformly Cauchy on the range of φ, there exists $K\in\mathbb{N}$ such that

$$\sup_{\substack{x\in X\\ \lambda\in\Sigma}}|\widehat{v_{k_n}}(\varphi(x,\lambda))-\widehat{v_{k_m}}(\varphi(x,\lambda))|<\varepsilon\delta^\alpha/4 \qquad (m,n>K).$$

On the other hand,

$$\begin{aligned}\sup_{\substack{x\in X\\ \lambda\in\Sigma}}|\widehat{v_{k_n}}(\varphi(x,\lambda))-\widehat{v_{k_m}}(\varphi(x,\lambda))| &= \sup_{x\in X}\sup_{\lambda\in\Sigma}|\lambda((Tv_{k_n}-Tv_{k_m})(x))|\\ &=\|Tv_{k_n}-Tv_{k_m}\|_\infty.\end{aligned}$$

Thus $\|Tv_{k_n}-Tv_{k_m}\|_\infty<\varepsilon\delta^\alpha/4$ for large enough n,m. Now if $x,y\in X$ and $0<d(x,y)<\delta$, then

$$\begin{aligned}&\frac{\|(Tv_{k_n}-Tv_{k_m})(x)-(Tv_{k_n}-Tv_{k_m})(y)\|}{d^\alpha(x,y)}\\ &\qquad=\sup_{\lambda\in\Sigma}\frac{|\lambda((Tv_{k_n}-Tv_{k_m})(x)-(Tv_{k_n}-Tv_{k_m})(y))|}{d^\alpha(x,y)}\\ &\qquad=\sup_{\lambda\in\Sigma}\frac{|\varphi_\lambda(x)(v_{k_n}-v_{k_m})-\varphi_\lambda(y)(v_{k_n}-v_{k_m})|}{d^\alpha(x,y)}\\ &\qquad\le\sup_{\lambda\in\Sigma}\frac{\|\varphi_\lambda(x)-\varphi_\lambda(y)\|}{d^\alpha(x,y)}\|v_{k_n}-v_{k_m}\|<2\frac{\varepsilon}{4}=\frac{\varepsilon}{2}.\end{aligned}$$

If $d(x,y)\ge\delta$, then

$$\begin{aligned}\frac{\|(Tv_{k_n}-Tv_{k_m})(x)-(Tv_{k_n}-Tv_{k_m})(y)\|}{d^\alpha(x,y)}&\le 2\frac{\|Tv_{k_n}-Tv_{k_m}\|_\infty}{\delta^\alpha}\\ &<\frac{2}{\delta^\alpha}\frac{\varepsilon\delta^\alpha}{4}=\frac{\varepsilon}{2}.\end{aligned}$$

Hence $p(Tv_{k_n} - Tv_{k_m}) < \varepsilon/2$ for large enough m, n. Therefore,

$$\|Tv_{k_n} - Tv_{k_m}\| = \|Tv_{k_n} - Tv_{k_m}\|_\infty + p(Tv_{k_n} - Tv_{k_m}) < \varepsilon$$

for large enough n, m, that is, $\{Tv_{k_n}\}$ is a Cauchy sequence in $\mathrm{lip}(X_\alpha, F)$, and now the completeness of this last space does the job.

(1)⇒(2). Let $T : E \to \mathrm{lip}(X_\alpha, F)$ be a compact linear operator. Then $T(\mathbb{B})$ is totally bounded in $\mathrm{lip}(X_\alpha, F)$. For $f \in \mathrm{lip}(X_\alpha, F)$ and $r > 0$, let

$$B(f, r) = \{g \in \mathrm{lip}(X_\alpha, F) : \|g - f\| < r\}.$$

Given $\varepsilon > 0$, there exist $f_1, \dots, f_n \in \mathrm{lip}(X_\alpha, F)$ such that

$$T(\mathbb{B}) \subset \bigcup_{i=1}^{n} B(f_i, \varepsilon/3).$$

One may choose $0 < \delta < \left(\frac{\varepsilon}{3\|T\|}\right)^{\alpha/(1-\alpha)}$ such that $\|f_i(x) - f_i(y)\|/d^\alpha(x, y) < \varepsilon/3$ for each $x, y \in X$ with $0 < d(x, y) < \delta$ and all $i \in \{1, \dots, n\}$. Given $v \in \mathbb{B}$, we have $Tv \in B(f_i, \varepsilon/3)$ for some $i \in \{1, \dots, n\}$. Thus, for every $x, y \in X$ and every $\lambda, \mu \in \Sigma$ with $0 < \rho((x, \lambda), (y, \mu)) < \delta$, we have

$$\begin{aligned}
\frac{\|\varphi(x,\lambda) - \varphi(y,\mu)\|}{\rho((x,\lambda),(y,\mu))} &= \sup_{\substack{v\in E\\ \|v\|=1}} \frac{|(\varphi(x,\lambda) - \varphi(y,\mu))(v)|}{\rho((x,\lambda),(y,\mu))} \\
&= \sup_{\substack{v\in E\\ \|v\|=1}} \frac{|\lambda(Tv(x)) - \mu(Tv(y))|}{\rho((x,\lambda),(y,\mu))} \\
&\le \sup_{\substack{v\in E\\ \|v\|=1}} \left(\frac{|\lambda(Tv(x)) - \lambda(Tv(y))|}{d^\alpha(x,y) + \|\lambda-\mu\|^\alpha} + \frac{|\lambda(Tv(y)) - \mu(Tv(y))|}{d^\alpha(x,y) + \|\lambda-\mu\|^\alpha} \right) \\
&= \sup_{\substack{v\in E\\ \|v\|=1}} \left(\left| \lambda\left(\frac{Tv(x) - Tv(y)}{d^\alpha(x,y) + \|\lambda-\mu\|^\alpha} \right) \right| + \left| \frac{\lambda-\mu}{d^\alpha(x,y) + \|\lambda-\mu\|^\alpha} (Tv(y)) \right| \right) \\
&\le \sup_{\substack{v\in E\\ \|v\|=1}} \left(\frac{\|Tv(x) - Tv(y)\|}{d^\alpha(x,y) + \|\lambda-\mu\|^\alpha} + \frac{\|\lambda-\mu\|}{d^\alpha(x,y) + \|\lambda-\mu\|^\alpha} \|Tv(y)\| \right) \\
&\le \sup_{\substack{v\in E\\ \|v\|=1}} \left(\frac{\|(Tv - f_i)(x) - (Tv - f_i)(y)\|}{d^\alpha(x,y)} + \frac{\|f_i(x) - f_i(y)\|}{d^\alpha(x,y)} \right. \\
&\qquad\qquad \left. + \|\lambda-\mu\|^{1-\alpha} \|Tv\|_\infty \right) \\
&\le \sup_{\substack{v\in E\\ \|v\|=1}} p(Tv - f_i) + \frac{\|f_i(x) - f_i(y)\|}{d^\alpha(x,y)} + \|\lambda-\mu\|^{1-\alpha}\|T\| \le \varepsilon.
\end{aligned}$$

Therefore, $\varphi \in \mathrm{lip}(X_\alpha \times \Sigma_\alpha, E^*)$. Since T is compact, so is T^*, by Schauder's theorem. In addition, the set $\{\lambda \otimes e_x : \lambda \in \Sigma, x \in X\}$ is included in the closed unit ball of $\mathrm{lip}(X_\alpha, F)^*$. Thus $\varphi(X\times\Sigma) = T^*(\{\lambda\otimes e_x : \lambda\in\Sigma,\ x\in X\})$ is a relatively compact subset of E^*.

(2)⇒(3). This is obvious. ■

Let us end the paper with a characterisation of weakly compact operators in a little Lipschitz space.

THEOREM 2.5. *A bounded linear operator* $T : E \to \mathrm{lip}(X_\alpha, F)$ $(0<\alpha<1)$ *is weakly compact if and only if the following conditions hold:*

(1) *The family* $\{\frac{\varphi_\lambda(x)-\varphi_\lambda(y)}{d^\alpha(x,y)}\}_\lambda$ *tends to zero uniformly with respect to the weak topology when* $d(x,y)\to 0$.
(2) *For every* $x\in X$, *the function* $\varphi_x : \Sigma \to E^*$ *is continuous with respect to the weak* topology on* Σ *and the weak topology on* E^*.

Proof. By [5, Theorem VI,4.2], T is weakly compact if and only if T^{**} maps E^{**} into $\mathrm{lip}(X_\alpha, F)$, in the sense that for every $\Lambda\in E^{**}$ there exists $f\in \mathrm{lip}(X_\alpha, F)$ such that $T^{**}(\Lambda) = \hat{f}$. Suppose that T is weakly compact. Given $\Lambda\in E^{**}$ there exists $f_\Lambda\in\mathrm{lip}(X_\alpha, F)$ such that $T^{**}(\Lambda) = \hat{f}_\Lambda$, so for every $\lambda\in\Sigma$ and any distinct $x, y\in X$ we get

$$\begin{aligned}\left|\Lambda\left(\frac{\varphi_\lambda(x)-\varphi_\lambda(y)}{d^\alpha(x,y)}\right)\right| &= \left|\frac{\Lambda(T^*(\lambda\otimes e_x) - \Lambda(T^*(\lambda\otimes e_y)}{d^\alpha(x,y)}\right| \\ &= \left|\frac{\hat{f}_\Lambda(\lambda\otimes e_x) - \hat{f}_\Lambda(\lambda\otimes e_y)}{d^\alpha(x,y)}\right| = \frac{|\lambda(f_\Lambda(x)-f_\Lambda(y))|}{d^\alpha(x,y)} \\ &\le \frac{\|f_\Lambda(x)-f_\Lambda(y)\|}{d^\alpha(x,y)}.\end{aligned}$$

Now, as $\lim_{d(x,y)\to 0}\|f_\Lambda(x)-f_\Lambda(y)\|/d^\alpha(x,y) = 0$ since $T(v)\in\mathrm{lip}(X_\alpha,F)$, condition (1) holds. On the other hand, since T is weakly compact, by [5, Lemma VI,4.7], $T^* : (\mathrm{lip}(X_\alpha,F)^*, w^*) \to (E^*, w)$ is continuous. Therefore, for every $x\in X$, the function

$$\varphi_x = T^*|_{\{\lambda\otimes e_x:\,\lambda\in\Sigma\}} : (\Sigma, w^*) \to (E^*, w)$$

is continuous and condition (2) holds.

Conversely, suppose that (1) and (2) hold. For every $x\in X$ and $\Lambda\in E^{**}$, let

$$f_{\Lambda,x} : F^*\to\mathbb{C},\qquad f_{\Lambda,x}(\lambda) = \begin{cases}\|\lambda\|\Lambda\circ\varphi_x(\lambda/\|\lambda\|), & \lambda\neq 0,\\ 0, & \lambda = 0.\end{cases}$$

If f_{Λ_0,x_0} is not weak*-continuous for some $x_0\in X$ and $\Lambda_0\in E^{**}$, then there exists a net $\{\lambda_i\}_{i\in I}$ in F^* such that $\lambda_i\to 0$ in the weak*-topology and $f_{\Lambda_0,x_0}(\lambda_i)\not\to 0$. Then there exists $\varepsilon>0$ and a subnet $\{\lambda_{i_j}\}_{j\in J}$ such that for every $j\in J$, $|f_{\Lambda_0,x_0}(\lambda_{i_j})|\ge\varepsilon$ and so

$$\|\lambda_{i_j}\|\,|\Lambda_0\circ\varphi_{x_0}(\lambda_{i_j}/\|\lambda_{i_j}\|)|\ge\varepsilon. \tag{2.2}$$

For every $\lambda \in F^*$,

$$|\Lambda_0 \circ \varphi_{x_0}(\lambda/\|\lambda\|)| = |\Lambda_0 \circ T^*(\lambda/\|\lambda\| \otimes e_{x_0})| \\ \leq \|\Lambda_0\|\,\|T^*\| \left\| \frac{\lambda}{\|\lambda\|} \otimes e_{x_0} \right\| = \|\Lambda_0\|\,\|T\|.$$

Thus $\frac{\varepsilon}{\|\Lambda_0\|\,\|T\|} \leq \|\lambda_{i_j}\|$ by (2.2). For every $u \in F$,

$$\left| \frac{\lambda_{i_j}}{\|\lambda_{i_j}\|}(u) \right| = \frac{|\lambda_{i_j}(u)|}{\|\lambda_{i_j}\|} \leq \frac{\|\Lambda_0\|\,\|T\|}{\varepsilon} |\lambda_{i_j}(u)| \to 0.$$

Thus $\lambda_{i_j}/\|\lambda_{i_j}\| \to 0$ in the weak*-topology. Then, by (2), $\varphi_{x_0}(\lambda_{i_j}/\|\lambda_{i_j}\|) \to 0$ in the weak topology, and so $\Lambda_0 \circ \varphi_{x_0}(\lambda_{i_j}/\|\lambda_{i_j}\|) \to 0$. This and (2.2) imply that $\lim_j \|\lambda_{i_j}\| = \infty$. Hence $\{\lambda_i\}$ is an unbounded net in E^* that converges to the zero in the weak*-topology, which contradicts the uniform boundedness principle. Therefore, for every $x_0 \in X$ and $\Lambda_0 \in E^{**}$ the function f_{Λ_0,x_0} is a weak*-continuous linear functional. Consequently, there exists $u_{\Lambda_0,x_0} \in F$ such that $f_{\Lambda_0,x_0} = \widehat{u}_{\Lambda_0,x_0}$. Now, let $\Lambda \in E^{**}$. Then the function

$$f_\Lambda : X \to F, \qquad f_\Lambda(x) = u_{\Lambda,x},$$

belongs to $\operatorname{lip}(X_\alpha, F)$. Indeed, by condition (1),

$$\begin{aligned}
\lim_{d(x,y)\to 0} \frac{\|f_\Lambda(x) - f_\Lambda(y)\|}{d^\alpha(x,y)} &= \lim_{d(x,y)\to 0} \frac{\|f_{\Lambda,x} - f_{\Lambda,y}\|}{d^\alpha(x,y)} \\
&= \lim_{d(x,y)\to 0} \frac{\sup_{\lambda\in\Sigma} |f_{\Lambda,x}(\lambda) - f_{\Lambda,y}(\lambda)|}{d^\alpha(x,y)} \\
&= \lim_{d(x,y)\to 0} \frac{\sup_{\lambda\in\Sigma} |\Lambda \circ \varphi_x(\lambda) - \Lambda \circ \varphi_y(\lambda)|}{d^\alpha(x,y)} \\
&= \lim_{d(x,y)\to 0} \sup_{\lambda\in\Sigma} \frac{|\Lambda(\varphi_\lambda(x) - \varphi_\lambda(y))|}{d^\alpha(x,y)} = 0.
\end{aligned}$$

In addition,

$$\begin{aligned}
\|f_\Lambda\|_\infty = \sup_{x\in X} \|f_\Lambda(x)\| &= \sup_{x\in X} \|u_{\Lambda,x}\| = \sup_{x\in X} \|f_{\Lambda,x}\| = \sup_{x\in X} \sup_{\lambda\in\Sigma} |f_{\Lambda,x}(\lambda)| \\
&= \sup_{\substack{x\in X \\ \lambda\in\Sigma}} |\Lambda \circ \varphi_x(\lambda)| = \sup_{\substack{x\in X \\ \lambda\in\Sigma}} |\Lambda \circ \varphi(x,\lambda)| \\
&= \sup_{\substack{x\in X \\ \lambda\in\Sigma}} |T^{**}(\Lambda)(\lambda \otimes e_x)| \leq \|T^{**}(\Lambda)\| \leq \|T\|\,\|\Lambda\|.
\end{aligned}$$

Since f_v is supercontractive and bounded it follows that f_v is Lipschitz, so $f_v \in \operatorname{lip}(X_\alpha, F)$. Also, for every $\lambda \in F^*$ and $x \in X$,

$$\Lambda \circ T^*(\lambda \otimes e_x) = \Lambda(\|\lambda\|\varphi(x, \lambda/\|\lambda\|)) = f_{\Lambda,x}(\lambda) = \widehat{u}_{\Lambda,x}(\lambda)$$
$$= \lambda(u_{\Lambda,x}) = \lambda(f_\Lambda(x)) = (\lambda \otimes e_x)(f_\Lambda) = \hat{f}_\Lambda(\lambda \otimes e_x).$$

Therefore, for every $\Lambda \in E^{**}$, $T^{**}(\Lambda) = \hat{f}_\Lambda$ for some $f_\Lambda \in \mathrm{lip}(X_\alpha, F)$. Consequently, by [5, Theorem VI,4.2], T is weakly compact and the proof is complete. ■

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