STUDIA MATHEMATICA
Online First version

Finite reflection groups and symmetric extensions of Laplacian

by

Krzysztof Stempak (Wrocław)

Abstract. Let W be a finite reflection group associated with a root system R in $\mathbb{R}^d$. Let C_+ denote a positive Weyl chamber. Consider an open subset Ω of $\mathbb{R}^d$, symmetric with respect to reflections from W. Let $\Omega_+ = \Omega \cap C_+$ be the positive part of Ω. We define a family $\{-\Delta_\eta^+\}$ of self-adjoint extensions of the Laplacian $-\Delta_{\Omega_+}$, labeled by homomorphisms $\eta\colon W \to \{1, -1\}$. In the construction of these η-Laplacians, η-symmetrization of functions on Ω is involved. The Neumann Laplacian $-\Delta_{N,\Omega_+}$ is included and corresponds to $\eta \equiv 1$. If $H^1(\Omega) = H_0^1(\Omega)$, then the Dirichlet Laplacian $-\Delta_{D,\Omega_+}$ is also included and corresponds to $\eta = \operatorname{sgn}$; otherwise the Dirichlet Laplacian is considered separately. Applying the spectral functional calculus we consider the pairs of operators $\Psi(-\Delta_{N,\Omega})$ and $\Psi(-\Delta_\eta^+)$, or $\Psi(-\Delta_{D,\Omega})$ and $\Psi(-\Delta_{D,\Omega_+})$, where Ψ is a Borel function on $[0,\infty)$. We prove relations between the integral kernels for the operators in these pairs, which are given in terms of symmetries governed by W.

1. Introduction. Let Ω be a non-empty open subset of $\mathbb{R}^d$, $d \geq 1$, and let $\Delta = \sum_{j=1}^d \partial_j^2$ denote the Laplacian. If not otherwise stated, $-\Delta_\Omega$ will mean the differential operator $f \mapsto -\Delta f$ with domain $C_c^\infty(\Omega)$ dense in $L^2(\Omega)$. Clearly, $-\Delta_\Omega$ is symmetric,

$$\langle(-\Delta_\Omega)f, g\rangle_{L^2(\Omega)} = \langle f, (-\Delta_\Omega)g\rangle_{L^2(\Omega)}, \qquad f, g \in \operatorname{Dom}(-\Delta_\Omega) = C_c^\infty(\Omega),$$

and non-negative, $\langle(-\Delta_\Omega)f, f\rangle_{L^2(\Omega)} \geq 0$ for $f \in \operatorname{Dom}(-\Delta_\Omega)$.

Let $\mathfrak{t}_\Omega$ be the sesquilinear form defined on the Sobolev space $H^1(\Omega)$ as its domain by

$$\mathfrak{t}_\Omega[f,g] = \int_\Omega (\nabla f)(x) \cdot \overline{(\nabla g)(x)}\,dx = \int_\Omega \sum_{j=1}^d \partial_j f(x)\,\overline{\partial_j g(x)}\,dx.$$

2020 *Mathematics Subject Classification*: Primary 35K08; Secondary 47B25.
Key words and phrases: Neumann Laplacian, Dirichlet Laplacian, self-adjoint operator, finite reflection group, Sobolev space, sesquilinear form, functional calculus, heat kernel.
Received 23 April 2020; revised 23 October 2020.
Published online 3 June 2021.

DOI: 10.4064/sm200423-19-11

The *Neumann Laplacian* on Ω, denoted $-\Delta_{N,\Omega}$, is defined as the operator on $L^2(\Omega)$ associated with the form $\mathfrak{t}_{N,\Omega} := \mathfrak{t}_\Omega$; in particular, $\mathrm{Dom}(-\Delta_{N,\Omega}) \subset \mathrm{Dom}(\mathfrak{t}_{N,\Omega}) := H^1(\Omega)$. On the other hand, the *Dirichlet Laplacian* on Ω, denoted $-\Delta_{D,\Omega}$, is defined as the operator on $L^2(\Omega)$ associated with the form $\mathfrak{t}_{D,\Omega}$, which is the restriction of $\mathfrak{t}_\Omega$ to $H^1_0(\Omega)$; in particular, $\mathrm{Dom}(-\Delta_{D,\Omega}) \subset \mathrm{Dom}(\mathfrak{t}_{D,\Omega}) := H^1_0(\Omega)$. Since the forms $\mathfrak{t}_{N,\Omega}$ and $\mathfrak{t}_{D,\Omega}$ are Hermitian, closed and non-negative, the associated operators are self-adjoint and non-negative. See [11, Chapter 10 and Section 3 of Chapter 12] and also [10, pp. 263 and 265], where comments on the definitions of $-\Delta_{N/D,\Omega}$ are gathered. Each of the operators $-\Delta_{N/D,\Omega}$ is indeed an extension of $-\Delta_\Omega$. This follows from the definitions in terms of forms, with an application of Gauss' formula for functions from Sobolev classes (which can be found, for instance, in [11, Appendix D]; see comments at the end of Section 4). We also mention that $-\Delta_{D,\Omega}$ coincides with the Friedrichs extension of $\overline{-\Delta_\Omega}$, the closure of $-\Delta_\Omega$. See [11, Section 10.6.1].

For a general open set Ω it is known (see, for instance, [11, Section 10.6.1]) that

$$\mathrm{Dom}(-\Delta_{D,\Omega,}) = H^\Delta(\Omega) \cap H^1_0(\Omega),$$

where $H^\Delta(\Omega) = \{f \in L^2(\Omega)\colon \Delta f \in L^2(\Omega)\}$ and for $f \in L^2(\Omega) \subset C^\infty_c(\Omega)'$, Δf is understood in the distributional sense, and

$$-\Delta_{D,\Omega} f = -\Delta f \qquad \text{for } f \in \mathrm{Dom}(-\Delta_{D,\Omega}).$$

Note that $H^2(\Omega) \subset H^\Delta(\Omega)$ but in general the inclusion may be proper. Compared to the case of the Dirichlet Laplacian much less is known about the explicit description of $\mathrm{Dom}(-\Delta_{N,\Omega})$, the domain of the Neumann Laplacian, for general $\Omega \subset \mathbb{R}^d$.

If Ω is an open bounded subset in $\mathbb{R}^d$, $d \geq 2$, with boundary $\partial\Omega$ of class C^2, or an open bounded subset of $\mathbb{R}$, there are much finer results concerning properties of $-\Delta_{D,\Omega}$ and $-\Delta_{N,\Omega}$. In particular, in this case the Dirichlet Laplacian refers to vanishing boundary values at $\partial\Omega$ and the Neumann Laplacian refers to vanishing directional normal derivatives at $\partial\Omega$ (see for instance [11, Theorems 10.19 and 10.20]).

In the special case $\Omega = \mathbb{R}^d$, one has $H^1(\mathbb{R}^d) = H^1_0(\mathbb{R}^d)$, and it follows that $-\Delta_{N,\mathbb{R}^d} = -\Delta_{D,\mathbb{R}^d}$. More precisely,

$$\mathrm{Dom}(-\Delta_{N/D,\mathbb{R}^d}) = H^2(\mathbb{R}^d) \quad \text{and} \quad -\Delta_{N/D,\mathbb{R}^d} f = -\Delta f \qquad \text{for } f \in H^2(\mathbb{R}^d).$$

We refer the reader to [1] for a comprehensive treatment of the theory of Sobolev spaces.

By the spectral theorem, for a Borel function Ψ on $[0,\infty)$, we consider the operators $\Psi(-\Delta_{N/D,\Omega})$ (recall that $-\Delta_{N/D,\Omega}$ are non-negative and hence their spectra are contained in $[0,\infty)$). In particular, we associate with $-\Delta_{D,\Omega}$ the semigroup $\{\exp(-t(-\Delta_{D,\Omega}))\}_{t>0}$ of bounded operators on $L^2(\Omega)$, called the *Dirichlet heat semigroup*. Each $\exp(-t(-\Delta_{D,\Omega}))$, $t>0$, is an integral operator with kernel $p_t^{D,\Omega}(x,y)$, that is, for every $f \in L^2(\Omega)$,

$$\exp(-t(-\Delta_{D,\Omega}))f(x) = \int_\Omega p_t^{D,\Omega}(x,y)f(y)\,dy, \qquad x\text{-a.e.}$$

Moreover, as a function on $(0,\infty)\times\Omega\times\Omega$, $p_t^{D,\Omega}(x,y)$ is C^∞ and strictly positive (see [5, Theorem 5.2.1]). The family $\{p_t^{D,\Omega}(x,y)\}_{t>0}$ is called the *Dirichlet heat kernel on* Ω.

Analogously, we consider $\{\exp(-t(-\Delta_{N,\Omega}))\}_{t>0}$, the *Neumann heat semigroup* associated with $-\Delta_{N,\Omega}$. As before, each $\exp(-t(-\Delta_{N,\Omega}))$, $t>0$, is an integral operator with kernel $p_t^{N,\Omega}(x,y)$ which, as a function on $(0,\infty)\times\Omega\times\Omega$, is C^∞ and strictly positive. The family $\{p_t^{N,\Omega}(x,y)\}_{t>0}$ is called the *Neumann heat kernel on* Ω.

Let R be a *normalized root system* in $\mathbb{R}^d$, that is, a finite set of unit vectors such that $\sigma_\alpha(R)=R$ for every $\alpha\in R$, where σ_α denotes the orthogonal reflection in $\langle\alpha\rangle^\perp$, the hyperplane orthogonal to α,

$$\sigma_\alpha(x) = x - 2\langle\alpha,x\rangle\alpha, \qquad x\in\mathbb{R}^d.$$

Clearly, $R\cap\mathbb{R}\alpha=\{\alpha,-\alpha\}$ for every $\alpha\in R$. The *finite reflection group* $W=W(R)$ *associated with* R (the group is indeed finite as simple arguments show) is the subgroup of $O(\mathbb{R}^d)$ generated by the reflections σ_α, $\alpha\in R$. The set $\mathbb{R}^d\setminus\bigcup_{\alpha\in R}\langle\alpha\rangle^\perp$ splits into an even number (equal to $|W|$) of connected components which are open polyhedral cones called the *Weyl chambers*. The group W acts on the set of Weyl chambers, the action is simply transitive, and hence the Weyl chambers are mutually congruent. A choice of $\check{\alpha}\in\mathbb{R}^d$ such that $\langle\alpha,\check{\alpha}\rangle\neq 0$ for every $\alpha\in R$ gives the partition $R=R_+\sqcup(-R_+)$, where $R_+=\{\alpha\in R\colon \langle\alpha,\check{\alpha}\rangle>0\}$. Then R_+ is referred to as the *set of positive roots*. The partition distinguishes the chamber $C_+=\{x\in\mathbb{R}^d\colon \forall\alpha\in R_+\ \langle x,\alpha\rangle>0\}$, called the *positive Weyl chamber*.

For a comprehensive treatment of the general theory of finite reflection groups the reader is referred to [8], [7] and [6, Chapter 4].

Given $\Omega\subset\mathbb{R}^d$, we say it is W*-symmetric* if $\sigma_\alpha(\Omega)=\Omega$ for $\alpha\in R$ (which implies that $g(\Omega)=\Omega$ for $g\in W$). Then we distinguish the *positive part of* Ω by setting $\Omega_+:=\Omega\cap C_+$.

In the particular case of $\Omega=\mathbb{R}^d$ and R_+ consisting of a single vector α, say $\alpha=(0,\dots,0,1)$, so that $\Omega_+=\{x\colon x_d>0\}$ is the half-space $\mathbb{R}^{d-1}\times(0,\infty)$, the following *reflection principles* for the Neumann and Dirichlet heat kernels on the half-space are well-known ('$+$' for N[eumann], '$-$' for D[irichlet])

$$p_t^{N/D,\Omega_+}(x,y) = p_t(x,y)\pm p_t(\tilde{x},y), \qquad x,y\in\Omega_+.$$

Here $\{p_t\}_{t>0}$ denotes the Gauss–Weierstrass kernel on $\mathbb{R}^d$, $\{p_t^{N/D,\Omega_+}\}_{t>0}$ denote the Neumann and Dirichlet heat kernels on Ω_+, respectively, and the point $\tilde{x} = (x_1, x_2, \dots, -x_d)$ is the reflection of $x = (x_1, x_2, \dots, x_d)$ in the hyperplane $x_d = 0$. In [9] these formulas were proved to hold in a general context of an arbitrary open subset of $\mathbb{R}^d$, symmetric in $\langle\alpha\rangle^\perp$. Moreover, the case of a finite number of symmetries with respect to mutually orthogonal hyperplanes was also considered and similar formulas with natural modifications were deduced by a recursion argument. To be precise heat kernels were considered in a more general context of action of functions on operators by using spectral functional calculus. In its simplest form, the reflection principles extracted from [9, Corollary 4.1] for the heat kernels $\{p_t^{N/D,\Omega_+}\}_{t>0}$, corresponding to the Neumann and Dirichlet Laplacians on $\Omega_+ = \Omega \cap \mathbb{R}^d_+$, the positive part of an open subset Ω of $\mathbb{R}^d$ symmetric in $\langle e_j\rangle^\perp$, $j = 1, \dots, d$, where $\mathbb{R}^d_+ = (0,\infty)^d$, read as follows:

$$p_t^{N,\Omega_+}(x,y) = \sum_{\varepsilon\in\{-1,1\}^d} p_t^{N,\Omega}(\varepsilon x, y), \qquad x,y \in \Omega_+,$$

$$p_t^{D,\Omega_+}(x,y) = \sum_{\varepsilon\in\{-1,1\}^d} \operatorname{sgn}(\varepsilon) p_t^{D,\Omega}(\varepsilon x, y), \quad x,y \in \Omega_+.$$

Here for $x = (x_1, \dots, x_d) \in \mathbb{R}^d_+$ and $\varepsilon \in \{-1,1\}^d$ we write $\operatorname{sgn}(\varepsilon) = \prod_{i=1}^d \varepsilon_i$ and $\varepsilon x = (\varepsilon_1 x_1, \dots, \varepsilon_d x_d)$.

It then became clear that the results obtained for a single reflection or in a slightly more general setting of reflections with respect to mutually orthogonal hyperplanes should find their final and complete expression in the context of symmetries governed by an arbitrary finite reflection group. This was, in fact, the main motivation for the present research. However, a closer look into the problem revealed that using constant or alternating signs to label summands on the right hand sides of expressions for $p_t^{N/D,\Omega_+}(x,y)$ meant using one of the two special homomorphisms of $W \simeq \{-1,1\}^d$ into $\widehat{\mathbb{Z}}_2 = \{-1,1\}$ with multiplication. This motivated us to introduce and investigate a family of natural self-adjoint extensions of $-\Delta_{\Omega_+}$, the Laplacian on Ω_+, by means of homomorphisms $\eta \in \operatorname{Hom}(W, \widehat{\mathbb{Z}}_2)$, and to connect them in terms of functional calculus to the Neumann Laplacian on Ω. The main ingredient of the construction of the η-Laplacian $-\Delta_\eta^+$ is η-symmetrization of functions on Ω. See Section 4 for definitions. For $\eta = \mathbf{1}$ the corresponding $\mathbf{1}$-Laplacian is the Neumann Laplacian on Ω_+. For $\eta = \operatorname{sgn}$, when $H_0^1(\Omega) = H^1(\Omega)$, the corresponding sgn-Laplacian is the Dirichlet Laplacian on Ω_+. Here and later on, sgn denotes the function $\operatorname{sgn}(g) := \det(g) \in \{\pm 1\}$ on W. The case when $H_0^1(\Omega) \neq H^1(\Omega)$ (for instance, for any bounded Ω) is treated separately and then analogous connections are established between $-\Delta_{D,\Omega_+}$ and $-\Delta_{D,\Omega}$, the Dirichlet Laplacians on Ω_+ and Ω, respectively.

We prove that in the general setting of an open W-symmetric $\Omega \subset \mathbb{R}^d$, where $W \subset O(\mathbb{R}^d)$ is an arbitrary finite reflection group, the integral kernels of operators emerging in spectral calculus applied to the η-Laplacians or Dirichlet Laplacians on Ω_+ are related to the corresponding integral kernels of Neumann or Dirichlet Laplacians on Ω. More specifically, we prove the following.

THEOREM 1.1. *Let Ω be an open W-symmetric subset of $\mathbb{R}^d$ with positive part Ω_+. Let Ψ be a Borel function on $[0,\infty)$ and $\eta \in \mathrm{Hom}(W, \widehat{\mathbb{Z}}_2)$. Assume that $\Psi(-\Delta_{N,\Omega})$ is an integral operator with kernel $K^{\Psi}_{-\Delta_{N,\Omega}}$. Then $\Psi(-\Delta_\eta^+)$ is also an integral operator with kernel*

$$K^{\Psi}_{-\Delta_\eta^+}(x,y) = \sum_{g\in W} \eta(g) K^{\Psi}_{-\Delta_{N,\Omega}}(gx,y), \qquad x,y \in \Omega_+. \tag{1.1}$$

Similarly, if $\Psi(-\Delta_{D,\Omega})$ is an integral operator with kernel $K^{\Psi}_{-\Delta_{D,\Omega}}$, then $\Psi(-\Delta_{D,\Omega_+})$ is also an integral operator with kernel

$$K^{\Psi}_{-\Delta_{D,\Omega_+}}(x,y) = \sum_{g\in W} \mathrm{sgn}(g) K^{\Psi}_{-\Delta_{D,\Omega}}(gx,y), \qquad x,y \in \Omega_+. \tag{1.2}$$

As a direct corollary we obtain the following result for heat kernels.

COROLLARY 1.2. *Let Ω be an open W-symmetric subset of $\mathbb{R}^d$ and let p_t^{η,Ω_+} and p_t^{D,Ω_+}, and $p_t^{N,\Omega}$ and $p_t^{D,\Omega}$, denote the η and Dirichlet heat kernels on Ω_+, and the Neumann and Dirichlet heat kernels on Ω, respectively. Then*

$$p_t^{\eta,\Omega_+}(x,y) = \sum_{g\in W} \eta(g) p_t^{N,\Omega}(gx,y), \qquad x,y \in \Omega_+,\ t>0,$$
$$p_t^{D,\Omega_+}(x,y) = \sum_{g\in W} \mathrm{sgn}(g) p_t^{D,\Omega}(gx,y), \qquad x,y \in \Omega_+,\ t>0.$$

We point out that analogous formulas hold for other important kernels including resolvent kernels and Riesz potential kernels.

The novelty of this work is twofold. First, as already mentioned, an arbitrary finite reflection group is admitted. Note that the case of a finite number of orthogonal mirrors considered in [9] was the ‘product case’. For an arbitrary finite reflection group, where in general there is no orthogonality between the mirrors completely new arguments are required. This include weighted extension and averaging operators as well as some additional technical tools. See Sections 2 and 3 for details. The generality permits us, for instance, to obtain closed formulas for the Neumann and Dirichlet heat kernels on cones on the plane with apertures π/n, $n \geq 3$; up to our knowledge these formulas are new for $n \neq 2^j$. The same can be said about analogous truncated cones (intersections of these infinite cones with the unit disc centered at the origin) on the plane, and also about polyhedral cones in $\mathbb{R}^d$

which are positive Weyl chambers corresponding to a finite reflection group and their truncated versions.

Secondly, the new idea of using homomorphisms from $\mathrm{Hom}(W, \widehat{\mathbb{Z}}_2)$ in constructing self-adjoint extensions of $-\Delta_{\Omega_+}$ is presented. Heat kernels associated with these extensions are useful tools in solving mixed Neumann-Dirichlet initial-boundary value problems on cones which are positive Weyl chambers. This issue is outlined in Section 6.1 and deserves further studies.

Finally we mention one more new aspect. Theorem 1.1 allows one to obtain heat kernel formulas for sets with less regular boundary from heat kernel formulas for more regular sets. Often a general theory applies to sets Ω with, say, $C^{1,1}$ boundary while the set $\Omega_+ = \Omega \cap C_+$ may have only Lipschitz boundary.

The paper is organized as follows. Section 2 contains definitions of weighted extension and averaging operators and relevant facts. Section 3 is devoted to the proofs of auxiliary results which are used in the proof of the main result in Section 5. Section 4 contains the definition of and basic results on η-Laplacians, which are self-adjoint extensions in $L^2(\Omega_+)$ of the Laplacian $-\Delta_{\Omega_+}$. Finally, in Section 6 we discuss two basic examples, the case of a finite reflection group associated with an orthonormal root system and the case of the dihedral group.

2. Preliminaries. From now on till Section 5, Ω is a fixed open subset of $\mathbb{R}^d$, symmetric with respect to a given finite reflection group W associated with a fixed root system R. Consequently, Ω_+ will denote its positive part related to a fixed subsystem R_+ of positive roots. Since the case $d = 1$ is simple (and is contained in [9]), we assume that $d \geq 2$. Throughout this and the next sections we use small letters (like $\phi, f, \ldots$) to denote functions on Ω_+, and capital letters (like Φ, $F, \ldots$) to denote functions on Ω. The symbol φ is reserved for a compactly supported C^∞ function with support either in Ω or Ω_+ (this will always be clear from the context). For $g \in W$ and a function F on Ω we denote by F_g the function $F_g(x) := F(gx)$, $x \in \Omega$. Also, recall that the norm $\|\cdot\|_{H^1(\Omega)}$ is given by

$$\|F\|_{H^1(\Omega)} = \Big(\|F\|^2_{L^2(\Omega)} + \sum_{i=1}^d \|\partial_i F\|^2_{L^2(\Omega)}\Big)^{1/2},$$

where $\partial_i F$ are weak partial derivatives of F on Ω, and similarly for $\|\cdot\|_{H^1(\Omega_+)}$.

Let $\omega \colon W \to \mathbb{C}$ be a function on W, called a *weight* henceforth. Recall that W acts on the set of Weyl chambers simply transitively.

DEFINITION 2.1. The *weighted extension operator* $\mathcal{E}^\omega$ acts on functions ϕ defined on Ω_+ by

$$\mathcal{E}^\omega \phi(gx) = \omega(g)\phi(x), \qquad x \in \Omega_+,\ g \in W.$$

The *weighted averaging operator* $\mathcal{A}_\omega$ acts on functions Φ defined on Ω by

$$\mathcal{A}_\omega\Phi(y) = \frac{1}{|W|}\sum_{g\in W}\omega(g)\Phi(gy), \qquad y\in\Omega.$$

Clearly, the resulting functions $\mathcal{E}^\omega\phi$ and $\mathcal{A}_\omega\Phi$ live on Ω (to be precise, $\mathcal{E}^\omega\phi$ is defined on Ω up to a set of Lebesgue measure zero). Also, for a function Φ on Ω, we let Φ^+ be the restriction of Φ to Ω_+, $\Phi^+ = \Phi|_{\Omega_+}$.

For the weight $\omega\equiv 1$ we shall simply write $\mathcal{E}$ and $\mathcal{A}$ rather than $\mathcal{E}^1$ and $\mathcal{A}_1$. Then $\mathcal{E}\phi$ and $\mathcal{E}^{\mathrm{sgn}}\phi$ can be called the *even* and *odd extensions* of ϕ onto Ω, respectively.

It is easily seen that the following averaging invariance properties hold:

$$\mathcal{A}(\mathcal{E}\phi) = \mathcal{E}\phi, \qquad \mathcal{A}_{\mathrm{sgn}}(\mathcal{E}^{\mathrm{sgn}}\phi) = \mathcal{E}^{\mathrm{sgn}}\phi.$$

LEMMA 2.2. *For suitable functions ϕ and Φ on Ω_+ and Ω, respectively* (*i.e. such that the first integral below makes sense*), *we have*

$$\int_\Omega \mathcal{E}^\omega\phi\cdot\Phi = |W|\int_{\Omega_+}\phi\cdot(\mathcal{A}_\omega\Phi)^+. \tag{2.1}$$

Proof. We write, using the change of variables $y:=gx$, $y\in g\Omega_+$, $x\in\Omega_+$,

$$\begin{aligned}\int_\Omega \mathcal{E}^\omega\phi\cdot\Phi &= \sum_{g\in W}\int_{g\Omega_+}\mathcal{E}^\omega\phi(y)\cdot\Phi(y)\,dy = \sum_{g\in W}\int_{\Omega_+}\omega(g)\phi(x)\cdot\Phi(gx)\,dx\\ &= \int_{\Omega_+}\phi(x)\cdot\Big(\sum_{g\in W}\omega(g)\Phi(gx)\Big)\,dx = |W|\int_{\Omega_+}\phi\cdot\big(\mathcal{A}_\omega\Phi\big)^+. \ \blacksquare\end{aligned}$$

For $g\in W$ let $[g_{ij}]$ be the matrix of the linear transformation $x\mapsto gx$. This means that for $x=(x_1,\ldots,x_d)$ and $gx=((gx)_1,\ldots,(gx)_d)$,

$$(gx)_i = \sum_j g_{ij}x_j, \qquad i=1,\ldots,d.$$

Clearly, $[g_{ij}]$ is also the Jacobi matrix of this transformation,

$$g_{ij} = \partial_j((gx)_i).$$

Moreover, since $[g_{ij}]$ is an orthogonal matrix, we have the orthogonality of rows and columns, $\sum_j g_{ij}g_{mj} = \delta_{im}$ and $\sum_i g_{ij}g_{im} = \delta_{jm}$.

LEMMA 2.3. *The operator $\mathcal{A}_\omega$ acts boundedly on $H^1(\Omega)$. Moreover, $\mathcal{A}_\omega$ leaves $H^1_0(\Omega)$ invariant.*

Proof. If $F\in H^1(\Omega)$, then clearly for any $g\in W$ the function F_g is also in $H^1(\Omega)$, and since $(\partial_i F_g)(x) = \sum_{j=1}^d \partial_j F(gx)g_{ji}$, the mapping $F\mapsto F_g$ is bounded. (See [1, Theorem 3.41] for a general result.) Consequently, $\mathcal{A}_\omega$ maps $H^1(\Omega)$ into itself and is bounded. The invariance claim is obvious. $\blacksquare$

In what follows, for any given $j, i \in \{1, \ldots, d\}$, by writing $\mathcal{A}_{g_{ji}}$ or $\mathcal{A}_{\omega \cdot g_{ji}}$ we shall mean the weighted averaging operators with weight functions $g \mapsto g_{ji}$ or $g \mapsto \omega(g) g_{ji}$ on W, respectively, and similarly for $\mathcal{E}^{g_{ji}}$ and $\mathcal{E}^{\omega \cdot g_{ji}}$.

LEMMA 2.4. *For any $F \in H^1(\Omega)$ we have*

$$\mathcal{A}_\omega(\partial_j F) = \sum_{i=1}^{d} \partial_i(\mathcal{A}_{\omega \cdot g_{ji}} F), \tag{2.2}$$

$$\partial_i(\mathcal{A}_\omega F) = \sum_{j=1}^{d} \mathcal{A}_{\omega \cdot g_{ji}}(\partial_j F). \tag{2.3}$$

Proof. First, assume $F \in C^1(\Omega)$. By the very definition,

$$\mathcal{A}_{\omega \cdot g_{ji}} F(y) = \frac{1}{|W|} \sum_{g \in W} \omega(g) g_{ji} F(gy), \qquad y \in \Omega,$$

and hence

$$\begin{aligned} \partial_i(\mathcal{A}_{\omega \cdot g_{ji}} F)(y) &= \frac{1}{|W|} \sum_{g \in W} \omega(g) g_{ji} \partial_i \big(F(gy)\big) \\ &= \frac{1}{|W|} \sum_{g} \omega(g) g_{ji} \sum_{m} \partial_m F(gy) \cdot g_{mi}. \end{aligned}$$

For a given j, summing over i gives (2.2).

For a general $F \in H^1(\Omega)$ find a sequence $\{F_n\} \subset C^1(\Omega) \cap H^1(\Omega)$ such that $F_n \to F$ in $H^1(\Omega)$ (see [1, Theorem 3.17]). This means, in particular, that $F_n \to F$ in $L^2(\Omega)$, and since $\mathcal{A}_\omega$ is bounded on $L^2(\Omega)$, also $\mathcal{A}_\omega F_n \to \mathcal{A}_\omega F$ in $L^2(\Omega)$. On the other hand, $\mathcal{A}_{\omega \cdot g_{ji}}$ is bounded on $H^1(\Omega)$, so $\mathcal{A}_{\omega \cdot g_{ji}} F_n \to \mathcal{A}_{\omega \cdot g_{ji}} F$ in $H^1(\Omega)$, and hence $\partial_i(\mathcal{A}_{\omega \cdot g_{ji}} F_n) \to \partial_i(\mathcal{A}_{\omega \cdot g_{ji}} F)$ in $L^2(\Omega)$, as $n \to \infty$. Since (2.2) holds for any such F_n, it holds in general.

The verification of (2.3) is analogous. ∎

3. Auxiliary results. We shall need an explicit description of the boundary of $\overline{C_+}$, the closure of the positive Weyl chamber C_+. For this purpose we recall the notion of the *system of simple roots* (see [8], [6] or [7]). This is the unique subset $\Sigma = \{\alpha_1, \ldots, \alpha_m\}$ of R_+ which is a basis of $\operatorname{lin}\{\alpha \colon \alpha \in R_+\}$ (so $1 \le m \le d$) and each $\alpha \in R_+$ is a linear combination of $\alpha_1, \ldots, \alpha_m$ with non-negative coefficients. Consequently,

$$C_+ = \{x \in \Omega \colon \langle x, \alpha_k \rangle > 0,\ k = 1, \ldots, m\},$$

and the closure $\overline{C_+}$ has exactly m facets $\overline{C_+} \cap \langle \alpha_k \rangle^\perp$, $k = 1, \ldots, m$. These are closed $(d-1)$-dimensional infinite cones in the hyperplanes $\langle \alpha_k \rangle^\perp$, respectively. For $m = 1$, the single cone coincides with $\langle \alpha_1 \rangle^\perp$, while for $m \ge 2$

these cones are proper. Finally, we mention that the 'simple' reflections σ_{α_k}, $k = 1, \ldots, m$, generate W.

In what follows, if V is a linear space of functions on Ω, we shall write V^+ for the linear space of their restrictions to Ω_+, $V^+ := \{\Phi^+ \colon \Phi \in V\}$. Analogously, $\mathcal{A}_\omega V$ is the space of (weighted) averages of functions from V, $\mathcal{A}_\omega V := \{\mathcal{A}_\omega \Phi \colon \Phi \in V\}$.

PROPOSITION 3.1. *Let $f \in H^1(\Omega_+)$. Then $\mathcal{E}f \in H^1(\Omega)$ and*

$$\partial_j(\mathcal{E}f) = \sum_{i=1}^{d} \mathcal{E}^{g_{ji}}(\partial_i f). \tag{3.1}$$

Proof. If $f \in C_c^1(\Omega_+)$, then obviously $\mathcal{E}f \in C_c^1(\Omega)$ (we set $\mathcal{E}f(x) = 0$ for $x \in \Omega \cap \bigcup_{\alpha \in R} \langle \alpha \rangle^\perp$ provided this set is non-empty) and we directly check that

$$\partial_j(\mathcal{E}f)(gx) = \sum_{i=1}^{d} g_{ji} \partial_i f(x), \qquad x \in \Omega_+,\ g \in W,$$

which is just the explicit form of (3.1).

For general $f \in H^1(\Omega_+)$ we shall verify that for any $\varphi \in C_c^\infty(\Omega)$,

$$\int_{\Omega_+} f \Big(\sum_i \partial_i (\mathcal{A}_{g_{ji}} \varphi) \Big)^+ = - \int_{\Omega_+} \sum_i \partial_i f \, (\mathcal{A}_{g_{ji}} \varphi)^+. \tag{3.2}$$

This will be sufficient to deduce that $\partial_j(\mathcal{E}f)$ exist and (3.1) holds true, which also means that $\mathcal{E}f \in H^1(\Omega)$. Indeed, to check (3.1) with the aid of (3.2) we use successively Lemma 2.2, (2.2) for $\omega \equiv 1$, (3.2) and Lemma 2.2 again, to write

$$\begin{aligned} \int_\Omega \mathcal{E}f \, \partial_j \varphi &= |W| \int_{\Omega_+} f \, (\mathcal{A}(\partial_j \varphi))^+ = |W| \int_{\Omega_+} f \Big(\sum_i \partial_i (\mathcal{A}_{g_{ji}} \varphi) \Big)^+ \\ &= -|W| \int_{\Omega_+} \sum_i \partial_i f \, (\mathcal{A}_{g_{ji}} \varphi)^+ = - \int_\Omega \Big(\sum_i \mathcal{E}^{g_{ji}} (\partial_i f) \Big) \varphi. \end{aligned}$$

It remains to verify (3.2). Assume $f \in H^1(\Omega_+)$ and $\varphi \in C_c^\infty(\Omega)$ are fixed and recall that $\overline{C_+}$ has exactly m facets: $\overline{C_+} \cap \langle \alpha_k \rangle^\perp$, $k = 1, \ldots, m$. We now choose $\widetilde{\Omega}$, a bounded open set with C^1 boundary (piecewise C^1 would be sufficient as well), which is W-symmetric and such that $\operatorname{supp} \varphi \subset \widetilde{\Omega} \subset \Omega$. The assumption on the support of φ and the symmetry of $\widetilde{\Omega}$ imply that also $\operatorname{supp} \mathcal{A}_\omega \varphi \subset \widetilde{\Omega}$ for any weight ω. Observe that verification of (3.2) reduces to checking that for any fixed $j \in \{1, \ldots, d\}$ and $\widetilde{\Omega}_+ := \widetilde{\Omega} \cap C_+$ (recall that $\Omega_+ = \Omega \cap C_+$),

$$\int_{\widetilde{\Omega}_+} f \Big(\sum_i \partial_i (\mathcal{A}_{g_{ji}} \varphi) \Big)^+ = - \int_{\widetilde{\Omega}_+} \sum_i \partial_i f \, (\mathcal{A}_{g_{ji}} \varphi)^+. \tag{3.3}$$

Since $\widetilde{\Omega}_+$ is a bounded open set with piecewise C^1 boundary, a general theory applies, and for any $i \in \{1, \dots, d\}$ the following integration by parts formula holds:

$$(3.4)\qquad \int_{\widetilde{\Omega}_+} g\,\partial_i h = -\int_{\widetilde{\Omega}_+} \partial_i g\, h + \int_{\partial(\widetilde{\Omega}_+)} g h \nu_i\, d\sigma, \qquad g, h \in H^1(\widetilde{\Omega}_+).$$

Here $\nu = \nu(x) = (\nu_1, \dots, \nu_d)$ denotes the outward unit normal vector at $x \in \partial(\widetilde{\Omega}_+)$ (whenever it exists) and $d\sigma$ is the surface measure on $\partial(\widetilde{\Omega}_+)$. See [2, pp. 263–271], where (3.4) is stated with a weaker assumption, for bounded sets with Lipschitz boundary, or [11, Theorem D.8], where the assumptions on the boundary are stronger. It should also be pointed out that in (3.4) the boundary values of g and h (*traces* of g and h) are well defined $L^2(\partial(\widetilde{\Omega}_+), d\sigma)$ functions; this is the content of the *trace theorem*, see [2, A8.6] or [11, Theorem D.6].

Now, substituting in (3.4) f for g and $\mathcal{A}_{g_{ji}}\varphi$ for h, and summing over i, we obtain an identity like (3.3) but with the additional term

$$(3.5)\qquad \int_{\partial(\widetilde{\Omega}_+)} f \Big(\sum_i \mathcal{A}_{g_{ji}}\varphi \cdot \nu_i \Big)\, d\sigma$$

on the right hand side. To be precise, in fact we substitute in (3.4) the restrictions $f|_{\widetilde{\Omega}_+}$ and $(\mathcal{A}_{g_{ji}}\varphi)|_{\widetilde{\Omega}_+}$ for g and h, respectively. But both restrictions are indeed in $H^1(\widetilde{\Omega}_+)$, and moreover $\partial_i(f|_{\widetilde{\Omega}_+}) = (\partial_i f)|_{\widetilde{\Omega}_+}$ and $\partial_i((\mathcal{A}_{g_{ji}}\varphi)|_{\widetilde{\Omega}_+}) = (\partial_i(\mathcal{A}_{g_{ji}}\varphi))|_{\widetilde{\Omega}_+}$. Also, f in (3.5) is understood as the trace of $f|_{\widetilde{\Omega}_+}$ on $\partial(\widetilde{\Omega}_+)$. The trace exists since $f|_{\widetilde{\Omega}_+}$ is in $H^1(\widetilde{\Omega}_+)$ and $\widetilde{\Omega}_+$ is bounded with Lipschitz boundary.

To deduce (3.3), we will check that the quantity in (3.5) vanishes.

The boundary $\partial(\widetilde{\Omega}_+)$ consists of at most m 'flat' parts, $\overline{\widetilde{\Omega}_+} \cap \langle\alpha_k\rangle^\perp$, $k \in \{1, \dots, m\}$ (some of these sets or even all may be empty) and the 'irregular' part $\partial\widetilde{\Omega} \cap C_+$. On the latter part all $\mathcal{A}_{g_{ji}}\varphi$, $i = 1, \dots, d$, vanish and the corresponding part of the integral vanishes. In fact, also on each non-empty 'flat' part of the boundary the expression $\sum_i \mathcal{A}_{g_{ji}}\varphi \cdot \nu_i$ vanishes. Indeed, note that ν, the outward unit normal vector to the 'flat' part $\overline{\widetilde{\Omega}_+} \cap \langle\alpha_k\rangle^\perp$, which we assume to be non-empty, coincides with α_k. Let $\alpha_k = (\alpha_{k1}, \dots, \alpha_{kd})$. Our aim is to check (slightly more than required, with φ extended by zero outside Ω) that

$$(3.6)\qquad \sum_i \mathcal{A}_{g_{ji}}\varphi(x)\, \alpha_{ki} = 0, \qquad x \in \langle\alpha_k\rangle^\perp.$$

Let $W^+ = \ker(\mathrm{sgn}) = \{g \in W : \mathrm{sgn}(g) = 1\}$. Then, due to the partition

$W = W^+ \sqcup (W^+ \cdot \sigma_{\alpha_k})$, for $x \in \langle \alpha_k \rangle^\perp$ we have

$$\begin{aligned} |W| \sum_i \mathcal{A}_{g_{ji}} \varphi(x)\, \alpha_{ki} &= \sum_i \Big(\sum_{g\in W} g_{ji}\varphi(gx) \Big)\, \alpha_{ki} \\ &= \sum_i \Big(\sum_{g\in W^+} g_{ji}\varphi(gx) + \sum_{g\in W^+} (g\sigma_{\alpha_k})_{ji}\varphi(g\sigma_{\alpha_k}x) \Big)\, \alpha_{ki} \\ &= \sum_i \Big(\sum_{g\in W^+} \varphi(gx)(g_{ji} + (g\sigma_{\alpha_k})_{ji}) \Big)\, \alpha_{ki} \\ &= \sum_{g\in W^+} \varphi(gx) \sum_i (g_{ji} + (g\sigma_{\alpha_k})_{ji})\alpha_{ki}. \end{aligned}$$

Since φ is arbitrary, to reach (3.6), we expect the last, innermost sum to vanish. This is indeed the case. For $g \in W^+$ (and $k \in \{1,\ldots,m\}$ fixed) an easy matrix calculation shows that

$$(g\sigma_{\alpha_k})_{ji} = g_{ji} - 2\alpha_{ki} \sum_s g_{js}\alpha_{ks},$$

and hence

$$\sum_i (g_{ji} + (g\sigma_{\alpha_k})_{ji})\alpha_{ki} = 2\Big(\sum_i g_{ji}\alpha_{ki} - \sum_i \alpha_{ki}^2 \cdot \sum_s g_{js}\alpha_{ks} \Big).$$

The last expression vanishes since $\alpha_k = (\alpha_{k1},\ldots,\alpha_{kd})$ is a unit vector. ∎

The following proposition is an immediate consequence.

Proposition 3.2. *We have*

$$H^1(\Omega_+) = \mathcal{A}(H^1(\Omega))^+. \tag{3.7}$$

Proof. The inclusion $\supset$ is obvious: if $F \in H^1(\Omega)$, then, by Lemma 2.3, $\mathcal{A}F \in H^1(\Omega)$ and hence $(\mathcal{A}F)^+ \in H^1(\Omega_+)$. To prove $\subset$ take $f \in H^1(\Omega_+)$. By Proposition 3.1 we have $\mathcal{E}f \in H^1(\Omega)$, and hence $f = (\mathcal{E}f)^+ = (\mathcal{A}(\mathcal{E}f))^+$ is in $\mathcal{A}(H^1(\Omega))^+$. ∎

Clearly, for an arbitrary weight the result of Proposition 3.1 is not true. However, with a stronger assumption we have the following.

Proposition 3.3. *Let $f \in H_0^1(\Omega_+)$ and ω be a weight on W. Then $\mathcal{E}^\omega f \in H_0^1(\Omega)$ and*

$$\partial_j(\mathcal{E}^\omega f) = \sum_{i=1}^d \mathcal{E}^{\omega\cdot g_{ji}}(\partial_i f). \tag{3.8}$$

Proof. If $f \in C_c^1(\Omega_+)$, then obviously $\mathcal{E}^\omega f \in C_c^1(\Omega)$ and we directly check that

$$\partial_j(\mathcal{E}^\omega f)(gx) = \sum_{i=1}^d \omega(g) g_{ji}\, \partial_i f(x), \qquad x \in \Omega_+,\ g \in W,$$

which is the explicit form of (3.8).

For general $f \in H_0^1(\Omega_+)$ we shall verify that for any weight ω' and $\varphi \in C_c^\infty(\Omega)$,

$$\int_{\Omega_+} f\partial_i(\mathcal{A}_{\omega'}\varphi)^+ = -\int_{\Omega_+} \partial_i f\,(\mathcal{A}_{\omega'}\varphi)^+, \qquad i=1,\dots,d. \tag{3.9}$$

This will be sufficient to deduce that $\partial_j(\mathcal{E}^\omega f)$ exist and (3.8) holds, which also means that $\mathcal{E}^\omega f \in H^1(\Omega)$. Indeed, to check (3.8) with the aid of (3.9), we use Lemma 2.2 twice, (2.2), and (3.9) applied to $\omega' = \omega\cdot g_{ji}$, to write

$$\int_\Omega \mathcal{E}^\omega f\,\partial_j\varphi = |W|\int_{\Omega_+} f\,(\mathcal{A}_\omega(\partial_j\varphi))^+ = |W|\int_{\Omega_+} f\Big(\sum_i \partial_i(\mathcal{A}_{\omega\cdot g_{ji}}\varphi)\Big)^+$$
$$= -|W|\sum_i\int_{\Omega_+}\partial_i f\,(\mathcal{A}_{\omega\cdot g_{ji}}\varphi)^+ = -\int_\Omega\Big(\sum_i \mathcal{E}^{\omega\cdot g_{ji}}(\partial_i f)\Big)\varphi.$$

We now prove (3.9). By density of $C_c^\infty(\Omega_+)$ in $H_0^1(\Omega_+)$, it suffices to verify this identity for $f \in C_c^\infty(\Omega_+)$.

Assume therefore that $f \in C_c^\infty(\Omega_+)$ and $\varphi \in C_c^\infty(\Omega)$ are fixed, choose a bounded open set $\Omega_0 \subset \Omega_+$ with C^1 boundary such that $\operatorname{supp} f \subset \Omega_0$, and observe that verification of (3.9) reduces to checking that

$$\int_{\Omega_0} f\partial_i(\mathcal{A}_{\omega'}\varphi)^+ = -\int_{\Omega_0}\partial_i f\,(\mathcal{A}_{\omega'}\varphi)^+. \tag{3.10}$$

By integration by parts, checking (3.10) further reduces to verification that the quantity as in (3.5) but with $\mathcal{A}_{g_{ji}}$ replaced by $\mathcal{A}_{\omega'}$ and $\widetilde{\Omega}_+$ replaced by Ω_0 vanishes. But this is obvious since f vanishes on $\partial\Omega_0$.

Finally, we show that in fact, for $f \in H_0^1(\Omega_+)$ we have $\mathcal{E}^\omega f \in H_0^1(\Omega)$. Indeed, take $\{\varphi_n\} \subset C_c^\infty(\Omega_+)$ such that $\varphi_n \to f$ in $H_0^1(\Omega_+)$. Since $\{\varphi_n\}$ is a Cauchy sequence in $H^1(\Omega_+)$, it follows that $\{\mathcal{E}^\omega\varphi_n\}$ is a Cauchy sequence in $H^1(\Omega)$. (Clearly $\mathcal{E}^\omega(\varphi_n-\varphi_m)\to 0$ in $L^2(\Omega)$ as $n,m\to\infty$, so verification that for every $j\in\{1,\dots,d\}$, also $\partial_j(\mathcal{E}^\omega(\varphi_n-\varphi_m))\to\infty$ in $L^2(\Omega)$ reduces, by using (3.8), to noting that for every $i\in\{1,\dots,d\}$, $\mathcal{E}^{\omega\cdot g_{ji}}(\partial_j(\varphi_n-\varphi_m))\to 0$ in $L^2(\Omega)$.) Let $F=\lim_{n\to\infty}\mathcal{E}^\omega\varphi_n$ in $H^1(\Omega)$. Obviously, $\{\mathcal{E}^\omega\varphi_n\}\subset C_c^\infty(\Omega)$ and hence $F\in H_0^1(\Omega)$ and $F|_{\Omega_+}=f$. ∎

Proposition 3.4. *We have*

$$H_0^1(\Omega_+) = \mathcal{A}_{\mathrm{sgn}}(H_0^1(\Omega))^+. \tag{3.11}$$

Proof. To prove $\subset$, take $f\in H_0^1(\Omega_+)$. Proposition 3.3 says that $\mathcal{E}^{\mathrm{sgn}}f\in H_0^1(\Omega)$, and hence $f=(\mathcal{E}^{\mathrm{sgn}}f)^+=\big(\mathcal{A}_{\mathrm{sgn}}(\mathcal{E}^{\mathrm{sgn}}f)\big)^+$ is in $\mathcal{A}_{\mathrm{sgn}}(H_0^1(\Omega))^+$.

To prove $\supset$, take $F\in H_0^1(\Omega)$. Then, by Lemma 2.3, $\mathcal{A}_{\mathrm{sgn}}F\in H_0^1(\Omega)$ and hence $(\mathcal{A}_{\mathrm{sgn}}F)^+\in H^1(\Omega_+)$. We will show that, in fact, $(\mathcal{A}_{\mathrm{sgn}}F)^+\in H_0^1(\Omega_+)$. Let $\{\Phi_n\}\subset C_c^\infty(\Omega)$ be such that $\Phi_n\to F$ in $H^1(\Omega)$. Then, again by Lemma 2.3, also $\mathcal{A}_{\mathrm{sgn}}\Phi_n\to\mathcal{A}_{\mathrm{sgn}}F$ in $H^1(\Omega)$ and hence $(\mathcal{A}_{\mathrm{sgn}}\Phi_n)^+\to$

$(\mathcal{A}_{\rm sgn}F)^+$ in $H^1(\Omega_+)$. Since $\{\mathcal{A}_{\rm sgn}\Phi_n\}\subset C_c^\infty(\Omega)$ and each $\mathcal{A}_{\rm sgn}\Phi_n$ for every $\alpha\in R$ satisfies $\mathcal{A}_{\rm sgn}\Phi_n(\sigma_\alpha y)=-\mathcal{A}_{\rm sgn}\Phi_n(y)$, $y\in\Omega$, it suffices to verify that each restriction $(\mathcal{A}_{\rm sgn}\Phi_n)^+$ can be approximated in the $H^1(\Omega_+)$ norm by $C_c^\infty(\Omega_+)$ functions.

In fact we shall prove a slightly more general claim: for every $\Phi\in C_c^\infty(\Omega)$ that vanishes on $\langle\alpha_k\rangle^\perp$ for every $k=1,\dots,m$, Φ^+ can be approximated in $H^1(\Omega_+)$ by $C_c^\infty(\Omega_+)$ functions.

Let $\beta\in C^\infty(0,\infty)$ with $\beta(t)=0$ for $0<t<1/2$ and $\beta(t)=1$ for $t>1$ and $\|\beta\|_\infty=1$. We define the sequence of functions $\widetilde{\beta_N}$ on Ω_+, $N\in\mathbb{N}$, by

$$\widetilde{\beta_N}(x)=\prod_{k=1}^m\beta(N\delta_k(x)),\qquad x\in\Omega_+,$$

where $\delta_k(x)=\sum_{i=1}^d\alpha_{ki}x_i$, $\alpha_k=(\alpha_{k1},\dots,\alpha_{kd})$, denotes the ℓ^1 distance from $x\in\Omega_+$ to the hyperplane $\langle\alpha_k\rangle^\perp$. Next we define

$$\widetilde{\Phi_N}(x)=\widetilde{\beta_N}(x)\Phi(x),\qquad x\in\Omega_+.$$

Clearly, $\widetilde{\Phi_N}\in C_c^\infty(\Omega_+)$. We shall verify that $\widetilde{\Phi_N}\to\Phi^+$ in $H^1(\Omega_+)$ as $N\to\infty$. We have

$$\|\widetilde{\Phi_N}-\Phi^+\|^2_{H^1(\Omega_+)}=\|\Phi^+(\widetilde{\beta_N}-1)\|^2_{L^2(\Omega_+)}+\sum_{j=1}^d\|\partial_j(\Phi^+(\widetilde{\beta_N}-1))\|^2_{L^2(\Omega_+)}.$$

Obviously, the first summand on the right hand side goes to 0 as $N\to\infty$. It remains to check the same for the remaining d summands. We have

$$\partial_j(\Phi^+(\widetilde{\beta_N}-1))=\partial_j\Phi^+(\widetilde{\beta_N}-1)+\Phi^+\partial_j\widetilde{\beta_N},$$

so we need to consider the second summand only (note that $\partial_j\Phi^+=(\partial_j\Phi)^+$). For simplicity, consider $j=1$. Then

$$\partial_1\widetilde{\beta_N}(x)=N\sum_{k=1}^m\partial_1\delta_k(x)\beta'(N\delta_k(x))\prod_{s\neq k}\beta(N\delta_s(x)).$$

Note that $\partial_1\delta_k(x)$ is a constant not depending on $x\in\Omega_+$. We are therefore reduced to showing that for any $k\in\{1,\dots,m\}$ the quantity

$$N^2\int_{\Omega_+}\Big|\Phi^+(x)\beta'(N\delta_k(x))\prod_{s\neq k}\beta(N\delta_s(x))\Big|^2\,dx \tag{3.12}$$

tends to 0 as $N\to\infty$. Assume that $\operatorname{supp}\Phi\subset B_R(0)$ with some $R>0$ and let $\|\beta'\|_\infty=C$. There exists a constant M such that for $x\in\Omega_+$ we have

$$|\Phi(x)|\le M\min_{1\le l\le m}\delta_l(x).$$

This follows from the mean value theorem and the fact that Φ is compactly supported and vanishes on each facet $\langle\alpha_l\rangle^\perp$. Therefore (3.12) is estimated from above by

$$N^2C^2M^2 \int\limits_{A(N,R)} \Big(\min_{1\le l\le m} \delta_l(x)\Big)^2\,dx,$$

where $A(N,R)=B_R(0)\cap\{x\in C_+\colon \min_{1\le l\le m}\delta_l(x)\le 1/N\}$. It is now clear that the last quantity tends to 0 as $N\to\infty$. ■

To end this section we note that Propositions 3.1 and 3.3 can be regarded as *extension theorems* in the sense of [1, Section 5]. Although it is not contained in the statements of the propositions, in fact the proofs show that both extension operators are bounded, that is, $\|\mathcal{E}f\|_{H^1(\Omega)}\lesssim\|f\|_{H^1(\Omega_+)}$, and similarly for $\mathcal{E}^\omega$.

4. η-extensions of the Laplacian. Let $\eta\colon W\to\widehat{\mathbb{Z}}_2$ be a homomorphism. A function F on Ω is said to be *η-symmetric* if $F(gy)=\eta(g)F(y)$ for every $g\in W$ and $y\in\Omega$. We call the averaging operator $\mathcal{A}_\eta$ the *η-symmetrization* operator. Obviously, F is η-symmetric if and only if $\mathcal{A}_\eta F=F$. Clearly, $\mathcal{A}_\eta F$ and $\mathcal{E}^\eta\phi$ are η-symmetric whatever F and ϕ are, and hence

$$\mathcal{A}_\eta(\mathcal{A}_\eta F)=\mathcal{A}_\eta F \quad\text{and}\quad \mathcal{A}_\eta(\mathcal{E}^\eta\phi)=\mathcal{E}^\eta\phi.$$

Recall that for any given $i,j\in\{1,\dots,d\}$, $\mathcal{A}_{\eta\cdot g_{ji}}$ is the weighted averaging operator with weight function $g\mapsto\eta(g)g_{ij}$ on W, and similarly for $\mathcal{E}^{\eta\cdot g_{ij}}$. Also, if V is a linear space of functions on Ω, then we write V^+ for the space of their restrictions to Ω_+, $V^+:=\{\Phi^+\colon\Phi\in V\}$, and $\mathcal{A}_\eta V$ for the space of their η-symmetrizations, $\mathcal{A}_\eta V:=\{\mathcal{A}_\eta\Phi\colon\Phi\in V\}$.

It will be convenient to use the notation

$$H^1_\eta(\Omega_+):=(\mathcal{A}_\eta H^1(\Omega))^+.$$

Obviously, since $\mathcal{A}_\eta$ maps $H^1(\Omega)$ into itself, $H^1_\eta(\Omega_+)$ is a subspace in $H^1(\Omega_+)$. It is easily seen that $H^1_\eta(\Omega_+)$ should be regarded as the space of restrictions to Ω_+ of η-symmetric functions from $H^1(\Omega)$, or as the space of functions from $L^2(\Omega_+)$ with η-symmetric extensions in $H^1(\Omega)$,

$$H^1_\eta(\Omega_+)=\{F\in H^1(\Omega)\colon F=\mathcal{A}_\eta F\}^+=\{f\in L^2(\Omega_+)\colon \mathcal{E}^\eta f\in H^1(\Omega)\}.$$

Until the end of this section, $\eta\in\operatorname{Hom}(W,\widehat{\mathbb{Z}}_2)$ is a fixed homomorphism. The following result is similar to Propositions 3.1 and 3.3; however, its assumptions are much stronger and hence the proof is much simpler.

PROPOSITION 4.1. *Let $f\in H^1_\eta(\Omega_+)$. Then*

$$\partial_j(\mathcal{E}^\eta f)=\sum_{i=1}^d \mathcal{E}^{\eta\cdot g_{ji}}(\partial_i f). \tag{4.1}$$

Proof. Take $f\in H^1_\eta(\Omega_+)$. Then $\mathcal{E}^\eta f\in H^1(\Omega)$, in particular $f\in H^1(\Omega_+)$, and hence both sides of (4.1) make sense. It suffices to check that both sides coincide on each Weyl chamber $g\Omega_+$, $g \in W$. But

$$\partial_j(\mathcal{E}^\eta f)|_{g\Omega_+} = \partial_j\big((\mathcal{E}^\eta f)|_{g\Omega_+}\big)$$

and hence the task reduces to checking that the jth weak partial derivative of the function $gx \mapsto \eta(g)f(x)$ defined on $g\Omega_+$, $x \in \Omega_+$, coincides with the restriction to $g\Omega_+$ of the right hand side of (4.1), i.e. the function $gx \mapsto \eta(g)\sum_{i=1}^d g_{ji}\partial_i f(x)$. This follows by an easy calculation. ∎

The following result will be crucial in the definition of $-\Delta_+^\eta$, the η-Laplacian on Ω_+.

PROPOSITION 4.2. *$H^1_\eta(\Omega_+)$ is a closed subspace in $H^1(\Omega_+)$.*

Proof. We begin by remarking that if $h \in L^2(\Omega_+)$ and $a \in \mathbb{R}$, then

$$\|\mathcal{E}^{\eta\cdot a}h\|_{L^2(\Omega)} = |W|^{d/2}|a|\,\|h\|_{L^2(\Omega_+)}. \tag{4.2}$$

Let $\{f_n\}$ be a sequence in $H^1_\eta(\Omega_+)$ converging to f in $H^1(\Omega_+)$. We claim, and this is sufficient for our purposes, that $\mathcal{E}^\eta f \in H^1(\Omega)$ and $\mathcal{E}^\eta f_n \to \mathcal{E}^\eta f$ in $H^1(\Omega)$. Indeed, by assumption and (4.2) it follows that $\mathcal{E}^\eta f_n \to \mathcal{E}^\eta f$ in $L^2(\Omega)$. Also by assumption, for any $j = 1,\dots,d$, $\{\partial_j f_n\}$ is a Cauchy sequence in $L^2(\Omega_+)$. Hence, by (4.2) and (4.1), $\{\partial_j(\mathcal{E}^\eta f_n)\}$ is a Cauchy sequence in $L^2(\Omega)$. Now, let $\partial_j(\mathcal{E}^\eta f_n) \to F_j$ in $L^2(\Omega)$. It follows that $\mathcal{E}^\eta f \in H^1(\Omega)$ and $\partial_j(\mathcal{E}^\eta f) = F_j$. The claim is proved. ∎

It is worth mentioning here that obviously $C_c^\infty(\Omega_+) \subset H^1_\eta(\Omega_+)$, which implies that $H^1_\eta(\Omega_+)$ is dense in $L^2(\Omega_+)$. Moreover, the question when the spaces $H^1_\eta(\Omega_+)$ are different for different η's requires a comment. It turns out that this heavily depends on the geometric relations between Ω and the hyperplanes $\langle\alpha_k\rangle^\perp$, $k = 1,\dots,m$. Namely, suppose that there exists $k \in \{1,\dots,m\}$ such that $B \setminus \langle\alpha_k\rangle^\perp \subset \Omega$, where $B = B(x_0,r)$ is a Euclidean ball centered at $x_0 \in \langle\alpha_k\rangle^\perp$. We can assume that $x_0 \neq 0$ and r is such that $B \cap \langle\alpha_j\rangle^\perp = \emptyset$ for $j \neq k$. Then, for $\eta,\eta' \in \mathrm{Hom}(W,\widehat{\mathbb{Z}}_2)$ such that $\eta(\sigma_{\alpha_k}) \neq \eta'(\sigma_{\alpha_k})$, we have $H^1_\eta(\Omega_+) \neq H^1_{\eta'}(\Omega_+)$. Indeed, suppose that, for instance, $\eta(\sigma_{\alpha_k}) = -1$ and $\eta'(\sigma_{\alpha_k}) = 1$. Choose a C^∞ function φ, symmetric in the hyperplane $\langle\alpha_k\rangle^\perp$, with support in B and equal to 1 on $\frac{1}{2}B$. It is easily seen that $\varphi^+ \in H^1_{\eta'}(\Omega_+)$, while $\varphi^+ \notin H^1_\eta(\Omega_+)$.

Thus, for instance when $\Omega = \mathbb{R}^d$ and hence $\Omega_+ = C_+$, different η's give different spaces $H^1_\eta(\Omega_+)$. This is a consequence of the fact that $\{\sigma_{\alpha_k}\}_{k=1}^m$ generates W.

Let $\mathfrak{t} = \mathfrak{t}_\Omega$ denote the sesquilinear form on $H^1(\Omega)$ defined by

$$\mathfrak{t}[F,G] = \int_\Omega \sum_{j=1}^d \partial_j F(x)\overline{\partial_j G(x)}\,dx, \qquad F,G \in H^1(\Omega).$$

Let $\mathfrak{t}_0$ denote $\mathfrak{t}$ restricted to $H^1_0(\Omega)$. Analogously, let $\mathfrak{t}^+ = \mathfrak{t}_{\Omega_+}$ denote the sesquilinear form on $H^1(\Omega_+)$ defined by

$$\mathfrak{t}^+[f,g] = \int_{\Omega_+} \sum_{j=1}^{d} \partial_j f(x)\overline{\partial_j g(x)}\,dx, \qquad f,g \in H^1(\Omega_+),$$

and let $\mathfrak{t}^+_0$ denote $\mathfrak{t}^+$ restricted to $H^1_0(\Omega_+)$. We denote by $\mathfrak{t}^+_\eta$ the restriction of $\mathfrak{t}^+$ to $H^1_\eta(\Omega_+)$. Thus $\mathrm{Dom}(\mathfrak{t}^+_\eta) = H^1_\eta(\Omega_+) \subset H^1(\Omega_+)$.

Recall that for a sesquilinear form $\mathfrak{s}$ with dense domain $\mathrm{Dom}(\mathfrak{s})$ on a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$, the associated operator $A_\mathfrak{s}$ is defined by first determining its domain,

$$\mathrm{Dom}(A_\mathfrak{s}) = \{h \in \mathrm{Dom}(\mathfrak{s})\colon \exists u_h \in \mathcal{H}\ \forall h' \in \mathrm{Dom}(\mathfrak{s})\ \mathfrak{s}[h,h'] = \langle u_h, h'\rangle\},$$

and then by setting its action on $h \in \mathrm{Dom}(A_\mathfrak{s})$ to be $A_\mathfrak{s} h = u_h$. If $\mathfrak{s}$ is Hermitian and closed, then $A_\mathfrak{s}$ is self-adjoint. If, in addition, $\mathfrak{s}$ is non-negative, then $A_\mathfrak{s}$ is non-negative. See [11, Chapter 10 and Section 3 of Chapter 12]. Also recall that closedness of a non-negative form $\mathfrak{s}$ means that the norm $\|x\|_\mathfrak{s} := (\mathfrak{s}[x,x] + \langle x,x\rangle^2)^{1/2}$ defined on $\mathrm{Dom}(\mathfrak{s})$ is complete.

Since the forms $\mathfrak{t}$, $\mathfrak{t}_0$, and $\mathfrak{t}^+$, $\mathfrak{t}^+_0$ and $\mathfrak{t}^+_\eta$, defined on $L^2(\Omega)$ and $L^2(\Omega_+)$, respectively, are Hermitian, closed and non-negative, the associated operators are self-adjoint and non-negative. Clearly, closedness of $\mathfrak{t}$ and $\mathfrak{t}_0$, and $\mathfrak{t}^+$ and $\mathfrak{t}^+_0$ is a consequence of the fact that the norms $\|\cdot\|_{H^1(\Omega)}$ and $\|\cdot\|_{H^1(\Omega_+)}$ are complete and $H^1_0(\Omega)$, $H^1_0(\Omega_+)$ are closed subspaces in $H^1(\Omega)$, $H^1(\Omega_+)$, respectively. Closedness of $\mathfrak{t}^+_\eta$ is a consequence of Proposition 4.2.

Let $-\Delta_{N,\Omega}$ and $-\Delta_{D,\Omega}$ be the Neumann and the Dirichlet Laplacians on Ω, which are the operators associated with $\mathfrak{t}$ and $\mathfrak{t}_0$, respectively. Let $-\Delta^+$, $-\Delta^+_0$ and $-\Delta^+_\eta$ denote the operators associated with $\mathfrak{t}^+$, $\mathfrak{t}^+_0$ and $\mathfrak{t}^+_\eta$, respectively. We shall call $-\Delta^+_\eta$ the *η-Laplacian* on Ω_+. It is clear that for $\eta \equiv 1$ the corresponding operator is the Neumann Laplacian on Ω_+, while $-\Delta^+_0$ is the Dirichlet Laplacian on Ω_+. Note, however, that if Ω is such that $H^1(\Omega) = H^1_0(\Omega)$, then $-\Delta^+_{\mathrm{sgn}}$ coincides with $-\Delta^+_0$.

In addition, $-\Delta^+_0$ and each of the operators $-\Delta^+_\eta$ is indeed an extension of $-\Delta_{\Omega_+}$; we postpone explanation of this fact till the end of this section. Thus we have a number of natural self-adjoint extensions of the differential operator $-\Delta_{\Omega_+}$ with the Neumann Laplacian and the Dirichlet Laplacian included.

The following result is crucial. It relates the domains $\mathrm{Dom}(-\Delta^+_\eta)$ and $\mathrm{Dom}(-\Delta_{N,\Omega})$, or $\mathrm{Dom}(-\Delta^+_0)$ and $-\Delta_{D,\Omega}$, as well as the actions of $-\Delta^+_\eta$ and $-\Delta_{N,\Omega}$, or $-\Delta^+_0$ and $-\Delta_{D,\Omega}$, respectively. Recall that if Ω is such that $H^1(\Omega) = H^1_0(\Omega)$, then the case of Dirichlet Laplacians below, in Corollary 4.4 and in the proof of the main theorem in Section 5, is covered by the general case.

PROPOSITION 4.3. *We have*

$$\mathrm{Dom}(-\Delta_\eta^+) = \mathcal{A}_\eta(\mathrm{Dom}(-\Delta_{N,\Omega}))^+ \tag{4.3}$$

and

$$(-\Delta_\eta^+)\big((\mathcal{A}_\eta F)^+\big) = \mathcal{A}_\eta(-\Delta_{N,\Omega}F)^+, \qquad F \in \mathrm{Dom}(-\Delta_{N,\Omega}). \tag{4.4}$$

Analogous identities also hold when $-\Delta_\eta^+$ and $-\Delta_{N,\Omega}$ are replaced by $-\Delta_{D,\Omega_+}$ and $-\Delta_{D,\Omega}$, respectively, and $\mathcal{A}_\eta$ is substituted by $\mathcal{A}_{\mathrm{sgn}}$.

Proof. We first prove the inclusion $\subset$ in (4.3). For brevity we write $-\Delta$ in place of $-\Delta_{N,\Omega}$. Take $f \in \mathrm{Dom}(-\Delta_\eta^+)$. Hence $f \in H_\eta^1(\Omega_+)$, that is, $f = F^+$, where $F \in H^1(\Omega)$ and $F = \mathcal{A}_\eta F$, and there is $u_f \in L^2(\Omega_+)$ such that

$$\forall g \in H_\eta^1(\Omega_+) \qquad \mathfrak{t}_\eta^+[f,g] = \langle u_f, g\rangle_{L^2(\Omega_+)}, \tag{4.5}$$

which also means that $(-\Delta_\eta^+)f = u_f$. Note that $\mathcal{E}^\eta f = F$ and thus $\mathcal{E}^\eta f \in H^1(\Omega)$. We shall verify that

$$\forall G \in H^1(\Omega) \qquad \mathfrak{t}[\mathcal{E}^\eta f, G] = \langle \mathcal{E}^\eta(u_f), G\rangle_{L^2(\Omega)}, \tag{4.6}$$

which means that $\mathcal{E}^\eta f \in \mathrm{Dom}(-\Delta)$ and hence $f = (\mathcal{E}^\eta f)^+ = \mathcal{A}_\eta(\mathcal{E}^\eta f)^+ \in \mathcal{A}_\eta\big(\mathrm{Dom}(-\Delta_\eta^+)\big)^+$, and also that

$$(-\Delta)(\mathcal{E}^\eta f) = \mathcal{E}^\eta((-\Delta_\eta^+)f) \quad \text{ for } f \in \mathrm{Dom}(-\Delta_\eta^+).$$

For any $G \in H^1(\Omega)$, using (4.1), Lemma 2.2 and (2.3) we have

$$\begin{aligned}
\mathfrak{t}[\mathcal{E}^\eta f, G] &= \sum_j \int_\Omega \partial_j(\mathcal{E}^\eta f)\overline{\partial_j G} = \sum_j \int_\Omega \Big(\sum_i \mathcal{E}^{\eta\cdot g_{ji}}(\partial_i f)\Big)\overline{\partial_j G} \\
&= |W| \sum_i \int_{\Omega_+} \partial_i f\, \overline{\Big(\sum_j \mathcal{A}_{\eta\cdot g_{ji}}(\partial_j G)^+\Big)} = |W| \sum_i \int_{\Omega_+} \partial_i f\, \overline{\partial_i(\mathcal{A}_\eta G)^+} \\
&= |W|\mathfrak{t}_\eta^+[f, (\mathcal{A}_\eta G)^+].
\end{aligned}$$

On the other hand,

$$\langle \mathcal{E}^\eta(u_f), G\rangle_{L^2(\Omega)} = |W|\langle u_f, (\mathcal{A}_\eta G)^+\rangle_{L^2(\Omega_+)},$$

and hence inserting $(\mathcal{A}_\eta G)^+$ for g in (4.5) gives (4.6).

To prove the opposite inclusion in (4.3), take $F \in \mathrm{Dom}(-\Delta)$. In particular, $F \in H^1(\Omega)$, and moreover there is $U_F \in L^2(\Omega)$ such that

$$\forall G \in H^1(\Omega) \qquad \mathfrak{t}[F,G] = \langle U_F, G\rangle_{L^2(\Omega)}, \tag{4.7}$$

which also means that $(-\Delta)F = U_F$. We shall verify that

$$\forall g \in H_\eta^1(\Omega_+) \qquad \mathfrak{t}_\eta^+[(\mathcal{A}_\eta F)^+, g] = \langle \mathcal{A}_\eta(U_F)^+, g\rangle_{L^2(\Omega_+)}, \tag{4.8}$$

which means that $(\mathcal{A}_\eta F)^+ \in \mathrm{Dom}(-\Delta_\eta^+)$ and completes the proof of (4.3), proving in addition that $(-\Delta_\eta^+)\big((\mathcal{A}_\eta F)^+\big) = \mathcal{A}_\eta((-\Delta)F)^+$ for $F \in \mathrm{Dom}(-\Delta)$, which completes the proof of (4.4).

Let $g \in H^1_\eta(\Omega_+)$, where $g = G^+$ and $G = \mathcal{A}_\eta G$. Using (2.3) and Lemma 2.2 we have

$$\begin{aligned}\mathfrak{t}^+_\eta[(\mathcal{A}_\eta F)^+, g] &= \sum_j \int_{\Omega_+} \partial_j((\mathcal{A}_\eta F)^+)\,\overline{\partial_j g} = \sum_j \int_{\Omega_+} \Big(\sum_i \mathcal{A}_{\eta\cdot g_{ij}}(\partial_i F)^+\Big)\,\overline{\partial_j g} \\ &= |W|^{-d} \sum_i \int_{\Omega} \partial_i F\, \overline{\Big(\sum_j \mathcal{E}^{\eta\cdot g_{ji}}(\partial_j g)\Big)} = |W|^{-d} \sum_i \int_{\Omega} \partial_i F\, \overline{\partial_i(\mathcal{E}^\eta g)} \\ &= |W|^{-d}\mathfrak{t}[F, \mathcal{E}^\eta g].\end{aligned}$$

On the other hand,

$$|W|\langle \mathcal{A}_\eta (U_F)^+, g\rangle_{L^2(\Omega_+)} = \langle U_F, \mathcal{E}^\eta g\rangle_{L^2(\Omega)},$$

and hence inserting $\mathcal{E}^\eta g$ for G in (4.7) gives (4.8); note that $\mathcal{E}^\eta g \in H^1(\Omega)$ by the assumption on g. This completes the proof of (4.8) and thus of the conclusion following it, and hence finishes the proof of (4.3) and (4.4).

We now comment on the changes to be made in the above reasoning in order to prove the last claim of the proposition. We shall write $-\Delta$, $-\Delta_+$ instead of $-\Delta_{D,\Omega}$, $-\Delta_{D,\Omega_+}$, respectively. For the inclusion $\subset$ in the modified (4.3) we copy *mutatis mutandis* the proof of $\subset$ in (4.3). Clearly, we replace H^1 by H^1_0, note that $\mathcal{E}^{\mathrm{sgn}} f \in H^1_0(\Omega)$ provided $f \in H^1_0(\Omega_+)$, and replace $\mathcal{E}^\eta$, $\mathcal{E}^{\eta\cdot g_{ji}}$, $\mathcal{A}_\eta$, $\mathcal{A}_{\eta\cdot g_{ji}}$ by $\mathcal{E}^{\mathrm{sgn}}$, $\mathcal{E}^{\mathrm{sgn}\cdot g_{ji}}$, $\mathcal{A}_{\mathrm{sgn}}$, $\mathcal{A}_{\mathrm{sgn}\cdot g_{ji}}$, respectively. The resulting identity is

$$\mathfrak{t}[\mathcal{E}^{\mathrm{sgn}} f, G] = |W|\mathfrak{t}^+[f, (\mathcal{A}_{\mathrm{sgn}} G)^+],$$

which together with

$$\langle \mathcal{E}^{\mathrm{sgn}}(u_f), G\rangle_{L^2(\Omega)} = |W|\langle u_f, (\mathcal{A}_{\mathrm{sgn}} G)^+\rangle_{L^2(\Omega_+)}$$

shows an appropriate version of (4.6) and its consequence. Note that we have used the fact that $(\mathcal{A}_{\mathrm{sgn}} G)^+ \in H^1_0(\Omega_+)$. For the opposite inclusion in the modified (4.3), again we copy *mutatis mutandis* the relevant part of the earlier reasoning with all previous symbol substitutions to conclude that

$$\mathfrak{t}^+[(\mathcal{A}_{\mathrm{sgn}} F)^+, g] = \frac{1}{|W|}\mathfrak{t}[F, \mathcal{E}^{\mathrm{sgn}} g],$$

and together with

$$|W|\langle \mathcal{A}_{\mathrm{sgn}}(U_F)^+, g\rangle_{L^2(\Omega_+)} = \langle U_F, \mathcal{E}^{\mathrm{sgn}} g\rangle_{L^2(\Omega)},$$

we end up with an appropriate version of (4.8) and the subsequent conclusions. ∎

For later reference it is convenient to single out the following from the proof of Proposition 4.3.

Corollary 4.4. *We have*

$$\mathcal{E}^\eta\big(\mathrm{Dom}(-\Delta_+^\eta)\big) \subset \mathrm{Dom}(-\Delta_{N,\Omega})$$

and

$$\mathcal{E}^\eta\big((-\Delta_+^\eta)f\big) = (-\Delta_{N,\Omega})(\mathcal{E}^\eta f), \qquad f \in \mathrm{Dom}(-\Delta_\eta^+).$$

Analogous identities also hold when $-\Delta_\eta^+$ and $-\Delta_{N,\Omega}$ are replaced by $-\Delta_{D,\Omega_+}$ and $-\Delta_{D,\Omega}$, respectively, and $\mathcal{E}^\eta$ is replaced by $\mathcal{E}^{\mathrm{sgn}}$.

We now come back to verification that $-\Delta_+^\eta$ and $-\Delta_0^+$ extend $-\Delta_{\Omega_+}$. This will follow from the definitions in terms of forms, with an application of Gauss' formula for functions from Sobolev classes. Indeed, we claim that $C_c^2(\Omega_+) \subset \mathrm{Dom}(-\Delta_+^\eta)$ and $(-\Delta_+^\eta)f = -\Delta_{\Omega_+}f$ for $f \in C_c^2(\Omega_+)$. To check this, take $f \in C_c^2(\Omega_+)$ and let Ω_0 be a bounded subset in Ω_+ with smooth boundary such that $\operatorname{supp} f \subset \Omega_0$. Recall that $\mathrm{Dom}(-\Delta_+^\eta)$ coincides with

$$\{h \in \mathrm{Dom}(\mathfrak{t}_\eta^+)\colon \exists u_h \in L^2(\Omega_+)\ \forall g \in \mathrm{Dom}(\mathfrak{t}_\eta^+)\ \mathfrak{t}_\eta^+[h,g] = \langle u_h, g\rangle_{L^2(\Omega_+)}\},$$

where $\mathfrak{t}_\eta^+[h,g] = \langle \nabla h, \overline{\nabla g}\rangle_{L^2(\Omega_+)}$. Clearly, $f \in \mathrm{Dom}(\mathfrak{t}_\eta^+)$, thus for every $g \in \mathrm{Dom}(\mathfrak{t}_\eta^+)$, since f vanishes on $\partial\,\Omega_0$, by using [11, (D.4), Appendix D] (note that $f, g \in H^1(\Omega_+)$), we obtain

$$\begin{aligned}\mathfrak{t}_\eta^+[f,g] &= \int_{\Omega_+} \Big(\sum_j \partial_j f\, \overline{\partial_j g}\Big)\, dx = \int_{\Omega_0} \Big(\sum_j \partial_j f \overline{\partial_j g}\Big)\, dx \\ &= \int_{\Omega_0} (-\Delta_{\Omega_+})\, f\overline{g}\, dx = \langle (-\Delta_{\Omega_+})f, g\rangle_{L^2(\Omega_+)}.\end{aligned}$$

This proves the claim.

Finally, the question when the operators $-\Delta_\eta^+$ are different for different η's also requires a comment. We focus on looking at the domains $\mathrm{Dom}(-\Delta_+^\eta)$ and apply an argument similar to that used in the analysis of differences between the spaces $H_\eta^1(\Omega_+)$. As in that case, the geometric relations between Ω and the hyperplanes $\langle \alpha_k\rangle^\perp$, $k = 1, \ldots, m$, are essential. Therefore, suppose that there exists $k \in \{1, \ldots, m\}$ such that $B \subset \Omega$, where B is a Euclidean ball centered at $0 \neq x_0 \in \langle \alpha_k\rangle^\perp$ and $B \cap \langle \alpha_j\rangle^\perp = \emptyset$ for $j \neq k$. Then for $\eta, \eta' \in \mathrm{Hom}(W, \widehat{\mathbb{Z}}_2)$ such that $\eta(\sigma_{\alpha_k}) \neq \eta'(\sigma_{\alpha_k})$, we have $\mathrm{Dom}(-\Delta_+^\eta) \neq \mathrm{Dom}(-\Delta_+^{\eta'})$. Indeed, suppose that, for instance, $\eta(\sigma_{\alpha_k}) = -1$ and $\eta'(\sigma_{\alpha_k}) = 1$. Choose a C^∞ function φ with support in B and equal to 1 on $\frac{1}{2}B$, which is η'-symmetric. Then $\varphi \in \mathrm{Dom}(-\Delta_{N,\Omega})$, and consequently, by (4.3), $\varphi^+ \in \mathrm{Dom}(-\Delta_+^{\eta'})$. On the other hand, it is clear that $\varphi^+ \notin H_\eta^1(\Omega_+)$ and hence $\varphi^+ \notin \mathrm{Dom}(-\Delta_+^\eta)$.

Thus, for instance in the case of $\Omega = \mathbb{R}^d$ and hence $\Omega_+ = C_+$, different η's give different operators $-\Delta_+^\eta$. Again this is a consequence of the fact that $\{\sigma_{\alpha_k}\}_{k=1}^m$ generates W.

5. Proof of the main result. The following *commuting property* of the spectral functional calculus is well known: if A is a self-adjoint operator on a Hilbert space $\mathcal{H}$ and $B\colon \mathcal{H} \to \mathcal{H}$ is a bounded operator such that $BA \subset AB$, then also $B\Psi(A) \subset \Psi(A)B$ for any Borel function Ψ on $\mathbb{R}$. At least as a folklore, the following two-Hilbert-space and two-operator version of this property is also known: if A_1 and A_2 are self-adjoint operators on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and $B\colon \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded operator such that $BA_1 \subset A_2B$, then also $B\Psi(A_1) \subset \Psi(A_2)B$ for any Borel function Ψ on $\mathbb{R}$; see [9] for additional comments. We shall refer to this property as the *intertwining property*. Recall that the latter inclusion precisely means the inclusion of domains (which is equivalent to the statement that $B(\mathrm{Dom}(\Psi(A_1))) \subset \mathrm{Dom}(\Psi(A_2)))$ and the identity $B\big(\Psi(A_1)x\big) = \Psi(A_2)\big(Bx\big)$ for all $x \in \mathrm{Dom}(\Psi(A_1))$.

Passing to the proof of Theorem 1.1 consider first the case of the η-Laplacians on Ω_+ and prove (1.1). As in the proof of Proposition 4.3, for brevity, if not otherwise stated, we write $-\Delta$ instead of $-\Delta_{N,\Omega}$, and consequently, $\Psi(-\Delta)$ instead of $\Psi(-\Delta_{N,\Omega})$. Analogously, we write K^Ψ for $K^\Psi_{-\Delta_{N,\Omega}}$.

It is intuitively clear and easily seen by using the definition of $-\Delta$ in terms of the sesquilinear form $\mathfrak{t}$ that for any $g \in W$, the operator $T_g\colon F \mapsto F_g$, $F_g(x) = F(gx)$, which is bounded on $L^2(\Omega)$, commutes with $-\Delta$. This implies that

$$(\Psi(-\Delta)F)_g = \Psi(-\Delta)(F_g), \qquad F \in \mathrm{Dom}(\Psi(-\Delta)).$$

Consequently, since $\mathrm{Dom}(\Psi(-\Delta))$ is dense in $L^2(\Omega)$, the kernel K^Ψ satisfies

$$K^\Psi(gx, y) = K^\Psi(x, g^{-1}y), \qquad (x, y) \in \Omega \times \Omega \text{ a.e.} \tag{5.1}$$

On the other hand, (4.4) says that the operator $\mathcal{A}^+_\eta\colon F \mapsto (\mathcal{A}_\eta F)^+$, which is bounded from $L^2(\Omega)$ into $L^2(\Omega_+)$, intertwines the operators $-\Delta$ and $-\Delta^+_\eta$ in the sense that $\mathcal{A}^+_\eta \circ (-\Delta) \subset (-\Delta^+_\eta) \circ \mathcal{A}^+_\eta$. By applying the aforementioned intertwining property to $\mathcal{H}_1 = L^2(\Omega)$ and $\mathcal{H}_2 = L^2(\Omega_+)$, $A_1 = -\Delta$ and $A_2 = -\Delta^+_\eta$, and $B = \mathcal{A}^+_\eta$, it follows that also $\mathcal{A}^+_\eta$ intertwines the operators $\Psi(-\Delta)$ and $\Psi(-\Delta^+_\eta)$; in particular,

$$\mathcal{A}_\eta(\Psi(-\Delta)F)^+ = \Psi(-\Delta^+_\eta)((\mathcal{A}_\eta F)^+) \qquad \text{for } F \in \mathrm{Dom}(\Psi(-\Delta)). \tag{5.2}$$

On the other hand, Corollary 4.4 says that the operator $\mathcal{E}^\eta\colon f \mapsto \mathcal{E}^\eta f$, which is bounded from $L^2(\Omega_+)$ into $L^2(\Omega)$, intertwines $-\Delta^+_\eta$ and $-\Delta$. It follows that $\mathcal{E}^\eta$ also intertwines the operators $\Psi(-\Delta^+_\eta)$ and $\Psi(-\Delta)$, and hence

$$\mathcal{E}^\eta(\mathrm{Dom}(\Psi(-\Delta^+_\eta))) \subset \mathrm{Dom}(\Psi(-\Delta)). \tag{5.3}$$

Here the intertwining property was applied to $\mathcal{H}_1 = L^2(\Omega_+)$ and $\mathcal{H}_2 = L^2(\Omega)$, $A_1 = -\Delta^+_\eta$ and $A_2 = -\Delta$, and $B = \mathcal{E}^\eta$.

Now take $f \in \mathrm{Dom}(\Psi(-\Delta_\eta^+))$ and let $F = \mathcal{E}^\eta f$. Then, by (5.3), $F \in \mathrm{Dom}(\Psi(-\Delta))$, and since F is η-invariant, $f = F^+ = (\mathcal{A}_\eta F)^+$. Hence, by (5.2), for $x \in \Omega_+$ we have

$$\begin{aligned}\Psi(-\Delta_\eta^+)f(x) &= \frac{1}{|W|}\sum_{g\in W}\eta(g)\Psi(-\Delta)F(gx)\\ &= \frac{1}{|W|}\sum_{g\in W}\eta(g)\int_\Omega K^\Psi(gx,y)F(y)\,dy\\ &= \frac{1}{|W|}\sum_{g\in W}\eta(g)\sum_{g'\in W}\eta(g')\int_{\Omega_+}K^\Psi(gx,g'y)f(y)\,dy\\ &= \int_{\Omega_+}\Big(\sum_{g''\in W}\eta(g'')K^\Psi(g''x,y)\Big)f(y)\,dy,\end{aligned}$$

where, for the last identity, we have used (5.1) to obtain

$$\begin{aligned}\sum_{g\in W}\eta(g)\sum_{g'\in W}\eta((g')^{-1})K^\Psi(gx,g'y) &= \sum_{g\in W}\eta(g)\sum_{g'\in W}\eta((g')^{-1})K^\Psi((g')^{-1}gx,y)\\ &= |W|\sum_{g''\in W}\eta(g'')K^\Psi(g''x,y).\end{aligned}$$

This means that $\Psi(-\Delta_\eta^+)$ has an integral kernel and (1.1) holds.

We now comment on the changes to be made in order to prove (1.2). For brevity, we now write $-\Delta$, $-\Delta_+$, $\Psi(-\Delta)$, $\Psi(-\Delta_+)$ and K^Ψ instead of $-\Delta_{D,\Omega}$, $-\Delta_{D,\Omega_+}$, $\Psi(-\Delta_{D,\Omega})$, $\Psi(-\Delta_{D,\Omega_+})$, $K^\Psi_{-\Delta_{D,\Omega}}$, respectively. Then we repeat the arguments we used in the previous case so that we again obtain (5.1) and by using a version of (4.4) with $\eta = \mathrm{sgn}$ (and $-\Delta$ replacing $-\Delta_{D,\Omega_+}$) we arrive at a version of (5.2) with $\mathcal{A}_\eta$ replaced by $\mathcal{A}_{\mathrm{sgn}}$. Consequently, picking $f \in \mathrm{Dom}(\Psi(-\Delta_+))$ and $F \in \mathrm{Dom}(\Psi(-\Delta))$ such that $(\mathcal{A}_{\mathrm{sgn}}F)^+ = f$ and considering $F = \mathcal{E}^{\mathrm{sgn}}f$, we follow the main calculation of the previous case and end up with

$$\Psi(-\Delta_+)f(x) = \int_{\Omega_+}\Big(\sum_{g''\in W}\mathrm{sgn}(g'')K^\Psi(g''x,y)\Big)f(y)\,dy, \qquad x\in\Omega_+.$$

The proof of Theorem 1.1 is completed.

6. Examples

6.1. Groups associated with orthogonal root systems. This example is related to the one discussed in [9, Section 4.1]. There, for an orthogonal root system and an arbitrary open and appropriately symmetric $\Omega \subset \mathbb{R}^d$, an inductive argument was applied to obtain relations between relevant kernels. Here we focus on $\Omega = \mathbb{R}^d$ and consider a Weyl chamber coming from an orthogonal root system. Then we directly apply our main theorem to obtain relations between kernels labeled by a variety of relevant homomorphisms. In addition, properties of η-heat kernels are discussed together with their application to solving some mixed Neumann–Dirichlet initial-boundary value problems.

Let R be an orthonormal root system in $\mathbb{R}^d$ with a chosen subsystem R_+ of positive roots (orthonormality of R means that R_+ is orthonormal as a set of vectors). Without any loss of generality, possibly by rotating and reflecting the coordinate axes, we can assume that $R_+ = \{e_{j_1}, \ldots, e_{j_k}\}$, where $1 = j_1 < \cdots < j_k \le d$, and e_j is the jth coordinate unit vector. Thus, given $1 \le k \le d$ and $J = (j_1, \ldots, j_k)$ as above, let $R_+^{(k,J)} = \{e_{j_s} : s = 1, \ldots, k\}$ be the system of positive roots so that $R^{(k,J)} = R_+^{(k,J)} \sqcup (-R_+^{(k,J)})$ is the orthonormal root system in $\mathbb{R}^d$. In what follows to fix ideas we consider only the case when $k = d$; the other cases, $1 \le k \le d-1$, can be treated analogously.

The positive Weyl chamber is then $\mathbb{R}_+^d = (0,\infty)^d$ and we identify the finite reflection group corresponding to $R^{(d,J_d)}$, $J_d = (1, \ldots, d)$, with $\widehat{\mathbb{Z}}_2^d$, and the action of $\varepsilon \in \widehat{\mathbb{Z}}_2^d$ on $\mathbb{R}^d$ is through $x \mapsto \varepsilon x = (\varepsilon_1 x_1, \ldots, \varepsilon_d x_d)$. Next, we identify $\mathrm{Hom}(\widehat{\mathbb{Z}}_2^d, \widehat{\mathbb{Z}}_2)$ with $\mathbb{Z}_2^d$, where, this time, $\mathbb{Z}_2 = \{0,1\}$ with addition modulo 2. In this identification a homomorphism $\eta \in \mathbb{Z}_2^d$ of the finite reflection group represented by $\widehat{\mathbb{Z}}_2^d$ into $\widehat{\mathbb{Z}}_2$ acts through $\varepsilon \mapsto \varepsilon^\eta := \prod_{j=1}^d \varepsilon_j^{\eta_j}$ for $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_d)$.

Also, after these identifications, the η-symmetrization operator is

$$\mathcal{A}_\eta F(x) = \frac{1}{2^d} \sum_{\varepsilon \in \{-1,1\}^d} \varepsilon^\eta F(\varepsilon x), \qquad x \in \mathbb{R}^d.$$

Moreover, (2.3) simplifies to $\partial_j(\mathcal{A}_\eta F) = \mathcal{A}_{\tau_j(\eta)}(\partial_j F)$, where for $\eta = (\eta_1, \ldots, \eta_d) \in \mathbb{Z}_2^d$ and $j \in \{1, \ldots, d\}$ we write $\tau_j(\eta) = (\eta_1, \ldots, 1-\eta_j, \ldots, \eta_d)$ (this is the '$\mathbb{Z}_2$-reflection' on the jth coordinate).

Let $\{\exp(-t(-\Delta))\}_{t>0}$ denote the *heat semigroup* associated with $-\Delta$, the Neumann Laplacian on $\mathbb{R}^d$ (that coincides with the Dirichlet Laplacian). For every $t > 0$, $\exp(-t(-\Delta))$ is a bounded on $L^2(\mathbb{R}^d)$ operator with kernel $p_t^{-\Delta}(x,y) = p_t^{(d)}(x-y)$, $x, y \in \mathbb{R}^d$, where

$$p_t^{(d)}(w) = (4\pi t)^{-d/2} \exp\big(-\|w\|^2/(4t)\big), \qquad w \in \mathbb{R}^d,$$

is the d-dimensional Gauss–Weierstrass kernel.

Similarly, let $\{\exp(-t(-\Delta_\eta^+))\}_{t>0}$ denote the *η-heat semigroup* associated with the η-Laplacian $-\Delta_\eta^+$, a semigroup of bounded operators on $L^2(\mathbb{R}_+^d)$, and let $p_t^{\eta,+}$ denote the corresponding kernel. Using $p_t^{(d)}(w) = \prod_{j=1}^d p_t^{(1)}(w_j)$, as an immediate consequence of Corollary 1.2 we obtain the following.

COROLLARY 6.1. *For every $t>0$, $\exp(-t(-\Delta^+_\eta))$ is an integral operator in $L^2(\mathbb{R}^d_+)$ with kernel*

$$p^{\eta,+}_t(x,y)=\prod_{j=1}^{d}\big(p^{(1)}_t(x_j-y_j)+(-1)^{\eta_j}p^{(1)}_t(x_j+y_j)\big),\qquad x,y\in\mathbb{R}^d_+.$$

Note that the (expected) properties of the η-*heat kernel* $\{p^{\eta,+}_t\}_{t>0}$, analogous to those that correspond to the Neumann and Dirichlet heat kernels on open subsets of $\mathbb{R}^d$, readily follow by inspection. Indeed, smoothness of $p^{\eta,+}_t(x,y)$ jointly in the variables $t>0$, $x,y\in\mathbb{R}^d_+$, is obvious. Equally obvious is its symmetry in the $x,y\in\mathbb{R}^d_+$ variables. Also it is easy to see that for every fixed $y\in\mathbb{R}^d_+$ the function $v(t,x):=p^{\eta,+}_t(x,y)$ satisfies the equation $(\partial_t-\Delta_{\mathbb{R}^d_+})v=0$ on $(0,\infty)\times\mathbb{R}^d_+$, and $p^{\eta,+}_t(x,y)>0$ for $x,y\in\mathbb{R}^d_+$. For the qualitative properties of $p^{\eta,+}_t(x,y)$ one easily observes that for every fixed $x\in\mathbb{R}^d_+$,

(a) we have
$$\lim_{t\to0^+}\int_{\mathbb{R}^d_+}p^{\eta,+}_t(x,y)\,dy=1;$$

(b) for $t>0$ we have
$$0<\int_{\mathbb{R}^d_+}p^{\eta,+}_t(x,y)\,dy\le1;$$
moreover, for $\eta=\mathbf{0}$ the integral equals 1; otherwise for every fixed $x\in\mathbb{R}^d_+$ and $t>0$ the integral is strictly less than 1;

(c) for every $\epsilon>0$ and $0<\delta<\operatorname{dist}(x,\partial\mathbb{R}^d_+)$ there exists $t_0>0$ such that for all $0<t<t_0$,
$$\int_{\mathbb{R}^d_+\setminus B(x,\delta)}p^{\eta,+}_t(x,y)\,dy\le\epsilon.$$

Let us focus, therefore, on the 'boundary' properties. First, notice that the boundary of $\mathbb{R}^d_+$ consists of the d facets

$$\mathcal{F}_j=\{x\in\mathbb{R}^d\colon x_j=0\text{ and }x_k\ge0\text{ for }k\ne j\},\qquad j=1,\dots,d.$$

It is now easy to check that $p^{\eta,+}_t(x,y)$ satisfies mixed Neumann–Dirichlet conditions: for every fixed $y\in\mathbb{R}^d_+$, the outward normal derivative of $p^\eta_t(\cdot,y)$ vanishes on any facet $\mathcal{F}_j$ (to be precise, on the 'interior' of $\mathcal{F}_j$, where the derivative exists) where $\eta_j=0$, and $p^\eta_t(\cdot,y)$ vanishes on any facet $\mathcal{F}_j$ where $\eta_j=1$. Clearly, this agrees with the Neumann boundary conditions when $\eta=\mathbf{0}$ and with the Dirichlet boundary conditions when $\eta=\mathbf{1}$.

To simplify the notation, given $\eta\in\mathbb{Z}^d_2$ we split the boundary $\partial\mathbb{R}^d_+$ into $(\partial\mathbb{R}^d_+)_{\eta,N}$ and $(\partial\mathbb{R}^d_+)_{\eta,D}$, where the former part includes all facets $\mathcal{F}_j$ where

$\eta_j = 0$, and the latter part includes all facets $\mathcal{F}_j$ where $\eta_j = 1$. We now come to the following initial-boundary value problem:

$$\begin{aligned} &(\partial_t - \Delta_{\mathbb{R}^d_+})u(t,x) = 0, && (t,x) \in (0,\infty) \times \mathbb{R}^d_+, \\ &u(0,x) = f(x), && x \in \mathbb{R}^d_+, \\ &\frac{\partial u}{\partial \nu}(t,\overline{x}) = 0 \quad \text{for } \overline{x} \in (\partial\mathbb{R}^d_+)_{\eta,N},\ t>0, \\ &u(t,\overline{x}) = 0 \quad\quad \text{for } \overline{x} \in (\partial\mathbb{R}^d_+)_{\eta,D},\ t>0. \end{aligned} \tag{6.1}$$

Here f is a given (suitable) function on $\mathbb{R}^d_+$ and ν denotes the normal outward vector on $\partial\mathbb{R}^d_+$ (whenever it exists). As one can expect, the solution to this problem is given by

$$u(t,x) = \int_{\mathbb{R}^d_+} p_t^{\eta,+}(x,y) f(y)\, dy, \qquad (t,x) \in (0,\infty) \times \overline{\mathbb{R}^d_+}. \tag{6.2}$$

Note that $p_t^{\eta,+}(\overline{x},y)$, $y \in \mathbb{R}^d_+$, and thus also $u(t,\overline{x})$ are well defined for $\overline{x} \in \partial\mathbb{R}^d_+$.

A typical result that can be reached by employing classical arguments and using the aforementioned properties of the η-heat kernel reads as follows (we leave the details of the proof to the interested readers).

Proposition 6.2. *Assume that f is continuous and bounded on $\overline{\mathbb{R}^d_+}$ and let u be given by* (6.2). *Then*

(1) *u is C^∞ on $(0,\infty) \times \mathbb{R}^d_+$;*
(2) *u satisfies the equation $(\partial_t - \Delta_{\mathbb{R}^d_+})u(t,x) = 0$ on $(0,\infty) \times \mathbb{R}^d_+$;*
(3) *u satisfies the initial condition $u(0,x) = f(x)$, $x \in \mathbb{R}^d_+$, in the sense that*

$$\lim_{t\to 0^+} u(t,x) = f(x) \qquad \textit{for every } x \in \mathbb{R}^d_+;$$

(4) *u satisfies the η-boundary conditions* (6.1).

6.2. Dihedral groups. Special cases (dyadic cones) of the examples considered in this subsection were discussed in [9, Section 4.4.2].

Let D_n, $n \ge 3$, be the reflection group associated with the root system $R = \{z_j : j = 0, \dots, 2n-1\}$ in $\mathbb{R}^2 \simeq \mathbb{C}$, where $z_j = e^{i\pi j/n}$. D_n is called the *dihedral group* and geometrically it is the group of isometries of the regular n-gon centered at the origin with one of the vertices at $i \simeq (0,1) \in \mathbb{R}^2$. The group $D_n \subset O(2)$ is isomorphic to the semidirect product $\mathbb{Z}_n \rtimes \widehat{\mathbb{Z}}_2$ and $|D_n| = 2n$. In this isomorphism $(1,1)$ corresponds to r, the counterclockwise rotation by $2\pi/n$ around the origin, while $(0,-1)$ corresponds to σ, reflection in the line orthogonal to z_0. Moreover,

$$D_n = \{1, r, \dots, r^{n-1}, \sigma, r\sigma, \dots, r^{n-1}\sigma\},$$

and $1, r, \dots, r^{n-1}$ are all rotations in D_n, while $\sigma, r\sigma, \dots, r^{n-1}\sigma$ are all reflections in D_n.

The structure of normal subgroups of index 2 in D_n is known. Namely, $\mathcal{N}_0 = \langle r \rangle$ is the unique such normal subgroup for n odd, while for n even, apart from $\mathcal{N}_0$, there are two more normal subgroups, $\mathcal{N}_1 = \langle r^2, \sigma \rangle$ and $\mathcal{N}_2 = \langle r^2, r\sigma \rangle$. Clearly, if $\mathcal{N}$ is any normal subgroup of index 2 in D_n, then

$$\eta_{\mathcal{N}}(g) = \begin{cases} 1 & \text{for } g \in \mathcal{N}, \\ -1 & \text{for } g \notin \mathcal{N}, \end{cases}$$

is a homomorphism of D_n onto $\widehat{\mathbb{Z}}_2$ with $\ker(\eta_{\mathcal{N}}) = \mathcal{N}$, and every non-trivial homomorphism of D_n into $\widehat{\mathbb{Z}}_2$ is of this form. In this notation, $\eta_{\mathcal{N}_0} = \operatorname{sgn}$. Therefore, the complete list of homomorphisms of D_n into $\widehat{\mathbb{Z}}_2$ is as follows: for n odd the only such homomorphisms are $\mathbf{1}$ and sgn, while for n even there are two additional ones, $\eta_{\mathcal{N}_1}$ and $\eta_{\mathcal{N}_2}$.

We shall write down the explicit form of the Neumann and Dirichlet heat kernels, as well as the two exceptional η-heat kernels, for the open cone on the plane with vertex at the origin and aperture π/n. This cone,

$$C_+^{(n)} = \begin{cases} \{\rho e^{i\theta} \colon \rho > 0,\, 0 < \theta < \pi/n\}, & n \text{ even}, \\ \{\rho e^{i\theta} \colon \rho > 0,\, |\theta| < \pi/2n\}, & n \text{ odd}, \end{cases}$$

is the positive Weyl chamber for the following selection of positive roots: if $n = 2k$, $k \geq 2$, then $R_+ = \{z_{3k+1}, z_{3k+2}, \dots, z_{4k-1}, z_0, z_1, \dots, z_k\}$ with $\Sigma = \{z_{3k+1}, z_k\}$ as the set of simple roots; if $n = 2k+1$, $k \geq 1$, then $R_+ = \{z_{3k+2}, \dots, z_{4k+1}, z_0, \dots, z_k\}$ with $\Sigma = \{z_{3k+2}, z_k\}$ as the simple roots.

For brevity we now write p_t for the two-dimensional Gauss–Weierstrass kernel $p_t^{(2)}$.

The Neumann heat kernel on $C_+^{(n)}$, which is the η-heat kernel for $\eta \equiv 1$, is given by

$$p_t^{N,\pi/n}(x,y) = \sum_{g \in D_n} p_t(gx - y) = \sum_{m=0}^{n-1} \big(p_t(r^m x - y) + p_t(r^m \sigma x - y)\big);$$

here $x, y \in C_+^{(n)}$. The Dirichlet heat kernel on $C_+^{(n)}$, which is the η-heat kernel for $\eta = \operatorname{sgn}$ (and corresponds to $\ker \eta = \mathcal{N}_0$), is for $x, y \in C_+^{(n)}$ given by

$$p_t^{D,\pi/n}(x,y) = \sum_{g \in D_n} \det(g) p_t(gx - y) = \sum_{m=0}^{n-1} \big(p_t(r^m x - y) - p_t(r^m \sigma x - y)\big).$$

It is worth mentioning here that the formula for $p_t^{D,\pi/n}(x,y)$ in terms of a convergent series is well known (in fact any aperture is admitted) and goes back to an old formula of Carslaw and Jaeger. See [3], where this formula was generalized to higher dimensions (and [9], where the analogous formula is stated in the Neumann case). The formula we have in mind, for n even reads: in polar coordinates, for $x=\rho e^{i\theta}\in C_+^{(n)}$, $y=re^{i\xi}\in C_+^{(n)}$,

$$p_t^{D,\pi/n}(x,y)=\frac{n}{2\pi t}\exp\Big(-\frac{\rho^2+r^2}{4t}\Big)\sum_{j=1}^{\infty}I_{jn}\Big(\frac{\rho r}{2t}\Big)2\sin(jn\theta)\sin(jn\xi).$$

Here $I_\nu(z)$ denotes the modified Bessel function of order ν (in the Neumann case sines are replaced by cosines). For n odd, θ and ξ have to be replaced by $\theta+\pi/(2n)$ and $\xi+\pi/(2n)$; for such n, $C_+^{(n)}$ rotated counterclockwise by $\pi/(2n)$ takes the form of $C_+^{(n)}$ for n even.

In the exceptional cases, when $n=2k$, $k\geq 2$, we have

$$\mathcal{N}_1=\{1,r^2,\dots,r^{2(k-1)},\sigma,r^2\sigma,\dots,r^{2(k-1)}\sigma\},$$
$$\mathcal{N}_2=\{1,r^2,\dots,r^{2(k-1)},r\sigma,r^3\sigma,\dots,r^{2k-1}\sigma\}.$$

Therefore the corresponding η_j-heat kernels, $j=1,2$, $x,y\in C_+^{(n)}$, are

$$p_t^{\eta_1,\pi/n}(x,y)=\sum_{g\in\mathcal{N}_1}\big(p_t(gx-y)-p_t(rgx-y)\big)$$
$$=\sum_{m=0}^{k-1}\big(p_t(r^{2m}x-y)-p_t(r^{2m+1}x-y)+p_t(r^{2m}\sigma x-y)-p_t(r^{2m+1}\sigma x-y)\big)$$

and

$$p_t^{\eta_2,\pi/n}(x,y)=\sum_{g\in\mathcal{N}_2}\big(p_t(gx-y)-p_t(g\sigma x-y)\big)$$
$$=\sum_{m=0}^{k-1}\big(p_t(r^{2m}x-y)-p_t(r^{2m+1}x-y)+p_t(r^{2m+1}\sigma x-y)-p_t(r^{2m}\sigma x-y)\big).$$

Acknowledgements. The author thanks Jacek Małecki for discussions.

This research was supported by funds of Faculty of Pure and Applied Mathematics, Wrocław University of Science and Technology, #0401/0155/18.

References

[1] R. A. Adams and J. J. F. Fournier, *Sobolev Spaces*, 2nd ed., Academic Press, 2003.
[2] H. W. Alt, *Linear Functional Analysis*, Springer, 2016.
[3] R. Bañuelos and R. G. Smits, *Brownian motion in cones*, Probab. Theory Related Fields 108 (1997), 299–319.
[4] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Universitext, Springer, 2011.

[5] E. B. Davies, *Heat Kernels and Spectral Theory*, Cambridge Tracts in Math. 92, Cambridge Univ. Press, 1989.
[6] C. Dunkl and Y. Xu, *Orthogonal Polynomials of Several Variables*, Encyclopedia Math. Appl. 81, Cambridge Univ. Press, 2001.
[7] G. Heckman, *Coxeter groups* (*unfinished notes*), www.math.ru.nl/heckman/∼CoxeterGroups.pdf, 2018.
[8] J. E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge Univ. Press, 1990.
[9] J. Małecki and K. Stempak, *Reflection principles for functions of Neumann and Dirichlet Laplacians on open reflection invariant subsets of* $\mathbb{R}^d$, Studia Math. 251 (2020), 171–193.
[10] M. Reed and B. Simon, *Methods of Modern Mathematical Physics IV. Analysis of Operators*, Academic Press, New York, 1978.
[11] K. Schmüdgen, *Unbounded Self-adjoint Operators on Hilbert Space*, Grad. Texts in Math. 265, Springer, 2012.

Krzysztof Stempak
Wydział Matematyki
Politechnika Wrocławska
Wybrzeże Wyspiańskiego 27
50-370 Wrocław, Poland
E-mail: Krzysztof.Stempak@pwr.edu.pl