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Controllable systems with vanishing energy

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Abstract. The paper surveys results on controllable systems with vanishing energy introduced by E. Priola and J. Zabczyk [SIAM J. Control Optim. 42 (2003), 1013–1032]. Applications to space travels and to partial differential equations are discussed.

1. Introduction. A large part of control theory is devoted to linear control systems which can be described as solutions to differential equations of the form

$$\frac{dy}{dt} = Ay(t) + Bu(t), \qquad y(0) = x \in H, \tag{1}$$

where A and B are linear operators, not necessarily continuous. The operator A acts on a Hilbert space H, and B acts from H to a Hilbert space U. The spaces H and U are called respectively the *state space* and the *space of control parameters.* In general they could be Banach spaces, but we restrict ourselves to Hilbert space modelling. A *strategy* is a function $u(t)$, $t \in [0, T]$, with values in U. To simplify the presentation we will assume that strategies belong to $L^2(0, T; U)$, the Hilbert space of square integrable U-valued functions defined on $[0, T]$.

The solution to (1) with initial condition x will be denoted as

$$y^{x,u}(t), \qquad t \in [0, T].$$

System (1) is *exactly controllable* (EC) if, for any $a, b \in H$, there exist $T > 0$ and $u(\cdot) \in L^2(0, T; H)$ such that

$$y^{a,u}(T) = b.$$

Similarly system (1) is *null controllable* (NC) if, for any $a \in H$, there exist

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$T > 0$ and $u(\cdot) \in L^2(0,T;H)$ such that

$$y^{a,u}(T) = 0.$$

There exists a vast literature devoted to exactly controllable and null controllable systems. We refer to Wonham [11], Curtain and Zwart [1] and Zabczyk [12] for the first reading.

In this paper we will be concerned with a related concept of controllability with vanishing energy.

System (1) is *exactly controllable with vanishing energy* (ECVE) if, for any $a, b \in H$ and any $\varepsilon > 0$, there exist $T > 0$ and $u(\cdot) \in L^2(0,T;H)$ such that

$$y^{a,u}(T) = b, \qquad \int_0^T |u(s)|^2\,ds < \varepsilon.$$

System (1) is *null controllable with vanishing energy* (NCVE) if, for any $a \in H$ and any $\varepsilon > 0$, there exist $T > 0$ and $u(\cdot) \in L^2(0,T;H)$ such that

$$y^{a,u}(T) = 0, \qquad \int_0^T |u(s)|^2\,ds < \varepsilon.$$

The paper is devoted to characterizations of linear control systems which are controllable with vanishing energy and to some applications of those characterizations.

The concept was introduced by E. Priola and the author in [8], where also basic characterizations were obtained. It turns out that control systems are controllable with vanishing energy if, and in many cases only if, the drift operator A has specific spectral properties. To cover heating systems where the control action is applied to the boundary of the heated object, it was necessary to extend the abstract results to systems where both operators A and B are discontinuous. The generalization was presented by Pandolfi, Priola and the author [7]. A discrete time version was elaborated by Ichikawa [3], [4]. A generalization of some results to the Banach space setting was given by Van Neerven [6].

Shibata and Ichikawa [10] applied results on NCVE to the orbital transfer problem for space travels. An interplay between controllable systems with vanishing energy and Liouville's theorem on bounded harmonic functions was discovered by Priola and the author [9]. In [8] general results on infinite-dimensional systems were applied to parabolic, hyperbolic and delay systems. Applications to boundary control systems are treated in [7].

Our presentation is divided into two sections devoted respectively to classical finite-dimensional systems and to infinite-dimensional ones with regular and irregular control operators.

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2. Classical control systems

2.1. Spectral characterizations. We assume that H and U are finite-dimensional spaces, say $H = \mathbb{R}^n$, $U = \mathbb{R}^m$. The operators A, B can be identified with $n \times n$ and $n \times m$ matrices. The equation (1) is then an ordinary differential equation. Its solution is of the form

$$y^{x,u}(t) = e^{At}x + \int_0^t e^{A(t-s)}Bu(s)\,ds, \qquad t \geq 0, \tag{2}$$

where the exponential function

$$e^{At} = \sum_{k=0}^{\infty} \frac{t^k}{k!}A^k, \qquad t \geq 0,$$

is its fundamental solution.

It is well known that in the present case the system (1) is exactly controllable if and only if it is null controllable and if and only if Kalman's rank condition holds: the $n \times nm$ matrix

$$(B, AB, \ldots, A^{n-1}B)$$

is of rank n. In this case we say that system (1) is *controllable.*

We denote by $\sigma(A)$ the set of all eigenvalues of the matrix A, that is, the set of all λ such that $\det(\lambda I - A) = 0$. The real and the imaginary parts of a complex number λ are denoted by $\operatorname{Re}\lambda$ and $\operatorname{Im}\lambda$ respectively.

THEOREM 1. *Assume that the finite-dimensional system* (1) *is controllable. Then*

(i) *system* (1) *is NCVE if and only if* $\operatorname{Re}\lambda \leq 0$ *for* $\lambda \in \sigma(A)$,
(ii) *system* (1) *is CVE if and only if* $\operatorname{Re}\lambda = 0$ *for* $\lambda \in \sigma(A)$.

2.2. Riccati-type equation characterization. A basic tool in the proof of the theorem is the following characterization of null controllable systems.

THEOREM 2. *A finite-dimensional system* (1) *is NCVE if and only if*

(i) *the system is controllable,*
(ii) *the matrix* $P = 0$ *is the only non-negative solution of the algebraic Riccati equation*

$$A^*P + PA - PBB^*P = 0.$$

2.3. Proofs and construction of strategies. Let us indicate some steps in the proof of Theorem 2. First, one shows that for each $t > 0$ there exists a positive definite matrix P_t such that

$$\inf\Big\{\int_0^t |u(s)|^2\,ds : y^{x,u}(t) = 0\Big\} = \langle P_t x, x\rangle, \qquad x \in \mathbb{R}^n. \tag{3}$$

Moreover

$$P_t = e^{tA^*}Q_t^{-1}e^{tA}, \quad \text{where} \quad Q_t = \int_0^t e^{tA}BB^*e^{sA^*}\,ds, \quad t \geq 0.$$

The main observation is that the matrix-valued function P_t is a solution of the following Riccati-type equation:

$$\frac{d}{dt}P_t = A^*P_t + P_tA - P_tBB^*P_t. \tag{4}$$

Using similar ideas one can construct strategies in *feedback* form which steer the system asymptotically to 0 with arbitrarily small energy. Namely, we have

THEOREM 3. *Assume that system* (1) *is NCVE.*

(i) *For every* $\epsilon > 0$ *there exists exactly one non-negative solution* $P(\epsilon)$ *of the equation*

$$A^*P + PA - PBB^*P + \epsilon I = 0$$

and $P(\epsilon)$ *decreases to* 0 *as* $\epsilon \downarrow 0$.

(ii) *If*

$$u_\epsilon(t) = -B^*P(\epsilon)y_\epsilon(t), \qquad t \geq 0,$$

where

$$\dot{y}_\epsilon(t) = Ay_\epsilon(t) + Bu_\epsilon(t) = (A - BB^*P_\epsilon)y_\epsilon(t), \qquad y_\epsilon(0) = a,$$

then

$$\lim_{t\to\infty} y_\epsilon(t) = 0 \quad \text{and} \quad \int_0^\infty |u_\epsilon(s)|^2\,ds \leq \langle P_\epsilon a, a\rangle.$$

2.4. A transfer problem. The general results of the previous sections were successfully applied by Shibata and Ichikawa [10] to a transfer problem for space travel. Consider an orbital station moving around the Earth and a spacecraft in its neighbourhood. Let us call them the *target* and the *chaser*, respectively. It is of interest to find a control which moves the chaser from a given orbit to a new one using as little energy as possible. If the final orbit coincides with the position of the target, one arrives at the landing problem with small energy.

Assume that the station moves around the Earth along a circle of radius R_0. If M denotes the mass of the Earth, G the gravitational constant,

T the time length of one rotation of the station around the Earth, and $\mu = GM$, then

$$\omega = \left(\frac{\mu}{R_0^3}\right)^{1/2} = \frac{2\pi}{T}$$

is the station's rotation rate. Let us consider the right-handed coordinate system (x, z) fixed at the centre of mass of the target. If the x-axis is along the radial direction and the z-axis is along the flight direction of the target then the Newton equations of two-dimensional motion are

$$\begin{aligned}\ddot{x} &= 2\omega\dot{z} + \omega^2(R_0 + x) - \frac{\mu(R_0 + x)}{((R_0 + x)^2 + z^2)^{1/2}},\\ \ddot{z} &= -2\omega\dot{x} + \omega^2 z - \frac{\mu z}{((R_0 + x)^2 + z^2)^{1/2}}.\end{aligned}$$

Linearization of the system around the origin leads to the equations

$$\begin{aligned}\ddot{x} &= 2\omega\dot{z} + 3\omega^2 x + u_1,\\ \ddot{z} &= -2\omega\dot{x} + u_2,\end{aligned}$$

with initial position $x_0, \dot{x}_0, z_0, \dot{z}_0$. We have added the steering functions u_1, u_2.

If $x_1 = x$, $x_2 = \dot{x}$, $x_3 = z$, $x_4 = \dot{z}$ and y is the state vector with coordinates x_1, x_2, x_3, x_4, then

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= 2\omega x_4 + 3\omega^2 x_1 + u_1,\\ \dot{x}_3 &= x_4,\\ \dot{x}_4 &= 2\omega x_2 + u_2,\end{aligned} \tag{5}$$

and

$$y(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix}, \qquad \dot{y} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & 2\omega & 0 & 0 \end{pmatrix} y(t) + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ u_2 \end{pmatrix}.$$

Thus for the control system under consideration,

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 2\omega & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since

$$\lambda^4 + \omega^2\lambda^2 \text{ is the characteristic polynomial of } A,$$

the eigenvalues of A are $\{0, \omega i, -\omega i\}$ and lie on the imaginary axis. Moreover the matrix

$$(B, AB) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 2\omega & 0 \end{pmatrix}$$

is non-singular, and thus the matrix

$$(B, AB, A^2B, A^3B)$$

is of full rank 4.

Consequently, the transfer system is controllable with vanishing energy and the theory of the previous section applies. In [10] the authors show in particular how to construct transfer strategies with small energy based on Theorem 3.

2.5. NCVE and generalized Liouville's theorem. There exists a surprising connection between controllable systems with vanishing energy and a generalization of Liouville's theorem on harmonic functions. Let us recall that a twice differentiable function $h : \mathbb{R}^n \to \mathbb{R}^1$ is *harmonic* if

$$\Delta h(x) = \sum_{k=1}^{n} \frac{\partial^2 h}{\partial x_k^2}(x) = 0, \qquad x \in \mathbb{R}^n,$$

and the classical Liouville theorem states that

- Bounded harmonic functions are constant.

The extension is concerned with Ornstein–Uhlenbeck operators $\mathcal{L}$ on $\mathbb{R}^n$ acting on functions h by the formula

$$\mathcal{L}h(x) = \langle Ax, Dh(x)\rangle + \mathrm{Tr}(B^*D^2h(x)B), \qquad x \in \mathbb{R}^n. \tag{6}$$

Here Dh and D^2h denote respectively the gradient and the Hessian of h:

$$Dh(x) = \left(\frac{\partial h}{\partial x_k}(x)\right)_{k=1}^{n}, \qquad D^2h(x) = \left(\frac{\partial^2 h}{\partial x_k \partial x_l}(x)\right)_{k,l=1}^{n}.$$

If $A = 0$, $B = I$, then $\mathcal{L} = \Delta$.

If $A = \left(\begin{smallmatrix} 0 & 1 \\ a & 0 \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$, then

$$\mathcal{L}h(x) = x_2 \frac{\partial h}{\partial x_1}(x) + a x_1 \frac{\partial h}{\partial x_2}(x) + \frac{\partial^2 h}{\partial x_2^2}(x), \qquad x = (x_1, x_2) \in \mathbb{R}^2, \tag{7}$$

and $\mathcal{L}$ is called the *Kolmogorov operator.*

The above mentioned extension of Liouville's theorem is the following result of E. Priola and the author [9].

THEOREM 4. *Bounded solutions h of the equation*

$$\mathcal{L}h(x) = 0, \qquad x \in \mathbb{R}^n,$$

where $\mathcal{L}$ is the Ornstein–Uhlenbeck operator (6), *are constant if and only if the control system* (A, B) *is NCVE.*

Note that if $\mathcal{L} = \varDelta$, then the pair (A, B) is controllable and $\sigma(A) = \{0\}$, and thus the system (A, B) is NCVE and Liouville's theorem holds. In the case of the Kolmogorov operator one easily checks that (A, B) is controllable for any $a \in \mathbb{R}^1$ and

$$\sigma(A) = \begin{cases} \{\sqrt{a}, -\sqrt{a}\} & \text{if } a > 0, \\ \{-i\sqrt{a}, i\sqrt{a}\} & \text{if } a \leq 0. \end{cases}$$

Consequently, bounded functions h for which $\mathcal{L}h = 0$, where $\mathcal{L}$ is the Kolmogorov operator (7), are constant if and only if $a \leq 0$.

3. Infinite-dimensional control systems. We now present extensions of the characterizations from the previous section to the situation when the state space H and the space U might be infinite-dimensional and also the operators A and B might be discontinuous. The models considered contain as special cases control systems with dynamics described by partial differential equations.

3.1. Systems with continuous control operators. We restrict our considerations to systems

$$\frac{dy}{dt} = Ay(t) + Bu(t), \qquad y(0) = x \in H, \tag{8}$$

where the operator A is the generator of a C_0-semigroup of linear operators $S(t)$, $t \geq 0$, acting on H. Such semigroups are generalizations of fundamental solutions of linear differential equations

$$\frac{dy}{dt} = Ay(t), \qquad y(0) = x \in H, \tag{9}$$

as well as of the exponential functions e^{tA}, $t \geq 0$. One requires that

$$S(t+s) = S(t)S(s), \quad t, s \geq 0, \qquad S(0) = I, \tag{10}$$

$$\lim_{t \downarrow 0} S(t)x = x \qquad \text{for } x \in H. \tag{11}$$

The *generator* of $S(t)$, $t \geq 0$, has domain $D(A)$ consisting of all those $x \in H$ for which the limit

$$\lim_{h \downarrow 0} \frac{S(h)x - x}{h} \tag{12}$$

exists, and the operator A itself is given by the formula

$$Ax = \lim_{h \downarrow 0} \frac{S(h)x - x}{h}, \qquad x \in D(A). \tag{13}$$

The solutions to (8) are given by the familiar formula (2):

$$y^{x,u}(t) = S(t)x + \int_0^t S(t-s)Bu(s)\,ds, \qquad t \geq 0,$$

where $u(\cdot)$ is a U-valued control function.

As is customary, the semigroup $S(t)$, $t \geq 0$, will often be denoted as e^{tA}, $t \geq 0$.

One denotes by $\rho(A)$ the set of all *regular values* of the operator A, that is, of all complex numbers λ such that the operator $\lambda I - A$ from $D(A)$ to H is one-to-one and onto and its inverse $(\lambda I - A)^{-1}$ is a continuous operator. The *spectrum* $\sigma(A)$ is the complement $\mathbb{C} \setminus \rho(A)$ of $\rho(A)$.

3.1.1. *Heating system.* A typical example is provided by a controlled heat equation on the interval $(0,1)$:

$$\frac{\partial y}{\partial t}(t,v) = \left(\frac{\partial^2 y}{\partial v^2}(t,v) + \gamma y(t,v)\right) + u(t,v), \qquad (t,v) \in (0,T) \times (0,1), \tag{14}$$

with the damping parameter γ and the initial and boundary conditions

$$y(0,v)=x(v), \quad v\in(0,1), \quad y(t,0)=y(t,1)=0, \quad x\in H,\ (t,v)\in(0,T)\times(0,1).$$

To represent the heating system in the abstract form (8) one takes $H = U = L^2[0,1]$ and $B = I$. The corresponding generator is $A = d^2/dv^2 + \gamma I$. Its domain $D(A)$ consists of functions $x(\cdot)$ absolutely continuous on $[0,1]$, equal to zero at 0 and 1, with absolutely continuous first derivative dx/dv, and the second derivative d^2x/dv^2 belonging to H (see e.g. [12]).

In the case when $\gamma = 0$, the corresponding semigroup is the classical heat semigroup:

$$S(t)x(v) = \int_0^1 p(t,w,v)x(w)\,dw, \qquad x \in H,\ v \in (0,1),\ t > 0,$$

where

$$p(t,w,v) = \frac{1}{\sqrt{4\pi t}} \sum_{k=-\infty}^{\infty} \left(e^{-(\frac{v-w+2k}{4t})^2} - e^{-(\frac{v+w+2k}{4t})^2}\right), \qquad t > 0,\ w,v \in (0,1).$$

3.1.2. *Riccati-type equation characterization.* The following two theorems are due to E. Priola and the author [8]. Theorem 5 is an infinite-dimensional counterpart of Theorem 2 concerned with classical control systems.

THEOREM 5. *A null controllable system* (1) *is NCVE if and only if the algebraic Riccati equation*

$$PA + A^*P - PBB^*P = 0, \qquad P \geq 0, \tag{15}$$

has a unique, bounded, non-negative solution $P = 0$.

In (15) the operator A is not everywhere defined so the meaning of its solution is not obvious. A positve definite operator P is a *solution* to (15) if

$$\langle Px, Ay\rangle + \langle PAx, y\rangle - \langle PBB^*Px, y\rangle = 0, \qquad x, y \in D(A).$$

The basic tool in the proof of Theorem 5 is the following result from [8].

THEOREM 6. *If system* (1) *is null controllable for any* $T > 0$*, then for every bounded linear operator* $R \geq 0$ *on* H*, there exists a maximal solution* $P \geq 0$ *of the equation*

$$PA + A^*P - PBB^*P + R = 0, \qquad P \geq 0, \tag{16}$$

and it is given by the formula

$$\langle Px, x\rangle = \inf_{t\geq T>0} \inf_{\substack{u\in L^2(0,t;U) \\ y^{x,u}(t)=0}} \int_0^t (\langle Ry^{x,u}(s), y^{x,u}(s)\rangle + |u(s)|^2)\, ds.$$

3.1.3. *Spectral characterizations.* For spectral characterizations of controllable systems with vanishing energy additional conditions should be imposed. They are formulated as the following hypotheses:

HYPOTHESIS (H1)

(i) There exists a sequence $(\lambda_n) \subset \sigma(A)$ such that each λ_n is isolated in $\sigma(A)$ and

$$\lim_{n\to\infty} \operatorname{Re}\lambda_n = s(A) = \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\}.$$

(ii) There exist closed linear subspaces H_s and H_u, invariant for e^{tA}, $t \geq 0$, such that their direct sum is H and

(a) e^{tA} restricted to H_s is exponentially stable on H_s, that is, there exist $M, \omega > 0$ such that

$$|e^{tA}x| \leq Me^{-t\omega}|x|, \qquad x \in H_s,\ t > 0,$$

(b) the generalized eigenvectors of A contained in H_u are linearly dense in H_u.

A vector $x \in H$, $x \neq 0$, is a *generalized eigenvector* of A if there exist $\lambda \in \sigma(A)$ and a natural k such that

$$(\lambda - A)^k x = 0$$

THEOREM 7. *Assume* (1) *is null controllable and* (H1) *holds. Then the following statements are equivalent:*

(a) *system* (1) *is NCVE,*
(b) $s(A) = \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\} \leq 0$.

Theorem 8. *Assume* (1) *is exactly controllable and* (H1) *holds. Then the following statements are equivalent:*

(a) *system* (1) *is ECVE,*
(b) $\sigma(A) \subset \{i\lambda : \lambda \in \mathbb{R}\}$.

Hypothesis (H1) holds in the following cases:

(i) H is finite-dimensional,
(ii) for some $t > 0$ the operator e^{tA} is compact,
(iii) system (1) is null controllable and the operator B is compact.

3.1.4. *Examples.* For the heating system (§3.1.1) the spectrum of the operator A consists of eigenvalues only, given by the formula $\lambda_n = -\pi^2 n^2 + \gamma$, $n = 1, 2, \ldots$.

Since $B = I$, the system is null controllable. In fact, by direct calculation one checks that the control

$$u(s) = -\frac{1}{T} e^{sA} x, \qquad s \in (0, T),$$

steers each state x to 0. By Theorem 7 the system is null controllable with vanishing energy if and only if $\gamma \leq 0$.

Applications to general parabolic, hyperbolic and delay systems can be found in [8].

3.2. Systems with discontinous control operators. The two-dimensional, parabolic coupled control system

$$\begin{cases} \frac{\partial y}{\partial t}(t,v) - \frac{\partial^2 y}{\partial v^2}(t,v) = A_0 y & \text{in } Q = (0,T) \times (0,\pi), \\ y(t,0) = B_0 u(t), \quad y(t,\pi) = 0, & t \in (0,T), \\ y(0,v) = x(v), & v \in (0,\pi), \end{cases} \tag{17}$$

was studied by E. Fernández-Cara, M. González-Burgos and L. de Teresa [2]. Here A_0 is a 2×2 real matrix, $B_0 \in \mathbb{R}^2$, $u \in L^2_{\mathrm{loc}}(0, \infty)$, and the function y takes values in $\mathbb{R}^2$: $y(t,v) \in \mathbb{R}^2$ for $t \in [0,T]$, $v \in [0,\pi]$.

The controllability of the system cannot be treated by the theory described in the previous section. In fact, the control action $u(\cdot)$ is applied at 0 rather than inside the interval $(0,\pi)$ and thus the control operator B is not continuous.

We will use an abstract setting, developed by I. Lasiecka and R. Triggiani in [5], allowing a successful treatment of such systems. Before formulating the setting and chararacterizations we present answers for system (17).

The following necessary and sufficient conditions for the null controllability of (17) were proved by E. Fernández-Cara, M. González-Burgos and L. de Teresa [2].

THEOREM 9. *System* (17) *is null controllable if and only if*

$$\operatorname{rank}[B_0, A_0B_0] = 2, \qquad \mu_1 - \mu_2 \neq j^2 - k^2, \quad j \neq k,\ j,k = 1,2,\ldots,$$

where $\sigma(A_0) = \{\mu_1, \mu_2\}$.

Using Theorem 9 and the general results by L. Pandolfi, E. Priola and the author [7] to be presented below, the following result was deduced in [7].

THEOREM 10. *Assume that* (17) *is null controllable. Then* (17) *is NCVE if and only if*

$$\operatorname{Re}\mu_1 \leq 1, \qquad \operatorname{Re}\mu_2 \leq 1.$$

3.2.1. *General setting and characterization theorems.* Let A be the generator of a C_0-semigroup e^{tA} on a Hilbert space H. Let us fix $\lambda \in \varrho(A)$ and define H_{-1} as the completion of H with respect to the norm $|(\lambda - A)^{-1} \cdot |$.

We are concerned with the control system

$$\text{(18)} \qquad \frac{dy}{dt} = \mathcal{A}y + Bu, \quad y(0) = y_0 \in H, \quad u \in L^2_{\mathrm{loc}}(0,\infty;U),$$

where the operator $\mathcal{A}$ is the generator of a semigroup $e^{t\mathcal{A}}$ on H_{-1} identical with the extension of e^{tA} to H_{-1}, and $B \in L(U, H_{-1})$.

Define the *transition operator* by the formula

$$\mathcal{L}u(t) = \int_0^t e^{\mathcal{A}(t-s)}Bu(s)\,ds, \qquad t \in [0,t],\ u \in L^2(0,T;U).$$

As in general the values of $\mathcal{L}$ might be discontinuous functions with values in H, we have the following definition:

An element $a \in H$ *can be steered to the origin in time* T if there exists $u \in L^2_{\mathrm{loc}}(0,\infty;U)$ with support in $[0,T]$ such that the support of

$$e^{t\mathcal{A}}a + \mathcal{L}u(t), \qquad t \geq 0,$$

is contained in $[0,T]$. If every element of H can be steered to the origin in some time, then the system is, by definition, null controllable.

Two hypotheses are needed.

HYPHOTESIS (H2). For every $T > 0$ the transformation $\mathcal{L}$ is continuous from $L^2(0,T;U)$ into $L^2(0,T;H)$.

HYPHOTESIS (H3). There exist closed linear subspaces H_s, H_u of H such that their direct sum is H and

(a) $\lim_{t\to\infty} e^{At}x = 0$ for $x \in H_s$,
(b) H_u is invariant for e^{At}, $t \geq 0$, and the generalized eigenvectors of A contained in H_u are linearly dense in H_u.

The following two theorems were established in [7] and we refer there for more details and proofs.

Theorem 11. *Assume* (H2) *and that there exists a subspace* $E \subset H$, $E \neq \{0\}$, *invariant for* e^{At}, $t \geq 0$, *such that* e^{-At}, $t \geq 0$, *is exponentially stable on* E. *Then the control system* (18) *is not NCVE.*

Theorem 12. *Assume* (H2) *and* (H3) *and that* $s(A) \leq 0$. *If system* (18) *is null controllable then it is NCVE.*

Theorem 10 is a direct consequence of Theorems 9, 11 and 12.

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