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Square roots of the Bessel operators and the related Littlewood–Paley estimates

by

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Abstract. Let Δ_λ and S_λ, $\lambda \in \mathbb{R}_+ := (0, +\infty)$, be the two Bessel operators studied by Muckenhoupt–Stein (1965). We prove that the square root of the Bessel operators and the corresponding "gradient" operators are equivalent in L^p spaces for $1 < p < \infty$. Moreover, by using holomorphic functional calculus, we establish the characterizations of boundedness on L^p spaces associated with Bessel operators in terms of the Littlewood–Paley g-function with respect to the square root of the Bessel operator. Also, we study boundedness properties of Littlewood–Paley g-function associated with the square root of the Bessel operator on the odd BMO space BMO_+ and the atomic Hardy space H^1.

1. Introduction. The gradient operator ∇ and the square root of the Laplace operator $\sqrt{-\Delta}$ in the Euclidean space $\mathbb{R}^n$ are two fundamental operators in harmonic analysis and partial differential equations. The nature of the operator ∇ is local and geometric while $\sqrt{-\Delta}$ is non-local and more analytic. It is well known that the seminorms $\|\nabla f\|_{L^p}$ and $\|\sqrt{-\Delta}\, f\|_{L^p}$ are equivalent for all $1 < p < \infty$ (see for example [34]). Some extension of this kind of equivalence for more general differential operators and square roots of self-adjoint operators have been studied extensively, for example the well-known Kato conjecture was proved in [5] (see also [3, 26]).

In 1965, Muckenhoupt and Stein [31] introduced a notion of conjugacy associated with the Bessel operators Δ_λ defined by

$$\Delta_\lambda f(x) := -D^2 f(x) - \frac{2\lambda}{x} Df(x) = -\left(D + \frac{2\lambda}{x}\right) Df(x) =: -D^* Df(x), \tag{1.1}$$

where $\lambda > 0$, $D := d/dx$, $x \in \mathbb{R}_+$. They started a study of the setting

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of Δ_λ which is parallel to the classical case associated to Δ. Since then, the Bessel operators have been extensively studied [2, 10, 7, 11, 12, 28, 33, 35, 36, 38]. The operator Δ_λ is non-negative self-adjoint and a basic question is whether the norms $\|\sqrt{\Delta_\lambda}\, f\|_{L^p(\mathbb{R}_+,dm_\lambda)}$ and $\|Df\|_{L^p(\mathbb{R}_+,dm_\lambda)}$ are equivalent for all $p \in (1,\infty)$, where $dm_\lambda(x) := x^{2\lambda}dx$ and dx is the Lebesgue measure. It is worth noting that the solution to the standard Kato square root problem is not applicable to the square root of the Bessel operator because of the lower order term with unbounded coefficients.

For $p \in [1,\infty)$ and $k \in \mathbb{N}^* := \{1,2,\dots\}$, define the Sobolev space $L^p_k(\mathbb{R}_+, dm_\lambda)$ to be the closure of $C_0^\infty(\mathbb{R}_+)$ under the norm $\|f\|_{L^p_k(\mathbb{R}_+,dm_\lambda)} = \sum_{|\alpha|\le k} \|D^\alpha f\|_{L^p(\mathbb{R}_+,dm_\lambda)}$.

Our first aim is to study the relationship between $\sqrt{\Delta_\lambda}$ and D in terms of the $L^p(\mathbb{R}_+, dm_\lambda)$ norm.

Theorem 1.1. *Let $\lambda > 0$, and let Δ_λ be defined as in* (1.1). *Then*

$$\|\sqrt{\Delta_\lambda}\, f\|_{L^p(\mathbb{R}_+,dm_\lambda)} \sim \|Df\|_{L^p(\mathbb{R}_+,dm_\lambda)} \quad \text{for } 1 < p < \infty. \tag{1.2}$$

Furthermore, the operator $\sqrt{\Delta_\lambda}$, a priori defined on $C_0^\infty(\mathbb{R}_+)$, extends to a bounded and invertible operator from $L^p_1(\mathbb{R}_+, dm_\lambda)$ onto $L^p(\mathbb{R}_+, dm_\lambda)$.

Remark 1.2. The $L^p(\mathbb{R}_+, dm_\lambda)$-boundedness $(1 < p < \infty)$ of the Bessel Riesz transform $D(\sqrt{\Delta_\lambda})^{-1}$ was studied by Muckenhoupt–Stein [31], Betancor et al. [10] and Villani [36], leading to

$$\|D(\sqrt{\Delta_\lambda})^{-1} f\|_{L^p(\mathbb{R}_+,dm_\lambda)} \le C\|f\|_{L^p(\mathbb{R}_+,dm_\lambda)}.$$

This implies that

$$\|Df\|_{L^p(\mathbb{R}_+,dm_\lambda)} \le C\|\sqrt{\Delta_\lambda}\, f\|_{L^p(\mathbb{R}_+,dm_\lambda)}, \tag{1.3}$$

for f in some dense subspace of $L^p_1(\mathbb{R}_+, dm_\lambda)$.

Moreover, the lower bound of the Bessel Riesz transform was obtained in [36, Theorem 3] (see also [31]), which implies a reverse direction of the inequality (1.3). Here we provide a direct proof via holomorphic functional calculus, which is different from [36].

In 2004, Auscher, Duong and McIntosh [4] introduced the Littlewood–Paley theory associated with the operator L which has a bounded H^∞-functional calculus on $L^2(X)$ where X is a space of homogeneous type. They pointed out that the square function norms of L are equivalent to the original norms on $L^p(X)$ for $1 < p < \infty$.

The second purpose of this paper is to study the Littlewood–Paley g-function associated with the Bessel operators. Let

$$\psi(t\Delta_\lambda) := \sqrt{t\Delta_\lambda}\, e^{-t\Delta_\lambda} \tag{1.4}$$

and define the quadratic functional $g_{\Delta_\lambda}(f)$ for $f \in L^2(\mathbb{R}_+, dm_\lambda)$ by

$$g_{\Delta_\lambda}(f)(x) := \Big(\int_0^\infty |\psi(t\Delta_\lambda)f(x)|^2 \frac{dt}{t}\Big)^{1/2}. \tag{1.5}$$

It follows from the result in [17, Theorem 6.6] that the $L^p(\mathbb{R}_+, dm_\lambda)$ norms of $g_{\Delta_\lambda}(f)$ and f are equivalent for $1 < p < \infty$. In this paper, we study the endpoint estimate when $p = 1$ and obtain the weak type $(1,1)$ estimate for g_{Δ_λ} (Theorem 1.3) and its boundedness from the Hardy space (associated to Δ_λ) to $L^1(\mathbb{R}_+, dm_\lambda)$ (Theorem 1.6).

THEOREM 1.3. *Let g_{Δ_λ} be defined as in* (1.5). *Then g_{Δ_λ} is of weak type* $(1,1)$.

We note that by using Theorem 1.3, Marcinkiewicz interpolation and duality, we then recover the following known result of [17].

PROPOSITION 1.4. *Let g_{Δ_λ} be defined as in* (1.5) *and $p \in (1, \infty)$. Then for all $f \in L^p(\mathbb{R}_+, dm_\lambda)$,*

$$\|g_{\Delta_\lambda}(f)\|_{L^p(\mathbb{R}_+, dm_\lambda)} \sim \|f\|_{L^p(\mathbb{R}_+, dm_\lambda)}.$$

We now recall the notion of a $(1, 2, M)$-atom from [27], which is associated to the operator Δ_λ on the space $(\mathbb{R}_+, dm_\lambda)$.

DEFINITION 1.5. A function $a \in L^2(\mathbb{R}_+, dm_\lambda)$ is called a $(1, 2, M)$-*atom* associated to the operator Δ_λ if there exist a function $b \in \mathcal{D}(\Delta_\lambda^M)$ and an interval I such that

(i) $a = \Delta_\lambda^M b$;
(ii) $\operatorname{supp} \Delta_\lambda^k b \subset I$, $k = 0, 1, \ldots, M$;
(iii) $\|(l^2\Delta_\lambda)^k b\|_{L^2(\mathbb{R}_+, dm_\lambda)} \le l^{2M} m_\lambda(I)^{-1/2}$, $k = 0, 1, \ldots, M$;

here $\mathcal{D}(\Delta_\lambda)$ denotes the domain of the operator Δ_λ, Δ_λ^k denotes the k-fold composition of Δ_λ with itself and l denotes the length of I.

The atomic Hardy space $H^1(\mathbb{R}_+, dm_\lambda)$ is defined as

$$H^1(\mathbb{R}_+, dm_\lambda) := \Big\{\sum_{j=0}^\infty \alpha_j a_j : \{\alpha_j\}_{j=0}^\infty \in \ell^1 \text{ and } a_j\text{'s are } (1, 2, M)\text{-atoms}\Big\},$$

where the series converges in $L^1(\mathbb{R}_+, dm_\lambda)$, with the norm given by

$$\|f\|_{H^1(\mathbb{R}_+, dm_\lambda)} := \inf\Big\{\sum_{j=0}^\infty |\alpha_j| : f = \sum_{j=0}^\infty \alpha_j a_j,\ \{\alpha_j\}_{j=0}^\infty \in \ell^1 \text{ and } a_j\text{'s are } (1, 2, M)\text{-atoms}\Big\}.$$

It is well known that the Hardy space is a good substitute for L^1 space for the study of singular integrals. For the boundedness of g_{Δ_λ} into $L^1(\mathbb{R}_+, dm_\lambda)$, we get the following.

THEOREM 1.6. *Let g_{Δ_λ} be defined as in* (1.5). *Then g_{Δ_λ} is bounded from $H^1(\mathbb{R}_+, dm_\lambda)$ into $L^1(\mathbb{R}_+, dm_\lambda)$.*

The other part of this paper is to establish a version of the above theorems in the following Bessel operator context, also introduced in [31]. For $\lambda > 0$ we consider the Bessel operator on $\mathbb{R}_+$,

$$S_\lambda := -D^2 + \frac{\lambda(\lambda-1)}{x^2}.$$

The operator S_λ is a non-negative self-adjoint operator in $L^2(\mathbb{R}_+, dx)$ and it can be written in divergence form as

$$S_\lambda = x^{-\lambda} D x^{2\lambda} D x^{-\lambda} =: A_\lambda^* A_\lambda, \tag{1.6}$$

where $A_\lambda^* := x^{-\lambda} D x^{\lambda}$ is the adjoint of $A_\lambda := x^{\lambda} D x^{-\lambda}$ (see [31]).

Our next result is to prove the equivalence of the norms $\|\sqrt{S_\lambda}\, f\|_{L^p(\mathbb{R}_+,dx)}$ and $\|A_\lambda f\|_{L^p(\mathbb{R}_+,dx)}$ for $f \in L_1^p(\mathbb{R}_+, dx)$, where $L_1^p(\mathbb{R}_+, dx)$ is the Sobolev space, studied in [24].

THEOREM 1.7. *Let $\lambda > 0$, and let S_λ be defined as in* (1.6). *Then*

$$\|\sqrt{S_\lambda}\, f\|_{L^p(\mathbb{R}_+,dx)} \sim \|A_\lambda f\|_{L^p(\mathbb{R}_+,dx)} \quad \text{for } 1 < p < \infty.$$

Furthermore, the operator $\sqrt{S_\lambda}$, a priori defined on $C_0^\infty(\mathbb{R}_+)$, extends to a bounded and invertible operator from $L_1^p(\mathbb{R}_+, dx)$ onto $L^p(\mathbb{R}_+, dx)$.

REMARK 1.8. Betancor et al. [10] and Villani [36] investigated $L^p(\mathbb{R}_+, dx)$ boundedness ($1 < p < \infty$) of the Riesz transforms associated with the Bessel operators S_λ, which implies that

$$\|A_\lambda f\|_{L^p(\mathbb{R}_+,dx)} \le C \|\sqrt{S_\lambda}\, f\|_{L^p(\mathbb{R}_+,dx)}, \tag{1.7}$$

for f in some dense subspace of $L_1^p(\mathbb{R}_+, dx)$.

Moreover, the lower bound of the Riesz transforms associated with S_λ was obtained [36, Theorem 8], which implies a reverse direction of the inequality (1.7). Here we provide a different, direct proof.

For $f \in L^2(\mathbb{R}_+, dx)$, let

$$g_{S_\lambda}(f)(x) := \left(\int_0^\infty |\sqrt{tS_\lambda}\, e^{-tS_\lambda} f(x)|^2 \, \frac{dt}{t} \right)^{1/2}. \tag{1.8}$$

According to [17], the $L^p(\mathbb{R}_+, dx)$ norms of $g_{S_\lambda}(f)$ and f are equivalent for $1 < p < \infty$. We also have a result similar to Theorem 1.3, by following the technique of Calderón–Zygmund decomposition originated from [20]. Parallel to Theorem 1.6, we will also have the endpoint estimate for g_{S_λ} on the

Hardy space associated to S_λ (see [9, 22] for the definition of Hardy spaces associated with S_λ) and a weak type estimate.

Theorem 1.9. *Let g_{S_λ} be defined as in* (1.8). *Then g_{S_λ} is of weak type* $(1,1)$. *Moreover, g_{S_λ} is bounded from $H^1_{S_\lambda}(\mathbb{R}_+, dx)$ into $L^1(\mathbb{R}_+, dx)$.*

Our next result is the boundedness of the operator g_{S_λ} at the end-point $p=\infty$ with L^∞ replaced by BMO_+, the space of all those $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ such that the odd extension f_0 of f to $\mathbb{R}$ is in $\mathrm{BMO}(\mathbb{R})$.

Theorem 1.10. *Let g_{S_λ} be defined as in* (1.8). *There exists $C>0$ such that for any $f \in \mathrm{BMO}_+$,*

$$\|g_{S_\lambda}(f)\|_{\mathrm{BMO}_+} \le C\|f\|_{\mathrm{BMO}_+}.$$

Betancor et al. [6] obtained the boundedness on Hardy and the odd BMO spaces for certain Littlewood–Paley g-functions of order $\beta>0$ and related to the Poisson semigroup associated with the Bessel operator S_λ. The BMO-boundedness of the Littlewood–Paley g-function associated with S_λ, $\big(\int_0^\infty |tS_\lambda e^{-tS_\lambda} f(x)|^2 \frac{dt}{t}\big)^{1/2}$, was studied in [7]. We point out that the boundedness on BMO_+ of g_{S_λ} (defined as in (1.8)) is not covered by [6] or [7]. Compared to [7], our technique is different in the kernel estimate, due to the square root of S_λ.

Remark 1.11. We note that our results in Theorems 1.3 and 1.6, and in Theorems 1.9 and 1.10, also hold for the Littlewood–Paley square functions of the forms

$$\Big(\int_0^\infty |(t\Delta_\lambda)^{k/2} e^{-t\Delta_\lambda} f(x)|^2 \frac{dt}{t}\Big)^{1/2} \quad \text{and} \quad \Big(\int_0^\infty |(tS_\lambda)^{k/2} e^{-tS_\lambda} f(x)|^2 \frac{dt}{t}\Big)^{1/2},$$

respectively. Specifically, if we denote Δ_λ or S_λ by L, it is easy to see that the estimation for the heat kernel is also true for $(tL)^n e^{-tL}$ with $n \in \mathbb{N}^*$. This ensures that we can characterize the operator $\sqrt{tL}\,(tL)^n e^{-tL}$, same as $\sqrt{tL}\,e^{-tL}$ in the proofs of the above theorems. Betancor et al. have studied the boundedness of (L^p, L^p), weak type $(1,1)$ and (H^1, L^1) for

$$\Big(\int_0^\infty |(t\sqrt{\Delta_\lambda})^k e^{-t\sqrt{\Delta_\lambda}} f(x)|^2 \frac{dt}{t}\Big)^{1/2}$$

in [11, Theorem 1.4].

Parallel to Theorem 1.10, it is natural to consider the boundedness of the Littlewood–Paley square function g_{Δ_λ} defined in (1.5), as well as of the form $\big(\int_0^\infty |(t\Delta_\lambda)^{k/2} e^{-t\Delta_\lambda} f(x)|^2 \frac{dt}{t}\big)^{1/2}$, on the end-point BMO space $\mathrm{BMO}_{\Delta_\lambda}(\mathbb{R}_+, dm_\lambda)$. However, the expected outcome and proof will be different from Theorem 1.10 above. We will study this estimate in a subsequent paper.

The paper is organized as follows. In Section 2, we work in the setting of Δ_λ. We first show the inverse inequality of the L^p-boundedness of the Riesz transform in Section 2.1, and next, we establish the characterizations of L^p space associated with Bessel operators in terms of the Littlewood–Paley g-function in Section 2.2. Finally, we study boundedness of Littlewood–Paley g-function associated with Δ_λ from H^1 into L^1 in Section 2.3. In Section 3, we consider S_λ and give the proofs of Theorems 1.7, 1.9 and 1.10. As an appendix, we verify the Hölder regularity estimate of the vector-valued kernel of the Littlewood–Paley g-function associated with Δ_λ in Section 4.

Throughout this paper, p' denotes the dual exponent of p, and c and C always denote some positive constants that may be different at each occurrence.

2. Square root and Littlewood–Paley square function associated with Δ_λ. To begin, we recall some necessary notation.

By using the formula $s^{-a} = \frac{1}{\Gamma(a)}\int_0^\infty e^{-ts}t^a\,\frac{dt}{t}$ $(s, a > 0)$, the square root operator of Δ_λ can be written in several ways. We use the following resolution of the square root $\sqrt{\Delta_\lambda}$:

$$\sqrt{\Delta_\lambda}\, f = \frac{8}{\Gamma(3)} \int_0^\infty t^3 \Delta_\lambda^2 e^{-2t\sqrt{\Delta_\lambda}} f \,\frac{dt}{t}, \tag{2.1}$$

where the Poisson semigroup $\{e^{-t\sqrt{\Delta_\lambda}}\}_{t>0}$ generated by the operator $-\sqrt{\Delta_\lambda}$ is a contraction semigroup on $L^r(\mathbb{R}_+, dm_\lambda)$ for all $r \in [1,\infty]$ (see [38, p. 361]). We write the Poisson semigroup as follows:

$$P_t^{[\lambda]} f(x) = e^{-t\sqrt{\Delta_\lambda}} f(x) = \int_0^\infty P_t^{[\lambda]}(x,y) f(y)\, dm_\lambda(y),$$

with the associated Poisson kernel

$$P_t^{[\lambda]}(x,y) = \frac{2\lambda t}{\pi} \int_0^\pi \frac{(\sin\theta)^{2\lambda-1}}{(x^2+y^2+t^2-2xy\cos\theta)^{\lambda+1}}\, d\theta, \qquad t, x, y \in \mathbb{R}_+.$$

Also, we recall the definition of the heat semigroup $\{W_t^{[\lambda]}\}_{t>0}$ generated by $-\Delta_\lambda$,

$$W_t^{[\lambda]} f(x) = e^{-t\Delta_\lambda} f(x) = \int_0^\infty W_t^{[\lambda]}(x,y) f(y)\, dm_\lambda(y),$$

where the heat kernel associated with the Bessel operator is given by

$$W_t^{[\lambda]}(x,y) = \frac{(xy)^{1/2-\lambda}}{2t} e^{-\frac{x^2+y^2}{4t}} I_{\lambda-1/2}\left(\frac{xy}{2t}\right), \qquad t, x, y \in \mathbb{R}_+, \tag{2.2}$$

and I_ν represents the modified Bessel function of the first kind and order ν.

We first recall some properties of the first kind modified Bessel function in the complex version $I_\nu(z)$, $\nu \in (-1,\infty)$ and $z \in \mathbb{C}_+$ (see [29, Chap. 5]):

$$\partial_z(z^{-\nu}I_\nu(z)) = z^{-\nu}I_{\nu+1}(z), \tag{2.3}$$

$$\lim_{z\to 0} z^{-\nu}I_\nu(z) = \frac{1}{2^\nu \Gamma(\nu+1)}, \tag{2.4}$$

and for $|\arg(z)| \le \pi/2-\varepsilon$, $\varepsilon \in (0,\pi/2]$,

$$I_\nu(z) = \frac{e^z}{\sqrt{2\pi z}}\Big(\sum_{k=0}^{n}(-1)^k[\nu,k](2z)^{-k} + \mathcal{O}(|z|^{-n-1})\Big), \tag{2.5}$$

where $n \in \mathbb{N}$, $[\nu,0] := 1$ and $[\nu,k] := \frac{(4\nu^2-1)\cdots(4\nu^2-(2k-1)^2)}{2^{2k}\Gamma(k+1)}$, $k \in \mathbb{N}^*$.

Based on the expression of $W_t^{[\lambda]}$ as in (2.2), we see that it extends to a smooth kernel $W_z^{[\lambda]}(x,y)$ for $x,y \in \mathbb{R}_+$ and it is analytic in $z \in \mathbb{C}_+$,

$$W_z^{[\lambda]}(x,y) = \frac{(xy)^{1/2-\lambda}}{2z}e^{-\frac{x^2+y^2}{4z}} I_{\lambda-1/2}\Big(\frac{xy}{2z}\Big). \tag{2.6}$$

It is known that the heat kernel $W_t^{[\lambda]}(x,y)$ satisfies the following upper bound (see for example [12, (10)]):

$$|W_t^{[\lambda]}(x,y)| \le Ct^{-\lambda-1/2}e^{-\frac{|x-y|^2}{4t}} =: H_t^{[\lambda]}(x,y). \tag{2.7}$$

From (2.7), by applying the Coulhon–Sikora result of [16, Theorem 4.1], we find that

$$|W_z^{[\lambda]}(x,y)| \le C|z|^{-\lambda-1/2}e^{-\mathrm{Re}\frac{|x-y|^2}{cz}}. \tag{2.8}$$

2.1. Proof of Theorem 1.1. We may assume that $f \in C_0^\infty(\mathbb{R}_+)$. Take $g \in C_0^\infty(\mathbb{R}_+)$ such that $\|g\|_{L^{p'}(\mathbb{R}_+,dm_\lambda)} = 1$. By using (2.1) and the Hölder inequality, we get

$$\begin{aligned}\Big|\int_{\mathbb{R}_+} \sqrt{\Delta_\lambda}\, f(x)g(x)\, dm_\lambda(x)\Big| &= \Big|\frac{8}{\Gamma(3)}\int_0^\infty\int_{\mathbb{R}_+} t^3\Delta_\lambda^2 e^{-2t\sqrt{\Delta_\lambda}} f(x)g(x)\, dm_\lambda(x)\,\frac{dt}{t}\Big| \\ &\le C\int_{\mathbb{R}_+}\Big(\int_0^\infty |e^{-t\sqrt{\Delta_\lambda}} t\Delta_\lambda f(x)|^2\,\frac{dt}{t}\Big)^{1/2} \\ &\quad\times\Big(\int_0^\infty |e^{-t\sqrt{\Delta_\lambda}} t^2\Delta_\lambda g(x)|^2\,\frac{dt}{t}\Big)^{1/2} dm_\lambda(x)\end{aligned}$$

$$\leq C\Big\|\Big(\int_0^\infty |e^{-t\sqrt{\Delta_\lambda}}tD^*Df(\cdot)|^2\,\frac{dt}{t}\Big)^{1/2}\Big\|_{L^p(\mathbb{R}_+,dm_\lambda)}$$
$$\times\Big\|\Big(\int_0^\infty |e^{-t\sqrt{\Delta_\lambda}}t^2\Delta_\lambda g(\cdot)|^2\,\frac{dt}{t}\Big)^{1/2}\Big\|_{L^{p'}(\mathbb{R}_+,dm_\lambda)}.$$

It is well known that for $1 < p < \infty$ (see [35]),

$$\Big\|\Big(\int_0^\infty |t^2\Delta_\lambda e^{-t\sqrt{\Delta_\lambda}}g|^2\,\frac{dt}{t}\Big)^{1/2}\Big\|_{L^{p'}(\mathbb{R}_+,dm_\lambda)}$$
$$= \Big\|\Big(\int_0^\infty |t^2\partial_t^2 e^{-t\sqrt{\Delta_\lambda}}g|^2\,\frac{dt}{t}\Big)^{1/2}\Big\|_{L^{p'}(\mathbb{R}_+,dm_\lambda)} \leq C\|g\|_{L^{p'}(\mathbb{R}_+,dm_\lambda)}.$$

Thus, to obtain the estimate (1.2), let $h := Df$; it suffices to verify that for $p \in (1,\infty)$,

$$\Big\|\Big(\int_0^\infty |e^{-t\sqrt{\Delta_\lambda}}tD^*h|^2\,\frac{dt}{t}\Big)^{1/2}\Big\|_{L^p(\mathbb{R}_+,dm_\lambda)} \leq C\|h\|_{L^p(\mathbb{R}_+,dm_\lambda)}. \tag{2.9}$$

We use integration by parts to deduce that for any $x \in \mathbb{R}_+$,

$$\begin{aligned} e^{-t\sqrt{\Delta_\lambda}}tD^*h(x) &= \int_0^\infty P_t^{[\lambda]}(x,y)t(D+2\lambda/y)h(y)\,dm_\lambda(y) \\ &= \int_0^\infty P_t^{[\lambda]}(x,y)tDh(y)\,dm_\lambda(y) + P_t^{[\lambda]}(x,y)th(y)y^{2\lambda}\Big|_0^\infty \\ &\quad - \int_0^\infty DP_t^{[\lambda]}(x,y)th(y)y^{2\lambda}\,dy - \int_0^\infty P_t^{[\lambda]}(x,y)tDh(y)y^{2\lambda}\,dy \\ &= -\int_0^\infty DP_t^{[\lambda]}(x,y)th(y)\,dm_\lambda(y) =: -Q_t(h)(x). \end{aligned} \tag{2.10}$$

It is well known (see [19, Theorem 2.12] or [38, Theorem 1.1]) that

$$\Big\|\Big(\int_0^\infty |Q_t(h)|^2\,\frac{dt}{t}\Big)^{1/2}\Big\|_{L^p(\mathbb{R}_+,dm_\lambda)} \leq C\|h\|_{L^p(\mathbb{R}_+,dm_\lambda)}, \qquad 1 < p < \infty. \tag{2.11}$$

Combining (2.9)–(2.11), we obtain

$$\Big\|\Big(\int_0^\infty |e^{-t\sqrt{\Delta_\lambda}}tD^*h|^2\,\frac{dt}{t}\Big)^{1/2}\Big\|_{L^p(\mathbb{R}_+,dm_\lambda)}.$$

The proof of Theorem 1.1 is complete.

2.2. Proofs of Theorem 1.3 and Proposition 1.4. We first need an auxiliary kernel estimate of the resolvent $(\Delta_\lambda - \zeta\mathcal{I})^{-1}$. We recall the holomorphic functional calculus introduced in [30]. For $0 \le \omega < v < \pi$, we consider the closed sector $S_\omega := \{z \in \mathbb{C} : |\arg z| \le \omega\} \cup \{0\}$ and let S_ω^0 be the interior of S_ω. We employ the following subspaces of the space $H(S_v^0)$ of all holomorphic functions on S_v^0: $H_\infty(S_v^0) := \{b \in H(S_v^0) : \|b\|_\infty < \infty\}$, where $\|b\|_\infty := \sup\{|b(z)| : z \in S_v^0\}$, and

$$\Psi(S_v^0) := \{\phi \in H(S_v^0) : \exists s > 0,\ |\phi(z)| \le C|z|^s(1 + |z|^{2s})^{-1}\}. \tag{2.12}$$

Let $0 \le \omega < \pi$. An operator L is said to be of type ω if its spectrum $\sigma(L)$ is contained in S_ω and for each $v > \omega$ there exists a constant c_v such that

$$\|(L - z\mathcal{I})^{-1}\| \le c_v|z|^{-1}, \qquad z \notin S_v.$$

If L is of type ω and $\phi \in \Psi(S_v^0)$, then we define $\phi(L) \in \mathcal{L}(L^2, L^2)$ by

$$\phi(L) := \frac{1}{2\pi i} \int_\Gamma (L - \zeta\mathcal{I})^{-1}\phi(\zeta)\, d\zeta,$$

where Γ is the contour

$$\Gamma = \Gamma_+ \cup \Gamma_- := \{Re^{i\mu} : R \in \mathbb{R}_+\} \cup \{Re^{-i\mu} : R \in \mathbb{R}_+\} \tag{2.13}$$

parameterized clockwise around S_ω and $\omega < \mu < v$. If L is a non-negative self-adjoint operator, then L is of type ω for any $\omega \in (0, \pi)$ (see [1, p. 88] or [18]).

Let $G_\zeta(x, y)$ be the kernel of $(\Delta_\lambda - \zeta\mathcal{I})^{-1}$, $\Psi(S_v^0)$ be defined as in (2.12), $\phi \in \Psi(S_v^0)$, $\phi_t(x) := \phi(tx)$ and

$$k_\phi(x, y) = \int_\Gamma G_\zeta(x, y)\phi_t(\zeta)\, d\zeta, \tag{2.14}$$

where Γ is the contour as in (2.13). By using the technique via holomorphic functional calculus, we can obtain the following lemma.

LEMMA 2.1. *Let $\lambda > 0$. There exists a positive constant C such that for any distinct $x, y \in \mathbb{R}_+$,*

$$\left(\int_0^\infty |k_\phi(x, y)|^2 \frac{dt}{t}\right)^{1/2} \le C \frac{1}{|x - y| \max\{x^{2\lambda}, y^{2\lambda}\}},$$

where $k_\phi(x, y)$ is as in (2.14).

Proof. To deduce the bound of the Green function $G_\zeta(x, y)$ for $\zeta \in \Gamma$ and $\omega < \mu < v$, we write

$$(\Delta_\lambda - \zeta\mathcal{I})^{-1} = \int_{\gamma_\zeta} e^{\zeta z} e^{-z\Delta_\lambda}\, dz,$$

where

$$\gamma_\zeta = \begin{cases} \{re^{i\theta} : r \in \mathbb{R}_+\} & \text{if } \zeta \in \Gamma_+, \\ \{re^{-i\theta} : r \in \mathbb{R}_+\} & \text{if } \zeta \in \Gamma_-, \end{cases} \tag{2.15}$$

and θ is chosen to satisfy $\theta \in (0, \pi/2)$ and $\mu + \theta > \pi/2$. Then we write

$$G_\zeta(x,y) = \int_{\gamma_\zeta} e^{\zeta z} W_z^{[\lambda]}(x,y)\, dz.$$

By using (2.14), we have

$$|k_\phi(x,y)| = \Big|\int_\Gamma \int_{\gamma_\zeta} e^{\zeta z} W_z^{[\lambda]}(x,y)\, dz \cdot \phi(t\zeta)\, d\zeta\Big|.$$

First, we claim that for any $z \in \gamma_\zeta$ and $\zeta \in \Gamma$,

$$\int_0^\infty |e^{\zeta z}\phi(t\zeta)|\, d|\zeta| \le C\frac{1}{t}\Big(\frac{t}{r}\Big)^k, \qquad k = s+1 \text{ or } 1-\alpha, \tag{2.16}$$

where $\alpha \in (0,s) \cap (0,1/2)$.

In fact, for any $z \in \gamma_\zeta$ and $\zeta \in \Gamma$, $\mathrm{Re}(z\zeta) = rR\cos(\theta+\mu) < 0$. Then by (2.12), we see that

$$\int_0^\infty |e^{\zeta z}\phi(t\zeta)|\, d|\zeta| \le C\int_0^\infty e^{rR\cos(\theta+\mu)} \frac{(tR)^s}{1+(tR)^{2s}}\, dR. \tag{2.17}$$

Notice that

$$\frac{(tR)^s}{1+(tR)^{2s}} \le \min\{(tR)^s, (tR)^{-\alpha}\} \qquad \text{for } \alpha \in (0,s) \cap (0,1/2). \tag{2.18}$$

Inserting (2.18) into (2.17), by a change of variable, we get (2.16). Now, from (2.8) and (2.16) and making the change $r = tu$ we have

$$\begin{aligned} |k_\phi(x,y)| &\le C\int_0^\infty r^{-\lambda-1/2} e^{-\frac{|x-y|^2}{cr}} \frac{1}{t}\Big(\frac{t}{r}\Big)^k dr \\ &= Ct^{-\lambda-1/2}\int_0^\infty u^{-\lambda-1/2-k} e^{-c\frac{|x-y|^2}{tu}}\, du \le C\frac{t^{k-1}}{|x-y|^{2\lambda-1+2k}}. \end{aligned}$$

Taking $k = s+1$ and $k = 1-\alpha$, $\alpha \in (0,s) \cap (0,1/2)$ respectively we obtain for any distinct x, y,

$$|k_\phi(x,y)| \le C\frac{t^s}{|x-y|^{2\lambda+1+2s}}, \tag{2.19}$$

$$|k_\phi(x,y)| \le C\frac{1}{t^\alpha |x-y|^{2\lambda+1-2\alpha}}. \tag{2.20}$$

On the other hand, by (2.4)–(2.6) and (2.16) we can write

$$|k_\phi(x,y)| \le C\Big(\frac{1}{(xy)^\lambda}\int_0^{\frac{xy}{2}} \frac{e^{-c\frac{|x-y|^2}{r}}}{\sqrt{r}}\,\frac{1}{t}\Big(\frac{t}{r}\Big)^k dr + \int_{\frac{xy}{2}}^\infty \frac{1}{r^{\lambda+1/2}} e^{-c\frac{x^2+y^2}{r}}\,\frac{1}{t}\Big(\frac{t}{r}\Big)^k dr\Big)$$
$$\le C\Big(\frac{1}{(xy)^\lambda\sqrt{t}}\int_0^{\frac{xy}{2t}} \frac{e^{-c\frac{|x-y|^2}{tu}}}{u^{k+1/2}}\,du + \frac{1}{t^{\lambda+1/2}}\int_{\frac{xy}{2t}}^\infty \frac{e^{-c\frac{x^2+y^2}{tu}}}{u^{\lambda+1/2+k}}\,du\Big)$$
$$\le C\frac{1}{(xy)^\lambda\sqrt{t}}\int_0^\infty \frac{e^{-c\frac{|x-y|^2}{tu}}}{u^{k+1/2}}\,du,$$

where in the last inequality we have taken into account that

$$\frac{e^{-c\frac{x^2+y^2}{tu}}}{(tu)^\lambda} \le C\frac{e^{-c\frac{x^2+y^2}{tu}}}{(x^2+y^2)^\lambda} \le C\frac{e^{-c\frac{|x-y|^2}{tu}}}{(xy)^\lambda}, \qquad x,y,t,u\in\mathbb{R}_+. \tag{2.21}$$

Now arguing as before, taking $k = s+1$ and $k = 1-\alpha$, $\alpha\in(0,s)\cap(0,1/2)$ respectively, we get

$$|k_\phi(x,y)| \le C\frac{1}{(xy)^\lambda\sqrt{t}}\int_0^\infty \frac{e^{-c\frac{|x-y|^2}{tu}}}{u^{s+3/2}}\,du \le C\frac{t^s}{(xy)^\lambda|x-y|^{1+2s}}, \tag{2.22}$$

$$|k_\phi(x,y)| \le C\frac{1}{(xy)^\lambda\sqrt{t}}\int_0^\infty \frac{e^{-c\frac{|x-y|^2}{tu}}}{u^{3/2-\alpha}}\,du \le C\frac{1}{t^\alpha(xy)^\lambda|x-y|^{1-2\alpha}}. \tag{2.23}$$

From (2.19) and (2.22), it follows that

$$|k_\phi(x,y)| \le C\frac{t^s}{|x-y|^{1+2s}\max\{x^{2\lambda},y^{2\lambda}\}}, \tag{2.24}$$

and by (2.20) and (2.23),

$$|k_\phi(x,y)| \le C\frac{1}{t^\alpha|x-y|^{1-2\alpha}\max\{x^{2\lambda},y^{2\lambda}\}}. \tag{2.25}$$

Therefore, according to (2.24) and (2.25), we obtain

$$\int_0^\infty |k_\phi(x,y)|^2\,\frac{dt}{t} \le C\int_0^{|x-y|^2} \frac{t^{2s-1}}{\max\{x^{4\lambda},y^{4\lambda}\}|x-y|^{2+4s}}\,dt$$
$$+ C\int_{|x-y|^2}^\infty \frac{1}{t^{2\alpha+1}\max\{x^{4\lambda},y^{4\lambda}\}|x-y|^{2-4\alpha}}\,dt$$
$$\le C\frac{1}{\max\{x^{4\lambda},y^{4\lambda}\}|x-y|^2}.$$

The proof of Lemma 2.1 is complete. ∎

Proof of Theorem 1.3. According to [17, Theorem 6.6], it is easy to see that g_{Δ_λ} is bounded on $L^2(\mathbb{R}_+, dm_\lambda)$. Here we provide an explicit estimate of

$$\|g_{\Delta_\lambda} f\|_{L^2(\mathbb{R}_+,dm_\lambda)} = \frac{1}{\sqrt{2}}\|f\|_{L^2(\mathbb{R}_+,dm_\lambda)}. \tag{2.26}$$

In fact, since Δ_λ is a non-negative self-adjoint operator,

$$\begin{aligned}\|g_{\Delta_\lambda} f\|^2_{L^2(\mathbb{R}_+,dm_\lambda)} &= \int_0^\infty\int_0^\infty |\sqrt{\Delta_\lambda}e^{-t\Delta_\lambda} f|^2\,dt\,dm_\lambda \\ &= \int_0^\infty\int_0^\infty \Delta_\lambda e^{-t\Delta_\lambda} f\cdot e^{-t\Delta_\lambda} f\,dm_\lambda\,dt \\ &= -\int_0^\infty\int_0^\infty \partial_t e^{-t\Delta_\lambda} f\cdot e^{-t\Delta_\lambda} f\,dm_\lambda\,dt \\ &= -\frac{1}{2}\int_0^\infty \partial_t\Big(\int_0^\infty |e^{-t\Delta_\lambda} f|^2\,dm_\lambda\Big)\,dt \\ &= -\frac{1}{2}\|e^{-t\Delta_\lambda} f\|^2_{L^2(\mathbb{R}_+,\,dm_\lambda)}\Big|_0^\infty = \frac{1}{2}\|f\|^2_{L^2(\mathbb{R}_+,dm_\lambda)}.\end{aligned}$$

Now, we turn to the weak $(1,1)$ bound of g_{Δ_λ}. We will prove that for any $f \in L^1(\mathbb{R}_+, dm_\lambda)\cap L^2(\mathbb{R}_+, dm_\lambda)$ and $\eta > 0$,

$$m_\lambda\bigg(\bigg\{x\in\mathbb{R}_+ : \bigg[\int_0^\infty |\psi(t\Delta_\lambda) f|^2\,\frac{dt}{t}\bigg]^{1/2} > \eta\bigg\}\bigg) \le C\eta^{-1}\|f\|_{L^1(\mathbb{R}_+,dm_\lambda)}, \tag{2.27}$$

where $\psi(t\Delta_\lambda)$ is as defined in (1.4). We use the Calderón–Zygmund decomposition, established in [14]. Denote by M the Hardy–Littlewood maximal operator,

$$Mf(x) := \sup_I \frac{1}{m_\lambda(I)}\int_I |f(y)|\,dm_\lambda(y),$$

where the supremum is taken over all intervals I in $\mathbb{R}_+$ containing x. Then there exists a collection $\{I_j\}_j$ of pairwise disjoint intervals such that

$$\{x\in\mathbb{R}_+ : Mf(x) > \eta\} = \bigcup_j I_j.$$

We can decompose $f = g + b = g + \sum_j b_j$, where

$$g(x) := f(x)\mathcal{X}_{\mathbb{R}_+\setminus\bigcup_j I_j} + \sum_j\bigg(\frac{1}{m_\lambda(I_j)}\int_{I_j} f(y)\,dm_\lambda(y)\bigg)\mathcal{X}_{I_j}(x),$$

$$b_j(x) := \bigg(f(x) - \frac{1}{m_\lambda(I_j)}\int_{I_j} f(y)\,dm_\lambda(y)\bigg)\mathcal{X}_{I_j}(x).$$

Moreover,

(i) $\|g\|_{L^\infty(\mathbb{R}_+,dm_\lambda)} \le C\eta$ and $\|g\|_{L^1(\mathbb{R}_+,dm_\lambda)} \le C\|f\|_{L^1(\mathbb{R}_+,dm_\lambda)}$;

(ii) for each j, $\int_{\mathbb{R}_+} |b_j(x)|\,dm_\lambda(x) \le C\eta m_\lambda(I_j)$, and

$$\sum_j \|b_j\|_{L^1(\mathbb{R}_+,dm_\lambda)} \le C\|f\|_{L^1(\mathbb{R}_+,dm_\lambda)};$$

(iii) $\sum_j m_\lambda(I_j) \le C\eta^{-1}\|f\|_{L^1(\mathbb{R}_+,dm_\lambda)}$.

Then it follows from the Calderón–Zygmund decomposition that

$$\begin{aligned} m_\lambda\bigg(\bigg\{x : \bigg[\int_0^\infty |\psi(t\varDelta_\lambda)f(x)|^2\,\frac{dt}{t}\bigg]^{1/2} > \eta\bigg\}\bigg) \\ &\le m_\lambda\bigg(\bigg\{x : \bigg[\int_0^\infty |\psi(t\varDelta_\lambda)g(x)|^2\,\frac{dt}{t}\bigg]^{1/2} > \frac{\eta}{2}\bigg\}\bigg) \\ &\quad + m_\lambda\bigg(\bigg\{x : \bigg[\int_0^\infty |\psi(t\varDelta_\lambda)b(x)|^2\,\frac{dt}{t}\bigg]^{1/2} > \frac{\eta}{2}\bigg\}\bigg) \\ &=: \mathrm{Term}_I + \mathrm{Term}_{II}. \end{aligned}$$

For Term_I, by (2.26) and (i), we get

$$\begin{aligned} m_\lambda\bigg(\bigg\{x : \bigg[\int_0^\infty |\psi(t\varDelta_\lambda)g(x)|^2\,\frac{dt}{t}\bigg]^{1/2} > \frac{\eta}{2}\bigg\}\bigg) \\ &\le C\eta^{-2}\int_0^\infty\int_0^\infty |\psi(t\varDelta_\lambda)g(x)|^2\,\frac{dt}{t}\,dm_\lambda(x) \\ &\le C\eta^{-2}\int_0^\infty |g(x)|^2\,dm_\lambda(x) \le C\eta^{-1}\int_0^\infty |f(x)|\,dm_\lambda(x). \end{aligned}$$

We now turn to Term_{II}. Let us fix a b_j whose support is contained in I_j, and $\tau_j := l_j^2$, where l_j stands for the length of the interval I_j. We decompose

$$\psi(t\varDelta_\lambda)b_j = \psi(t\varDelta_\lambda)e^{-\tau_j\varDelta_\lambda}b_j + \psi(t\varDelta_\lambda)(\mathcal{I} - e^{-\tau_j\varDelta_\lambda})b_j.$$

Also, denote $I_j^* := 2I_j \cap \mathbb{R}_+$ and $E^* := \mathbb{R}_+ \setminus \bigcup_j I_j^*$. Then

$$\mathrm{Term}_{II} \le m_\lambda\Big(\bigcup_j I_j^*\Big) + m_\lambda\bigg(\bigg\{x \in E^* : \bigg[\int_0^\infty \Big|\psi(t\varDelta_\lambda)\sum_j b_j(x)\Big|^2\,\frac{dt}{t}\bigg]^{1/2} > \frac{\eta}{2}\bigg\}\bigg)$$

$$\le m_\lambda\Big(\bigcup_j I_j^*\Big) + m_\lambda\bigg(\bigg\{x\in E^* : \bigg[\int_0^\infty |\psi(t\Delta_\lambda)\sum_j e^{-\tau_j\Delta_\lambda}b_j(x)|^2\,\frac{dt}{t}\bigg]^{1/2} > \frac{\eta}{4}\bigg\}\bigg)$$
$$+ m_\lambda\bigg(\bigg\{x\in E^* : \bigg[\int_0^\infty |\psi(t\Delta_\lambda)\sum_j (\mathcal{I}-e^{-\tau_j\Delta_\lambda})b_j(x)|^2\,\frac{dt}{t}\bigg]^{1/2} > \frac{\eta}{4}\bigg\}\bigg)$$
$$=: \mathrm{Term}_{II,1} + \mathrm{Term}_{II,2} + \mathrm{Term}_{II,3}.$$

From the definition of I_j^* and (iii) we see that

$$\mathrm{Term}_{II,1} \le C\sum_j m_\lambda(I_j) \le C\eta^{-1}\|f\|_{L^1(\mathbb{R}_+,dm_\lambda)}.$$

Next, to estimate $\mathrm{Term}_{II,2}$, let $H_t^{[\lambda]}$ be as in (2.7). We claim that

$$\sup_{y\in I_j} H_{\tau_j}^{[\lambda]}(x,y) \le C\inf_{y\in I_j} H_{4\tau_j}^{[\lambda]}(x,y). \tag{2.28}$$

In fact, if $x\in I_j^*$ then $|x-y|\le C\sqrt{\tau_j}$, hence (2.28) holds directly; if $x\in\mathbb{R}_+\setminus I_j^*$, one has

$$|x-y_1| \le |x-y_2| + |y_1-y_2| \le |x-y_2| + l_j \le 2|x-y_2|$$

for all $y_1,y_2\in I_j$. Hence, for $\tau_j = l_j^2$,

$$H_{\tau_j}^{[\lambda]}(x,y_2) \le C(4\tau_j)^{-\lambda-1/2}e^{-\frac{|x-y_1|^2}{16\tau_j}} \le CH_{4\tau_j}^{[\lambda]}(x,y_1).$$

See also [21, Proposition 2.5].

By using (2.26) we obtain

$$\mathrm{Term}_{II,2} \le C\eta^{-2}\int_0^\infty \Big|\sum_j e^{-\tau_j\Delta_\lambda}b_j(x)\Big|^2\,dm_\lambda(x). \tag{2.29}$$

According to (ii), (2.28) and L^2-boundedness of M, we obtain

$$\int_0^\infty \Big|\sum_j e^{-\tau_j\Delta_\lambda}b_j(x)\Big|^2\,dm_\lambda(x) \tag{2.30}$$
$$= \sup_{\|v\|_{L^2(\mathbb{R}_+,dm_\lambda)}\le 1}\Big|\int_0^\infty v(x)\cdot\sum_j e^{-\tau_j\Delta_\lambda}b_j(x)\,dm_\lambda(x)\Big|^2$$
$$\le C\sup_{\|v\|_{L^2(\mathbb{R}_+,dm_\lambda)}\le 1}\Big[\sum_j\int_0^\infty |v(x)|\int_{I_j} H_{\tau_j}^{[\lambda]}(x,y)\cdot|b_j(y)|\,dm_\lambda(y)\,dm_\lambda(x)\Big]^2$$
$$\le C\sup_{\|v\|_{L^2(\mathbb{R}_+,dm_\lambda)}\le 1}\Big[\sum_j\int_0^\infty |v(x)|\cdot\sup_{y\in I_j} H_{\tau_j}^{[\lambda]}(x,y)\cdot\|b_j\|_{L^1(\mathbb{R}_+,dm_\lambda)}\,dm_\lambda(x)\Big]^2$$

$$\le C \sup_{\|v\|_{L^2(\mathbb{R}_+,dm_\lambda)}\le 1} \eta^2\Big[\sum_j \int_0^\infty |v(x)| \int_0^\infty H^{[\lambda]}_{4\tau_j}(x,y)\mathcal{X}_{I_j}(y)\,dm_\lambda(y)\,dm_\lambda(x)\Big]^2$$
$$\le C \sup_{\|v\|_{L^2(\mathbb{R}_+,dm_\lambda)}\le 1} \eta^2\Big[\int_0^\infty M(|v|)(y)\sum_j \mathcal{X}_{I_j}(y)\,dm_\lambda(y)\Big]^2$$
$$\le C\eta^2\Big\|\sum_j \mathcal{X}_{I_j}\Big\|^2_{L^2(\mathbb{R}_+,dm_\lambda)}.$$

Thus, using (2.29), (2.30), (iii) and the disjointness of $\{I_j\}$, we obtain

$$\mathrm{Term}_{II,2} \le C\sum_j m_\lambda(I_j) \le C\eta^{-1}\|f\|_{L^1(\mathbb{R}_+,dm_\lambda)}.$$

We now estimate $\mathrm{Term}_{II,3}$. Denote by $\delta_{t,\tau_j}(x,y)$ the kernel of the operator $\psi(t\Delta_\lambda)(\mathcal{I}-e^{-\tau_j\Delta_\lambda})$. Then, using Chebyshev's inequality, we have

$$\begin{aligned}\mathrm{Term}_{II,3}&\\ &\le \frac{C}{\eta}\sum_j\int_{E^*}\Big[\int_0^\infty |\psi(t\Delta_\lambda)(\mathcal{I}-e^{-\tau_j\Delta_\lambda})b_j(x)|^2\,\frac{dt}{t}\Big]^{1/2} dm_\lambda(x)\\ &\le \frac{C}{\eta}\sum_j\int_{E^*}\int_0^\infty\Big[\int_0^\infty |\delta_{t,\tau_j}(x,y)b_j(y)|^2\,\frac{dt}{t}\Big]^{1/2} dm_\lambda(y)\,dm_\lambda(x)\\ &\le \frac{C}{\eta}\sum_j\int_0^\infty |b_j(y)|\Big\{\int_{|x-y|>\tau_j^{1/2}}\Big(\int_0^\infty |\delta_{t,\tau_j}(x,y)|^2\,\frac{dt}{t}\Big)^{1/2} dm_\lambda(x)\Big\}\,dm_\lambda(y).\end{aligned}$$

We claim that for any $\tau,x,y\in\mathbb{R}_+$,

$$\int_{|x-y|>\tau^{1/2}}\Big(\int_0^\infty |\delta_{t,\tau}(x,y)|^2\,\frac{dt}{t}\Big)^{1/2} dm_\lambda(x)\le C. \tag{2.31}$$

Assume the above claim for the moment. Then by using (ii) and (iii), we obtain

$$\mathrm{Term}_{II,3}\le C\sum_j \eta^{-1}\|b_j\|_{L^1(\mathbb{R}_+,dm_\lambda)}\le C\sum_j m_\lambda(I_j)\le C\eta^{-1}\|f\|_{L^1(\mathbb{R}_+,dm_\lambda)}.$$

Thus, combining this with the estimate for Term_I and Term_{II}, we finish the proof of (2.27).

It remains to prove the claim (2.31). By using functional calculus, we write

$$\delta_{t,\tau}(x,y)=\int_\Gamma\int_{\gamma_\zeta} W_z^{[\lambda]}(x,y)e^{\zeta z}\varphi_{t,\tau}(\zeta)\,dz\,d\zeta$$

where $\varphi_{t,\tau}(\zeta)=(t\zeta)^{1/2}e^{-t\zeta}(1-e^{\tau\zeta})$, and Γ and γ_ζ are as in (2.13) and (2.15).

Recall the estimate [3, (7.3)],

$$\int_0^\infty |e^{\zeta z}\varphi_{t,\tau}(\zeta)|\,d|\zeta| \le C\frac{t^{1/2}}{(r+t)^{3/2}}\min\Big\{1,\frac{\tau}{r+t}\Big\}. \tag{2.32}$$

First, from (2.8) and (2.32), we have

$$\begin{aligned}|\delta_{t,\tau}(x,y)| &\le C\int_0^\infty r^{-\lambda-1/2}e^{-\frac{|x-y|^2}{cr}}\frac{t^{1/2}}{(t+r)^{3/2}}\min\Big\{1,\frac{\tau}{t+r}\Big\}\,dr\\ &\le Ct^{-\lambda-1/2}\int_0^\infty u^{-\lambda-1/2}e^{-c\frac{|x-y|^2}{tu}}\frac{1}{(1+u)^{3/2}}\min\Big\{1,\frac{\tau}{t(1+u)}\Big\}\,du\\ &\le C\tau t^{-\lambda-3/2}\int_0^\infty u^{-\lambda-1/2}\frac{e^{-c\frac{|x-y|^2}{tu}}}{(1+u)^{5/2}}\,du.\end{aligned}$$

Now, using $(1+u)^{-5/2}\le Cu^{-5/2}$ and $(1+u)^{-5/2}\le Cu^{-1}, u\in\mathbb{R}_+$, we obtain, for any distinct x,y,

$$|\delta_{t,\tau}(x,y)| \le C\tau t^{-\lambda-3/2}\int_0^\infty u^{-\lambda-3}e^{-c\frac{|x-y|^2}{tu}}\,du \le C\frac{\tau\sqrt{t}}{|x-y|^{2\lambda+4}}, \tag{2.33}$$

$$|\delta_{t,\tau}(x,y)| \le C\tau t^{-\lambda-3/2}\int_0^\infty u^{-\lambda-3/2}e^{-c\frac{|x-y|^2}{tu}}\,du \le C\frac{\tau}{t|x-y|^{2\lambda+1}}. \tag{2.34}$$

On the other hand, by (2.4)–(2.6) and (2.32),

$$\begin{aligned}&|\delta_{t,\tau}(x,y)|\\ &\le C\bigg(\frac{\tau}{(xy)^\lambda t^{3/2}}\int_0^{\frac{xy}{2t}}\frac{e^{-c\frac{|x-y|^2}{tu}}}{\sqrt{u}\,(1+u)^{5/2}}\,du + \tau t^{-\lambda-3/2}\int_{\frac{xy}{2t}}^\infty \frac{e^{-c\frac{x^2+y^2}{tu}}}{u^{\lambda+1/2}(1+u)^{5/2}}\,du\bigg)\\ &\le C\frac{\tau}{(xy)^\lambda t^{3/2}}\int_0^\infty\frac{e^{-c\frac{|x-y|^2}{tu}}}{\sqrt{u}\,(1+u)^{5/2}}\,du,\end{aligned}$$

where the last inequality follows from (2.21).

Now arguing as before, using $(1+u)^{-5/2}\le Cu^{-5/2}$ and $(1+u)^{-5/2}\le Cu^{-1}$, $u\in\mathbb{R}_+$, we get

$$|\delta_{t,\tau}(x,y)| \le C\frac{\tau}{(xy)^\lambda t^{3/2}}\int_0^\infty \frac{e^{-c\frac{|x-y|^2}{tu}}}{u^3}\,du \le C\frac{\tau\sqrt{t}}{(xy)^\lambda|x-y|^4}, \tag{2.35}$$

$$|\delta_{t,\tau}(x,y)| \le C\frac{\tau}{(xy)^\lambda t^{3/2}}\int_0^\infty \frac{e^{-c\frac{|x-y|^2}{tu}}}{u^{3/2}}\,du \le C\frac{\tau}{t(xy)^\lambda|x-y|}. \tag{2.36}$$

From (2.33) and (2.35), it follows that

$$|\delta_{t,\tau}(x,y)| \le C\frac{\tau\sqrt{t}}{|x-y|^4}\min\left\{\frac{1}{(xy)^\lambda}, \frac{1}{|x-y|^{2\lambda}}\right\},$$

and by (2.34) and (2.36),

$$|\delta_{t,\tau}(x,y)| \le C\frac{\tau}{t|x-y|}\min\left\{\frac{1}{(xy)^\lambda}, \frac{1}{|x-y|^{2\lambda}}\right\}.$$

We deduce that

$$|\delta_{t,\tau}(x,y)| \le C\frac{\tau\sqrt{t}}{|x-y|^4\max\{x^{2\lambda}, y^{2\lambda}\}}, \tag{2.37}$$

$$|\delta_{t,\tau}(x,y)| \le C\frac{\tau}{t|x-y|\max\{x^{2\lambda}, y^{2\lambda}\}}. \tag{2.38}$$

Therefore, according to (2.37) and (2.38), we obtain

$$\begin{aligned}\int_0^\infty |\delta_{t,\tau}(x,y)|^2\,\frac{dt}{t} &\le C\int_0^{|x-y|^2}\frac{\tau^2}{\max\{x^{4\lambda}, y^{4\lambda}\}|x-y|^8}\,dt \\ &\quad + C\int_{|x-y|^2}^\infty \frac{\tau^2}{t^3\max\{x^{4\lambda}, y^{4\lambda}\}|x-y|^2}\,dt \\ &\le C\frac{\tau^2}{\max\{x^{4\lambda}, y^{4\lambda}\}|x-y|^6}.\end{aligned} \tag{2.39}$$

Inserting (2.39) into the left side of (2.31), we have

$$\begin{aligned}&\int_{|x-y|>\tau^{1/2}}\left(\int_0^\infty |\delta_{t,\tau}(x,y)|^2\,\frac{dt}{t}\right)^{1/2} dm_\lambda(x) \\ &\qquad \le C\int_{|x-y|>\tau^{1/2}}\frac{\tau}{\max\{x^{2\lambda}, y^{2\lambda}\}|x-y|^3}\,dm_\lambda(x) \\ &\qquad \le C\tau\int_{|x-y|>\tau^{1/2}}\frac{dx}{|x-y|^3} \le C,\end{aligned}$$

where $2^k\widetilde{I} := I(y, 2^k\tau^{1/2})\cap\mathbb{R}_+$.

The proof of Theorem 1.3 is complete. ■

Proof of Proposition 1.4. Since the square function g_{Δ_λ} is bounded on $L^2(\mathbb{R}_+, dm_\lambda)$ and is of weak type $(1,1)$, by interpolating we have

$$\|g_{\Delta_\lambda} f\|_{L^p(\mathbb{R}_+, dm_\lambda)} \le C\|f\|_{L^p(\mathbb{R}_+, dm_\lambda)} \tag{2.40}$$

for $1 < p \le 2$.

We now prove the boundedness of g_{Δ_λ} on $L^q(\mathbb{R}_+, dm_\lambda)$ for $2<q<\infty$. We will use the standard duality approach via vector-valued singular integrals (see for example [25, Proposition 4.5.7]), as well as our estimates as in the proof of Theorem 1.3 above.

To begin, following [25, Section 4.5.3], let $\mathbb{B}$ be a Hilbert space over the field of real numbers with the norm

$$\|F_t(x)\|_{\mathbb{B}} = \left\{ \int_0^\infty |F_t(x)|^2 \frac{dt}{t} \right\}^{1/2}.$$

For $1 \le p < \infty$, we have $L^p(\mathbb{B})$, which consists of functions F_t such that

$$\|F_t\|_{L^p(\mathbb{B})} = \left(\int_0^\infty \|F_t(x)\|_{\mathbb{B}}^p \, dm_\lambda(x) \right)^{1/p}.$$

Then from (2.40) we find that the $\mathbb{B}$-valued operator $\psi(t\Delta_\lambda)$ (as defined in (1.4)) is bounded from $L^p(\mathbb{R}_+, dm_\lambda)$ to $L^p(\mathbb{B})$ for $1 < p \le 2$. We now show that it is also bounded from $L^q(\mathbb{R}_+, dm_\lambda)$ to $L^q(\mathbb{B})$ for $q > 2$.

For every $f \in L^q(\mathbb{R}_+, dm_\lambda) \cap L^2(\mathbb{R}_+, dm_\lambda)$ and $G_t \in L^{q'}(\mathbb{B}) \cap L^2(\mathbb{B})$, we have

$$\begin{aligned}\langle \psi(t\Delta_\lambda) f, G_t \rangle &:= \int_0^\infty \int_0^\infty \psi(t\Delta_\lambda) f(x) G_t(x) \frac{dt}{t} \, dm_\lambda(x) \\ &= \int_0^\infty \int_0^\infty f(y) \psi(t\Delta_\lambda)(G_t)(y) \frac{dt}{t} \, dm_\lambda(y),\end{aligned}$$

where the second equality follows from the fact that $\psi(t\Delta_\lambda)$ is self-adjoint since Δ_λ is self-adjoint.

As a consequence,

$$\begin{aligned}|\langle \psi(t\Delta_\lambda) f, G_t \rangle| &\le \int_0^\infty |f(y)| \cdot \left| \int_0^\infty \psi(t\Delta_\lambda)(G_t)(y) \frac{dt}{t} \right| dm_\lambda(y) \\ &\le \|f\|_{L^q(\mathbb{R}_+, dm_\lambda)} \left\| \int_0^\infty \psi(t\Delta_\lambda) G_t \frac{dt}{t} \right\|_{L^{q'}(\mathbb{R}_+, dm_\lambda)}.\end{aligned}$$

Now it suffices to prove that the operator

$$\vec{T}(G_t)(y) := \int_0^\infty \psi(t\Delta_\lambda)(G_t)(y) \frac{dt}{t}$$

is bounded from $L^{q'}(\mathbb{B})$ to $L^{q'}(\mathbb{R}_+, dm_\lambda)$ for $1 < q' \le 2$. By interpolation, it suffices to prove that $\vec{T}(G_t)$ is bounded from $L^2(\mathbb{B})$ to $L^2(\mathbb{R}_+, dm_\lambda)$ and from $L^1(\mathbb{B})$ to $L^{1,\infty}(\mathbb{R}_+, dm_\lambda)$.

In fact, for any $h \in L^2(\mathbb{R}_+, dm_\lambda)$ with $\|h\|_{L^2(\mathbb{R}_+, dm_\lambda)} = 1$, we have

$$\begin{aligned}|\langle \vec{T}(G_t), h \rangle| &= \left| \int_0^\infty \int_0^\infty \psi(t\Delta_\lambda) h(x) G_t(x) \frac{dt}{t} \, dm_\lambda(x) \right| \\ &\le \|g_{\Delta_\lambda} h\|_{L^2(\mathbb{R}_+, dm_\lambda)} \|G_t\|_{L^2(\mathbb{B})} \le C \|G_t\|_{L^2(\mathbb{B})},\end{aligned}$$

which shows that

$$\|\vec{T}(G_t)\|_{L^2(\mathbb{R}_+,dm_\lambda)} \le C\|G_t\|_{L^2(\mathbb{B})}.$$

To see that $\vec{T}(G_t)$ maps $L^1(\mathbb{B})$ into $L^{1,\infty}(\mathbb{R}_+, dm_\lambda)$, for any G_t in $L^1(\mathbb{B})$, we consider the Calderón–Zygmund decomposition just as in the scalar case for the function $y \mapsto \|G_t(y)\|_{L^1(\mathbb{B})}$, as pointed out in the proof of [25, Theorem 4.6.1]. We get the decomposition $G_t = G_{t,\mathrm{good}} + G_{t,\mathrm{bad}} = G_{t,\mathrm{good}} + \sum_j G_{t,\mathrm{bad},j}$ such that

(i)′ $\|G_{t,\mathrm{good}}\|_{L^\infty(\mathbb{B})} \le C\eta$ and $\|G_{t,\mathrm{good}}\|_{L^1(\mathbb{B})} \le C\|G_t\|_{L^1(\mathbb{B})}$;
(ii)′ $\|G_{t,\mathrm{bad},j}\|_{L^1(\mathbb{B})} \le C\eta m_\lambda(I_j)$ and $\sum_j \|G_{t,\mathrm{bad},j}\|_{L^1(\mathbb{B})} \le C\|G_t\|_{L^1(\mathbb{B})}$;
(iii)′ $\sum_j m_\lambda(I_j) \le C\eta^{-1}\|G_t\|_{L^1(\mathbb{B})}$.

The $L^1(\mathbb{B})$ to $L^{1,\infty}(\mathbb{R}_+, dm_\lambda)$ estimate of $\vec{T}(G_t)$ follows from the proof of (2.27) with minor modifications in the vector-valued setting.

As a consequence, the $\mathbb{B}$-valued operator $\psi(t\Delta_\lambda)$ is bounded from $L^q(\mathbb{R}_+, dm_\lambda)$ to $L^q(\mathbb{B})$ for $q > 2$, which implies that

$$\|g_{\Delta_\lambda} f\|_{L^q(\mathbb{R}_+,dm_\lambda)} \le C\|f\|_{L^q(\mathbb{R}_+,dm_\lambda)}.$$

Finally, to complete the proof of Proposition 1.4, it remains to prove the reverse inequality

$$\|f\|_{L^p(\mathbb{R}_+,dm_\lambda)} \le C\|g_{\Delta_\lambda} f\|_{L^p(\mathbb{R}_+,dm_\lambda)}, \qquad 1 < p < \infty.$$

By the fact that

$$\int_0^\infty te^{-t}\,\frac{dt}{t} = 1$$

and functional calculus, for any $f \in L^p(\mathbb{R}_+, dm_\lambda) \cap L^2(\mathbb{R}_+, dm_\lambda)$ and $h \in L^{p'}(\mathbb{R}_+, dm_\lambda) \cap L^2(\mathbb{R}_+, dm_\lambda)$ we obtain

$$\begin{aligned}\Big|\int_0^\infty f(x)h(x)\,dm_\lambda(x)\Big| &= C\Big|\int_0^\infty\int_0^\infty t\Delta_\lambda e^{-2t\Delta_\lambda} f(x)h(x)\,dm_\lambda(x)\,\frac{dt}{t}\Big| \\ &= C\Big|\int_0^\infty\int_0^\infty \psi(t\Delta_\lambda)f(x)\cdot\psi(t\Delta_\lambda)h(x)\,\frac{dt}{t}\,dm_\lambda(x)\Big| \\ &\le C\int_0^\infty g_{\Delta_\lambda}(f)(x)g_{\Delta_\lambda}(h)(x)\,dm_\lambda(x) \\ &\le C\|g_{\Delta_\lambda}(f)\|_{L^p(\mathbb{R}_+,dm_\lambda)}\|g_{\Delta_\lambda}(h)\|_{L^{p'}(\mathbb{R}_+,dm_\lambda)},\end{aligned}$$

where the last two inequalities follow from Hölder's inequality. Noting that

$\|g_{\Delta_\lambda}(h)\|_{L^{p'}(\mathbb{R}_+,dm_\lambda)} \le C\|h\|_{L^{p'}(\mathbb{R}_+,dm_\lambda)}$, we further have

$$\begin{aligned}\|f\|_{L^p(\mathbb{R}_+,dm_\lambda)} &\le C \sup_{\|h\|_{L^{p'}(\mathbb{R}_+,dm_\lambda)}\le 1} \Big|\int_0^\infty f(x)h(x)\,dm_\lambda(x)\Big| \\ &\le C\|g_{\Delta_\lambda}(f)\|_{L^p(\mathbb{R}_+,dm_\lambda)}.\end{aligned}$$

The proof of Proposition 1.4 is complete. ∎

Remark 2.2. We also point out that, by verifying directly the Hölder regularity estimate for the vector-valued kernel of $g_{\Delta_\lambda}(f)$, we find that it is a vector-valued Calderón–Zygmund operator of non-convolution type (for the details, we refer to Proposition 4.1 in the appendix). And hence, the results in Theorem 1.3 and Proposition 1.4 follow from the classical result (see for example [25, Section 4.6.1]). However, our technique here is of independent interest, since we do not require the kernel to satisfy Hölder's regularity estimate, and hence the method can be adapted to a more general setting.

2.3. Proof of Theorem 1.6. Combining the estimates (2.24)–(2.25) in the proof of Lemma 2.1, one can obtain the following lemma.

Lemma 2.3. *Let $n \in \mathbb{N}^*$ and let $k^{[\lambda]}_{t,n}(x,y)$ be the kernel of the operator $(\sqrt{t\Delta_\lambda})^n e^{-t\Delta_\lambda}$. There is a positive constant C such that for some $\alpha \in (0,1/2)$ and all $t > 0$,*

$$|k^{[\lambda]}_{t,n}(x,y)| \le C \min\bigg\{\frac{t^{n/2}}{\max\{x^{2\lambda},y^{2\lambda}\}|x-y|^{1+n}}, \\ \frac{1}{t^\alpha \max\{x^{2\lambda},y^{2\lambda}\}|x-y|^{1-2\alpha}}\bigg\}.$$

Proof of Theorem 1.6. Let a be a $(1,2,1)$-atom supported in an interval $I := I(x_0,l)$ and $I^* := 2I \cap \mathbb{R}_+$. We have the following decomposition:

$$\int_0^\infty |g_{\Delta_\lambda}(a)|\,dm_\lambda = \int_{I^*} |g_{\Delta_\lambda}(a)|\,dm_\lambda + \int_{\mathbb{R}_+\setminus I^*} |g_{\Delta_\lambda}(a)|\,dm_\lambda.$$

By the Hölder inequality, (2.26) and Definition 1.5(iii), we have

$$\begin{aligned}\int_{I^*} |g_{\Delta_\lambda}(a)|\,dm_\lambda &\le m_\lambda(I^*)^{1/2}\Big(\int_0^\infty |g_{\Delta_\lambda}(a)|^2\,dm_\lambda\Big)^{1/2} \\ &\le Cm_\lambda(I^*)^{1/2}\Big(\int_0^\infty |a|^2\,dm_\lambda\Big)^{1/2} \\ &\le Cm_\lambda(I^*)^{1/2}\|\Delta_\lambda b\|_{L^2(\mathbb{R}_+,dm_\lambda)} \le C.\end{aligned} \tag{2.41}$$

For $x \in \mathbb{R}_+ \setminus I^*$, $y \in I$, we write

$$|g_{\Delta_\lambda}(a)(x)|^2 = \Big(\int_0^{l^2} + \int_{l^2}^{|x-y|^2} + \int_{|x-y|^2}^{\infty}\Big)|\sqrt{t\Delta_\lambda}\, e^{-t\Delta_\lambda} a(x)|^2 \,\frac{dt}{t}$$
$$=: G_1 + G_2 + G_3.$$

By Lemma 2.3 with $n = 1$, we have

$$\begin{aligned} G_1 &\le C \int_0^{l^2} \Big(\int_I |k_{t,1}^{[\lambda]}(x,y)|\,|a(y)|\,dm_\lambda(y)\Big)^2 \frac{dt}{t} \\ &\le C \int_0^{l^2} \Big(\frac{t^{1/2}}{\max\{x^{2\lambda}, y^{2\lambda}\}|x-y|^2}\Big)^2 \|a\|^2_{L^1(\mathbb{R}_+, dm_\lambda)} \frac{dt}{t} \\ &\le C \frac{1}{\max\{x^{2\lambda}, y^{2\lambda}\}^2 |x-y|^4} \|a\|^2_{L^1(\mathbb{R}_+, dm_\lambda)} \cdot l^2. \end{aligned}$$

For G_2, we write $a = \Delta_\lambda b$; then by Lemma 2.3 with $n = 3$,

$$\begin{aligned} G_2 &= \int_{l^2}^{|x-y|^2} |(t\Delta_\lambda)^{3/2} e^{-t\Delta_\lambda} b(x)|^2 \frac{dt}{t^3} \\ &\le C \int_{l^2}^{|x-y|^2} \frac{t^3}{(\max\{x^{2\lambda}, y^{2\lambda}\}|x-y|^4)^2} \|b\|^2_{L^1(\mathbb{R}_+, dm_\lambda)} \frac{dt}{t^3} \\ &\le C \frac{1}{\max\{x^{2\lambda}, y^{2\lambda}\}^2 |x-y|^6} \|b\|^2_{L^1(\mathbb{R}_+, dm_\lambda)}. \end{aligned}$$

For G_3, it follows from Lemma 2.3 with $n = 3$ and $\alpha = 1/4$ that

$$\begin{aligned} G_3 &= \int_{|x-y|^2}^{\infty} |(t\Delta_\lambda)^{3/2} e^{-t\Delta_\lambda} b(x)|^2 \frac{dt}{t^3} \\ &\le C \frac{1}{\max\{x^{2\lambda}, y^{2\lambda}\}^2 |x-y|} \|b\|^2_{L^1(\mathbb{R}_+, dm_\lambda)} \int_{|x-y|^2}^{\infty} \frac{1}{t^{7/2}}\, dt \\ &\le C \frac{1}{\max\{x^{2\lambda}, y^{2\lambda}\}^2 |x-y|^6} \|b\|^2_{L^1(\mathbb{R}_+, dm_\lambda)}. \end{aligned}$$

Therefore, it follows from Definition 1.5(iii) that

$$\begin{aligned} g_{\Delta_\lambda}(a)(x) &\le Cl \cdot \frac{1}{\max\{x^{2\lambda}, y^{2\lambda}\}|x-y|^2} (\|a\|_{L^1(\mathbb{R}_+, dm_\lambda)} + \|b\|_{L^1(\mathbb{R}_+, dm_\lambda)} l^{-2}) \\ &\le C \frac{1}{m_\lambda(I(x, |x-y|))|x-y|}. \end{aligned}$$

Note that if $x \in \mathbb{R}_+ \setminus I^*$ and $y \in I$, then

$$\tfrac{1}{2}|x - x_0| \le |x - y| \le \tfrac{3}{2}|x - x_0|.$$

Thus, using the doubling property of m_λ, we have

$$\begin{aligned}(2.42)\qquad &\int\limits_{\mathbb{R}_+\setminus I^*} |g_{\Delta_\lambda}(a)(x)|\, dm_\lambda(x) \\ &\le Cl \int\limits_{\mathbb{R}_+\setminus I^*} \frac{1}{m_\lambda(I(x_0,|x-x_0|))|x-x_0|}\, dm_\lambda(x) \\ &\le Cl \sum_{j=1}^{\infty} \int\limits_{2^{j+1}I\setminus 2^j I} \frac{1}{m_\lambda(I(x_0,|x-x_0|))|x-x_0|}\, dm_\lambda(x) \\ &\le C \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{m_\lambda(2^{j+1}I)}{m_\lambda(2^j I)} \le C.\end{aligned}$$

Combining (2.41) with (2.42), we deduce that

$$\|g_{\Delta_\lambda}(a)\|_{L^1(\mathbb{R}_+,dm_\lambda)} \le C.$$

If for every $f = \sum_{j=0}^{\infty} \alpha_j a_j$, we have

$$(2.43)\qquad |g_{\Delta_\lambda}(f)(x)| \le \sum_{j=0}^{\infty} |\alpha_j| g_{\Delta_\lambda}(a_j)(x) \quad m_\lambda\text{-a.e. in } \mathbb{R}_+,$$

then, by Fatou's Lemma,

$$\begin{aligned}&\|g_{\Delta_\lambda}(f)\|_{L^1(\mathbb{R}_+,dm_\lambda)} \\ &\quad \le \sum_{j=0}^{\infty} |\alpha_j| \cdot \|g_{\Delta_\lambda}(a_j)\|_{L^1(\mathbb{R}_+,dm_\lambda)} \sim \sum_{j=0}^{\infty} |\alpha_j| = C\|f\|_{H^1(\mathbb{R}_+,dm_\lambda)}.\end{aligned}$$

To prove (2.43), recall that $\lim_{N\to\infty} \sum_{j=0}^{N} \alpha_j a_j = f$ in $L^1(\mathbb{R}_+, dm_\lambda)$ and g_{Δ_λ} is of weak type $(1,1)$, so

$$\lim_{N\to\infty} g_{\Delta_\lambda}\Big(f - \sum_{j=0}^{N} \alpha_j a_j\Big) = 0 \quad \text{in } L^{1,\infty}(\mathbb{R}_+, dm_\lambda).$$

There exists a subsequence, denoted by $\{\sum_{j=0}^{N} \alpha_j a_j\}_N$, such that

$$\lim_{N\to\infty} g_{\Delta_\lambda}\Big(f - \sum_{j=0}^{N} \alpha_j a_j\Big)(x) = 0 \quad m_\lambda\text{-a.e. in } \mathbb{R}_+.$$

Also,

$$\begin{aligned}
g_{\Delta_\lambda}(f)(x) &\le g_{\Delta_\lambda}\Big(f - \sum_{j=0}^{N} \alpha_j a_j\Big)(x) + g_{\Delta_\lambda}\Big(\sum_{j=0}^{N} \alpha_j a_j\Big)(x) \\
&\le g_{\Delta_\lambda}\Big(f - \sum_{j=0}^{N} \alpha_j a_j\Big)(x) + \sum_{j=0}^{N} |\alpha_j| g_{\Delta_\lambda}(a_j)(x).
\end{aligned}$$

Consequently, as $N \to \infty$,

$$g_{\Delta_\lambda}(f)(x) \le \sum_{j=0}^{\infty} |\alpha_j| g_{\Delta_\lambda}(a_j)(x) \qquad m_\lambda\text{-a.e. in } \mathbb{R}_+.$$

The proof of Theorem 1.6 is finished. ∎

3. Square root and Littlewood–Paley square function associated with S_λ

3.1. Proof of Theorem 1.7. We use some ideas from [15]. Let $f, g \in C_0^\infty(\mathbb{R}_+)$. Then (1.6) yields

$$\langle S_\lambda f, g\rangle = \langle A_\lambda f, A_\lambda g\rangle.$$

From the definition of the square root of the Bessel operator,

$$\langle S_\lambda f, g\rangle = \langle \sqrt{S_\lambda}\, f, \sqrt{S_\lambda}\, g\rangle.$$

Thus, we have

$$\langle \sqrt{S_\lambda}\, f, \sqrt{S_\lambda}\, g\rangle = \langle A_\lambda f, A_\lambda g\rangle. \tag{3.1}$$

It follows from (3.1), the Hölder inequality and (1.7) that

$$\begin{aligned}
|\langle \sqrt{S_\lambda}\, f, \sqrt{S_\lambda}\, g\rangle| &\le \|A_\lambda f\|_{L^p(\mathbb{R}_+,dx)} \|A_\lambda g\|_{L^{p'}(\mathbb{R}_+,dx)} \\
&\le C\|A_\lambda f\|_{L^p(\mathbb{R}_+,dx)} \|\sqrt{S_\lambda}\, g\|_{L^{p'}(\mathbb{R}_+,dx)}.
\end{aligned}$$

Using the fact that the set $\sqrt{S_\lambda}\, C_0^\infty$ is dense in L^p (see [32, Lemma 1]), we have

$$\|\sqrt{S_\lambda}\, f\|_{L^p(\mathbb{R}_+,dx)} = \sup_{\|\sqrt{S_\lambda} g\|_{L^{p'}(\mathbb{R}_+,dx)} \le 1} |\langle \sqrt{S_\lambda}\, f, \sqrt{S_\lambda} g\rangle| \le C\|A_\lambda f\|_{L^p(\mathbb{R}_+,dx)}.$$

3.2. Proof of Theorem 1.10. We write the heat semigroup $\{e^{-tS_\lambda}\}_{t>0}$ associated with S_λ as (see [8])

$$e^{-tS_\lambda} f(x) = \int_0^\infty \mathbb{W}_t^{[\lambda]}(x,y) f(y)\, dy, \qquad t, x \in \mathbb{R}_+,$$

where the heat kernel is given by

$$\mathbb{W}_t^{[\lambda]}(x,y) := \frac{(xy)^{1/2}}{2t} e^{-\frac{x^2+y^2}{4t}} I_{\lambda-1/2}\Big(\frac{xy}{2t}\Big), \qquad t,x,y \in \mathbb{R}_+. \tag{3.2}$$

It is well known that the kernel $\mathbb{W}_t^{[\lambda]}(x,y)$ satisfies the Gaussian upper bound [29, inequality (5.16.5)]:

$$|\mathbb{W}_t^{[\lambda]}(x,y)| \le \frac{C}{\sqrt{t}} e^{-\frac{|x-y|^2}{4t}}. \tag{3.3}$$

Based on the expression of $\mathbb{W}_t^{[\lambda]}(x,y)$ as in (3.2), we see that it extends to a smooth kernel

$$\mathbb{W}_z^{[\lambda]}(x,y) = \frac{(xy)^{1/2}}{2z} e^{-\frac{x^2+y^2}{4z}} I_{\lambda-1/2}\Big(\frac{xy}{2z}\Big) \tag{3.4}$$

for $x,y \in \mathbb{R}_+$ and it is analytic in $z \in \mathbb{C}_+$. For the analyticity of the Bessel function $I_{\lambda-1/2}(\frac{xy}{2z})$ in $\mathbb{C}_+$, we refer to [29].

Based on the upper bound of $\mathbb{W}_t^{[\lambda]}(x,y)$ as in (3.3), by applying Coulhon–Sikora's result [16, Theorem 1.2], we find that $\mathbb{W}_z^{[\lambda]}(x,y)$ satisfies

$$|\mathbb{W}_z^{[\lambda]}(x,y)| \le \frac{C}{\sqrt{|z|}} e^{-\mathrm{Re}\frac{|x-y|^2}{cz}}. \tag{3.5}$$

We recall that $f \in \mathrm{BMO}_+$ if and only if for all $p \in [1,\infty)$ (equivalently, for some $p \in [1,\infty)$) (see [7, inequalities (1) and (2)])

$$\bigg\{\frac{1}{b-a}\int_a^b |f(x) - f_{(a,b)}|^p\,dx\bigg\}^{1/p} \le C_p, \qquad 0 < a < b < \infty, \tag{3.6}$$

$$\bigg\{\frac{1}{b}\int_0^b |f(x)|^p\,dx\bigg\}^{1/p} \le C_p, \qquad 0 = a < b < \infty, \tag{3.7}$$

and

$$\|f\|_{\mathrm{BMO}_+} \sim \inf\{C_p > 0 : (3.6) \text{ and } (3.7) \text{ hold}\},$$

where $f_{(a,b)} := \frac{1}{b-a}\int_a^b f(y)\,dy$.

Next we recall that

$$g_{S_\lambda}(f)(x) := \bigg(\int_0^\infty \bigg|\int_0^\infty K_{\psi(tS_\lambda)}(x,y) f(y)\,dy\bigg|^2 \frac{dt}{t}\bigg)^{1/2}, \qquad x \in \mathbb{R}_+,$$

where $K_{\psi(tS_\lambda)}(x,y)$ is the kernel of $\psi(tS_\lambda) = \sqrt{tS_\lambda}\,e^{-tS_\lambda}$. By functional calculus, we can write

$$K_{\psi(tS_\lambda)}(x,y) = \frac{1}{2\pi i}\int_\Gamma\int_{\gamma_\zeta} \mathbb{W}_z^{[\lambda]}(x,y) e^{\zeta z}\psi(t\zeta)\,dz\,d\zeta, \qquad x,y \in \mathbb{R}_+,$$

where Γ and γ_ζ are as in (2.13) and (2.15).

Proof of Theorem 1.10. Let f_0 be the odd extension of f. We write

$$g_\Delta(f_0)(x) := \Big(\int_0^\infty \Big| \int_{-\infty}^\infty K_{\psi(t\Delta)}(x,y) f_0(y)\,dy \Big|^2 \frac{dt}{t}\Big)^{1/2}$$
$$= \Big\{\int_0^\infty \Big| \int_0^\infty [K_{\psi(t\Delta)}(x,y) - K_{\psi(t\Delta)}(x,-y)] f(y)\,dy \Big|^2 \frac{dt}{t}\Big\}^{1/2},$$

where $K_{\psi(t\Delta)}(x,y)$ is the kernel of $\sqrt{t\Delta}\,e^{-t\Delta}$, $\Delta := -\frac{d^2}{dx^2}$. That is,

$$K_{\psi(t\Delta)}(x,y) = \frac{1}{2\pi i} \int_\Gamma \int_{\gamma_\zeta} W_z(x,y) e^{\zeta z} \psi(t\zeta)\,dz\,d\zeta, \tag{3.8}$$

where $W_z(x,y) = \frac{1}{\sqrt{4\pi z}} e^{-|x-y|^2/(4z)}$ for $x,y \in \mathbb{R}$, $t \in \mathbb{R}_+$, $z \in \mathbb{C}_+$, and Γ, γ_ζ are as in (2.13) and (2.15).

We also denote

$$\widetilde{g}_{S_\lambda}(f)(x) := \Big(\int_0^\infty \Big| \int_{x/2}^{2x} K_{\psi(tS_\lambda)}(x,y) f(y)\,dy \Big|^2 \frac{dt}{t}\Big)^{1/2},$$
$$\widetilde{g}_\Delta(f)(x) := \Big(\int_0^\infty \Big| \int_{x/2}^{2x} K_{\psi(t\Delta)}(x,y) f(y)\,dy \Big|^2 \frac{dt}{t}\Big)^{1/2}.$$

It is easy to see that

$$\|g_{S_\lambda}(f)\|_{\mathrm{BMO}_+} \le \|g_{S_\lambda}(f) - g_\Delta(f_0)\|_{\mathrm{BMO}_+} + \|g_\Delta(f_0)\|_{\mathrm{BMO}_+} =: G_1 + G_2.$$

We point out that for the first term G_1, we actually prove that

$$\|g_{S_\lambda}(f) - g_\Delta(f_0)\|_\infty \le C\|f\|_{\mathrm{BMO}_+}.$$

For the second term G_2, we need to estimate the BMO_+ norm of $g_\Delta(f_0)$.

To be more precise, we now verify two estimates.

1. $G_1 \le C\|f\|_{\mathrm{BMO}_+}$.

By using Minkowski's inequality, we find that for each $x \in \mathbb{R}_+$,

$$\begin{aligned}|g_{S_\lambda}(f)(x) - g_\Delta(f_0)(x)| &\\ &\le |g_{S_\lambda}(f)(x) - \widetilde{g}_{S_\lambda}(f)(x)| + |\widetilde{g}_{S_\lambda}(f)(x) - \widetilde{g}_\Delta(f)(x)| \\ &\quad + |g_\Delta(f_0)(x) - \widetilde{g}_\Delta(f)(x)|\end{aligned}$$

$$\leq \Big(\int_0^{x/2} + \int_{2x}^{\infty}\Big)|f(y)|\cdot \|K_{\psi(tS_\lambda)}(x,y)\|_{L^2(\mathbb{R}_+,\frac{dt}{t})}\,dy$$
$$+ \int_{x/2}^{2x} |f(y)|\cdot \|K_{\psi(tS_\lambda)}(x,y) - K_{\psi(t\Delta)}(x,y)\|_{L^2(\mathbb{R}_+,\frac{dt}{t})}\,dy$$
$$+ \Big[\int_{x/2}^{2x} |f(y)|\cdot \|K_{\psi(t\Delta)}(x,-y)\|_{L^2(\mathbb{R}_+,\frac{dt}{t})}\,dy$$
$$+ \Big(\int_0^{x/2} + \int_{2x}^{\infty}\Big)|f(y)|\cdot \|K_{\psi(t\Delta)}(x,y) - K_{\psi(t\Delta)}(x,-y)\|_{L^2(\mathbb{R}_+,\frac{dt}{t})}\,dy\Big].$$

We claim that for any $x, y \in \mathbb{R}_+$,

$$\|K_{\psi(tS_\lambda)}(x,y) - \mathcal{X}_{\{0<x/2<y<2x\}}K_{\psi(t\Delta)}(x,y)\|_{L^2(\mathbb{R}_+,\frac{dt}{t})} \leq C\frac{(xy)^\lambda}{\max\{x,y\}^{2\lambda+1}}, \tag{3.9}$$

$$\|K_{\psi(t\Delta)}(x,-y)\|_{L^2(\mathbb{R}_+,\frac{dt}{t})} \leq C\frac{1}{x+y}, \tag{3.10}$$

$$\|K_{\psi(t\Delta)}(x,y) - K_{\psi(t\Delta)}(x,-y)\|_{L^2(\mathbb{R}_+,\frac{dt}{t})} \leq C\frac{xy}{|x-y|^3}. \tag{3.11}$$

Then, combining (3.9)–(3.11), we have

$$|g_{S_\lambda}(f)(x) - g_\Delta(f_0)(x)|$$
$$\leq C\Big\{\Big[\int_0^{x/2} \frac{y^\lambda}{x^{\lambda+1}}|f(y)|\,dy + \int_{2x}^{\infty} \frac{x^\lambda}{y^{\lambda+1}}|f(y)|\,dy\Big] + \int_{x/2}^{2x} \frac{|f(y)|}{x}\,dy$$
$$+ \Big(\int_0^{x/2} + \int_{2x}^{\infty}\Big)|f(y)|\frac{xy}{|x-y|^3}\,dy\Big\}$$
$$\leq C\Big\{\frac{1}{x}\int_0^{x}|f(y)|\,dy + x^\lambda\sum_{k=1}^{\infty}\int_{2k^{2/\lambda}x}^{2(k+1)^{2/\lambda}x} \frac{|f(y)|}{y^{\lambda+1}}\,dy + \frac{1}{x}\int_{x/2}^{2x}|f(y)|\,dy$$
$$+ \int_0^{x/2}|f(y)|\frac{y}{x^2}\,dy + x\sum_{k=1}^{\infty}\int_{2k^2x}^{2(k+1)^2x}\frac{|f(y)|}{y^2}\,dy\Big\}$$
$$\leq C\|f\|_{\mathrm{BMO}_+},$$

where the last inequality follows from (3.7).

Now we prove the inequalities (3.9)–(3.11).

First, we give the proof of (3.10). It is not hard to see that for any $0 < \beta < 3/2$,

$$
\int_0^\infty |e^{\zeta z}\psi(t\zeta)|\,d|\zeta| \le C \int_0^\infty e^{-cRr} \frac{(tR)^{1/2}}{1+(tR)^\beta}\,dR \le C\min\{t^{1/2-\beta}r^{\beta-3/2}, t^{1/2}r^{-3/2}\}. \tag{3.12}
$$

Then, for any x and y with $x/2 \le y \le 2x$, it follows from (3.8) and (3.12) that

$$
|K_{\psi(t\Delta)}(x,-y)| \le C\int_0^\infty \frac{1}{r^{1/2}} e^{-\frac{(x+y)^2}{cr}} (t^{1/2}r^{-3/2})\,dr \le C\frac{t^{1/2}}{(x+y)^2}.
$$

For $0 < \beta < 1$,

$$
|K_{\psi(t\Delta)}(x,-y)| \le Ct^{1/2-\beta}\int_0^\infty \frac{1}{r^{2-\beta}} e^{-\frac{(x+y)^2}{cr}}\,dr \le C\frac{t^{1/2-\beta}}{(x+y)^{2-2\beta}}.
$$

Then

$$
\begin{aligned}
&\|K_{\psi(t\Delta)}(x,-y)\|_{L^2(\mathbb{R}_+,\frac{dt}{t})} \\
&\quad\le C\bigg\{\bigg[\int_0^{(x+y)^2} \frac{1}{(x+y)^4}\,dt\bigg]^{1/2} + \bigg[\int_{(x+y)^2}^\infty \frac{t^{-2\beta}}{(x+y)^{4-4\beta}}\,dt\bigg]^{1/2}\bigg\} \le C\frac{1}{x+y}.
\end{aligned}
$$

Next, we estimate (3.11). By the mean value theorem,

$$
\bigg|\frac{1}{\sqrt{4\pi z}}\big(e^{-\frac{(x-y)^2}{4z}} - e^{-\frac{(x+y)^2}{4z}}\big)\bigg| \le C\frac{xy}{|z|^{3/2}} e^{-\frac{|x-y|^2}{c|z|}}. \tag{3.13}
$$

Then, by (3.12) and (3.13), we get

$$
\begin{aligned}
&|K_{\psi(t\Delta)}(x,y) - K_{\psi(t\Delta)}(x,-y)| \\
&\qquad = \bigg|\int_\Gamma\int_{\gamma_\zeta} [W_z(x,y) - W_z(x,-y)]e^{\zeta z}\psi(t\zeta)\,dz\,d\zeta\bigg| \\
&\qquad \le Ct^{1/2}\int_0^\infty \frac{xy}{r^3} e^{-\frac{|x-y|^2}{cr}}\,dr \le C\frac{xy}{|x-y|^4}t^{1/2},
\end{aligned}
$$

and for $0 < \beta < 3/2$,

$$
\begin{aligned}
|K_{\psi(t\Delta)}(x,y) - K_{\psi(t\Delta)}(x,-y)| &\le Ct^{1/2-\beta}\int_0^\infty \frac{xy}{r^{3-\beta}} e^{-\frac{|x-y|^2}{cr}}\,dr \\
&\le C\frac{xy}{|x-y|^{4-2\beta}}t^{1/2-\beta}.
\end{aligned}
$$

Hence

$$\left\{\int_0^\infty |K_{\psi(t\Delta)}(x,y) - K_{\psi(t\Delta)}(x,-y)|^2 \,\frac{dt}{t}\right\}^{1/2}$$

$$\le C\left\{\int_0^{|x-y|^2} \left[\frac{xy}{|x-y|^4}\right]^2 dt\right\}^{1/2} + C\left\{\int_{|x-y|^2}^\infty \left[\frac{xy}{|x-y|^{4-2\beta}}\right]^2 \frac{dt}{t^{2\beta}}\right\}^{1/2}$$

$$\le C\frac{xy}{|x-y|^3}.$$

Finally, let us prove (3.9). First we consider the off-diagonal region $0 < y < x/2$ and $2x < y < \infty$. By symmetry of the kernel $K_{\psi(tS_\lambda)}(x,y)$, it suffices to consider the cone $0 < y < x/2$, and we need to show that

$$\left\{\int_0^\infty |K_{\psi(tS_\lambda)}(x,y)|^2 \,\frac{dt}{t}\right\}^{1/2} \le C\frac{(xy)^\lambda}{x^{2\lambda+1}}. \tag{3.14}$$

By (2.4) and (3.4) we have, for $z \in \gamma_\zeta$ with $|z| > xy$,

$$|\mathbb{W}_z^{[\lambda]}(x,y)| \le C\frac{(xy)^\lambda}{|z|^{\lambda+1/2}} e^{-\frac{x^2+y^2}{c|z|}}. \tag{3.15}$$

We divide the left hand side of (3.14) into three parts. According to Minkowski's inequality, functional calculus, (3.5), (3.12) and (3.15), for $0 < y < x/2$,

$$\left\{\int_0^\infty |K_{\psi(tS_\lambda)}(x,y)|^2 \,\frac{dt}{t}\right\}^{1/2}$$

$$\le C\left\{\int_0^\infty \left|\int_\Gamma\int_{\gamma_\zeta} \mathbb{W}_z^{[\lambda]}(x,y) e^{\zeta z}\psi(t\zeta)\,dz\,d\zeta\right|^2 \frac{dt}{t}\right\}^{1/2}$$

$$\le C\left\{\int_0^{xy}\left[\int_0^{xy} \frac{1}{r^2} e^{-\frac{x^2}{cr}}\,dr\right]^2 dt\right\}^{1/2}$$

$$+ C\left\{\int_{xy}^\infty\left[\int_0^{xy} \frac{1}{\sqrt{r}} e^{-\frac{|x-y|^2}{cr}} \cdot t^{1/2-\beta} r^{\beta-3/2}\,dr\right]^2 \frac{dt}{t}\right\}^{1/2}$$

$$+ C\left\{\int_0^\infty\left[\int_{xy}^\infty \frac{(xy)^\lambda}{r^{\lambda+1/2}} e^{-\frac{x^2+y^2}{cr}} \int_0^\infty e^{-cRr}\frac{tR}{1+(tR)^2}\,dR\,dr\right]^2 \frac{dt}{t}\right\}^{1/2}$$

$$=: K_1 + K_2 + K_3.$$

For K_1, we obtain

$$K_1 \le C(xy)^{1/2}\int_0^{xy} \frac{1}{r^2}\left(\frac{r}{x^2}\right)^{\lambda+2} dr \le C\frac{(xy)^{\lambda+3/2}}{x^{2\lambda+4}} \le C\frac{(xy)^\lambda}{x^{2\lambda+1}}.$$

For K_2,

$$K_2 \le C\bigg\{\int\limits_{xy}^{\infty}\bigg[\int\limits_0^{xy}\frac{1}{r^{2-\beta}}e^{-\frac{x^2}{cr}}\,dr\bigg]^2\frac{dt}{t^{2\beta}}\bigg\}^{1/2}$$
$$\le C(xy)^{-\beta+1/2}\int\limits_0^{xy}\frac{1}{r^{2-\beta}}\bigg(\frac{r}{x^2}\bigg)^{\lambda+2}dr \le C\frac{(xy)^\lambda}{x^{2\lambda+1}}.$$

For K_3, by using generalized Minkowski's inequality,

$$K_3 \le C\int\limits_{xy}^{\infty}\frac{(xy)^\lambda}{r^{\lambda+1/2}}e^{-\frac{x^2+y^2}{cr}}\int\limits_0^{\infty}e^{-cRr}\bigg(\int\limits_0^{\infty}\frac{tR}{1+(tR)^2}\frac{dt}{t}\bigg)^{1/2}dR\,dr$$
$$\le C(xy)^\lambda\int\limits_{xy}^{\infty}\frac{1}{r^{\lambda+3/2}}e^{-\frac{x^2+y^2}{cr}}dr \le C\frac{(xy)^\lambda}{x^{2\lambda+1}}.$$

Next, we show that (3.9) holds in the diagonal region $0 < x/2 \le y \le 2x$. In this case, we just have to estimate

$$\bigg(\int\limits_0^{\infty}|K_{\psi(tS_\lambda)}(x,y) - K_{\psi(t\Delta)}(x,y)|^2\frac{dt}{t}\bigg)^{1/2} \le C\frac{1}{(xy)^{1/2}}. \tag{3.16}$$

It is easy to see that

$$\bigg|\int\limits_{\Gamma}e^{\zeta z}\psi(t\zeta)\,d\zeta\bigg| \le C\int\limits_0^{\infty}e^{-crR}(tR)^{1/2}e^{-tR}\,dR \le C\frac{t^{1/2}}{(t+r)^{3/2}}. \tag{3.17}$$

On the one hand, by (2.5) and (2.6), we conclude that

$$|\mathbb{W}_z^{[\lambda]}(x,y) - W_z(x,y)|$$
$$= \bigg|\frac{(xy)^{1/2}}{2z}e^{-\frac{x^2+y^2}{4z}}\bigg[\frac{e^{\frac{xy}{2z}}}{\sqrt{\pi\cdot\frac{xy}{z}}}\bigg(1+\mathcal{O}\bigg(\frac{z}{xy}\bigg)\bigg)\bigg] - \frac{1}{\sqrt{4\pi z}}e^{-\frac{|x-y|^2}{4z}}\bigg|$$
$$\le C\frac{1}{\sqrt{|z|}}e^{-\frac{|x-y|^2}{c|z|}}\mathcal{O}\bigg(\frac{|z|}{xy}\bigg)$$

for $|z| \le xy$ (see [12, Lemma 8, Case 1]). On the other hand, it follows from (3.15) that for $|z| > xy$,

$$|\mathbb{W}_z^{[\lambda]}(x,y) - W_z(x,y)| \le |\mathbb{W}_z^{[\lambda]}(x,y)| + |W_z(x,y)|$$
$$\le C\frac{(xy)^\lambda}{|z|^{\lambda+1/2}}e^{-\frac{x^2+y^2}{c|z|}} + C\frac{1}{|z|^{1/2}}e^{-\frac{|x-y|^2}{c|z|}}. \tag{3.18}$$

Then, by using (3.17)–(3.18) and Minkowski's inequality, we divide the left side of (3.16) into four parts:

$$\left(\int_0^\infty |K_{\psi(tS_\lambda)}(x,y)-K_{\psi(t\Delta)}(x,y)|^2\,\frac{dt}{t}\right)^{1/2}$$

$$\begin{aligned}
&=\left(\int_0^\infty\left|\int_\Gamma\int_{\gamma_\zeta}[\mathbb{W}_z^{[\lambda]}(x,y)-W_z(x,y)]e^{z\zeta}\psi(t\zeta)\,dz\,d\zeta\right|^2\frac{dt}{t}\right)^{1/2}\\
&\le C\left(\int_0^{xy}\left[\int_0^{xy}\frac{1}{\sqrt{r}}e^{-\frac{|x-y|^2}{cr}}\,\frac{r}{xy}\cdot\frac{1}{(r+t)^{3/2}}\,dr\right]^2dt\right)^{1/2}\\
&\quad+C\left(\int_{xy}^\infty\left[\int_0^{xy}\frac{1}{\sqrt{r}}e^{-\frac{|x-y|^2}{cr}}\,\frac{r}{xy}\cdot\frac{1}{(r+t)^{3/2}}\,dr\right]^2dt\right)^{1/2}\\
&\quad+C\left(\int_0^{xy}\left[\int_{xy}^\infty\left(\frac{(xy)^\lambda}{r^{\lambda+1/2}}e^{-\frac{x^2+y^2}{cr}}+\frac{1}{r^{1/2}}e^{-\frac{|x-y|^2}{cr}}\right)\frac{1}{(r+t)^{3/2}}\,dr\right]^2dt\right)^{1/2}\\
&\quad+C\left(\int_{xy}^\infty\left[\int_{xy}^\infty\left(\frac{(xy)^\lambda}{r^{\lambda+1/2}}e^{-\frac{x^2+y^2}{cr}}+\frac{1}{r^{1/2}}e^{-\frac{|x-y|^2}{cr}}\right)\frac{1}{(r+t)^{3/2}}\,dr\right]^2dt\right)^{1/2}\\
&=:\mathbb{K}_1+\mathbb{K}_2+\mathbb{K}_3+\mathbb{K}_4.
\end{aligned}$$

For $\mathbb{K}_1$,

$$\begin{aligned}
\mathbb{K}_1&\le C\int_0^{xy}\left[\int_0^{xy}\frac{1}{(r+t)^3}\,dt\right]^{1/2}\frac{\sqrt{r}}{xy}e^{-\frac{|x-y|^2}{cr}}\,dr\\
&\le C\frac{1}{xy}\int_0^{xy}\frac{1}{\sqrt{r}}e^{-\frac{|x-y|^2}{cr}}\,dr\le C\left(\frac{1}{xy}\right)^{1/2}.
\end{aligned}$$

For $\mathbb{K}_2$,

$$\mathbb{K}_2\le C\left(\int_{xy}^\infty\frac{dt}{t^3}\right)^{1/2}\cdot\int_0^{xy}e^{-\frac{|x-y|^2}{cr}}\frac{\sqrt{r}}{xy}\,dr\le C\left(\frac{1}{xy}\right)^{1/2}.$$

Now, we turn to estimate $\mathbb{K}_3$ and $\mathbb{K}_4$:

$$\begin{aligned}
\mathbb{K}_3&\le C(xy)^{1/2}\left(\int_{xy}^\infty\frac{(xy)^\lambda}{r^{\lambda+2}}e^{-\frac{x^2+y^2}{cr}}\,dr+\int_{xy}^\infty\frac{1}{r^2}e^{-\frac{|x-y|^2}{cr}}\,dr\right)\\
&\le C(xy)^{1/2}\left((xy)^\lambda\int_{xy}^\infty\frac{1}{r^{\lambda+2}}\,dr+\int_{xy}^\infty\frac{1}{r^2}\,dr\right)\le C\left(\frac{1}{xy}\right)^{1/2},
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{K}_4&\le C\int_{xy}^\infty\frac{(xy)^\lambda}{r^{\lambda+3/2}}e^{-\frac{x^2+y^2}{cr}}\,dr+C\int_{xy}^\infty\frac{1}{r^{3/2}}e^{-\frac{|x-y|^2}{cr}}\,dr\\
&\le C(xy)^\lambda\int_{xy}^\infty\frac{dr}{r^{\lambda+3/2}}+C\int_{xy}^\infty\frac{dr}{r^{3/2}}\le C\left(\frac{1}{xy}\right)^{1/2}.
\end{aligned}$$

The proof of (3.9) is finished.

2. $G_2 \le C\|f\|_{\mathrm{BMO}_+}$.

To see this, we need to verify that (3.6) and (3.7) hold for $g_\Delta(f_0)$ with $p = 1$, that is,

$$\frac{1}{b-a}\int_a^b |g_\Delta(f_0)(x) - g_\Delta(f_0)_{(a,b)}|\,dx \le C\|f\|_{\mathrm{BMO}_+}, \qquad 0<a<b<\infty, \tag{3.19}$$

$$\frac{1}{a}\int_0^a |g_\Delta(f_0)(x)|\,dx \le C\|f\|_{\mathrm{BMO}_+}, \qquad 0<a<\infty. \tag{3.20}$$

We first prove (3.20). Let $f_{(0,2a)} = \frac{1}{2a}\int_0^{2a} f(y)\,dy$. We write

$$\begin{aligned} f_0(x) &= f_{(0,2a)} + (f(x) - f_{(0,2a)})\mathcal{X}_{(0,2a)}(x) + (f(x) - f_{(0,2a)})\mathcal{X}_{(2a,\infty)}(x) \\ &=: f_1(x) + f_2(x) + f_3(x), \qquad x \in \mathbb{R}_+, \end{aligned}$$

and $f_i(x) = -f_i(-x)$, $x \in \mathbb{R}_- := (-\infty, 0)$, $i = 1,2,3$. By (1.5) and the formula $e^{-t\Delta}(\mathbf{1}) = 1$, we have

$$\begin{aligned} g_\Delta(f_1)(x) &= |f_1|\Big(\int_0^\infty |\sqrt{t\Delta}\,e^{-t\Delta}(\mathbf{1})(x)|^2\,\frac{dt}{t}\Big)^{1/2} \\ &= |f_1|\Big(\int_0^\infty |\sqrt{\Delta}(\mathbf{1})(x)|^2\,dt\Big)^{1/2}. \end{aligned}$$

Note that from [13], $\sqrt{\Delta}(f)(x) = C\int_{\mathbb{R}} \frac{f(x)-f(\xi)}{|x-\xi|^2}\,d\xi$ for all f with $\int_{\mathbb{R}} \frac{|f(x)|}{(1+|x|)^2}\,dx < +\infty$. We find that $\sqrt{\Delta}(\mathbf{1})(x) = 0$ and

$$g_\Delta(f_1)(x) \equiv 0 \quad \text{for } x \in \mathbb{R}. \tag{3.21}$$

It is well known that g_Δ is a bounded operator from $L^2(\mathbb{R})$ into itself (see [3]). Thus, it follows from (3.6) and $f \in \mathrm{BMO}_+$ that

$$\begin{aligned} \int_0^a |g_\Delta(f_2)(x)|\,dx &\le C\sqrt{a}\,\Big\{\int_0^{2a} |f(x) - f_{(0,2a)}|^2\,dx\Big\}^{1/2} \\ &\le Ca\|f\|_{\mathrm{BMO}_+}. \end{aligned} \tag{3.22}$$

Since $g_\Delta(f_2)(x)$ is even, we obtain $g_\Delta(f_2)(x) < \infty$ for a.e. $x \in (-a,a)$. Also from (3.11), we obtain

$$\begin{aligned} \int_0^a |g_\Delta(f_3)(x)|dx &\le C\int_0^a\int_{2a}^\infty \frac{x}{y^2}|f(y) - f_{(0,2a)}|\,dy\,dx \\ &\le C\int_0^a x\int_{2a}^\infty \frac{1}{y^2}|f(y)|\,dy\,dx + a|f_{(0,2a)}| \le Ca\|f\|_{\mathrm{BMO}_+}. \end{aligned} \tag{3.23}$$

It is obvious that $g_\Delta(f_3)(x) < \infty$ for a.e. $x \in (-a,a)$. We conclude that $g_\Delta(f_0)(x) < \infty$ for a.e. $x \in (-a,a)$.

Now combining (3.21)–(3.23), we see that (3.20) holds.

We now turn to the proof of (3.19). For simplicity, we denote by $I(x_I, r_I)$ the interval with center $x_I := (a+b)/2$ and length $r_I := (b-a)/2$, so $(a,b) = I(x_I, r_I)$. When $x_I \le 4r_I$, (3.19) follows from (3.20). Thus, we only need to prove (3.19) when $x_I > 4r_I$.

Let $I^* := I(x_I, 2r_I) \cap \mathbb{R}_+$ and $(I^*)^c := \mathbb{R}_+ \setminus I^*$. We rewrite

$$\begin{aligned} f_0(x) &= f_{I^*} + \big(f(x) - f_{I^*}\big)\mathcal{X}_{I^*}(x) + \big(f(x) - f_{I^*}\big)\mathcal{X}_{(I^*)^c}(x) \\ &=: \tilde{f}_1(x) + \tilde{f}_2(x) + \tilde{f}_3(x), \qquad x \in \mathbb{R}_+, \end{aligned}$$

and $\tilde{f}_i(x) = -\tilde{f}_i(-x)$, $x \in \mathbb{R}_-$, $i = 1,2,3$. Similar to the proof of (3.21) and (3.22), we obtain $g_\Delta(\tilde{f}_j)(x) < \infty$ for a.e. $x \in (a,b)$, $j = 1,2$ and

$$g_\Delta(\tilde{f}_1)(x) \equiv 0, \qquad \int_I |g_\Delta(\tilde{f}_2)(x)|\,dx \le C(b-a)\|f\|_{\mathrm{BMO}_+}. \tag{3.24}$$

Observe that

$$\int_{\mathbb{R}_+} \frac{|f(y) - f_J|}{l^2 + |x_0 - y|^2}\,dy \le \frac{C}{l}\|f\|_{\mathrm{BMO}_+}, \tag{3.25}$$

where $J = (x_0 - l, x_0 + l) \cap \mathbb{R}_+$ and $f_J = \frac{1}{|J|}\int_J f(x)\,dx$ (see [23, (1.2′)] or [37, (3.21)]).

Noticing that $x \in I$ and $y \in (I^*)^c$ and $x_I > 4r_I$ implies $|x-y| \sim |x_I - y| \ge 2r_I$ and $x \le x_I + r_I \le Cx_I$, by Minkowski's inequality, (3.11) and (3.25) we have

$$\begin{aligned} \int_I |g_\Delta(\tilde{f}_3)(x)|\,dx &\le C \int_I \int_{(I^*)^c} |f(y) - f_{I^*}| \\ &\qquad \times \bigg\{ \int_0^\infty |K_{\psi(t\Delta)}(x,y) - K_{\psi(t\Delta)}(x,-y)|^2 \frac{dt}{t} \bigg\}^{1/2} dy\,dx \\ &\le C \int_I \int_{(I^*)^c} |f(y) - f_{I^*}| \frac{xy}{|x-y|^3}\,dy\,dx \\ &\le C \int_I \bigg(\int_0^{x_I - 2r_I} + \int_{x_I + 2r_I}^{2x_I} + \int_{2x_I}^\infty \bigg) |f(y) - f_{I^*}| \frac{x_I y}{|x_I - y|^3}\,dy\,dx \\ &\le C(b-a)\bigg(\frac{x_I^2}{r_I} + x_I\bigg) \int_0^\infty \frac{|f(y) - f_{I^*}|}{|x_I - y|^2 + (2r_I)^2}\,dy \\ &\le C\frac{(b+a)^2}{b-a}\|f\|_{\mathrm{BMO}_+}. \end{aligned}$$

Therefore, $g_\Delta(\tilde{f}_3)(x) < \infty$ for a.e. $x \in (a,b)$. Thus, there exists $\tilde{x} \in (a,b)$ for which $g_\Delta(\tilde{f}_3)(\tilde{x}) < \infty$.

We claim that for $x \in (a,b)$,

$$|g_\Delta(\tilde{f}_3)(x) - g_\Delta(\tilde{f}_3)(\tilde{x})| \le C\|f\|_{\mathrm{BMO}_+}. \tag{3.26}$$

In fact, by Minkowski's inequality, we get

$$\begin{aligned}
|g_\Delta(\tilde{f}_3)(x) &- g_\Delta(\tilde{f}_3)(\tilde{x})| \\
&\le \bigg\{ \int_0^\infty \bigg| \int_{(I^*)^c} \big(K_{\psi(t\Delta)}(x,y) - K_{\psi(t\Delta)}(x,-y) - K_{\psi(t\Delta)}(\tilde{x},y) \\
&\qquad\qquad + K_{\psi(t\Delta)}(\tilde{x},-y)\big) \cdot [f(y) - f_{I^*}]\, dy \bigg|^2 \frac{dt}{t} \bigg\}^{1/2} \\
&\le \int_{(I^*)^c} |f(y) - f_{I^*}| \bigg\{ \int_0^\infty \big| K_{\psi(t\Delta)}(x,y) - K_{\psi(t\Delta)}(x,-y) \\
&\qquad\qquad - K_{\psi(t\Delta)}(\tilde{x},y) + K_{\psi(t\Delta)}(\tilde{x},-y)\big|^2 \frac{dt}{t} \bigg\}^{1/2} dy.
\end{aligned}$$

By Minkowski's inequality,

$$\begin{aligned}
&\bigg\{ \int_0^\infty \big| K_{\psi(t\Delta)}(x,y) - K_{\psi(t\Delta)}(x,-y) - K_{\psi(t\Delta)}(\tilde{x},y) + K_{\psi(t\Delta)}(\tilde{x},-y)\big|^2 \frac{dt}{t} \bigg\}^{1/2} \\
&\qquad \le \bigg\{ \int_0^\infty \big| K_{\psi(t\Delta)}(x,y) - K_{\psi(t\Delta)}(\tilde{x},y)\big|^2 \frac{dt}{t} \bigg\}^{1/2} \\
&\qquad\quad + \bigg\{ \int_0^\infty \big| K_{\psi(t\Delta)}(x,-y) - K_{\psi(t\Delta)}(\tilde{x},-y)\big|^2 \frac{dt}{t} \bigg\}^{1/2} \\
&\qquad =: R_1 + R_2.
\end{aligned} \tag{3.27}$$

By functional calculus, we can write

$$\begin{aligned}
|K_{\psi(t\Delta)}(x,y) &- K_{\psi(t\Delta)}(\tilde{x},y)| \\
&= \bigg| \int_\Gamma \int_{\gamma_\zeta} [W_z(x,y) - W_z(\tilde{x},y)] \cdot e^{\zeta z} \psi(t\zeta)\, dz\, d\zeta \bigg|,
\end{aligned} \tag{3.28}$$

and

$$\begin{aligned}
|K_{\psi(t\Delta)}(x,-y) &- K_{\psi(t\Delta)}(\tilde{x},-y)| \\
&= \bigg| \int_\Gamma \int_{\gamma_\zeta} [W_z(x,-y) - W_z(\tilde{x},-y)] \cdot e^{\zeta z} \psi(t\zeta)\, dz\, d\zeta \bigg|,
\end{aligned} \tag{3.29}$$

where Γ and γ_ζ are as in (2.13) and (2.15).

Since $\tilde{x}, x \in I$, $y \in (I^*)^c$, by the Newton–Leibniz formula we have

$$|W_z(x,y) - W_z(\tilde{x},y)| = \left| \int_{\tilde{x}}^{x} \frac{1}{\sqrt{4\pi z}} e^{-\frac{|s-y|^2}{4z}} \frac{|s-y|}{2z}\, ds \right| \le C \int_I \frac{|s-y|}{r^{3/2}} e^{-\frac{|s-y|^2}{cr}}\, ds = C(b-a)\frac{|x_I - y|}{r^{3/2}} e^{-\frac{|x_I-y|^2}{cr}}, \tag{3.30}$$

and

$$|W_z(x,-y) - W_z(\tilde{x},-y)| = \left| \int_{\tilde{x}}^{x} \frac{1}{\sqrt{4\pi z}} e^{-\frac{(s+y)^2}{4z}} \frac{s+y}{2z}\, ds \right| \le C \int_I \frac{|s+y|}{r^{3/2}} e^{-\frac{(s+y)^2}{cr}}\, ds = C(b-a)\frac{x_I + y}{r^{3/2}} e^{-\frac{(x_I+y)^2}{cr}} \tag{3.31}$$

for $x_I > 4r_I$. It follows from (3.12), (3.28) and (3.30) that

$$\begin{aligned} R_1 &\le C(b-a)\left\{ \int_0^{|x_I-y|^2} \left[t^{1/2} \int_0^\infty \frac{|x_I-y|}{r^3} e^{-\frac{|x_I-y|^2}{cr}}\, dr \right]^2 \frac{dt}{t} \right\}^{1/2} \\ &\quad + C(b-a)\left\{ \int_{|x_I-y|^2}^{\infty} \left[t^{1/2-\beta} \int_0^\infty \frac{|x_I-y|}{r^{3-\beta}} e^{-\frac{|x_I-y|^2}{cr}}\, dr \right]^2 \frac{dt}{t} \right\}^{1/2} \\ &\le C(b-a)\left\{ \int_0^{|x_I-y|^2} \left[\frac{1}{|x_I-y|^3} \right]^2 dt \right\}^{1/2} \\ &\quad + C(b-a)\left\{ \int_{|x_I-y|^2}^{\infty} \left[\frac{1}{|x_I-y|^{3-2\beta}} \right]^2 \frac{dt}{t^{2\beta}} \right\}^{1/2} \\ &\le C\frac{b-a}{|x_I-y|^2} \end{aligned} \tag{3.32}$$

for any $0 < \beta < 3/2$. Similarly, by (3.12), (3.29) and (3.31), we obtain

$$R_2 \le C\frac{b-a}{(x_I+y)^2} \le C\frac{b-a}{|x_I-y|^2}. \tag{3.33}$$

Therefore, by (3.27), (3.32), (3.33) and the fact $|x_I - y| \ge 2r_I$, we obtain

$$|g_\Delta(\tilde{f}_3)(x) - g_\Delta(\tilde{f}_3)(\tilde{x})| \le C(b-a) \int_{(I^*)^c} \frac{|f(y) - f_{I^*}|}{|x_I-y|^2 + (2r_I)^2}\, dy \le C\|f\|_{\mathrm{BMO}_+},$$

where the last inequality follows from (3.25). This completes the proof of (3.26).

By (3.24) and (3.26), we observe

$$\begin{aligned}\int_a^b |g_\Delta(f_0)(x) - (g_\Delta(f_0))_{(a,b)}|\,dx \\ &\le \int_a^b |g_\Delta(f_0)(x) - g_\Delta(\tilde{f}_3)(\tilde{x})|\,dx + \int_a^b |g_\Delta(\tilde{f}_3)(\tilde{x}) - (g_\Delta(f_0))_{(a,b)}|\,dx \\ &\le C\int_a^b |g_\Delta(\tilde{f}_2)(x)|\,dx + C\int_a^b |g_\Delta(\tilde{f}_3)(x) - g_\Delta(\tilde{f}_3)(\tilde{x})|\,dx \\ &\le C(b-a)\|f\|_{\mathrm{BMO}_+}.\end{aligned}$$

Thus, the proof of $G_2 \le C\|f\|_{\mathrm{BMO}_+}$ is finished. ∎

4. Appendix. As pointed out in Remark 2.2, we now verify the Hölder regularity estimate for the vector-valued kernel of $g_{\Delta_\lambda}(f)$.

PROPOSITION 4.1. *Let $\psi(t\Delta_\lambda)$ be as in (1.4) and $k_t^{[\lambda]}(x,y)$ be the kernel of $\psi(t\Delta_\lambda)$. There exists a positive constant C such that for all distinct $x, y \in \mathbb{R}_+$,*

$$\Big(\int_0^\infty [|\partial_x k_t^{[\lambda]}(x,y)|^2 + |\partial_y k_t^{[\lambda]}(x,y)|^2]\,\frac{dt}{t}\Big)^{1/2} \le \frac{C}{|x-y|^2 \max\{x^{2\lambda}, y^{2\lambda}\}}.$$

Proof. By symmetry of $k_t^{[\lambda]}(x,y)$, we only consider $\partial_x k_t^{[\lambda]}(x,y)$. The estimate of $\partial_y k_t^{[\lambda]}(x,y)$ is the same as that of $\partial_x k_t^{[\lambda]}(x,y)$ and is omitted. We write

$$|\partial_x k_t^{[\lambda]}(x,y)| = \Big|\int_\Gamma \int_{\gamma_\zeta} \partial_x W_z^{[\lambda]}(x,y)\cdot e^{\zeta z}\psi(t\zeta)\,dz\,d\zeta\Big|.$$

where Γ and γ_ζ are as in (2.13) and (2.15), respectively. Arguing as for (2.16), we have

$$\int_0^\infty |e^{\zeta z}\psi(t\zeta)|\,d|\zeta| \le C\frac{t^{k-1}}{r^k}, \qquad k = 3/2 \text{ or } 3/4.$$

Moreover, it follows from (2.3) that

$$\begin{aligned}\partial_x W_z^{[\lambda]}(x,y) &= -\frac{x}{4z^2} e^{-\frac{x^2+y^2}{4z}} (xy)^{-\lambda+1/2} I_{\lambda-1/2}\Big(\frac{xy}{2z}\Big) \\ &\quad + \frac{y}{4z^2} e^{-\frac{x^2+y^2}{4z}} (xy)^{-\lambda+1/2} I_{\lambda+1/2}\Big(\frac{xy}{2z}\Big).\end{aligned}$$

We write

$$|\partial_x k_t^{[\lambda]}(x,y)| \le \Big[\int\limits_{\frac{xy}{2|z|}<1} + \int\limits_{\frac{xy}{2|z|}\ge 1}\Big] |\partial_x W_z^{[\lambda]}(x,y)| \cdot \int\limits_0^\infty |e^{\zeta z}\psi(t\zeta)|\, d|\zeta|\, d|z|$$
$$=: E_1(x,y) + E_2(x,y).$$

Note that when $\frac{xy}{2|z|} < 1$, by (2.4),

$$|\partial_x W_z^{[\lambda]}(x,y)| \le C\frac{x+y}{|z|^{\lambda+3/2}} e^{-\frac{x^2+y^2}{c|z|}}.$$

We see that when $xy < 2|z|$,

$$\text{(4.1)} \qquad E_1(x,y) \le Ct^{k-1}\int\limits_{xy/2}^\infty \frac{x+y}{r^{\lambda+3/2+k}} e^{-\frac{x^2+y^2}{cr}}\, dr$$
$$\le Ct^{k-1}\frac{x+y}{(x^2+y^2)^{\lambda+1/2+k}}\int\limits_0^\infty u^{\lambda+k-1/2}e^{-u}\, du$$
$$\le C\begin{cases} \dfrac{t^{1/2}}{(x^2+y^2)^{\lambda+3/2}} & \text{if } k=3/2,\\ \dfrac{1}{t^{1/4}(x^2+y^2)^{\lambda+3/4}} & \text{if } k=3/4.\end{cases}$$

Now we estimate $E_2(x,y)$ by considering two cases.

(i) Suppose $y\in(x/2,2x)$. Note that when $\frac{xy}{2|z|}\ge 1$, by (2.5) we have

$$\text{(4.2)} \quad |\partial_x W_z^{[\lambda]}(x,y)| \le C\frac{|x-y|}{|z|^{3/2}}e^{-\frac{|x-y|^2}{c|z|}}(xy)^{-\lambda} + C\frac{x+y}{|z|^{1/2}}e^{-\frac{|x-y|^2}{c|z|}}(xy)^{-\lambda-1}.$$

When $x/2 < y < 2x$ and $xy \ge 2|z|$, we see that, for any $t,x,y\in\mathbb{R}_+$ with $x\ne y$,

$$\text{(4.3)} \qquad E_2(x,y) \le C\int\limits_0^{xy/2}\Big[\frac{|x-y|}{r^{3/2}}e^{-\frac{|x-y|^2}{4r}}(xy)^{-\lambda}$$
$$+ \frac{x+y}{r^{1/2}}e^{-\frac{|x-y|^2}{cr}}(xy)^{-\lambda-1}\Big]\frac{t^{k-1}}{r^k}\, dr$$
$$\le Ct^{k-1}\Big\{\frac{1}{(xy)^\lambda|x-y|^{2k}}\int\limits_0^\infty u^{k-1/2}e^{-u}\, du$$
$$+ \frac{x+y}{(xy)^{\lambda+1}|x-y|^{2k-1}}\int\limits_0^\infty u^{k-3/2}e^{-u}\, du\Big\}$$

$$\leq C\frac{t^{k-1}}{(xy)^{\lambda}|x-y|^{2k}} = C\begin{cases} \dfrac{t^{1/2}}{(xy)^{\lambda}|x-y|^{3}} & \text{if } k=3/2,\\ \dfrac{1}{t^{1/4}(xy)^{\lambda}|x-y|^{3/2}} & \text{if } k=3/4.\end{cases}$$

(ii) Suppose now $y \in (0, x/2] \cup [2x, \infty)$. When $xy \geq 2|z|$, it follows from (4.2) that

$$|\partial_x W_z^{[\lambda]}(x,y)| \leq C\frac{x+y}{|z|^{\lambda+3/2}} e^{-\frac{|x-y|^2}{c|z|}}.$$

Similarly, we have

$$\begin{aligned}(4.4)\qquad E_2(x,y) &\leq Ct^{k-1}\int_0^{xy/2} \frac{x+y}{r^{\lambda+3/2+k}} e^{-\frac{|x-y|^2}{cr}}\,dr\\ &\leq Ct^{k-1}\frac{x+y}{|x-y|^{2\lambda+2k+1}}\int_0^{\infty} u^{\lambda+k-1/2}e^{-u}\,du\\ &\leq C\begin{cases} \dfrac{t^{1/2}}{|x-y|^{2\lambda+3}} & \text{if } k=3/2,\\ \dfrac{1}{t^{1/4}|x-y|^{2\lambda+3/2}} & \text{if } k=3/4.\end{cases}\end{aligned}$$

From (4.3) and (4.4), it follows that

$$(4.5)\qquad E_2(x,y) \leq C\begin{cases} \dfrac{t^{1/2}}{|x-y|^{3}\max\{x^{2\lambda},y^{2\lambda}\}} & \text{if } k=3/2,\\ \dfrac{1}{t^{1/4}|x-y|^{3/2}\max\{x^{2\lambda},y^{2\lambda}\}} & \text{if } k=3/4.\end{cases}$$

We deduce from (4.1) and (4.5) that

$$|\partial_x k_t^{[\lambda]}(x,y)| \leq C\begin{cases} \dfrac{t^{1/2}}{|x-y|^{3}\max\{x^{2\lambda},y^{2\lambda}\}} & \text{if } k=3/2,\\ \dfrac{1}{t^{1/4}|x-y|^{3/2}\max\{x^{2\lambda},y^{2\lambda}\}} & \text{if } k=3/4.\end{cases}$$

Therefore,

$$\begin{aligned}\Big(\int_0^{\infty} |\partial_x k_t^{[\lambda]}(x,y)|^2\,\frac{dt}{t}\Big)^{1/2} &\leq C\frac{1}{|x-y|^{3}\max\{x^{2\lambda},y^{2\lambda}\}}\Big(\int_0^{|x-y|^2} dt\Big)^{1/2}\\ &\quad + C\frac{1}{|x-y|^{3/2}\max\{x^{2\lambda},y^{2\lambda}\}}\Big(\int_{|x-y|^2}^{\infty}\frac{1}{t^{3/2}}\,dt\Big)^{1/2}\\ &= C\frac{1}{\max(x^{2\lambda},y^{2\lambda})|x-y|^2}.\end{aligned}$$

The proof of Proposition 4.1 is complete. ∎

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