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E. Gordienko, J. Ruiz de Chávez and
P. Vázquez-Ortega (México City)

NOTE ON STABILITY OF THE RUIN TIME DENSITY IN A SPARRE ANDERSEN RISK MODEL WITH EXPONENTIAL CLAIM SIZES

Abstract. In this note, the Sparre Andersen risk process with exponential claim sizes is considered. We derive upper bounds for deviations of the ruin time density when approximating the inter-claim time distribution. In particular, we treat approximation by means of empirical densities.

In actuarial theory and practice, the significance of ruin probability is well-known (see e.g. [1, 15]). Additionally, in certain situations the ruin time distribution has to be estimated. This problem arises, for instance, in the cases of sudden natural disasters, when the flow of claims increases sharply.

For the Sparre Andersen risk model relatively simple methods of obtaining the distribution of the ruin time τ are known only for some particular distributions of claim sizes or/and of inter-claim times. (See, for instance, [1, 6, 7, 8, 11, 16, 18].)

In this paper we deal with the Sparre Andersen risk models with exponential claim sizes. For such models formula (0.2) below, established in [4], can be used to compute the density ρ_τ of τ. We apply this formula to obtain the stability inequalities in Section 2.

In the Sparre Andersen model the surplus process is given by

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i, \qquad t \geq 0, \tag{0.1}$$

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where $u \geq 0$ is the *initial surplus* (or *initial capital*), $c > 0$ is the *premium rate*, $\{X_i,\ i \geq 1\}$ is a sequence of positive i.i.d. random variables representing consecutive *claim sizes*, and $N(t)$, $t \geq 0$, is a *delayed renewal process*, independent of $\{X_i, i \geq 1\}$. The process $N(t)$ is generated by a sequence $T_0, T_1, T_2, \ldots$ of independent *inter-claim times*. We assume that T_0 has a density f_0, and the i.i.d. random variables T_j, $j \geq 1$, have a common density f.

We will consider a particular version of (0.1) where the random variable X_1 has exponential density with parameter λ. As shown in [4], the *ruin time*

$$\tau := \inf\{t > 0 : U(t) < 0\} \in (0, \infty]$$

has a (defective) density $p_\tau(t)$, $t \geq 0$, given by

$$\begin{aligned}(0.2)\qquad p_\tau(t) &= e^{-\lambda(u+ct)} \\ &\times \left[f_0(t) + \frac{1}{u+ct} \sum_{n=1}^{\infty} \frac{(\lambda(u+ct))^n}{n!} \int_0^t (u+cv) f^{*n}(t-v) f_0(v)\, dv \right]\end{aligned}$$

(see [4, proof of Theorem 1]). Here and in what follows, $f^{*n} = f^{*(n-1)} * f$ stands for the n-fold convolution of f with itself.

The formula (0.2) allows us to estimate (at least theoretically) the density of the ruin time τ in a rather general case. Nevertheless, in practical situations, the values of "governing parameters": λ, f_0 and f in (0.1) and (0.2), as a rule, are *unknown*. For this, one has to rely on certain *known* (available) approximations $\widetilde{\lambda}$, $\widetilde{f}_0$ and $\widetilde{f}$. In this way, a researcher would not deal with the "real process" (0.1), but rather with its "approximate version"

$$(0.3)\qquad \widetilde{U}(t) = u + ct - \sum_{i=1}^{\widetilde{N}(t)} \widetilde{X}_i, \qquad t \geq 0,$$

whose components are defined similarly to those in (0.1), but with parameters $\widetilde{\lambda}$, $\widetilde{f}_0$ and $\widetilde{f}$. The latter, as a rule, are obtained from statistical data.

We assume that the application of the corresponding version of (0.2) makes it feasible to evaluate the density $\widetilde{p}_\tau$ of the ruin time τ in the model (0.3). So the following question of *stability* arises: Under which conditions does $\widetilde{p}_\tau$ provide a good approximation of the unknown density p_τ (in the "real model" (0.1))? Proposition 2.1 and Theorem 2.2 suggest the answer to this question, in terms of the proximity of the densities involved. We will use suitable probability metrics to measure that proximity.

We emphasize that we do not propose any new techniques or algorithms to estimate ρ_τ when the densities f_0 and f are known. Our main goal is to study the case where $\widetilde{f}_0 = \widetilde{f}_0^{(n)}$ and $\widetilde{f} = \widetilde{f}^{(n)}$ are appropriate statistical estimators of the *unknown* densities f_0 and f. Here, n is the number of observed inter-claim times.

Theorem 2.2 and the results of Section 3 imply that under some restrictions on f_0, $\widetilde{f}_0$, f and $\widetilde{f}$,

$$E\Big(\sup_{t\geq 0}|\rho_\tau(t)-\widetilde{\rho}_\tau^{(n)}(t)|\Big)=O\bigg(\frac{\log(n)}{n}\bigg)^{2/5},$$

where $\widetilde{\rho}_\tau^{(n)}=\widetilde{\rho}_\tau$ is the density of ruin time for the risk process (0.3) governed by $\widetilde{f}_0^{(n)}$ and $\widetilde{f}^{(n)}$. For a large insurance company, and a relatively long period of data collection, the number n could be very large.

1. Some probability metrics and relevant results. The problem we are concerned with is to find upper bounds for $\sup_{t\geq 0}|p_\tau(t)-\widetilde{p}_\tau(t)|$, which are expressed in terms of $|\lambda-\widetilde{\lambda}|$ and certain distances between the densities. We will use the following distances:

$$d(f,\widetilde{f}):=\operatorname*{ess\,sup}_{t\geq 0}|f(t)-\widetilde{f}(t)| \quad \text{(the } L^\infty \textit{ metric}), \tag{1.1}$$

$$d_{\mathrm{TV}}(f,\widetilde{f}):=\int_0^\infty |f(t)-\widetilde{f}(t)|\,dt, \quad \text{(the \textit{total variation metric})}, \tag{1.2}$$

and

$$\mathbb{K}_2(f_a,\widetilde{f}_a):=2\int_{-\infty}^{\infty}|t|\,|F_a(t)-\widetilde{F}_a(t)|\,dt \tag{1.3}$$

(the *second difference pseudomoment*).

In (1.3) for a given $a\in\mathbb{R}$,

$$F_a(t)=\int_{-\infty}^{t} f(z+a)\,dz, \qquad \widetilde{F}_a(t)=\int_{-\infty}^{t}\widetilde{f}(z+a)\,dz. \tag{1.4}$$

Proposition 2.1 and Theorem 2.2 below give upper bounds for $d(p_\tau,\widetilde{p}_\tau)$. The relevant results on stability (continuity) of the infinite horizon ruin probability can be found, for instance, in [13, 17] for the Cramer–Lundberg risk process, and for more general risk model in [2, 3, 9, 10]. On the other hand, we are not aware of any continuity inequalities for densities of ruin time.

2. The main results. Let T_0 and T_1 ($\overset{d}{=} T_j,\ j\geq 1$) be the inter-claim times in the risk model (0.1), and, respectively, $\widetilde{T}_0$ and $\widetilde{T}_1$ ($\overset{d}{=}\widetilde{T}_j,\ j\geq 1$) be the inter-claim times in the approximating model (0.3).

Correspondingly, f_0 and $\widetilde{f}_0$ are the densities of T_0 and $\widetilde{T}_0$, while f and $\widetilde{f}$ are the densities of T_1 and $\widetilde{T}_1$. Let

$$M:=\sup_{t\geq 0} f_0(t)<\infty. \tag{2.1}$$

Proposition 2.1.

$$d(p_\tau,\widetilde{p}_\tau)\le d(f_0,\widetilde{f}_0)+\widetilde{\lambda}[u+cE\widetilde{T}_0]d(f,\widetilde{f})+2M(\lambda e)^{-1}|\lambda-\widetilde{\lambda}|, \tag{2.2}$$

where d is the metric defined in (1.1).

Note that if $\lambda=\widetilde{\lambda}$, then condition (2.1) can be omitted, and in this case the constant on the right-hand side of (2.2) does not involve any attribute of the unknown densities f_0 and f.

Since the initial capital u is involved in the right-hand side of (2.2), this inequality is relevant only for u close to zero.

The bound (2.4) given below does not have those shortcomings, but its proof requires more restrictions.

Assumption 2.1. *We suppose that $ET_1=E\widetilde{T}_1$ and the random variables T_1 and $\widetilde{T}_1$ have finite positive variances. Also we suppose that there is an integer $s\ge 1$ such that the density $g:=\widetilde{f}^{*s}$ has an absolutely continuous derivative g', and $\varphi(t):=tg''(t)\in\mathbb{L}_1[0,\infty)$.*

Let $\widetilde{M}:=\sup_{t\ge 0}\widetilde{f}_0(t)<\infty$.

Theorem 2.2. *Under Assumption* 2.1, *there exists a finite constant b, uniquely determined by $\widetilde{f}$, such that if*

$$\max\{d_{\mathrm{TV}}(f,\widetilde{f}),\tfrac{1}{2}\mathbb{K}_2(f,\widetilde{f})\}\le(2b)^{-1} \tag{2.3}$$

then

$$\begin{aligned} &d(p_\tau,\widetilde{p}_\tau)\\ &\le d(f_0,\widetilde{f}_0)+b\widetilde{M}\max\{d_{\mathrm{TV}}(f,\widetilde{f}),\inf_{a\in\mathbb{R}}\tfrac{1}{2}\mathbb{K}_2(f_a,\widetilde{f}_a)\}+2M(\lambda e)^{-1}|\lambda-\widetilde{\lambda}|.\end{aligned} \tag{2.4}$$

The metrics d_{TV} and $\mathbb{K}_2$ were defined in (1.2) *and* (1.3).

Again, when $\lambda=\widetilde{\lambda}$, the constant on the right-hand side of (2.4) depends only on the presumably known approximating process (0.3). In the proof of Theorem 2.2, to compare convolutions of densities, we use inequality (3.3) from [12]. This inequality involves the constant b that appears in (2.4). From [12] and the numerical calculations it follows that if, for example, $\widetilde{f}$ is a gamma density with $\alpha=4$ and an arbitrary λ, then $b<\max\{1,\,0.675\lambda^2\}$. Surely, such bounds are useful only in the cases where $\widetilde{f}$ functions as a "theoretical approximation" to f.

3. The case when $\widetilde{f}$ is a statistical estimator of f. Considering an ordinary renewal process with $f_0=f$, we assume that the density f in (0.2) is unknown, but n i.i.d. observations $T_1,\dots,T_n$ of the random variable T with density f are available. These statistical data can be used to construct an estimator $\widetilde{f}\equiv\widetilde{f}_n$, that is used as a density of the inter-claim times in the approximating risk process (0.3). In such situations, both parts of the inequalities (2.2) and (2.4) are random, and we should operate with expectations.

For sure, the statistical data $T_1, \dots, T_n$ can be used in many ways, and there exists a great amount of statistical literature on this topic.

Let us assume that the bounded density f has a bounded second derivative, and consider the metric d defined in (1.1). It is well-known (see e.g. [19]) that there are kernel estimators $\widetilde{f}_n$ such that

$$Ed(f, \widetilde{f}_n) = O[(\log n/n)^{2/5}] \quad \text{as } n \to \infty. \tag{3.1}$$

In the above mentioned class of densities the rate of convergence in (3.1) cannot be improved.

The situation is quite similar when the total variation metric d_{TV} defined in (1.2) is used. As shown in [5, Chapter 5], under certain additional conditions (on the "tail" of the density) there are kernel estimators $\widetilde{f}_n$ such that

$$Ed_{\mathrm{TV}}(f, \widetilde{f}_n) = O[(1/n)^{2/5}] \quad \text{as } n \to \infty, \tag{3.2}$$

and, again in the class of densities considered, $Ed_{\mathrm{TV}}(f, \widetilde{f}_n)$ cannot vanish faster.

As far as the authors are aware, there are no results concerning the rate of convergence of $E\mathbb{K}_2(f_a, \widetilde{f}_{a,n})$, where $\mathbb{K}_2$ is the second difference pseudomoment defined in (1.3).

Fix $a \in \mathbb{R}$, and let F_a and $\widetilde{F}_a$ be as defined in (1.4). Let $F \equiv F_a$ and

$$\hat{F}_n(t) := \frac{1}{n} \sum_{i=1}^{n} I_{\{T_i - a \le t\}}, \qquad t \in \mathbb{R}, \tag{3.3}$$

be the corresponding *empirical distribution function*. The standard approach to estimating the unknown F is to use $\hat{F}_n$. Next, we derive the following upper bound of $E\mathbb{K}_2(F, \hat{F}_n)$.

PROPOSITION 3.1. *Assume that* $\mathcal{H} := E|T_1 - a|^{4+\alpha} < \infty$ *for some* $\alpha > 0$. *Then*

$$E\mathbb{K}_2(F, \hat{F}_n) \le \frac{2}{\sqrt{n}} \left[\frac{1}{2} + \frac{4\mathcal{H}}{\alpha} \right], \qquad n = 1, 2, \dots. \tag{3.4}$$

Note that the convergence in (3.4) is faster than in (3.1) and (3.2).

4. The proofs

4.1. Proof of Proposition 2.1. Let d be the metric defined in (1.1), and f and $\widetilde{f}$ be the densities introduced at the beginning of Section 2.

First, we show by induction that

$$d(f^{*n}, \widetilde{f}^{*n}) \le n d(f, \widetilde{f}), \qquad n = 1, 2, \dots. \tag{4.1}$$

Assuming (4.1) we have, for any $t \geq 0$,

$$|f^{*(n+1)}(t) - \widetilde{f}^{*(n+1)}(t)| \leq \left|\int f^{*n}(t-v)f(v)\,dv - \int \widetilde{f}^{*n}(t-v)f(v)\,dv\right| + \left|\int \widetilde{f}^{*n}(t-v)f(v)\,dv - \int \widetilde{f}^{*n}(t-v)\widetilde{f}(v)\,dv\right| \leq d(f^{*n}, \widetilde{f}^{*n}) + d(f, \widetilde{f}) \leq (n+1)d(f, \widetilde{f}).$$

Fix $t \geq 0$ and let $u_t := u + ct$. Then, applying (0.2) and its version for the approximating risk process (0.3), we write

$$p_\tau(t) = e^{-\lambda u_t}\left[f_0(t) + \frac{1}{u_t}\sum_{n=1}^{\infty}\frac{(\lambda u_t)^n}{n!}\int_0^t u_v f^{*n}(t-v)f_0(v)\,dv\right],$$

$$\widetilde{p}_\tau(t) = e^{-\widetilde{\lambda} u_t}\left[\widetilde{f}_0(t) + \frac{1}{u_t}\sum_{n=1}^{\infty}\frac{(\widetilde{\lambda} u_t)^n}{n!}\int_0^t u_v \widetilde{f}^{*n}(t-v)\widetilde{f}_0(v)\,dv\right].$$

Adding and subtracting suitable terms in the expression of $|p_\tau(t) - \widetilde{p}_\tau(t)|$, we will obtain

$$|p_\tau(t) - \widetilde{p}_\tau(t)| \leq I_1(t) + I_2(t) + I_3(t), \tag{4.2}$$

where the summands are defined below. By applying (4.1), we obtain

$$\begin{aligned} I_1(t) &:= \Bigg| e^{-\widetilde{\lambda} u_t}\left[\widetilde{f}_0(t) + \frac{1}{u_t}\sum_{n=1}^{\infty}\frac{(\widetilde{\lambda} u_t)^n}{n!}\int_0^t u_v \widetilde{f}^{*n}(t-v)\widetilde{f}_0(v)\,dv\right] \\ &\qquad - e^{-\widetilde{\lambda} u_t}\left[\widetilde{f}_0(t) + \frac{1}{u_t}\sum_{n=1}^{\infty}\frac{(\widetilde{\lambda} u_t)^n}{n!}\int_0^t u_v f^{*n}(t-v)\widetilde{f}_0(v)\,dv\right]\Bigg| \\ &\leq e^{-\widetilde{\lambda} u_t}\frac{1}{u_t}\sum_{n=1}^{\infty}\frac{(\widetilde{\lambda} u_t)^n}{n!}\int_0^t u_v |\widetilde{f}^{*n}(t-v) - f^{*n}(t-v)|\widetilde{f}_0(v)\,dv \\ &\leq e^{-\widetilde{\lambda} u_t}\widetilde{\lambda}\sum_{n=1}^{\infty}\frac{(\widetilde{\lambda} u_t)^{n-1}}{(n-1)!}[u + cE\widetilde{T}_0]d(f, \widetilde{f}) \\ &= \widetilde{\lambda}[u + cE\widetilde{T}_0]d(f, \widetilde{f}). \end{aligned} \tag{4.3}$$

Since $v \leq t$ entails $u_v \leq u_t$, we get

$$\begin{aligned} I_2(t) &:= \Bigg| e^{-\widetilde{\lambda} u_t}\left[\widetilde{f}_0(t) + \frac{1}{u_t}\sum_{n=1}^{\infty}\frac{(\widetilde{\lambda} u_t)^n}{n!}\int_0^t u_v f^{*n}(t-v)\widetilde{f}_0(v)\,dv\right] \\ &\qquad - e^{-\widetilde{\lambda} u_t}\left[f_0(t) + \frac{1}{u_t}\sum_{n=1}^{\infty}\frac{(\widetilde{\lambda} u_t)^n}{n!}\int_0^t u_v f^{*n}(t-v)f_0(v)\,dv\right]\Bigg| \\ &\leq e^{-\widetilde{\lambda} u_t}d(f_0, \widetilde{f}_0) + e^{-\widetilde{\lambda} u_t}(e^{\widetilde{\lambda} u_t} - 1)d(f_0, \widetilde{f}_0) = d(f_0, \widetilde{f}_0). \end{aligned} \tag{4.4}$$

Finally,

$$\begin{aligned}I_3(t) &:= \bigg| e^{-\widetilde{\lambda} u_t}\bigg[f_0(t) + \frac{1}{u_t}\sum_{n=1}^{\infty}\frac{(\widetilde{\lambda} u_t)^n}{n!}\int_0^t u_v f^{*n}(t-v) f_0(v)\,dv\bigg] \\ &\qquad - e^{-\lambda u_t}\bigg[f_0(t) + \frac{1}{u_t}\sum_{n=1}^{\infty}\frac{(\lambda u_t)^n}{n!}\int_0^t u_v f^{*n}(t-v) f_0(v)\,dv\bigg]\bigg| \\ &\le M|e^{-\widetilde{\lambda} u_t} - e^{-\lambda u_t}| + M\bigg| e^{-\widetilde{\lambda} u_t}\sum_{n=1}^{\infty}\frac{(\widetilde{\lambda} u_t)^n}{n!} - e^{-\lambda u_t}\sum_{n=1}^{\infty}\frac{(\lambda u_t)^n}{n!}\bigg| \\ &= 2M|e^{-\widetilde{\lambda} u_t} - e^{-\lambda u_t}|.\end{aligned}$$

Let, for instance, $\widetilde{\lambda} > \lambda$. Then

(4.5)
$$I_3(t) \le 2Me^{-\lambda u_t}|e^{-(\widetilde{\lambda}-\lambda)u_t} - 1| \le 2Me^{-\lambda u_t}u_t|\widetilde{\lambda} - \lambda| \le 2M(\lambda e)^{-1}|\widetilde{\lambda} - \lambda|,$$

since $\sup_{x\ge 0} xe^{-x} = e^{-1}$. For $\lambda > \widetilde{\lambda}$, the inequality (4.5) is proved in an analogous way.

Note that the right-hand sides of (4.3)–(4.5) do not depend on $t \ge 0$. Combining them with (4.2), we obtain (2.2).

4.2. Proof of Theorem 2.2. The terms I_2 and I_3 in (4.2) are bounded exactly in the same way as in (4.4) and (4.5). As for I_1, the very first inequality in (4.3) shows that

$$\begin{aligned}I_1(t) &\le e^{-\widetilde{\lambda} u_t}\sum_{n=1}^{\infty}\frac{(\widetilde{\lambda} u_t)^n}{n!}\widetilde{M}\int_0^{\infty}|\widetilde{f}^{*n}(x) - f^{*n}(x)|\,dx \\ &= \widetilde{M}e^{-\widetilde{\lambda} u_t}\sum_{n=1}^{\infty}\frac{(\widetilde{\lambda} u_t)^n}{n!}d_{\mathrm{TV}}(\widetilde{f}^{*n}, f^{*n}).\end{aligned} \tag{4.6}$$

The paper [12] contains an upper bound of the total variation distance between two sums of i.i.d. random variables. Namely, Lemma 1 in [12] states that under the conditions of Theorem 2.2 for each $n = 1, 2, \dots,$

$$d_{\mathrm{TV}}(\widetilde{f}^{*n}, f^{*n}) \le b\varDelta, \tag{4.7}$$

provided that

$$\varDelta := \max\{d_{\mathrm{TV}}(\widetilde{f}, f), \zeta_2(\widetilde{f}, f)\} \le (2b)^{-1}. \tag{4.8}$$

In (4.8), ζ_2 is the Zolotarev metric of order 2.

By applying the well-known inequality $\zeta_2 \le \frac{1}{2}\mathbb{K}_2$ (see e.g. [14]), which holds under the equality of the first moments, we see that (2.3) ensures (4.8), and by (4.7),

$$d_{\mathrm{TV}}(\widetilde{f}^{*n}, f^{*n}) \le b\max\big\{d_{\mathrm{TV}}(\widetilde{f}, f), \tfrac{1}{2}\mathbb{K}_2(\widetilde{f}, f)\big\}. \tag{4.9}$$

Let us recall that f and $\widetilde{f}$ are the densities of the random variables T_1 and $\widetilde{T}_1$, respectively. Let a be an arbitrary number. Consider the random variables $X_1 = T_1 - a$ and $\widetilde{X}_1 = \widetilde{T}_1 - a$ with the corresponding densities $h_a(\cdot) = f(\cdot + a)$ and $\widetilde{h}_a(\cdot) = \widetilde{f}(\cdot + a)$. Inequality (4.9) holds true for the pair of the densities $(h_a, \widetilde{h}_a)$. From (1.2) it follows that $d_{\mathrm{TV}}(\widetilde{f}, f) = d_{\mathrm{TV}}(\widetilde{h}_a, h_a)$ and $d_{\mathrm{TV}}(\widetilde{f}^n, f^{*n}) = d_{\mathrm{TV}}(\widetilde{h}_a^{*n}, h_a^{*n})$. Therefore, for all $n \ge 1$,

$$d_{\mathrm{TV}}(\widetilde{f}^{*n}, f^{*n}) \le b \inf_{a \in \mathbb{R}} \max\{d_{\mathrm{TV}}(\widetilde{f}, f), \tfrac{1}{2}\mathbb{K}_2(\widetilde{h}_a, h_a)\}.$$

Combining this inequality with (4.6), we obtain

$$\sup_{t \ge 0} I_1(t) \le \widetilde{M} b \max\Big\{d_{\mathrm{TV}}(\widetilde{f}, f), \inf_{a \in \mathbb{R}} \tfrac{1}{2}\mathbb{K}_2(\widetilde{h}_a, h_a)\Big\},$$

which together with (4.2) and the bounds given in the proof of Proposition 2.1 provides the desired inequality (2.4).

4.3. Proof of Proposition 3.1. By (1.3) and Fubini's theorem,

$$\begin{aligned} E\mathbb{K}_2(F, \hat{F}_n) &= 2E \int_{-\infty}^{\infty} |x||F(x) - \hat{F}_n(x)|\, dx \\ &= 2 \int_{-\infty}^{\infty} |x| E|F(x) - \hat{F}_n(x)|\, dx \\ &\le 2 \int_{-\infty}^{\infty} |x| \big(E[F(x) - \hat{F}_n(x)]^2\big)^{1/2}\, dx. \end{aligned}$$

In view of (3.3), $E[F(x) - \hat{F}_n(x)]^2 = \frac{1}{n}[F(x)(1 - F(x))]$. Hence

$$\begin{aligned} E\mathbb{K}_2(F, \hat{F}_n) &\le \frac{2}{\sqrt{n}} \int_{-\infty}^{\infty} |x|[F(x)(1 - F(x))]^{1/2}\, dx \\ &= \frac{2}{\sqrt{n}} \int_{-1}^{1} |x|[F(x)(1 - F(x))]^{1/2}\, dx \\ &\quad + \frac{2}{\sqrt{n}} \int_{-\infty}^{-1} |x|[F(x)(1 - F(x))]^{1/2}\, dx \\ &\quad + \frac{2}{\sqrt{n}} \int_{1}^{\infty} x[F(x)(1 - F(x))]^{1/2}\, dx. \end{aligned} \tag{4.10}$$

By the Markov inequality, for $x \ge 1$,

$$F(x)(1 - F(x)) \le (1 - F(x)) \le P(|T_1 - a| > x) \le \frac{\mathcal{H}}{x^{4+\alpha}}.$$

Also, for $x \leq -1$,

$$F(x)(1-F(x)) \leq F(x) \leq P(|T_1 - a| \geq -x) \leq \frac{\mathcal{H}}{|x|^{4+\alpha}}.$$

Substituting these inequalities in (4.10) and integrating with $F(1-F) \leq 1/4$, we obtain (3.4).

5. Concluding remark. The bounds (2.2) and (2.4) could be applied in the following two situations. First, when the "parameters" of the real risk model (0.1) are unknown, and they are approximated by, say, statistical estimators. Second, when the densities f_0 and f in (0.1) are available, but they are too complicated to be used in (0.2). It might be practical to approximate f_0 and f by more tractable densities $\widetilde{f}_0$ and $\widetilde{f}$, which would make it possible to use (0.2).

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E. Gordienko, J. Ruiz de Chávez, P. Vázquez-Ortega
Departamento de Matemáticas
UAM–Iztapalapa
San Rafael Atlixco 186
09340 Iztapalapa, México City, México
E-mail: gord@xanum.uam.mx
jrch@xanum.uam.mx
patricia.v.ortega@gmail.com