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Lipschitz isomorphisms of compact quantum metric spaces

by

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Abstract. A compact quantum metric space is a complete order unit space endowed with a Lip-norm. We study morphisms and isomorphisms between compact quantum metric spaces. We give two sufficient conditions for a unital positive linear map of a compact quantum metric space with lower semicontinuous Lip-norm to be in fact a bi-Lipschitz map.

1. Introduction. In 1989, Connes initiated the program of noncommutative metric geometry in the framework of spectral triples [4, 5]. Let M be a connected compact, Riemannian and spin manifold, let $C(M)$ be the unital commutative C^*-algebra of complex-valued continuous functions on M, and let D be a Dirac type operator on the Hilbert space $L^2(M,S)$ of L^2 spinors on M with $C(M)$ acting by multiplication operators. Then $(C(M), L^2(M,S), D)$ is a spectral triple. Connes recovered the geodesic distance on M via the *-seminorm $\|[D,\cdot\,]\|$ on a dense *-subalgebra of $C(M)$. More precisely, for any two points p,q in the manifold M, the geodesic distance $\rho(p,q)$ between them can be computed by the following formula:

$$\rho(p,q) = \sup\{|f(p) - f(q)| : f \in C(M),\ \|[D,f]\| \le 1\}.$$

From Gelfand's fundamental theorem on commutative C^*-algebras, the points p and q can be considered as pure states (or characters) $\hat{p}$ and $\hat{q}$ of $C(M)$, and we can understand the above formula as

$$\rho(p,q) = \sup\{|\hat{p}(f) - \hat{q}(f)| : f \in C(M),\ \|[D,f]\| \le 1\}.$$

This observation led Connes to obtain an ordinary metric ρ_{L_D} on the state space $\mathcal{S}(A)$ of a unital noncommutative C^*-algebra A from a spectral triple

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$(A, \mathcal{H}, D)$ by means of the similar formula

$$\rho_{L_D}(\mu, \nu) = \sup\{|\mu(a) - \nu(a)| : a \in A,\ L_D(a) = \|[D, a]\| \leq 1\} \tag{1.1}$$

for $\mu, \nu \in \mathcal{S}(A)$, generalizing the Monge–Kantorovich metric on the space of probability measures on a compact Hausdorff space X (identified with the state space of $C(X)$). In this way, we can think of a spectral triple $(A, \mathcal{H}, D)$ as a noncommutative metric space. Moreover, in 1995 Connes defined an isometry of a spectral triple $(A, \mathcal{H}, D)$ as a unitary U on $\mathcal{H}$ with $UD = DU$ and $U^* A'' U = A''$ [6].

Let $d + d^*$ be the de Rham operator on the Hilbert space $L^2(\Lambda^*(M))$ of complex L^2-forms on a compact oriented Riemannian manifold (M, g), and let $C(M)$ act on $L^2(\Lambda^*(M))$ by pointwise multiplication. Then the triple $(C(M), L^2(\Lambda^*(M)), d + d^*)$ is a spectral triple, and Park showed in [23] that the Riemannian isometry group of (M, g) is isomorphic to the group

$$\{\phi_U \in \mathrm{Aut}(C(M)) : U \text{ is a unitary on } L^2(\Lambda^*(M)) \text{ such that } U(d + d^*) = (d + d^*)U \text{ and } \phi_U(a) = U^* a U \text{ for every } a \in C(M)\}.$$

Motivated by this observation, Park defined an isometry of a spectral triple $(A, \mathcal{H}, D)$ to be a *-automorphism $\phi \in \mathrm{Aut}(A)$ with a unitary U on $\mathcal{H}$ such that $UD = DU$ and $\phi(a) = U^* a U$ for any $a \in A$.

In [3], using countable sums of two-dimensional modules, Christensen and Ivan constructed two kinds of spectral triples for the unital C^*-algebra $C(X)$ of an ordinary compact metric space (X, ρ). One is finitely summable for any positive real number, and gives the original metric ρ back by the formula (1.1) exactly. The other one depends on a parameter $\delta > 0$, and its induced metric ρ_δ on X is within δ of the original metric, that is, these two metrics are bi-Lipschitz equivalent [33]. More concretely,

$$\rho(p, q) \leq \rho_\delta(p, q) \leq (1 + \delta)\rho(p, q), \quad p, q \in X.$$

Moreover, the second spectral triple reflects dimension properties of X such as upper Minkowski dimension.

In 1998, inspired by what happens for ordinary compact metric spaces, Rieffel defined a metric on a subset of the Banach space dual in a very rudimentary Banach space setting and gave necessary and sufficient conditions under which the topology of the metric agrees with the weak*-topology [24]. The metric data for a unital noncommutative C^*-algebra A was formulated by means of a suitable seminorm on the order unit space A_{sa} of all self-adjoint elements in A, which plays the role of the usual Lipschitz seminorms for ordinary compact metric spaces and gives $\mathcal{S}(A)$ the weak*-topology [29]. In particular, for Connes' spectral triple, the seminorm is given by $L_D(a) = \|[D, a]\|$. Many important examples of this situation have been constructed [1–4, 8, 18, 21, 22, 24, 26].

In general, if L is a *-seminorm defined on a dense *-subalgebra of a unital C^*-algebra A with $L(1_A) = 0$, we can obtain a metric ρ_L on the state space $\mathcal{S}(A)$ of A, much as Connes did, by

$$\rho_L(\mu, \nu) = \sup\{|\mu(a) - \nu(a)| : a \in A_{sa},\ L(a) \leq 1\}, \qquad \mu, \nu \in \mathcal{S}(A).$$

(Without further hypotheses ρ_L may take value $+\infty$.) When the metric topology on the state space $\mathcal{S}(A)$ induced by ρ_L agrees with the underlying weak*-topology, L is a Lip-norm on A_{sa}, and (A_{sa}, L) is a compact quantum metric space [29]. By introducing the notion of quantum Gromov–Hausdorff distance for compact quantum metric spaces, Rieffel was able to make sense of the assertion that a sequence of matrix algebras converges to the 2-sphere [27, 28], which appears in the literature of theoretical physics and string theory (see [27, 28] and references therein). See [11, 12, 14–17, 30–32, 34–36] for further investigations.

Motivated by the classical morphisms for the category of metric spaces [33], Kerr proposed morphisms of compact quantum metric spaces as unital positive linear maps satisfying a Lipschitz condition, and calculated metric dimension and dynamical entropy of some important examples of compact quantum metric spaces arising from compact group actions [10]. In 2011, the second author defined a unital C^*-algebra equipped with a Leibniz Lip-norm to be a C^*-metric algebra, and proved that if the Lip-norms concerned are lower semicontinuous, then any unital *-homomorphism from a C^*-metric algebra to another one is necessarily Lipschitz [37]. Recently, Latrémolière introduced the notion of quantum Lipschitz homomorphisms for the category of compact quantum metric spaces as unital *-homomorphisms from a compact quantum metric space (A, L_A) to another compact quantum metric space (B, L_B) such that their induced maps from the compact metric space $(\mathcal{S}(B), \rho_{L_B})$ to the compact metric space $(\mathcal{S}(A), \rho_{L_A})$ are Lipschitz as usual, and showed that unital *-morphisms, preserving Lipschitz elements, between compact quantum metric spaces with lower semicontinuous Lip-norms are Lipschitz [12–15].

For any compact quantum metric space (A, L), let L_{sa} denote the restriction of L to the order unit space A_{sa} of all self-adjoint elements in the unital C^*-algebra A. Then the resulting pair (A_{sa}, L_{sa}) is a compact quantum metric space in the much more general framework of order unit spaces [10, 27]. Let L^s_{sa} be the greatest lower semicontinuous Lip-norm on A_{sa} smaller than L_{sa}, and let (A^c_{sa}, L^c_{sa}) be the closure of (A_{sa}, L^s_{sa}) [25]. Rieffel considered an isometry of a compact quantum metric space (A_{sa}, L_{sa}) as a unital order isomorphism φ on A^c_{sa} such that $L^c_{sa} = L^c_{sa} \circ \varphi$, and demonstrated that the space of isometry equivalence classes of compact quantum metric spaces is a separable complete metric space for the quantum Gromov–Hausdorff distance [27].

In [20], the present authors proposed morphisms for the category of compact quantum metric spaces in the framework of order unit spaces, and proved that a unital positive linear map, preserving Lipschitz elements, between compact quantum metric spaces with lower semicontinuous Lip-norms is necessarily Lipschitz. Motivated by the asymptotically expansive (nonexpansive) maps for metric spaces, the authors defined asymptotically expansive (nonexpansive) maps between order unit spaces with Lipschitz seminorms, and showed that these maps are Lipschitz isometries under certain conditions.

In the present paper we continue the study of morphisms between compact quantum metric spaces, and go further by investigating the question of when a morphism on a compact quantum metric space is a Lipschitz isomorphism in the framework of order unit spaces. The organization of the paper is as follows. In Section 2 we obtain a new characterization of Lipschitz morphisms between compact quantum metric spaces with lower semicontinuous Lip-norms, and give several properties of Lipschitz number of the Lipschitz morphisms. In Section 3 for a positive real number k we define asymptotically k-nonexpansive maps between order unit spaces with Lipschitz seminorms, and show that a unital positive, asymptotically k-nonexpansive and isometric linear map on a compact quantum metric space (A, L) with lower semicontinuous Lip-norm is a bi-Lipschitz map from (A, L) onto itself. In Section 4 for $k > 0$ we define asymptotically k-expansive maps between order unit spaces with Lipschitz seminorms, and show that a unital positive and asymptotically k-expansive linear map with the set of Lipschitz elements invariant on a compact quantum metric space (A, L) with lower semicontinuous Lip-norm is a bi-Lipschitz map from (A, L) onto itself.

2. Lipschitz morphisms

Definition 2.1. An *order unit space* is a real partially ordered vector space A with a distinguished element 1_A, the *order unit*, which satisfies:

(1) For each $a \in A$ there is an $r \in \mathbb{R}$ such that $a \leq r1_A$.
(2) If $a \in A$ and if $a \leq r1_A$ for all $r \in \mathbb{R}^+$, then $a \leq 0$.

For any $a \in A$ its *order norm* is given by

$$\|a\| = \inf\{r \in \mathbb{R}^+ : -r1_A \leq a \leq r1_A\}. \tag{2.1}$$

Then $(A, \|\cdot\|)$ is a real normed linear space. If A is complete with respect to this norm, we say that A is a *complete order unit space*. As in [25], we will not assume the completeness from the outset.

Example 2.2. The real vector space A_{sa} of all self-adjoint elements in a unital C^*-algebra A is a complete order unit space with order unit being the identity element of A.

Example 2.3. The real vector space $\mathrm{Af}(X)$ of all continuous affine real-valued functions on a compact convex set X of a topological vector space, equipped with the usual order, is a complete order unit space with order unit being the constant function 1 on X. The order norm on $\mathrm{Af}(X)$ is the supremum norm. Indeed, by the Kadison representation theorem, every complete order unit space A is unital order isomorphic to the space $\mathrm{Af}(\mathcal{S}(A))$, where $\mathcal{S}(A)$ is the state space of A.

Let L be a seminorm on an order unit space A with order unit 1_A, which is permitted to take the value $+\infty$. If

(1) $L(1_A) = 0$;
(2) the set $\mathcal{A} = \{a \in A : L(a) < \infty\}$ of *Lipschitz elements* in A is a dense subspace of A,

then we can define an extended metric

$$\rho_L : \mathcal{S}(A) \times \mathcal{S}(A) \to [0, \infty]$$

on the state space $\mathcal{S}(A)$ of A by

$$\rho_L(\mu, \nu) = \sup\{|\mu(a) - \nu(a)| : a \in A,\ L(a) \le 1\}$$

for all $\mu, \nu \in \mathcal{S}(A)$.

If there is an element $a_0 \in A$ with $a_0 \notin \mathbb{R}1_A$ and $L(a_0) = 0$, then for any $\mu \in \mathcal{S}(A)$, we have $a_0 - \mu(a_0)1_A \neq 0$, and so there is a $\nu \in \mathcal{S}(A)$ such that $\nu(a_0 - \mu(a_0)1_A) \neq 0$. It follows that

$$\begin{aligned}\rho_L(\mu, \nu) &= \sup\{|\mu(a) - \nu(a)| : a \in A,\ L(a) \le 1\} \\ &\ge \sup\{|\mu(na_0) - \nu(na_0)| : n \in \mathbb{N}\} \\ &= \sup\{n|\nu(a_0 - \mu(a_0)1_A)| : n \in \mathbb{N}\} = \infty\end{aligned}$$

since $L(na_0) = nL(a_0) = 0$ for any $n \in \mathbb{N}$. On the other hand, if $(\mathcal{S}(A), \rho_L)$ is a bounded metric space, then for any $a \in A$ with $L(a) = 0$ we must have $a \in \mathbb{R}1_A$. Indeed, for any $n \in \mathbb{N}$ we have $L(na) = nL(a) = 0$. Thus for any $\mu, \nu \in \mathcal{S}(A)$,

$$|\mu(na) - \nu(na)| \le \rho_L(\mu, \nu) \le \mathrm{diam}(\mathcal{S}(A), \rho_L) < \infty.$$

Hence

$$|\mu(a) - \nu(a)| \le \frac{1}{n}\,\mathrm{diam}(\mathcal{S}(A), \rho_L) \to 0$$

for any $n \in \mathbb{N}$. It follows that $\mu(a) = \nu(a)$ for all $\mu, \nu \in \mathcal{S}(A)$. Now we fix a state $\mu_0 \in \mathcal{S}(A)$. Then

$$\mu(a - \mu_0(a)1_A) = \mu(a) - \mu_0(a) = 0$$

for all $\mu \in \mathcal{S}(A)$. Since the state space $\mathcal{S}(A)$ separates the elements of the order unit space A, we have $a = \mu_0(a)1_A \in \mathbb{R}1_A$. Therefore, we should assume that $L(a) = 0$ if and only if $a \in \mathbb{R}1_A$ to try to make ρ_L become a “real” metric on $\mathcal{S}(A)$.

A *Lipschitz seminorm* on an order unit space A with order unit 1_A is a seminorm $L : A \to [0, \infty]$ such that

(1) the set $\mathcal{A} = \{a \in A : L(a) < \infty\}$ is dense in A;
(2) $L(a) = 0$ if and only if $a \in \mathbb{R}1_A$.

A seminorm L on an order unit space A is said to be *lower semicontinuous* if for some and hence all $r > 0$ the set $\{a \in \mathcal{A} : L(a) \leq r\}$ is closed in A. Equivalently, for any sequence $\{a_n\}_{n\in\mathbb{N}}$ in A that converges in order norm to $a \in A$, we have $L(a) \leq \liminf_{n\to\infty} L(a_n)$. Recall the following useful result [25, 37].

Proposition 2.4. *Let L be a lower semicontinuous Lipschitz seminorm on a complete order unit space A. Then the set $\mathcal{A} = \{a \in A : L(a) < \infty\}$ is a real Banach space for the norm $\|\cdot\|_1$ defined by*

$$\|a\|_1 = \|a\| + L(a) \quad \text{for } a \in \mathcal{A}. \tag{2.2}$$

If a multiplication operation is defined on A such that A is a Banach algebra and L is Leibniz, that is, $L(ab) \leq L(a)\|b\| + \|a\|L(b)$ for all $a, b \in A$, then $\mathcal{A}$ is a Banach algebra for the norm $\|\cdot\|_1$. Moreover, if A is a Banach *-algebra with isometric involution and L is a *-seminorm, that is, $L(a) = L(a^*)$ for all $a \in A$, then $\mathcal{A}$ is a Banach *-algebra with isometric involution [21, 25, 37].

Let (X, ρ_X) and (Y, ρ_Y) be metric spaces. A map $f : X \to Y$ is *Lipschitz* if there exists a constant $\lambda \geq 0$ such that

$$\rho_Y(f(p), f(q)) \leq \lambda\rho_X(p, q) \quad \text{for all } p, q \in X.$$

The least such λ is called the *Lipschitz number* of f, and is denoted by $L(f)$. If f is invertible and both f and f^{-1} are Lipschitz, then we say that f is *bi-Lipschitz* [33]. If there is a bi-Lipschitz map between metric spaces (X, ρ_X) and (Y, ρ_Y), then they are *bi-Lipschitz equivalent* [20].

Motivated by [20, Theorem 2.6] and the definition of Lipschitz map between metric spaces, we define a Lipschitz map between order unit spaces with Lipschitz seminorms [10, 12–15, 18–21, 37].

Definition 2.5 ([20]). Let A and B be order unit spaces with Lipschitz seminorms L_A and L_B, respectively, and let ϕ be a map from A to B. If there is a constant $\lambda \geq 0$ such that

$$L_B(\phi(a)) \leq \lambda L_A(a) \quad \text{for all } a \in \mathcal{A},$$

we say that ϕ is *Lipschitz*. The least such constant λ is called the *Lipschitz number* of ϕ, and is denoted by $L(\phi)$. When ϕ is invertible and both ϕ and ϕ^{-1} are Lipschitz, we say that ϕ is *bi-Lipschitz*. If

$$L_B(\phi(a)) = L_A(a) \quad \text{for all } a \in A,$$

we say that ϕ is *Lipschitz isometric*.

Suppose that ϕ is a unital positive linear map from an order unit space A with Lipschitz seminorm L_A to another order unit space B with Lipschitz seminorm L_B. For any $r > 0$ with $-r1_A \leq a \leq r1_A$, we have $-r1_B \leq \phi(a) \leq r1_B$. So $\|\phi(a)\| \leq r$, and hence $\|\phi(a)\| \leq \|a\|$. From $\phi(1_A) = 1_B$, we see that $\|\phi\| = 1$. Let A' and B' be the order-norm dual Banach spaces of A and B. We then define the dual map $\hat{\phi} : B' \to A'$ of ϕ as follows:

$$\hat{\phi}(f)(a) = f(\phi(a)), \qquad a \in A,$$

for all $f \in B'$. For any $f \in B'$ with $\|f\| = 1$ and $a \in A$ with $\|a\| = 1$, we have

$$|\hat{\phi}(f)(a)| = |f(\phi(a))| \leq \|f\| \, \|\phi(a)\| \leq \|f\| \, \|\phi\| \, \|a\| = 1,$$

and so $\|\hat{\phi}\| \leq 1$. In particular, taking $a = 1_A$ and $f = \mu \in \mathcal{S}(B)$, we have

$$\hat{\phi}(\mu)(1_A) = \mu(\phi(1_A)) = \mu(1_B) = 1,$$

and hence $\|\hat{\phi}\| = 1$. Moreover, if $\mu \in \mathcal{S}(B)$, then it is easy to check that $\hat{\phi}(\mu) \in \mathcal{S}(A)$, and thus $\hat{\phi}$ is a weak*-continuous affine map from $\mathcal{S}(B)$ to $\mathcal{S}(A)$.

Let A'^o and B'^o be the dual Banach spaces of $A/(\mathbb{R}1_A)$ and $B/(\mathbb{R}1_B)$ for the quotient norms with respect to the order norms $\|\cdot\|_A$ and $\|\cdot\|_B$. Then A'^o and B'^o can be viewed as the subspaces of all elements $f \in A'$ and $g \in B'$ such that $f(1_A) = 0$ and $g(1_B) = 0$. Then $\hat{\phi}$ carries B'^o into A'^o. As with the usual formula for a dual norm, the dual norms of the quotient norms of L_A and L_B are defined by (see [25])

$$L'_A(f) = \sup\{|f(a)| : a \in A,\, L_A(a) \leq 1\}, \qquad f \in A'^o,$$
$$L'_B(g) = \sup\{|g(b)| : b \in B,\, L_B(b) \leq 1\}, \qquad g \in B'^o.$$

THEOREM 2.6. *Let A and B be complete order unit spaces with lower semicontinuous Lipschitz seminorms L_A and L_B. Let ϕ be a unital positive linear map from A to B. Consider the following statements:*

(1) *$\phi(\mathcal{A}) \subseteq \mathcal{B}$, that is, for every $a \in A$ with $L_A(a) < \infty$ we have $L_B(\phi(a)) < \infty$;*
(2) *ϕ is continuous from $(\mathcal{A}, \|\cdot\|_{A,1})$ to $(\mathcal{B}, \|\cdot\|_{B,1})$, where $\|\cdot\|_A$ and $\|\cdot\|_B$ are the order norms as defined in (2.1), and $\|\cdot\|_{A,1}$ and $\|\cdot\|_{B,1}$ are defined as in (2.2);*
(3) *ϕ is a Lipschitz map from A to B;*
(4) *$\hat{\phi}$ is a continuous linear map from (B'^o, L'_B) to (A'^o, L'_A);*
(5) *$\hat{\phi}$ is a Lipschitz map from $(\mathcal{S}(B), \rho_{L_B})$ to $(\mathcal{S}(A), \rho_{L_A})$.*

Then (3)$\Leftrightarrow$(4)$\Leftrightarrow$(5)$\Rightarrow$(1)$\Rightarrow$(2) in general. Moreover, if $(\mathcal{S}(A), \rho_{L_A})$ is a bounded metric space, then (2)$\Rightarrow$(3).

Proof. (3)⇒(4). Clearly $\hat{\phi}$ is linear. Suppose that there is a constant $\lambda > 0$ such that $L_B(\phi(a)) \le \lambda L_A(a)$ for all $a \in \mathcal{A}$. For any $f \in B'^o$ we have

$$\begin{aligned} L'_A(\hat{\phi}(f)) &= \sup\{|\hat{\phi}(f)(a)| : a \in A,\ L_A(a) \le 1\} \\ &= \sup\{|f(\phi(a))| : a \in A,\ L_A(a) \le 1\} \\ &\le \lambda \sup\{|f(b)| : b \in B,\ L_B(b) \le 1\} = \lambda L'_B(f). \end{aligned}$$

Thus $\hat{\phi}$ is continuous from (B'^o, L'_B) to (A'^o, L'_A).

(4)⇒(5). Since $\hat{\phi}$ is a continuous linear map from (B'^o, L'_B) to (A'^o, L'_A), there is a constant $\lambda > 0$ such that $L'_A(\hat{\phi}(f)) \le \lambda L'_B(f)$ for all $f \in B'^o$. By [25, Lemma 4.3], we have

$$\begin{aligned} \rho_{L_A}(\hat{\phi}(\mu), \hat{\phi}(\nu)) &= L'_A(\hat{\phi}(\mu) - \hat{\phi}(\nu)) = L'_A(\hat{\phi}(\mu - \nu)) \\ &\le \lambda L'_B(\mu - \nu) = \lambda \rho_{L_B}(\mu, \nu) \end{aligned}$$

for all $\mu, \nu \in \mathcal{S}(B)$, and so $\hat{\phi}$ is a Lipschitz map from $(\mathcal{S}(B), \rho_{L_B})$ to $(\mathcal{S}(A), \rho_{L_A})$.

(5)⇒(3). Since $\hat{\phi}$ is a Lipschitz map from $(\mathcal{S}(B), \rho_{L_B})$ to $(\mathcal{S}(A), \rho_{L_A})$, there is a constant $\lambda > 0$ such that $\rho_{L_A}(\hat{\phi}(\mu), \hat{\phi}(\nu)) \le \lambda \rho_{L_B}(\mu, \nu)$ for all $\mu, \nu \in \mathcal{S}(B)$. It follows that for any $a \in A$ and $\mu, \nu \in \mathcal{S}(B)$, we have

$$\begin{aligned} |\mu(\phi(a)) - \nu(\phi(a))| &= |\hat{\phi}(\mu)(a) - \hat{\phi}(\nu)(a)| \\ &\le L_A(a)\rho_{L_A}(\hat{\phi}(\mu), \hat{\phi}(\nu)) \le \lambda L_A(a) \rho_{L_B}(\mu, \nu). \end{aligned}$$

That the Lipschitz seminorm L_B is lower semicontinuous implies, by [25, Theorem 4.1], that L_B can be exactly recovered from the metric space $(\mathcal{S}(B), \rho_{L_B})$, that is,

$$L_B(b) = \sup\left\{ \frac{|\mu(b) - \nu(b)|}{\rho_{L_B}(\mu, \nu)} : \mu, \nu \in \mathcal{S}(B),\ \mu \ne \nu \right\}$$

for all $b \in B$. Thus we have $L_B(\phi(a)) \le \lambda L_A(a)$ for all $a \in \mathcal{A}$. Therefore, ϕ is Lipschitz.

(3)⇒(1). Since ϕ is a Lipschitz map, there is a constant $\lambda > 0$ such that

$$L_B(\phi(a)) \le \lambda L_A(a) < \infty$$

for all $a \in \mathcal{A}$, so we have $\phi(\mathcal{A}) \subseteq \mathcal{B}$.

(1)⇒(2). This can be proved just as in [20, Theorem 2.6(i)⇒(ii)].

(2)⇒(3). Suppose that $(\mathcal{S}(A), \rho_{L_A})$ is a bounded metric space. Then $\mathrm{diam}(\mathcal{S}(A), \rho_{L_A}) < \infty$, and one can argue as in [20, proof of Theorem 2.6(ii)⇒(iii)]. ∎

PROPOSITION 2.7. *Let A and B be order unit spaces with Lipschitz seminorms L_A and L_B. Then the set of all Lipschitz maps from A to B is a real vector space under pointwise addition and scalar multiplication. Moreover, if ϕ and ψ are Lipschitz maps from A to B, then*

(1) $L(\lambda\phi) = |\lambda| L(\phi)$ *for all* $\lambda \in \mathbb{R}$;
(2) $L(\phi + \psi) \le L(\phi) + L(\psi)$.

Proof. Suppose that ϕ and ψ are Lipschitz maps from A to B. Then for any $a \in \mathcal{A}$, we have

$$\begin{aligned} L_B((\phi+\psi)(a)) &= L_B(\phi(a)+\psi(a)) \le L_B(\phi(a)) + L_B(\psi(a)) \\ &\le L(\phi)L_A(a) + L(\psi)L_A(a) = (L(\phi)+L(\psi))L_A(a), \end{aligned}$$

which implies (2).

For any $\lambda \in \mathbb{R}$ and $a \in \mathcal{A}$, we have

$$L_B((\lambda\phi)(a)) = L_B(\lambda\phi(a)) = |\lambda| L_B(\phi(a)) \le |\lambda| L(\phi) L_A(a),$$

and hence

$$L(\lambda\phi) \le |\lambda| L(\phi).$$

If $\lambda \ne 0$, using the same procedure for $\frac{1}{\lambda}$ and $\lambda\phi$ we have

$$L(\phi) = L\left(\frac{1}{\lambda}(\lambda\phi)\right) \le \frac{1}{|\lambda|} L(\lambda\phi) \le \frac{1}{|\lambda|}|\lambda| L(\phi) = L(\phi),$$

i.e., $L(\lambda\phi) = |\lambda| L(\phi)$. When $\lambda = 0$, we have $\lambda\phi \equiv 0$ and $L(\lambda\phi) = L(0) = 0 = 0L(\phi)$. Thus (1) is proved.

In particular, these facts also imply that $\phi+\psi$ and $\lambda\phi$ are Lipschitz maps from A to B. This completes the proof. ∎

From this proposition we see that the Lipschitz number is a seminorm on the real vector space of all Lipschitz maps from an order unit space with a Lipschitz seminorm to another such space.

The following proposition gives a sufficient condition for the lower semicontinuity of the Lipschitz number with respect to the point-norm topology.

PROPOSITION 2.8. *Let A and B be order unit spaces with Lipschitz seminorms L_A and L_B. Suppose that ϕ and ϕ_n ($n \in \mathbb{N}$) are maps from A to B. If $\phi_n \to \phi$ as $n \to \infty$ in the point-norm topology and L_B is lower semicontinuous, then*

$$L(\phi) \le \liminf_{n\to\infty} L(\phi_n).$$

Proof. For any $a \in \mathcal{A}$, we have $\phi_n(a) \to \phi(a)$ as $n \to \infty$. Then it follows from the lower semicontinuity of L_B that

$$L_B(\phi(a)) \le \liminf_{n\to\infty} L_B(\phi_n(a)) \le \liminf_{n\to\infty} L(\phi_n) L_A(a),$$

and hence

$$L(\phi) \le \liminf_{n\to\infty} L(\phi_n). \ ∎$$

PROPOSITION 2.9. *Let A, B and C be order unit spaces with Lipschitz seminorms L_A, L_B and L_C. Suppose that $\phi : A \to B$ and $\psi : B \to C$ are*

Lipschitz maps. Then $\psi \circ \phi : A \to C$ is also Lipschitz. Moreover, $L(\psi \circ \phi) \leq L(\psi)L(\phi)$.

Proof. There are constants $\lambda, \gamma \geq 0$ such that

$$L_B(\phi(a)) \leq \lambda L_A(a) \quad \text{and} \quad L_C(\psi(b)) \leq \gamma L_B(b)$$

for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. It follows that for any $a \in \mathcal{A}$,

$$L_C(\psi \circ \phi(a)) = L_C(\psi(\phi(a))) \leq \gamma L_B(\phi(a)) \leq \lambda\gamma L_A(a) < \infty,$$

and hence $\psi \circ \phi$ is Lipschitz.

In particular, if we take $\lambda = L(\phi)$ and $\gamma = L(\psi)$, then by the definition of Lipschitz number we have $L(\psi \circ \phi) \leq L(\psi)L(\phi)$. ∎

From this proposition we see that the Lipschitz number is a submultiplicative seminorm under the composition of maps on the real vector space of all Lipschitz maps from an order unit space with a Lipschitz seminorm to itself.

PROPOSITION 2.10. *Let A and B be order unit spaces with Lipschitz seminorms L_A and L_B. Suppose that $\phi : A \to B$ is a unital positive linear map. Then $L(\hat{\phi}) \leq L(\phi)$. Moreover, if L_A and L_B are lower semicontinuous, then $L(\hat{\phi}) = L(\phi)$.*

Proof. By the definition of Lipschitz number we have

$$L_B(\phi(a)) \leq L(\phi)L_A(a)$$

for all $a \in \mathcal{A}$. It follows that for any $\mu, \nu \in \mathcal{S}(B)$, we have

$$\begin{aligned}
\rho_{L_A}(\hat{\phi}(\mu), \hat{\phi}(\nu)) &= \sup\{|\hat{\phi}(\mu)(a) - \hat{\phi}(\nu)(a)| : a \in A, L_A(a) \leq 1\} \\
&= \sup\{|\mu(\phi(a)) - \nu(\phi(a))| : a \in A, L_A(a) \leq 1\} \\
&\leq L(\phi) \sup\{|\mu(b) - \nu(b)| : b \in B, L_B(b) \leq 1\} \\
&= L(\phi)\rho_{L_B}(\mu, \nu),
\end{aligned}$$

and so $L(\hat{\phi}) \leq L(\phi)$.

Now suppose that the Lipschitz seminorms L_A and L_B are lower semicontinuous. Then for any $a \in A$ and $\mu, \nu \in \mathcal{S}(B)$, we have

$$\begin{aligned}
|\mu(\phi(a)) - \nu(\phi(a))| = |\hat{\phi}(\mu)(a) - \hat{\phi}(\nu)(a)| &\leq L_A(a)\rho_{L_A}(\hat{\phi}(\mu), \hat{\phi}(\nu)) \\
&\leq L(\hat{\phi})L_A(a)\rho_{L_B}(\mu, \nu).
\end{aligned}$$

Since L_B is lower semicontinuous, by [25, Theorem 4.1] it can be exactly recovered from $(\mathcal{S}(B), \rho_{L_B})$:

$$L_B(b) = \sup\left\{\frac{|\mu(b) - \nu(b)|}{\rho_{L_B}(\mu, \nu)} : \mu, \nu \in \mathcal{S}(B),\ \mu \neq \nu\right\} \quad \text{for all } b \in B.$$

Thus we have

$$L_B(\phi(a)) \leq L(\hat{\phi})L_A(a) \quad \text{for all } a \in \mathcal{A}.$$

Hence $L(\phi) \leq L(\hat{\phi})$. It follows that $L(\phi) = L(\hat{\phi})$. ∎

Let A be an order unit space, and let L be a Lipschitz seminorm on A. Denote

$$\mathcal{L}_1 = \{a \in A : L(a) \leq 1\}.$$

Let $\overline{\mathcal{L}_1}$ be the closure of $\mathcal{L}_1$ in $\bar{A}$, the completion of A for the order norm, and let $\bar{L}$ denote the corresponding "Minkowski functional" on $\bar{A}$:

$$\bar{L}(a) = \inf\{\beta > 0 : a \in \beta\overline{\mathcal{L}_1}\} \quad \text{for } a \in \bar{A}.$$

We call $\bar{L}$ the *closure* of L [25]. It is clear that $a \in \overline{\mathcal{L}_1}$ if and only if $\bar{L}(a) \leq 1$, i.e., $\bar{L}$ is lower semicontinuous on $\bar{A}$. For any $\varepsilon > 0$ and $a \in A$ with $L(a) < \infty$, we have

$$a \in (L(a) + \varepsilon)\mathcal{L}_1 \subseteq (L(a) + \varepsilon)\overline{\mathcal{L}_1}.$$

Thus $\bar{L}(a) \leq L(a) + \varepsilon$ for any $\varepsilon > 0$, and so $\bar{L}(a) \leq L(a)$. If L is lower semicontinuous, then for any $a \in A$ we have $\bar{L}(a) = L(a)$. Indeed, for any $\varepsilon > 0$ and $a \in A$ with $\bar{L}(a) < \infty$, we have $\bar{L}(a/(\bar{L}(a) + \varepsilon)) \leq 1$, and so there is a sequence $\{a_n\}_{n\in\mathbb{N}}$ in A with

$$\lim_{n\to\infty} a_n = \frac{a}{\bar{L}(a) + \varepsilon}$$

and $L(a_n) \leq 1$ for every n. Then from the lower semicontinuity of L,

$$L\left(\frac{a}{\bar{L}(a) + \varepsilon}\right) \leq \liminf_{n\to\infty} L(a_n) \leq 1,$$

that is, $L(a) \leq \bar{L}(a) + \varepsilon$ for any $\varepsilon > 0$, and thus $L(a) \leq \bar{L}(a)$.

The following proposition concerns the extension of a continuous Lipschitz linear map between order unit spaces with Lipschitz seminorms.

Proposition 2.11. *Let A and B be order unit spaces with Lipschitz seminorms L_A and L_B. Suppose that $\phi : A \to B$ is a continuous linear map. If ϕ is Lipschitz, then its unique uniformly continuous extension $\bar{\phi} : \bar{A} \to \bar{B}$ is Lipschitz and $L(\bar{\phi}) \leq L(\phi)$. Moreover, if L_A and L_B are lower semicontinuous, then $L(\bar{\phi}) = L(\phi)$.*

Proof. Since ϕ is a continuous linear map, it has a unique extension to a continuous linear map $\bar{\phi} : \bar{A} \to \bar{B}$. Now we need to check that $\bar{\phi}$ is Lipschitz and satisfies

$$L(\bar{\phi}) \leq L(\phi).$$

For any $\varepsilon > 0$ and $a \in \bar{A}$ with $\bar{L}_A(a) < \infty$, we have

$$\bar{L}_A\left(\frac{a}{\bar{L}_A(a) + \varepsilon}\right) \leq 1,$$

and so there is a sequence $\{a_n\}_{n\in\mathbb{N}}$ in A with

$$\lim_{n\to\infty} a_n = \frac{a}{\bar{L}_A(a) + \varepsilon}$$

and $L_A(a_n) \le 1$ for every n. Since $\bar{L}_A$ and $\bar{L}_B$ are lower semicontinuous, we have

$$\begin{aligned}\bar{L}_B\left(\bar{\phi}\left(\frac{a}{\bar{L}_A(a)+\varepsilon}\right)\right) &\le \liminf_{n\to\infty} \bar{L}_B(\phi(a_n)) \le \liminf_{n\to\infty} L_B(\phi(a_n)) \\ &\le L(\phi) \liminf_{n\to\infty} L_A(a_n) \le L(\phi),\end{aligned}$$

that is,

$$\bar{L}_B(\bar{\phi}(a)) \le L(\phi)(\bar{L}_A(a)+\varepsilon)$$

for any $\varepsilon > 0$, and hence

$$\bar{L}_B(\bar{\phi}(a)) \le L(\phi)\bar{L}_A(a)$$

for all $a \in \mathcal{A}$. It follows that $L(\bar{\phi}) \le L(\phi)$.

If L_A and L_B are lower semicontinuous, then for any $a \in \mathcal{A}$ we have

$$L_B(\phi(a)) = \bar{L}_B(\bar{\phi}(a)) \le L(\bar{\phi})\bar{L}_A(a) = L(\bar{\phi})L_A(a).$$

It follows that $L(\phi) \le L(\bar{\phi})$. Therefore, $L(\bar{\phi}) = L(\phi)$. ∎

3. Lipschitz isomorphisms from asymptotically nonexpansive maps. Let (X, ρ_X) and (Y, ρ_Y) be metric spaces, and let k be a positive real number. A map $f : X \to Y$ is said to be *k-nonexpansive* if

$$\rho_Y(f(p), f(q)) \le k\rho_X(p,q) \quad \text{for all } p, q \in X.$$

Definition 3.1. Let (X, ρ_X) be a metric space, and let $k > 0$. A map $f : X \to X$ is said to be *asymptotically k-nonexpansive* if there is a sequence $\{k_i\}_{i\in\mathbb{N}}$ of positive real numbers with $k_i \to k$ as $i \to \infty$ such that

$$\rho_X(f^i(p), f^i(q)) \le k_i\rho_X(p,q) \quad \text{for all } p, q \in X \text{ and } i \in \mathbb{N},$$

where f^i is the i-fold iteration of f.

From these definitions we can easily see that (asymptotically) k-nonexpansive maps between metric spaces are uniformly continuous. Note that (asymptotically) 1-nonexpansive maps are the usual (asymptotically) nonexpansive maps [7, 20].

Definition 3.2. Let $k > 0$, and let A and B be order unit spaces with Lipschitz seminorms L_A and L_B. A map $\phi : A \to B$ is said to be *k-nonexpansive* if for any $a \in \mathcal{A}$ we have

$$L_B(\phi(a)) \le kL_A(a).$$

A map $\phi : A \to A$ is said to be *asymptotically k-nonexpansive* if there is a sequence $\{k_i\}_{i\in\mathbb{N}}$ of positive real numbers with $k_i \to k$ as $i \to \infty$ such that

$$L_A(\phi^i(a)) \le k_iL_A(a) \quad \text{for all } a \in \mathcal{A} \text{ and } i \in \mathbb{N}.$$

Remark 3.3. Let $k > 0$. One can define a map f from a metric space (X, ρ) to itself to be *uniformly k-Lipschitz* if

$$\rho(f^i(p), f^i(q)) \le k\rho(p, q) \quad \text{for all } p, q \in X \text{ and } i \in \mathbb{N}.$$

Similarly, one can define a map ϕ from an order unit space A with a Lipschitz seminorm L to itself to be *uniformly k-Lipschitz* if

$$L(\phi^i(a)) \le kL(a) \quad \text{for all } a \in \mathcal{A} \text{ and } i \in \mathbb{N}.$$

If $\{k_i\}_{i\in\mathbb{N}}$ is a sequence of positive real numbers with $k_i \to k$ as $i \to \infty$, then

$$k_i \le \sup_{i\in\mathbb{N}} k_i$$

for all $i \in \mathbb{N}$, and hence each asymptotically k-nonexpansive map of a metric space (or an order unit space with a Lipschitz seminorm) is uniformly $\sup_{i\in\mathbb{N}} k_i$-Lipschitz. Conversely, each uniformly k-Lipschitz map of a metric space (or an order unit space with a Lipschitz seminorm) is asymptotically k-nonexpansive since one can take $k_i = k$ for all $i \in \mathbb{N}$. Therefore, these two definitions are equivalent in some sense.

In this paper we prefer the notion of asymptotically k-nonexpansive to that of uniformly k-Lipschitz. On the one hand, there are sequences $\{k_i\}_{i\in\mathbb{N}}$ with $k_i \to k$ such that

$$k < \sup_{i\in\mathbb{N}} k_i,$$

as in [20, Example 3.3]; on the other hand, one can get a better/sharp positive real number in Proposition 3.7 and Theorem 3.9 for asymptotically k-nonexpansive maps.

Proposition 3.4. *Let A be a complete order unit space with lower semicontinuous Lipschitz seminorm L, and let $k > 0$. Suppose that $\phi : A \to A$ is a unital positive linear map. If $(\mathcal{S}(A), \rho_L)$ is a bounded metric space, then the following statements are equivalent:*

(1) *ϕ is asymptotically k-nonexpansive;*
(2) *$\hat{\phi}$ is an asymptotically k-nonexpansive map from A'^o to itself with respect to L';*
(3) *$\hat{\phi}$ is asymptotically k-nonexpansive from $(\mathcal{S}(A), \rho_L)$ to itself;*
(4) *there is a $\lambda > 0$ such that ϕ is asymptotically λ-nonexpansive from $(\mathcal{A}, \|\cdot\|_1)$ to itself.*

Proof. (1)⇒(2). If there is a sequence $\{k_i\}_{i\in\mathbb{N}}$ with $k_i \to k$ as $i \to \infty$ such that

$$L(\phi^i(a)) \le k_i L(a)$$

for all $i \in \mathbb{N}$ and $a \in \mathcal{A}$, then for any $f \in A'^o$ we have

$$\begin{aligned} L'((\hat{\phi})^i(f)) &= \sup\{|(\hat{\phi})^i(f)(a)| : a \in A,\ L(a) \leq 1\} \\ &= \sup\{|f(\phi^i(a))| : a \in A,\ L(a) \leq 1\} \\ &\leq k_i \sup\{|f(b)| : b \in A,\ L(b) \leq 1\} = k_i L'(f), \end{aligned}$$

proving (2).

(2)⇒(3). If there is a sequence $\{k_i\}_{i\in\mathbb{N}}$ with $k_i \to k$ as $i \to \infty$ such that

$$L'((\hat{\phi})^i(f)) \leq k_i L'(f)$$

for all $i \in \mathbb{N}$ and $f \in A'^o$, then by [25, Lemma 4.3] we have

$$\begin{aligned} \rho_L((\hat{\phi})^i(\mu), (\hat{\phi})^i(\nu)) &= L'((\hat{\phi})^i(\mu) - (\hat{\phi})^i(\nu)) = L'((\hat{\phi})^i(\mu - \nu)) \\ &\leq k_i L'(\mu - \nu) = k_i \rho_L(\mu, \nu) \end{aligned}$$

for all $\mu, \nu \in \mathcal{S}(A)$, which yields (3).

(3)⇒(1). If there is a sequence $\{k_i\}_{i\in\mathbb{N}}$ with $k_i \to k$ as $i \to \infty$ such that

$$\rho_L((\hat{\phi})^i(\mu), (\hat{\phi})^i(\nu)) \leq k_i \rho_L(\mu, \nu)$$

for all $i \in \mathbb{N}$ and $\mu, \nu \in \mathcal{S}(A)$, then any $a \in \mathcal{A}$, $i \in \mathbb{N}$ and $\mu, \nu \in \mathcal{S}(A)$, we have

$$\begin{aligned} |\mu(\phi^i(a)) - \nu(\phi^i(a))| &= |(\hat{\phi})^i(\mu)(a) - (\hat{\phi})^i(\nu)(a)| \\ &\leq L(a)\rho_L((\hat{\phi})^i(\mu), (\hat{\phi})^i(\nu)) \leq k_i L(a) \rho_L(\mu, \nu). \end{aligned}$$

Since L is lower semicontinuous, by [25, Theorem 4.1] it can be exactly recovered from $(\mathcal{S}(A), \rho_L)$, that is,

$$L(b) = \sup\left\{ \frac{|\mu(b) - \nu(b)|}{\rho_L(\mu, \nu)} : \mu, \nu \in \mathcal{S}(A),\ \mu \neq \nu \right\}$$

for all $b \in A$. Thus

$$L(\phi^i(a)) \leq k_i L(a) < \infty$$

for all $a \in \mathcal{A}$ and $i \in \mathbb{N}$, proving (1).

(1)⇒(4). If there is a sequence $\{k_i\}_{i\in\mathbb{N}}$ with $k_i \to k$ as $i \to \infty$ such that

$$L(\phi^i(a)) \leq k_i L(a)$$

for all $i \in \mathbb{N}$ and $a \in \mathcal{A}$, then for any $i \in \mathbb{N}$ and $a \in \mathcal{A}$ we have

$$\begin{aligned} \|\phi^i(a)\|_1 &= \|\phi^i(a)\| + L(\phi^i(a)) \leq \|a\| + k_i L(a) \\ &\leq \max\{1, k_i\}(\|a\| + L(a)) = \max\{1, k_i\}\|a\|_1. \end{aligned}$$

Hence ϕ is asymptotically $\max\{1, k\}$-nonexpansive from $(\mathcal{A}, \|\cdot\|_1)$ to itself.

(4)⇒(1). If there is a sequence $\{\lambda_i\}_{i\in\mathbb{N}}$ with $\lambda_i \to \lambda$ as $i \to \infty$ such that

$$\|\phi^i(a)\|_1 \leq \lambda_i \|a\|_1$$

for all $i \in \mathbb{N}$ and $a \in \mathcal{A}$, then

$$L(\phi^i(a)) \leq \|\phi^i(a)\| + L(\phi^i(a)) \leq \lambda_i(\|a\| + L(a))$$

for all $i \in \mathbb{N}$ and $a \in \mathcal{A}$. It follows that for any $i \in \mathbb{N}$, $\gamma \in \mathbb{R}$ and $a \in \mathcal{A}$,

$$\begin{aligned}L(\phi^i(a)) &= L(\phi^i(a) + \gamma 1_A) = L(\phi^i(a + \gamma 1_A)) \\ &\le \|\phi^i(a+\gamma 1_A)\| + L(\phi^i(a+\gamma 1_A)) \\ &\le \lambda_i(\|a+\gamma 1_A\| + L(a+\gamma 1_A)) = \lambda_i(\|a+\gamma 1_A\| + L(a)).\end{aligned}$$

Hence

$$L(\phi^i(a)) \le \lambda_i(\|\tilde{a}\|^{\sim} + L(a))$$

for all $i \in \mathbb{N}$ and $a \in \mathcal{A}$, where $\|\cdot\|^{\sim}$ is the quotient norm on $\mathcal{A}/(\mathbb{R}1_A)$ with respect to the order norm $\|\cdot\|$. By [25, Proposition 2.2],

$$\|\tilde{a}\|^{\sim} \le L(a)\,\mathrm{diam}(\mathcal{S}(A), \rho_L) < \infty,$$

for all $a \in \mathcal{A}$ since $(\mathcal{S}(A), \rho_L)$ is a bounded metric space. Then

$$L(\phi^i(a)) \le \lambda_i(\|\tilde{a}\|^{\sim} + L(a)) \le \lambda_i(\mathrm{diam}(\mathcal{S}(A), \rho_L) + 1)L(a)$$

for all $i \in \mathbb{N}$ and $a \in \mathcal{A}$. Set

$$k_i = \lambda_i(\mathrm{diam}(\mathcal{S}(A), \rho_L) + 1) > 0.$$

Then

$$L(\phi^i(a)) \le k_i L(a)$$

for all $i \in \mathbb{N}$ and $a \in \mathcal{A}$, and

$$k_i \to k = \lambda(\mathrm{diam}(\mathcal{S}(A), \rho_L) + 1) > 0 \quad \text{as } i \to \infty. \ \blacksquare$$

Let (X, ρ_X) and (Y, ρ_Y) be metric spaces, and let $k > 0$. A map $f : X \to Y$ is said to be *k-expansive* if

$$\rho_Y(f(p), f(q)) \ge k\rho_X(p, q) \quad \text{for all } p, q \in X.$$

LEMMA 3.5. *Let (X, ρ) be a compact metric space, and let $f : X \to X$ be an asymptotically k-nonexpansive surjective map about ρ. Then f is $\frac{1}{k}$-expansive.*

Proof. There is a sequence $\{k_i\}_{i\in\mathbb{N}}$ with $k_i \to k$ as $i \to \infty$ such that for any $i \in \mathbb{N}$ and $x, y \in X$,

$$\rho(f^i(x), f^i(y)) \le k_i \rho(x, y).$$

For any $x_0, y_0 \in X$, since f is surjective, there are sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ in X such that $f(x_{n+1}) = x_n$ and $f(y_{n+1}) = y_n$ for $n = 0, 1, 2, \ldots$. Since (X, ρ) is compact, there exist convergent subsequences $\{x_{n_i}\}_{i\in\mathbb{N}}$ and $\{y_{n_i}\}_{i\in\mathbb{N}}$. For any $\varepsilon > 0$, there is an integer N such that for any $i \ge N$, we have

$$\rho(x_{n_i}, x_{n_{i+j}}) < \varepsilon \quad \text{and} \quad \rho(y_{n_i}, y_{n_{i+j}}) < \varepsilon,$$

for any $j \in \mathbb{N}$. Fix an $i \geq N$ and define $p_j = n_{i+j} - n_i$ for $j \in \mathbb{N}$. Obviously $p_1 < p_2 < \cdots$ and $p_j \to \infty$ as $j \to \infty$. Since

$$\begin{aligned}\rho(x_{n_i}, x_{n_{i+j}}) &= \rho(x_{n_i}, x_{n_i+p_j})\\ &\geq \frac{1}{k_{n_{i+j}}}\rho(f^{n_i+p_j}(x_{n_i}), f^{n_{i+j}}(x_{n_{i+j}})) = \frac{1}{k_{n_{i+j}}}\rho(f^{p_j}(x_0), x_0)\end{aligned}$$

for any $j \in \mathbb{N}$, we have

$$\rho(f^{p_j}(x_0), x_0) \leq k_{n_{i+j}}\rho(x_{n_i}, x_{n_{i+j}}) \leq \gamma\rho(x_{n_i}, x_{n_{i+j}}) < \gamma\varepsilon$$

for any $j \in \mathbb{N}$, where $\gamma = \sup\{k_i : i \in \mathbb{N}\}$. Similarly,

$$\rho(f^{p_j}(y_0), y_0) \leq \gamma\rho(y_{n_i}, y_{n_{i+j}}) < \gamma\varepsilon$$

for any $j \in \mathbb{N}$. Thus

$$\begin{aligned}\rho(x_0, y_0) &\leq \rho(x_0, f^{p_j}(x_0)) + \rho(f^{p_j}(x_0), f^{p_j}(y_0)) + \rho(f^{p_j}(y_0), y_0)\\ &\leq k_{p_j-1}\rho(f(x_0), f(y_0)) + 2\gamma\varepsilon.\end{aligned}$$

Letting $j \to \infty$ gives

$$\rho(x_0, y_0) \leq k\rho(f(x_0), f(y_0)) + 2\gamma\varepsilon$$

for any $\varepsilon > 0$, and hence $\rho(x_0, y_0) \leq k\rho(f(x_0), f(y_0))$. ∎

LEMMA 3.6. *Let (X, ρ) be a compact metric space, and let $f : X \to X$ be an asymptotically k-nonexpansive surjective map. Then f is k-nonexpansive.*

Proof. For any $x_0, y_0 \in X$ and $\varepsilon > 0$, as in the proof of Lemma 3.5, there is a sequence $\{p_i\}_{i\in\mathbb{N}}$ with $p_i \to \infty$ as $i \to \infty$ such that

$$\rho(x_0, f^{p_i}(x_0)) < \gamma\varepsilon \quad \text{and} \quad \rho(f^{p_i}(y_0), y_0) < \gamma\varepsilon,$$

for any $i \in \mathbb{N}$, where $\gamma = \sup\{k_i : i \in \mathbb{N}\}$. It follows that for any $i \in \mathbb{N}$,

$$\begin{aligned}\rho(f(x_0), f(y_0)) &\leq \rho(f(x_0), f^{p_i+1}(x_0)) + \rho(f^{p_i+1}(x_0), f^{p_i+1}(y_0))\\ &\quad + \rho(f^{p_i+1}(y_0), f(y_0))\\ &\leq k_1\rho(x_0, f^{p_i}(x_0)) + k_{p_i+1}\rho(x_0, y_0) + k_1\rho(y_0, f^{p_i}(y_0))\\ &\leq 2k_1\gamma\varepsilon + k_{p_i+1}\rho(x_0, y_0).\end{aligned}$$

Letting $i \to \infty$, we have

$$\rho(f(x_0), f(y_0)) \leq 2k_1\gamma\varepsilon + k\rho(x_0, y_0).$$

Since this holds for every $\varepsilon > 0$, we can see that $\rho(f(x_0), f(y_0)) \leq k\rho(x_0, y_0)$, as desired. ∎

From Lemmas 3.5 and 3.6 we immediately have the following:

PROPOSITION 3.7. *Let (X, ρ) be a compact metric space, and let $f : X \to X$ be an asymptotically k-nonexpansive surjective map.*

(1) *If $0 < k < 1$, then X is a singleton.*

(2) *If $k \geq 1$, then f is a bi-Lipschitz map. More precisely,*

$$\frac{1}{k}\rho(p,q) \leq \rho(f(p), f(q)) \leq k\rho(p,q) \quad \textit{for any } p, q \in X.$$

Proof. If $0 < k < 1$, then for any $p, q \in X$, by Lemmas 3.5 and 3.6 we have

$$\frac{1}{k}\rho(p,q) \leq \rho(f(p), f(q)) \leq k\rho(p,q).$$

This implies that $p = q$, and hence X is a singleton.

If $k \geq 1$, then the conclusion follows from Lemmas 3.5 and 3.6. ∎

Now we turn to the more general case of compact quantum metric spaces.

Recall in [10, 20, 25, 27, 29, 37] that a *Lip-norm* on an order unit space A is a Lipschitz seminorm L such that the topology on $\mathcal{S}(A)$ induced by the metric

$$\rho_L(\mu,\nu) = \sup\{|\mu(a) - \nu(a)| : a \in A,\ L(a) \leq 1\}, \qquad \mu,\nu \in \mathcal{S}(A),$$

coincides with the underlying weak*-topology.

DEFINITION 3.8. Let A be a complete order unit space. If there exists a Lip-norm L on A, we say that the pair (A, L) is a *compact quantum metric space.*

If (A, L) is a compact quantum metric space, then $(\mathcal{S}(A), \rho_L)$ is a compact metric space, and so it is bounded. Unfortunately, in general, the extended metric space $(\mathcal{S}(A), \rho_L)$ for a Lipschitz seminorm L on an order unit space A need not be bounded [24, 25].

THEOREM 3.9. *Let (A, L) be a compact quantum metric space with lower semicontinuous Lip-norm, and let $k \in [1, \infty)$. If $\phi : A \to A$ is a unital positive, asymptotically k-nonexpansive and isometric linear map, then ϕ is bi-Lipschitz from (A, L) onto itself. Moreover,*

$$\frac{1}{k}L(a) \leq L(\phi(a)) \leq kL(a) \quad \textit{for any } a \in \mathcal{A}.$$

Proof. Since (A, L) is a compact quantum metric space, $(\mathcal{S}(A), \rho_L)$ is a compact metric space. Thus by [20, Lemma 3.7] and Proposition 3.4, the induced map $\hat{\phi}$ is asymptotically k-nonexpansive map from the compact metric space $(\mathcal{S}(A), \rho_L)$ onto itself, and then by Proposition 3.7, $\hat{\phi}$ is bi-Lipschitz with

$$\frac{1}{k}\rho_L(\mu,\nu) \leq \rho_L(\hat{\phi}(\mu), \hat{\phi}(\nu)) \leq k\rho_L(\mu,\nu)$$

for all $\mu, \nu \in \mathcal{S}(A)$. In particular, $\hat{\phi}$ is an affine homeomorphism on $(\mathcal{S}(A), \rho_L)$.

For any $b \in A$ we have

$$\hat{b} \circ \hat{\phi}^{-1} \in \mathrm{Af}(\mathcal{S}(A)).$$

So there is an $a \in A$ such that $\hat{a} = \hat{b} \circ \hat{\phi}^{-1}$ by the Kadison representation theorem. Now for any $\mu \in \mathcal{S}(A)$, there is a $\nu \in \mathcal{S}(A)$ such that $\mu = \hat{\phi}(\nu)$ since $\hat{\phi}$ is surjective. Thus

$$\begin{aligned}\widehat{\phi(a)}(\mu) &= \widehat{\phi(a)}(\hat{\phi}(\nu)) = \hat{\phi}(\nu)(\phi(a)) = \hat{a}(\hat{\phi}(\nu) \circ \phi) = (\hat{b} \circ \hat{\phi}^{-1})(\hat{\phi}(\nu) \circ \phi) \\ &= \hat{b}(\nu \circ \phi) = \hat{b}(\hat{\phi}(\nu)) = \hat{b}(\mu),\end{aligned}$$

i.e., $\phi(a) = b$. Hence ϕ is surjective.

For any $a \in A$ and $\mu \in \mathcal{S}(A)$ we have

$$\widehat{\phi(a)}(\mu) = \mu(\phi(a)) = \hat{\phi}(\mu)(a) = \hat{a}(\hat{\phi}(\mu)).$$

So $\widehat{\phi(a)} \geq 0$ if and only if $\hat{a} \geq 0$. From the Kadison representation theorem we see that $\phi(a) \geq 0$ if and only if $a \geq 0$. Therefore, ϕ is a unital order isomorphism from A onto itself.

For any $a \in A$ and $\mu, \nu \in \mathcal{S}(A)$, we have

$$\begin{aligned}|\mu(\phi(a)) - \nu(\phi(a))| &= |\hat{\phi}(\mu)(a) - \hat{\phi}(\nu)(a)| \leq L(a)\rho_L(\hat{\phi}(\mu), \hat{\phi}(\nu)) \\ &\leq kL(a)\rho_L(\mu, \nu).\end{aligned}$$

Since L is lower semicontinuous, by [25, Theorem 4.1] it can be exactly recovered from $(\mathcal{S}(A), \rho_L)$:

$$L(b) = \sup\left\{\frac{|\mu(b) - \nu(b)|}{\rho_L(\mu, \nu)} : \mu, \nu \in \mathcal{S}(A),\ \mu \neq \nu\right\}$$

for all $b \in A$. Thus

$$L(\phi(a)) \leq kL(a) < \infty$$

for any $a \in \mathcal{A}$. It follows that ϕ is k-nonexpansive. Similarly, since the affine map $(\hat{\phi})^{-1} = \widehat{\phi^{-1}}$ is also k-nonexpansive from $(\mathcal{S}(A), \rho_L)$ to itself, we have

$$L(\phi^{-1}(a)) \leq kL(a)$$

for any $a \in \mathcal{A}$. So we have

$$\frac{1}{k}L(a) \leq L(\phi(a)) \leq kL(a)$$

for any $a \in \mathcal{A}$. Thus ϕ is a bi-Lipschitz map from (A, L) onto (A, L). ∎

Recall that for compact quantum metric spaces (A, L_A) and (B, L_B), if there is a unital order isomorphism ϕ from A to B such that both ϕ and ϕ^{-1} are Lipschitz isometric, we say that (A, L_A) and (B, L_B) are *Lipschitz isometric* [10, 18–21, 27, 37].

Corollary 3.10 ([20, Theorem 3.8]). *Let (A, L) be a compact quantum metric space with lower semicontinuous Lip-norm. If $\phi : A \to A$ is a unital positive and isometric linear map, then ϕ is asymptotically 1-nonexpansive if and only if it is a Lipschitz isometry from (A, L) onto itself.*

Proof. Suppose that ϕ is asymptotically 1-nonexpansive. By Theorem 3.9, ϕ is a bi-Lipschitz map from (A, L) onto itself. Moreover, for any $a \in \mathcal{A}$,

$$L(a) \leq L(\phi(a)) \leq L(a).$$

Thus ϕ is Lipschitz isometric. Similarly, ϕ^{-1} is also Lipschitz isometric. Therefore, ϕ is a Lipschitz isometry from (A, L) onto itself.

The proof in the other direction is obvious. ■

COROLLARY 3.11. *Let (A, L) be a compact quantum metric space with lower semicontinuous Lip-norm, and let $k \in [1, \infty)$. If $\phi : A \to A$ is a unital positive and asymptotically k-nonexpansive linear map such that $\hat{\phi}$ is surjective from $\mathcal{S}(A)$ to itself, then ϕ is bi-Lipschitz from (A, L) onto itself. Moreover,*

$$\frac{1}{k} L(a) \leq L(\phi(a)) \leq kL(a) \quad \textit{for any } a \in \mathcal{A}.$$

Proof. By [20, Lemma 3.7] the map ϕ is isometric, thus we have the conclusion by Theorem 3.9. ■

COROLLARY 3.12. *Let (A, L) be a compact quantum metric space with lower semicontinuous Lip-norm, and let $k \in [1, \infty)$. If $\phi : A \to A$ is a unital positive and isometric linear map such that $\hat{\phi}$ is asymptotically k-nonexpansive from $(\mathcal{S}(A), \rho_L)$ to itself, then ϕ is bi-Lipschitz from (A, L) onto itself. Moreover,*

$$\frac{1}{k} L(a) \leq L(\phi(a)) \leq kL(a) \quad \textit{for any } a \in \mathcal{A}.$$

Proof. Since (A, L) is a compact quantum metric space, $(\mathcal{S}(A), \rho_L)$ is a compact metric space. It follows from the proof of Proposition 3.4 that ϕ is an asymptotically k-nonexpansive map from the compact quantum metric space (A, L) to itself since $\hat{\phi}$ is asymptotically k-nonexpansive from $(\mathcal{S}(A), \rho_L)$ to itself. Thus we have the conclusion by Theorem 3.9. ■

COROLLARY 3.13. *Let (A, L) be a compact quantum metric space with lower semicontinuous Lip-norm, and let $k \in [1, \infty)$. If $\phi : A \to A$ is a unital positive linear map such that $\hat{\phi}$ is surjective and asymptotically k-nonexpansive from $(\mathcal{S}(A), \rho_L)$ to itself, then ϕ is bi-Lipschitz from (A, L) onto itself. Moreover,*

$$\frac{1}{k} L(a) \leq L(\phi(a)) \leq kL(a) \quad \textit{for any } a \in \mathcal{A}.$$

Proof. By [20, Lemma 3.7] the map ϕ is isometric, thus we have the conclusion by Corollary 3.12. ■

EXAMPLE 3.14. Let $(A, \mathcal{H}, D)$ be a spectral triple such that $(A, \|[D, \cdot]\|)$, or more precisely $(A_{sa}, \|[D, \cdot]\|)$, is a compact quantum metric space. Let $\phi : A \to A$ be a unital and injective *-homomorphism such that there is a sequence $\{k_i\}_{i\in\mathbb{N}}$ of positive real numbers with $k_i \to k \geq 1$ as $i \to \infty$ such that

$$L_D(\phi^i(a)) = \|[D, \phi^i(a)]\| \leq k_i L_D(a)$$

for all $a \in \mathcal{A}$ and $i \in \mathbb{N}$. By Theorem 3.9, ϕ is a bi-Lipschitz map from (A_{sa}, L_D) onto itself such that

$$\frac{1}{k} L_D(a) \leq L_D(\phi(a)) \leq k L_D(a)$$

for all $a \in \mathcal{A}_{sa}$. Moreover, ϕ is a *-isomorphism from A onto itself. Since L_D is a *-seminorm on A, we have

$$\frac{1}{2k} L_D(a) \leq L_D(\phi(a)) \leq 2k L_D(a)$$

for all $a \in \mathcal{A}$. For any $i \geq 2$ and $a \in \mathcal{A}$, we have

$$\frac{1}{k_i} L_D(a) \leq L_D(\phi^i(a)) \leq k_i L_D(a).$$

Thus the *-isomorphism ϕ induces a metrically equicontinuous action α of $\mathbb{Z}$ on the compact quantum metric space (A_{sa}, L_D) [8]. Therefore, by [8, Theorem 2.11], there is a compact quantum metric space structure on the reduced crossed product C^*-algebra $A \rtimes_{\alpha,r} \mathbb{Z}$.

4. Lipschitz isomorphisms from asymptotically expansive maps

Definition 4.1. Let (X, ρ_X) be a metric space, and let $k > 0$. A map $f : X \to X$ is said to be *asymptotically k-expansive* if there is a sequence $\{k_i\}_{i\in\mathbb{N}}$ of positive real numbers with $k_i \to k$ as $i \to \infty$ such that

$$\rho_X(f^i(p), f^i(q)) \geq k_i \rho_X(p, q) \quad \text{for all } p, q \in X \text{ and } i \in \mathbb{N}.$$

From the definition we can easily find that (asymptotically) k-expansive maps between metric spaces are injective. Note that asymptotically k-expansive maps were called H-expansive in the case of compact metric spaces [38], and that [asymptotically] 1-expansive maps between metric spaces are the usual [asymptotically] expansive maps [9, 20].

Definition 4.2. Let $k > 0$, and let A and B be order unit spaces with Lipschitz seminorms L_A and L_B. A map $\phi : A \to B$ is said to be *k-expansive* if

$$L_B(\phi(a)) \geq k L_A(a) \quad \text{for any } a \in \mathcal{A}.$$

A map $\phi : A \to A$ is said to be *asymptotically k-expansive* if there is a sequence $\{k_i\}_{i\in\mathbb{N}}$ of positive real numbers with $k_i \to k$ as $i \to \infty$ such that

$$L_A(\phi^i(a)) \geq k_i L_A(a) \quad \text{for all } a \in \mathcal{A} \text{ and } i \in \mathbb{N}.$$

Proposition 4.3. *Let $k > 0$, and let A and B be order unit spaces with lower semicontinuous Lipschitz seminorms L_A and L_B. Suppose that $\phi : A \to B$ is a unital positive linear map, and that $(\mathcal{S}(A), \rho_{L_A})$ and $(\mathcal{S}(B), \rho_{L_B})$*

are bounded metric spaces. Then the following statements are equivalent:

(1) $\hat{\phi}$ *is* k*-expansive from* B'^o *to* A'^o *relative to* L'_B *and* L'_A*;*
(2) $\hat{\phi}$ *is* k*-expansive from* $(\mathcal{S}(B), \rho_{L_B})$ *to* $(\mathcal{S}(A), \rho_{L_A})$*.*

Proof. Suppose that $\hat{\phi}$ is a k-expansive map from B'^o to A'^o relative to L'_B and L'_A. By [25, Lemma 4.3], we have

$$\begin{aligned}\rho_{L_A}(\hat{\phi}(\mu), \hat{\phi}(\nu)) &= L'_A(\hat{\phi}(\mu) - \hat{\phi}(\nu)) = L'_A(\hat{\phi}(\mu - \nu)) \\ &\geq kL'_B(\mu - \nu) = k\rho_{L_B}(\mu, \nu)\end{aligned}$$

for all $\mu, \nu \in \mathcal{S}(B)$, proving (2).

Conversely, suppose $\hat{\phi}$ is k-expansive from $(\mathcal{S}(B), \rho_{L_B})$ to $(\mathcal{S}(A), \rho_{L_A})$. For any $f \in B'^o$, we have $\|f/(1 + \|f\|)\| \leq 1$ and $f/(1 + \|f\|) \in D_2$, where D_2 is the ball of radius 2 about 0 in B'^o. By [25, Lemma 2.1] there are $\mu, \nu \in \mathcal{S}(B)$ such that $f/(1 + \|f\|) = \mu - \nu$. It follows from [25, Lemma 4.3] that

$$\begin{aligned}L'_A\bigg(\hat{\phi}\Big(\frac{f}{1 + \|f\|}\Big)\bigg) &= L'_A(\hat{\phi}(\mu - \nu)) = L'_A(\hat{\phi}(\mu) - \hat{\phi}(\nu)) = \rho_{L_A}(\hat{\phi}(\mu), \hat{\phi}(\nu)) \\ &\geq k\rho_{L_B}(\mu, \nu) = kL'_B(\mu - \nu) = kL'_B\Big(\frac{f}{1 + \|f\|}\Big).\end{aligned}$$

Thus $L'_A(\hat{\phi}(f)) \geq kL'_B(f)$ for all $f \in B'^o$, proving (1). ∎

PROPOSITION 4.4. *Let* $k > 0$*, and let* A *and* B *be order unit spaces with lower semicontinuous Lipschitz seminorms* L_A *and* L_B*. Suppose that* $\phi : A \to B$ *is a unital positive linear map, and that* $(\mathcal{S}(A), \rho_{L_A})$ *and* $(\mathcal{S}(B), \rho_{L_B})$ *are bounded metric spaces.*

(1) *If* ϕ *is* k*-expansive with* $\phi(\mathcal{A}) = \mathcal{B}$*, then* $\hat{\phi}$ *is* k*-expansive from* $(\mathcal{S}(B), \rho_{L_B})$ *to* $(\mathcal{S}(A), \rho_{L_A})$*.*
(2) *If* $\hat{\phi}$ *is* k*-expansive and surjective from* $(\mathcal{S}(B), \rho_{L_B})$ *to* $(\mathcal{S}(A), \rho_{L_A})$*, then* ϕ *is* k*-expansive.*

Proof. Suppose ϕ is k-expansive with $\phi(\mathcal{A}) = \mathcal{B}$. Then for any $f \in B'^o$,

$$\begin{aligned}L'_B(f) &= \sup\{|f(b)| : b \in B,\ L_B(b) \leq 1\} \\ &\leq \sup\{|f(\phi(a))| : a \in A,\ L_A(a) \leq 1/k\} \\ &= \sup\{|\hat{\phi}(f)(a)| : a \in A,\ L_A(a) \leq 1/k\} = \frac{1}{k}L'_A(\hat{\phi}(f)).\end{aligned}$$

Thus $\hat{\phi}$ is k-expansive from B'^o to A'^o relative to L'_B and L'_A. From Proposition 4.3 we see that $\hat{\phi}$ is k-expansive map from $(\mathcal{S}(B), \rho_{L_B})$ to $(\mathcal{S}(A), \rho_{L_A})$.

Suppose $\hat{\phi}$ is k-expansive and surjective from $(\mathcal{S}(B), \rho_{L_B})$ to $(\mathcal{S}(A), \rho_{L_A})$. Since the Lipschitz seminorms L_A and L_B are lower semicontinuous, by [25, Theorem 4.1] they can be exactly recovered from $(\mathcal{S}(A), \rho_{L_A})$ and $(\mathcal{S}(B), \rho_{L_B})$, respectively. Then

$$\begin{aligned}
kL_A(a) &= k\sup\left\{\frac{|\mu'(a)-\nu'(a)|}{\rho_{L_A}(\mu',\nu')} : \mu',\nu' \in \mathcal{S}(A),\ \mu' \neq \nu'\right\} \\
&= k\sup\left\{\frac{|\hat\phi(\mu)(a)-\hat\phi(\nu)(a)|}{\rho_{L_A}(\hat\phi(\mu),\hat\phi(\nu))} : \mu,\nu \in \mathcal{S}(B),\ \hat\phi(\mu) \neq \hat\phi(\nu)\right\} \\
&\le \sup\left\{\frac{|\mu(\phi(a))-\nu(\phi(a))|}{\rho_{L_A}(\hat\phi(\mu),\hat\phi(\nu))}\,\frac{\rho_{L_A}(\hat\phi(\mu),\hat\phi(\nu))}{\rho_{L_B}(\mu,\nu)} : \mu,\nu \in \mathcal{S}(B),\ \mu \neq \nu\right\} \\
&= \sup\left\{\frac{|\mu(\phi(a))-\nu(\phi(a))|}{\rho_{L_B}(\mu,\nu)} : \mu,\nu \in \mathcal{S}(B),\ \mu \neq \nu\right\} = L_B(\phi(a))
\end{aligned}$$

for all $a \in \mathcal{A}$. Hence ϕ is k-expansive. ∎

Proposition 4.5. *Let A be an order unit space with lower semicontinuous Lipschitz seminorm L, and let $k > 0$. Suppose that $\phi : A \to A$ is a unital positive linear map, and that $(\mathcal{S}(A), \rho_{L_A})$ is a bounded metric space. Then the following statements are equivalent:*

(1) *$\hat\phi$ is asymptotically k-expansive map from A'^o to itself relative to L'.*
(2) *$\hat\phi$ is asymptotically k-expansive from $(\mathcal{S}(A),\ \rho_L)$ to itself.*

Proof. If there is a sequence $\{k_i\}_{i\in\mathbb{N}}$ with $k_i \to k$ as $i \to \infty$ such that

$$L'((\hat\phi)^i(f)) \ge k_i L'(f),$$

for all $i \in \mathbb{N}$ and $f \in A'^o$, then by [25, Lemma 4.3] we have

$$\begin{aligned}
\rho_L((\hat\phi)^i(\mu),(\hat\phi)^i(\nu)) &= L'((\hat\phi)^i(\mu)-(\hat\phi)^i(\nu)) = L'((\hat\phi)^i(\mu-\nu)) \\
&\ge k_i L'(\mu-\nu) = k_i\rho_L(\mu,\nu)
\end{aligned}$$

for all $i \in \mathbb{N}$ and $\mu,\nu \in \mathcal{S}(A)$, proving (2).

Conversely, suppose there is a sequence $\{k_i\}_{i\in\mathbb{N}}$ with $k_i \to k$ as $i \to \infty$ such that

$$\rho_L((\hat\phi)^i(\mu),(\hat\phi)^i(\nu)) \ge k_i\rho_L(\mu,\nu)$$

for all $i \in \mathbb{N}$ and $\mu,\nu \in \mathcal{S}(A)$. For any $f \in A'^o$, we have $\|f/(1+\|f\|)\| \le 1$ and $f/(1+\|f\|) \in D_2$. By [25, Lemma 2.1] there are $\mu,\nu \in \mathcal{S}(A)$ such that $f/(1+\|f\|) = \mu-\nu$. It follows from [25, Lemma 4.3] that

$$\begin{aligned}
L'\left((\hat\phi)^i\left(\frac{f}{1+\|f\|}\right)\right) &= L'_A((\hat\phi)^i(\mu-\nu)) = L'((\hat\phi)^i(\mu)-(\hat\phi)^i(\nu)) \\
&= \rho_L((\hat\phi)^i(\mu),(\hat\phi)^i(\nu)) \\
&\ge k_i\rho_L(\mu,\nu) = k_i L'(\mu-\nu) = k_i L'\left(\frac{f}{1+\|f\|}\right).
\end{aligned}$$

Thus $L'((\hat\phi)^i(f)) \ge k_i L'(f)$ for all $i \in \mathbb{N}$ and $f \in A'^o$, proving (1). ∎

PROPOSITION 4.6. *Let A be an order unit space with lower semicontinuous Lipschitz seminorm L, and let $k > 0$. Suppose that $\phi : A \to A$ is a unital positive linear map, and that $(\mathcal{S}(A), \rho_L)$ is a bounded metric space.*

(1) *If ϕ is asymptotically k-expansive with $\phi(\mathcal{A}) = \mathcal{A}$, then $\hat{\phi}$ is asymptotically k-expansive from $(\mathcal{S}(A), \rho_L)$ to itself.*
(2) *If $\hat{\phi}$ is asymptotically k-expansive and surjective from $(\mathcal{S}(A), \rho_L)$ to itself, then ϕ is asymptotically k-expansive.*

Proof. Suppose that ϕ is asymptotically k-expansive with $\phi(\mathcal{A}) = \mathcal{A}$. Then there is a sequence $\{k_i\}_{i\in\mathbb{N}}$ with $k_i \to k$ as $i \to \infty$ such that

$$L(\phi^i(a)) \geq k_i L(a)$$

for all $i \in \mathbb{N}$ and $a \in \mathcal{A}$. Then for any $f \in A'^o$ we have

$$\begin{aligned} L'(f) &= \sup\{|f(b)| : b \in A,\ L(b) \leq 1\} \\ &\leq \sup\{|f(\phi^i(a))| : a \in A,\ L(a) \leq 1/k_i\} \\ &= \sup\{|(\hat{\phi})^i(f)(a)| : a \in A,\ L(a) \leq 1/k_i\} = \frac{1}{k_i} L'((\hat{\phi})^i(f)). \end{aligned}$$

Thus $\hat{\phi}$ is an asymptotically k-expansive map from A'^o to itself relative to L', and from Proposition 4.5 we see that $\hat{\phi}$ is also asymptotically k-expansive from $(\mathcal{S}(A), \rho_L)$ to itself.

Suppose that $\hat{\phi}$ is asymptotically k-expansive and surjective from $(\mathcal{S}(A), \rho_L)$ to itself. Then there is a sequence $\{k_i\}_{i\in\mathbb{N}}$ with $k_i \to k$ as $i \to \infty$ such that

$$\rho_L((\hat{\phi})^i(\mu), (\hat{\phi})^i(\nu)) \geq k_i \rho_L(\mu, \nu)$$

for all $i \in \mathbb{N}$ and $\mu, \nu \in \mathcal{S}(A)$. Since the Lipschitz seminorm L is lower semicontinuous, by [25, Theorem 4.1] it can be exactly recovered from the metric space $(\mathcal{S}(A), \rho_L)$. Then

$$\begin{aligned} k_i L(a) &= k_i \sup\left\{\frac{|\mu'(a) - \nu'(a)|}{\rho_L(\mu', \nu')} : \mu', \nu' \in \mathcal{S}(A),\ \mu' \neq \nu'\right\} \\ &= k_i \sup\left\{\frac{|(\hat{\phi})^i(\mu)(a) - (\hat{\phi})^i(\nu)(a)|}{\rho_L((\hat{\phi})^i, (\hat{\phi})^i(\nu))} : \mu, \nu \in \mathcal{S}(A),\ \hat{\phi}(\mu) \neq \hat{\phi}(\nu)\right\} \\ &\leq \sup\left\{\frac{|\mu(\phi^i(a)) - \nu(\phi^i(a))|}{\rho_L((\hat{\phi})^i(\mu), (\hat{\phi})^i(\nu))} \frac{\rho_L((\hat{\phi})^i(\mu), (\hat{\phi})^i(\nu))}{\rho_L(\mu, \nu)} : \mu, \nu \in \mathcal{S}(A),\ \mu \neq \nu\right\} \\ &= \sup\left\{\frac{|\mu(\phi^i(a)) - \nu(\phi^i(a))|}{\rho_L(\mu, \nu)} : \mu, \nu \in \mathcal{S}(A),\ \mu \neq \nu\right\} = L(\phi^i(a)) \end{aligned}$$

for all $a \in \mathcal{A}$. Hence ϕ is asymptotically k-expansive. ∎

LEMMA 4.7. *Let (X, ρ) be a compact metric space, and let $f : X \to X$ be an asymptotically k-expansive map. Then f is $1/k$-nonexpansive.*

Proof. There is a sequence $\{k_i\}_{i\in\mathbb{N}}$ with $k_i \to k$ as $i \to \infty$ such that for any $i \in \mathbb{N}$ and $x, y \in X$,

$$\rho(f^i(x), f^i(y)) \geq k_i \rho(x, y).$$

For any $x_0, y_0 \in X$, we define two sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ in X by $f(x_{n-1}) = x_n$ and $f(y_{n-1}) = y_n$ for $n \in \mathbb{N}$. Since (X, ρ) is compact, there exist convergent subsequences $\{x_{n_i}\}_{i\in\mathbb{N}}$ and $\{y_{n_i}\}_{i\in\mathbb{N}}$. For any $\varepsilon > 0$, there is an integer N such that for any $i \geq N$ we have

$$\rho(x_{n_i}, x_{n_{i+j}}) < \varepsilon \quad \text{and} \quad \rho(y_{n_i}, y_{n_{i+j}}) < \varepsilon,$$

for any $j \in \mathbb{N}$. Fix an $i \geq N$ and define $p_j = n_{i+j} - n_i$ for $j \in \mathbb{N}$. Obviously $p_1 < p_2 < \cdots$ and $p_j \to \infty$ as $j \to \infty$. Since

$$\begin{aligned}\rho(x_{n_i}, x_{n_{i+j}}) &= \rho(x_{n_i}, x_{n_i+p_j}) = \rho(f^{n_i}(x_0), f^{n_i}(x_{p_j}))\\ &\geq k_{n_i}\rho(x_0, x_{p_j}) = k_{n_i}\rho(x_0, f^{p_j}(x_0))\end{aligned}$$

for any $j \in \mathbb{N}$, we have

$$\rho(f^{p_j}(x_0), x_0) \leq \frac{1}{k_{n_i}}\rho(x_{n_i}, x_{n_{i+j}}) \leq \gamma\rho(x_{n_i}, x_{n_{i+j}}) < \gamma\varepsilon$$

for any $j \in \mathbb{N}$, where $\gamma = \sup\{1/k_i : i \in \mathbb{N}\}$. Similarly,

$$\rho(f^{p_j}(y_0), y_0) \leq \gamma\rho(y_{n_i}, y_{n_{i+j}}) < \gamma\varepsilon$$

for any $j \in \mathbb{N}$. Thus

$$\begin{aligned}\rho(f(x_0), f(y_0)) &\leq \frac{1}{k_{p_j-1}}\rho(f^{p_j-1}(f(x_0)), f^{p_j-1}(f(y_0)))\\ &= \frac{1}{k_{p_j-1}}\rho(f^{p_j}(x_0), f^{p_j}(y_0))\\ &\leq \frac{1}{k_{p_j-1}}(\rho(f^{p_j}(x_0), x_0) + \rho(x_0, y_0) + \rho(y_0, f^{p_j}(y_0)))\\ &\leq \frac{1}{k_{p_j-1}}(\rho(x_0, y_0) + 2\gamma\varepsilon)\end{aligned}$$

for any $j \in \mathbb{N}$. Letting $j \to \infty$ yields

$$\rho(f(x_0), f(y_0)) \leq \frac{1}{k}\rho(x_0, y_0) + \frac{2\gamma}{k}\varepsilon$$

for any $\varepsilon > 0$, and hence $\rho(f(x_0), f(y_0)) \leq \frac{1}{k}\rho(x_0, y_0)$, as desired. ∎

LEMMA 4.8. *Let (X, ρ) be a compact metric space, and let $f : X \to X$ be an asymptotically k-expansive map. Then f is k-expansive.*

Proof. For any $x_0, y_0 \in X$ and $\varepsilon > 0$, as in the proof of Lemma 4.7, there is a sequence $\{p_i\}_{i\in\mathbb{N}}$ with $p_i \to \infty$ as $i \to \infty$ such that

$$\rho(x_0, f^{p_i}(x_0)) < \gamma\varepsilon \quad \text{and} \quad \rho(f^{p_i}(y_0), y_0) < \gamma\varepsilon,$$

for any $i \in \mathbb{N}$, where $\gamma = \sup\{1/k_i : i \in \mathbb{N}\}$. By Lemma 4.7, f is $1/k$-non-expansive, thus

$$\begin{aligned}\rho(f(x_0), f(y_0)) &\geq \rho(f^{p_i}(f(x_0)), f^{p_i}(f(y_0))) - \rho(f^{p_i+1}(x_0), f(x_0)) \\ &\quad - \rho(f^{p_i+1}(y_0), f(y_0)) \\ &\geq k_{p_i+1}\rho(x_0, y_0) - \rho(f^{p_i+1}(x_0), f(x_0)) - \rho(f^{p_i+1}(y_0), f(y_0)) \\ &\geq k_{p_i+1}\rho(x_0, y_0) - \frac{1}{k}\rho(f^{p_i}(x_0), x_0) - \frac{1}{k}\rho(f^{p_i}(y_0), y_0) \\ &\geq k_{p_i+1}\rho(x_0, y_0) - \frac{2}{k}\gamma\varepsilon.\end{aligned}$$

Letting $i \to \infty$, we have

$$\rho(f(x_0), f(y_0)) \geq k\rho(x_0, y_0) - \frac{2}{k}\gamma\varepsilon.$$

Since this holds for every $\varepsilon > 0$, we see that $\rho(f(x_0), f(y_0)) \geq k\rho(x_0, y_0)$, as desired. ∎

PROPOSITION 4.9. *Let (X, ρ) be a compact metric space, and let $f : X \to X$ be an asymptotically k-expansive map. Then for any $p, q \in X$,*

$$k\rho(p, q) \leq \rho(f(p), f(q)) \leq \frac{1}{k}\rho(p, q).$$

Proof. The conclusion follows from Lemmas 4.7 and 4.8. ∎

Moreover, we have the following result.

LEMMA 4.10. *Let (X, ρ) be a compact metric space, and let $f : X \to X$ be an asymptotically k-expansive map. Then f is surjective.*

Proof. Since X is compact, by Lemma 4.7, $f(X)$ is also compact, and hence it is a closed subset of X.

Suppose that $x_0 \in X \backslash f(X) \neq \emptyset$. For any integer $n \geq 0$, let $x_n = f^n(x_0)$. We prove by induction that $x_n \notin f^{n+1}(X)$ for all $n \geq 0$. For $n = 0$, this is clear. If $x_n \in f^{n+1}(X)$ for some $n \geq 1$, then there is a y in X such that $x_n = f^n(x_0) = f^{n+1}(y)$. By Lemma 4.8, f is k-expansive, and hence injective. It follows that $x_{n-1} = f^{n-1}(x_0) = f^n(y)$, and so x_{n-1} is in $f^n(X)$, a contradiction. This completes the induction. Thus for all $n \geq 0$, we have $x_n \in f^n(X) \setminus f^{n+1}(X)$.

Let

$$\gamma = \inf\{\rho(x_0, f(x)) : x \in X\}.$$

Since $x_0 \notin f(X)$ and $f(X)$ is compact, we have $\gamma > 0$.

For all positive integers n and m, since f is asymptotically k-expansive, we have

$$\begin{aligned}\rho(x_n, x_{n+m}) &= \rho(f^n(x_0), f^{n+m}(x_0)) \\ &\geq \inf\{\rho(f^n(x_0), f^{n+m}(x)) : x \in X\} \\ &\geq k_n \inf\{\rho(x_0, f^m(x)) : x \in X\} \\ &\geq k_n \inf\{\rho(x_0, f(x)) : x \in X\} = k_n\gamma \\ &\geq \inf\{k_n : n \in \mathbb{N}\}\gamma > 0.\end{aligned}$$

This proves that $\{x_i\}_{i\in\mathbb{N}}$ has no convergent subsequence, which contradicts the compactness of X. ∎

Proposition 4.11 ([38, Theorem 2]). *Let (X, ρ) be a compact metric space, and let $f : X \to X$ be an asymptotically k-expansive map.*

(1) *If $k > 1$, then X is a singleton.*
(2) *If $0 < k \leq 1$, then f is a bi-Lipschitz map. More precisely,*

$$k\rho(p, q) \leq \rho(f(p), f(q)) \leq \frac{1}{k}\rho(p, q) \quad \text{for any } p, q \in X.$$

Proof. For any $p, q \in X$, by Lemmas 4.7 and 4.8 we have

$$k\rho(p, q) \leq \rho(f(p), f(q)) \leq \frac{1}{k}\rho(p, q).$$

If $k > 1$, then for any $p, q \in X$,

$$k^2\rho(p, q) \leq \rho(p, q).$$

This implies that $p = q$, and thus X is a singleton.

If $0 < k \leq 1$, the conclusion follows from Lemmas 4.7, 4.8 and 4.10. ∎

Now we prove the main result of this section.

Theorem 4.12. *Let (A, L) be a compact quantum metric space with lower semicontinuous Lip-norm, and let $k \in (0, 1]$. If $\phi : A \to A$ is a unital positive and asymptotically k-expansive linear map with $\phi(\mathcal{A}) = \mathcal{A}$, then ϕ is a bi-Lipschitz map from (A, L) onto itself.*

Proof. Since (A, L) is a compact quantum metric space, $(\mathcal{S}(A), \rho_L)$ is a compact metric space. Thus by Proposition 4.6 the induced map $\hat{\phi}$ is asymptotically k-expansive from the compact metric space $(\mathcal{S}(A), \rho_L)$ to itself, and then from Proposition 4.11 we see that $\hat{\phi}$ is bi-Lipschitz such that

$$k\rho_L(\mu, \nu) \leq \rho_L(\hat{\phi}(\mu), \hat{\phi}(\nu)) \leq \frac{1}{k}\rho_L(\mu, \nu)$$

for any $\mu, \nu \in \mathcal{S}(A)$. In particular, $\hat{\phi}$ is an affine homeomorphism on $(\mathcal{S}(A), \rho_L)$. From [20, Lemma 3.7] we see that ϕ is an isometry. Therefore, as in the proof of Theorem 3.9 we find that ϕ is a unital order isomorphism from A onto itself.

For any $a \in A$ and $\mu, \nu \in \mathcal{S}(A)$, we have

$$|\mu(\phi(a)) - \nu(\phi(a))| = |\hat{\phi}(\mu)(a) - \hat{\phi}(\nu)(a)| \leq L(a)\rho_L(\hat{\phi}(\mu), \hat{\phi}(\nu)) \\ \leq \frac{1}{k}L(a)\rho_L(\mu, \nu).$$

Since the Lipschitz seminorm L is lower semicontinuous, by [25, Theorem 4.1] it can be exactly recovered from $(\mathcal{S}(A), \rho_L)$, that is,

$$L(b) = \sup\left\{\frac{|\mu(b) - \nu(b)|}{\rho_L(\mu, \nu)} : \mu, \nu \in \mathcal{S}(A),\ \mu \neq \nu\right\}$$

for all $b \in A$. Thus

$$L(\phi(a)) \leq \frac{1}{k}L(a) < \infty$$

for any $a \in \mathcal{A}$. It follows that ϕ is $1/k$-nonexpansive. Similarly, since the affine map $(\hat{\phi})^{-1} = \widehat{\phi^{-1}}$ is also $1/k$-nonexpansive from $(\mathcal{S}(A), \rho_L)$ to itself, we have

$$L(\phi^{-1}(a)) \leq \frac{1}{k}L(a)$$

for any $a \in \mathcal{A}$. So we have

$$kL(a) \leq L(\phi(a)) \leq \frac{1}{k}L(a)$$

for any $a \in \mathcal{A}$. Thus ϕ is bi-Lipschitz from (A, L) onto itself. ∎

COROLLARY 4.13 ([20, Theorem 4.7]). *Let (A, L) be a compact quantum metric space with lower semicontinuous Lip-norm. If $\phi : A \to A$ is a unital positive linear map with $\phi(\mathcal{A}) = \mathcal{A}$, then ϕ is asymptotically 1-expansive if and only if it is a Lipschitz isometry from (A, L) onto itself.*

Proof. Suppose that ϕ is asymptotically 1-expansive. By Theorem 4.12, ϕ is bi-Lipschitz from (A, L) onto itself. Moreover, for any $a \in \mathcal{A}$,

$$L(a) \leq L(\phi(a)) \leq L(a).$$

Thus ϕ is Lipschitz isometric. Similarly, ϕ^{-1} is also Lipschitz isometric. Therefore, ϕ is a Lipschitz isometry from (A, L) onto itself.

The proof in the other direction is trivial. ∎

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